Indistinguishable Macroscopic Behaviour of Palatini Gravities and General Relativity

Baojiu Li[1], David F. Mota[2] and Douglas J. Shaw[3]

[1]DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

[2]Institut für Theoretische Physik, Universität Heidelberg, D-69120 Heidelberg, Germany

[3]School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, UK

Abstract. We show that, within some modified gravity theories, such as the Palatini models, the non-linear nature of the field equations implies that the usual naïve averaging procedure (replacing the microscopic energy-momentum by its cosmological average) could be invalid. As a consequence, the relative motion of particles in Palatini theories is actually indistinguishable from that predicted by General Relativity. Moreover, there is no WEP violation. Our new and most important result is that the cosmology and astrophysics, or put more generally, the behaviours on macroscopic scales, predicted by these two theories are the same, and as a result the naturalness problems associated with the cosmological constant are not alleviated. Palatini gravity does however predict alterations to the internal structure of particles and the particle physics laws, *e.g.*, corrections to the hydrogen energy levels. Measurements of which place strong constraints on the properties of viable Palatini gravities.

PACS numbers: 04.50.+h

Extensions of General Relativity (GR) have always received a great deal of attention. Such theories are motivated by quantum gravity models and by the wish to find phenomenological alternatives to the standard paradigm of dark matter and dark energy [1].

Modifications to Einstein gravity generally result in non-linear (in energy momentum tensor $T_{\mu\nu}$) corrections to the field equations. The application of these equations equations to macroscopic (*e.g.* cosmological) scales involves an implicit coarse-graining over the microscopic structure of particles. However, when there are extra non-linear terms in the field equations, *a priori*, the validity of the usual coarse-graining procedure can no longer be taken for granted, as first proposed by [2]. Hence, as discussed in [2], it may be important to take into account the microscopic structure of matter when applying the field equations to macroscopic scales.

Unfortunately, to date, that has not been the common practice [3]. This is probably because, in GR, as in Newtonian gravity, the microscopic structure of matter is not particularly important on macroscopic scales. It is standard practice to replace the metric, $g_{\mu\nu}$, and the energy momentum tensor, $T_{\mu\nu}$, with some average of them that coarse-grains over the microscopic structure. This simple procedure only works, however, because on microscopic scales the equations of GR are approximately linear.

In this *Letter* we show that such an approach *cannot* simply be applied to modified gravity without a detailed first analysis of the energy-momentum microstructure. Indeed, naïvely averaging over the microscopic structure will generally lead one to make incorrect predictions, and inaccurate conclusions as to the validity of the theory [2]. Indeed, it is possible that a theory deviates *significantly* from GR at the level of the microscopic field equations to be *indistinguishable* from the latter when correctly coarse grained over macroscopic, *e.g.*, cosmological, scales.

We illustrate this point for a class of modified gravity theories in which the Ricci scalar, R, in the Einstein-Hilbert action is replaced by some function $f(R, R^{\mu\nu}R_{\mu\nu})$, *i.e.*:

$$\frac{1}{2\kappa}\int \mathrm{d}^4\,x\,\sqrt{-\bar{g}}R \rightarrow \frac{1}{2\kappa}\int \mathrm{d}^4\,x\,\sqrt{-\bar{g}}f(R, R^{\mu\nu}R_{\mu\nu}).$$

The field equations for this action can be derived according to two *inequivalent* variational approaches: metric and Palatini. In the former, $R_{\mu\nu}$ and R are taken to be constructed from the matter metric $\bar{g}_{\mu\nu}$, which couples to matter and governs the conservation of energy momentum tensor, and the field equations are found by minimizing the action with respect to variations in $\bar{g}_{\mu\nu}$. In the alternative, Palatini approach, $R = R_{\mu\nu}\bar{g}^{\mu\nu}$ where $R_{\mu\nu}$ is a function of some connection field $\Gamma^{\mu}_{\nu\rho}$ which is, *a priori*, treated as being independent of $\bar{g}_{\mu\nu}$. The field equations are then found by minimizing the action with respect to both $\Gamma^{\mu}_{\nu\rho}$ and $\bar{g}_{\mu\nu}$. If $f(R, R^{\mu\nu}R_{\mu\nu}) = R - 2\Lambda$ (*i.e.* GR with a cosmological constant) then the two approaches result in the same field equations. Otherwise they are generally different. Note that the Palatini $f(R)$ field equations are mathematically equivalent to an $\omega = -\frac{3}{2}$ Brans-Dicke theory with a potential, and thus represents more general modified gravity theories.

Within the metric approach to the $f(R)$ gravity theories, averaging over *microscopic* scales is generally no less straightforward than it is in GR; this is because in both cases all degrees of freedom are dynamical. These dynamics normally ensure that the field equations, for all degrees of freedom, are approximately linear (in energy momentum tensor $T_{\mu\nu}$) on small-scale structures [4]. In contrast, averaging in Palatini models is not so trivial as the new degree of freedom is *non-dynamical*, and so its field equation remains non-linear (in energy momentum tensor $T_{\mu\nu}$) even on the smallest scales. Consequently the cosmological behaviour of these theories can be very different from what has been suggested in the literature [3, 5, 6], which adopt the same averaging procedure as in the metric approach. Also, although GR plus a cosmological constant is a special case of Palatini $f(R)$ (and also of metric $f(R)$!) theories, the averaging problem for general Palatini theories dose not arise there because the GR field equations are linear (and algebraic) in curvature R and energy momentum tensor components, while in contrast for general Palatini theories the curvature depends nonlinearly (but also algebraically) on energy momentum tensor. We emphasize again that it is *this* "nonlinearity" that makes the averaging of general Palatini theories not as trivial as that in GR.

Palatini $f(R^{\mu\nu}R_{\mu\nu})$ theories are similar to the $f(R)$ ones in many aspects, but their study is more complicated. In what follows we shall mainly focus on the latter and state the results of the former when appropriate, referring more details to [4]. To avoid confusion later, we now take $R \to \mathcal{R}$, where $\mathcal{R} = R_{\mu\nu}(\Gamma)\bar{g}^{\mu\nu}$.

Varying the $f(\mathcal{R})$ action with respect to $\Gamma^{\mu}_{\nu\rho}$ gives that $\Gamma^{\mu}_{\nu\rho}$ is the Levi-Cività connection of $g_{\mu\nu} = f'(\mathcal{R})\bar{g}_{\mu\nu}$. Varying the action with respect to $\bar{g}_{\mu\nu}$ gives:

$$G^{\mu}{}_{\nu}(g) = \frac{1}{f'^{2}(\Phi)}\left[\kappa\mathcal{T}^{\mu}{}_{\nu} - V(\Phi)\delta^{\mu}{}_{\nu}\right], \tag{1}$$

$$\kappa\mathcal{T}^{\mu}{}_{\mu} \;\; \equiv \kappa\mathcal{T} \;\; = \;\; f'(\Phi)\Phi - 2f(\Phi), \tag{2}$$

where we have defined $\Phi \equiv \mathcal{R}$, $R \equiv R_{\mu\nu}g^{\mu\nu} = \Phi/f'(\Phi)$, $G_{\mu\nu}(g) = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and:

$$V(\Phi) = \frac{1}{2}\left(f'(\Phi)\Phi - f(\Phi)\right), \qquad \mathcal{T}^{\mu}{}_{\nu} = -\frac{2\bar{g}^{\mu\rho}}{\sqrt{-\bar{g}}}\frac{\delta S_{\mathrm{matter}}}{\delta\bar{g}^{\rho\nu}}.$$

Also $\nabla^{(\bar{g})}_{\mu}\mathcal{T}^{\mu}{}_{\nu} = 0$, where $\nabla^{(\bar{g})}_{\mu}\bar{g}_{\nu\rho} = 0$ defines $\nabla^{(\bar{g})}_{\mu}$. In $f(R^{\mu\nu}R_{\mu\nu})$ gravity the field equation is similar to Eq. (1), but the metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are *disformal*, and V depends *algebraically* on $\mathcal{T}_{\mu\nu}$ rather than simply on $\mathcal{T}$ [4].

It is important to bear in mind that Eqs. (1) and (2) are *microscopic* field equations, and they are only definitely valid when all the microscopic structure in the distribution of energy and momentum is taken into account. Nonetheless, the cosmological and astrophysical behaviours of these theories have, to date, been studied by simply replacing $g_{\mu\nu}$, Φ and $\mathcal{T}^{\mu}{}_{\nu}$ in Eqs. (1, 2) by some coarse-grained averages of them [5, 3, 6]. However, in Palatini $f(\mathcal{R})$ theories, a deviation from GR requires that f depends *nonlinearly* on Φ. This nonlinearity introduces an averaging problem for the Palatini theories. As we shall show, this means that the standard averaging procedure is no longer valid and generally results in incorrect physical and mathematical predictions. Furthermore, when the microscopic structure of matter is taken into account, the late-time cosmology of essentially *all* $f(\mathcal{R})$ Palatini theories is indistinguishable from that of standard GR with a cosmological constant. We also show that these these theories do *not* violate the weak equivalence principle (WEP). Palatini $f(R)$ gravity *is* however a different theory from GR and does produce alterations to microscopic particle physics that could be detected.

The microscopic structure of the space-time distribution of matter energy density, ρ, will not affect the macroscopic, or coarse-grained, behaviour of the theory if and only if the field equations of the theory are *linear in* ρ. Here is a simple example: consider a region of space with average density $\langle\rho\rangle$ and volume $\mathcal{V}$, which contains N particles each with density ρ_c and volume $\mathcal{V}_p$. The space in between the particles is empty and so $\langle\rho\rangle\,\mathcal{V} = N\rho_c\mathcal{V}_p$. Now consider some quantity $Q(\rho)$. Inside the particles,

$\rho = \rho_c$ and so $Q = Q_c \equiv Q(\rho_c)$; outside $\rho = 0$ and so $Q = Q_0 \equiv Q(0)$. The average value of Q (by volume) is:

$$\langle Q \rangle = Q_0 \left(1 - \frac{N\mathcal{V}_p}{\mathcal{V}}\right) + Q_c \frac{N\mathcal{V}_p}{\mathcal{V}} = Q_0 + \frac{Q_c - Q_0}{\rho_c} \langle \rho \rangle \tag{3}$$

It is clear that, irrespective of how Q depends on ρ, $\langle Q \rangle$ depends on $\langle \rho \rangle$ linearly. Thus if Q depends nonlinearly on ρ, we have $\langle Q(\rho) \rangle \neq Q(\langle \rho \rangle)$. In Palatini theories, $V(\Phi)$ generally exhibits highly non-linear dependence on ρ, and so $\langle V(\Phi(\rho)) \rangle \neq V(\Phi(\langle \rho \rangle))$.

It is also interesting to know when naïvely coarse-grained equations are valid to a good approximation over the length scales and density scales where we have observations. This only occurs in Palatini theories when the inherently nonlinear modifications to GR have sub-leading effect. In all these cases the theory will reduce to GR up to the order where the naïvely coarse-grained equations are valid. In this work we are concerned with the how Palatini theories deviate from GR, even when those deviations are sub-leading order. We must then always take account of those nonlinear terms in the equations which determine how Palatini theories deviate from GR. The crucial rôle played by these nonlinear terms means that the naïve coarse-graining procedure will always fail to accurately describe the differences between Palatini theories and GR.

To uncover how Palatini gravities behave on macroscopic scales, we consider how a set of microscopic particles evolve under such a modified gravity. Consider a single, spherical particle for which $\mathcal{T}^{\mu}_{\nu} \neq 0$ for $R < R_p$ but vanishes otherwise. The metric for such a particle is [5]:

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -W(r)e^{2\chi(r)}\mathrm{d}t^2 + \frac{1}{W(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2,$$

with

$$W(r) = 1 - \frac{2GM(r)}{r} - \frac{V(\Phi_0)}{3f'^2(\Phi_0)}r^2, \tag{4}$$

$$2GM(r) = \kappa \int_0^r \mathrm{d}r'\, r'^2 \left[\frac{\rho}{f'^2(\Phi)} + \frac{\Delta V(\Phi)}{\kappa f'^2(\Phi)}\right], \tag{5}$$

$$\chi = \frac{\kappa}{2} \int_{R_p}^r \mathrm{d}r'\, r' \frac{(\rho + p_I + p_A)}{W(r')f'^2(\Phi)}, \tag{6}$$

where

$$\Delta V \equiv V(\Phi) - f'^2(\Phi)V(\Phi_0)/f'^2(\Phi_0),$$
$$\mathcal{T}^{\mu}{}_{\nu} = (\rho + p_I)\, u^{\mu}u^{\rho}\bar{g}_{\rho\nu} + p_I\delta^{\mu}{}_{\nu} + \pi^{\mu}{}_{\nu},$$

$u^{\mu}u^{\nu}\bar{g}_{\mu\nu} = -1$ and $\pi^{\mu}{}_{\nu}$ is the anisotropic stress satisfying $\pi^{\mu}{}_{\nu}u^{\nu} = 0$. In the rest frame of the particle

$$\pi^{\mu}{}_{\nu} = \mathrm{diag}(0, p_A, -p_A/2, -p_A/2), \qquad \Phi_0 \equiv \Phi(\mathcal{T} = 0).$$

The presentation of this solution in Ref. [5] was, however, incomplete as it failed to note that the $\theta\theta$ component of Eq. (1) results in the condition:

$$r\frac{\mathrm{d}}{\mathrm{d}r}P_{\mathrm{eff}} + \frac{\rho + p_I + p_A}{f'^2(\Phi)rW(r)}Y(r) = -3\frac{p_A}{f'^2(\Phi)} \tag{7}$$

where

$$Y(r) = \left(4\pi G P_{\mathrm{eff}} r^3 + GM - \frac{\Lambda_{\mathrm{eff}} r^3}{3}\right),$$

$\Lambda_{\rm eff} = V(\Phi_0)/f'^{\,2}(\Phi_0)$ and

$$P_{\rm eff} = (p_I + p_A - \Delta V/\kappa)/f'^{\,2}.$$

Eq. (7) implies that at $R = R_p$ one has $P_{\rm eff} = 0$.

Outside the particle, $\chi = 0$ and

$$W(r) = 1 - GM_p/r - \Lambda_{\rm eff} r^2/3$$

where $M_p = M(R_p) = $ const and so the metric is precisely that of a Schwarzschild de-Sitter spacetime with gravitational mass M_p and an effective cosmological constant $\Lambda_{\rm eff}$. A similar Schwarzschild de-Sitter solution is found for the Palatini $f(R^{\mu\nu}R_{\mu\nu})$ case [4], with V replaced by a more complicated function of $\mathcal{T}^{\mu}{}_{\nu}$. Thus we see that in Palatini models the external metric of a single particle is precisely what it would be in General Relativity with a cosmological constant.

We now consider a spacetime containing many such particles. We define $\mathbf{x}_{(I)}(t)$ to be the position of a particle I and $v^2_{(I)} = \dot{x}^2_{(I)}$. Making a weak field approximation with respect to $g_{\mu\nu}$ and using the above solution, we find that inside the particle labeled K and to $\mathcal{O}(\epsilon^2)$, where $\epsilon \sim \max(|v_{(K)}|, \sqrt{(1-W(r))})$:

$$\begin{aligned} g_{\mu\nu}{\rm d}x^{\mu}{\rm d}x^{\nu} =& -\left(W(r) + 2\chi(r) - 2U(x)\right){\rm d}t^2 + \left(1 + 2U(x)\right){\rm d}x^k{\rm d}x^k \\ &+ \frac{(1-W(r))}{r^2}\Delta x^i_{(K)}\Delta x^j_{(K)}{\rm d}x^i{\rm d}x^j, \\ U(x) =& \sum_{I\neq K}\frac{Gm_{A\,(I)}}{|x^i - x^i_{(I)}(t)|}, \end{aligned}$$

$\Delta x^i_{(K)} = x^i - x^i_{(K)}(t)$ and $r = |\Delta x^i_{(K)}|$; $m_{A\,(I)}$ is the active gravitational mass of each particle given by $M(R_p)$.

Let $u^{\mu}_{(K)}$ be the 4-velocity of the $K^{\rm th}$ particle which satisfies $u^{\mu}_{(K)}u^{\nu}_{(K)}\bar{g}_{\mu\nu} = -1$. To the order $\mathcal{O}(\epsilon)$,

$$u^{\mu}_{(K)} = f'^{\,1/2}(\Phi)\left(1, {\rm d}x^i/{\rm d}t\right).$$

Now $\mathcal{T}^{\mu}{}_{\nu}$ is conserved with respect to $\bar{g}_{\mu\nu}$ *i.e.* $\nabla^{(\bar{g})}_{\mu}\mathcal{T}^{\mu}{}_{\nu} = 0$. Evaluating this equation at the centre of the particle we have $u^{\mu}_{K}\nabla^{(\bar{g})}_{\mu}u^{\nu}_{K} = 0$, *i.e.*

$$\frac{{\rm d}^2x^i_{(K)}}{{\rm d}\tau^2} = \frac{1}{2}\bar{g}^{ij}\bar{g}_{00,j}u^0u^0 = -f'(\Phi)U_{,i}. \tag{8}$$

where τ is the proper time along the worldline of the particle, and so $\partial t/\partial\tau = f'^{\,1/2}(\Phi)$ (because the metric $-d\tau^2 = \frac{1}{f'^2}(-dt^2 + d\mathbf{x}^2)$ means $1 = \frac{1}{f'^2}\left[(dt/d\tau)^2 + (d\mathbf{x}/d\tau)^2\right] \approx \frac{1}{f'^2}(dt/d\tau)^2$).

The internal configuration of the particle can be static or non-static, and here for simplicity we assume it to be static and will comment on the non-static case below. This requires that all gradients in Φ cancel with gradients in the pressure and also that $\mathcal{T}$ and hence Φ are conserved along particle worldlines, *i.e.*, $u^{\mu}\partial_{\mu}\mathcal{T} = u^{\mu}\partial_{\mu}\Phi = 0$. Equation (8) is then equivalent to:

$$a^i_{(K)} \equiv \frac{{\rm d}^2x^i_{(K)}}{{\rm d}t^2} = -\left.U_{,i}\right|_{\mathbf{x}=\mathbf{x}_{(K)}(t)}. \tag{9}$$

The acceleration with respect to τ depends both on the gravitational field, $U_{,i}$, and $f'(\Phi)$. The acceleration measured with respect to t, however, depends only on $U_{,i}$.

The relative acceleration of two particles labeled 1 and 2 as measured by a third, labeled 3 say, with proper time $\tau_{(3)}$ is therefore:

$$\Delta a^i_{12} = \frac{\mathrm{d}^2(x^i_{(1)} - x^i_{(2)})}{\mathrm{d}^2\tau_{(3)}} = f'(\Phi_{(3)}) \left(a^i_{(1)} - a^i_{(2)}\right),$$

where $\partial t/\partial\tau_{(3)} = f'^{\,1/2}(\Phi_{(3)})$. In a uniform gravitational field: $U_{,i} = \mathrm{const}$ and so

$$a^i_{(1)} = a^i_{(2)} \Rightarrow \Delta a^i_{12} = 0.$$

It follows that an observer (*i.e.* particle 3) sees any two other particles, 1 and 2, accelerate at the same rate in a uniform gravitational field. This is precisely what is required by the Weak Equivalence Principle.

Since the internal configuration of the particles is static and the centres of two particles do not have relative acceleration, there will be no relative acceleration between the two whole particles. In Ref. [5] it was suggested that internal gradients in Φ would lead to WEP violations. In fact, because those gradients all vanish outside the particle, they cannot affect the overall motion of the particle [8] (this is related to the fact that the extra force, which arises due to the modification to GR, has zero range; as we shall discuss in more detail below). In this case, hydrostatic equilibrium, Eq. (7), ensures that gradients in Φ are cancelled by pressure gradients.

The absence of WEP violation in the dynamics of particles ensures that the inertial and passive gravitational mass of particles are equal. Moreover, the inertial mass and the active gravitational mass of particles are also equal in these theories. Let us define

$$\kappa T^{\mu}_{\mathrm{eff}\,\nu} = G^{\mu}{}_{\nu}(g) + \Lambda_{\mathrm{eff}}\delta^{\mu}{}_{\nu}. \tag{10}$$

By the modified Einstein equation, Eq. (1), and the contracted Bianchi identity: $\nabla^{(g)}_{\mu} T^{\mu}{}_{\mathrm{eff}\,\nu} = 0$ where $\nabla^{(g)}_{\mu} g_{\nu\rho} = 0$. Outside an isolated particle $T^{\mu}{}_{\mathrm{eff}\,\nu} = 0$ and, as we have seen above, the metric $g_{\mu\nu}$ outside this quasi-static (*i.e.* $v_I^2 \ll 1$) isolated particle is Schwarzschild-de-Sitter. In Ref. [9], Tolman shows that, under these conditions, the inertial and active gravitational mass of an isolated (not necessarily spherically symmetric) system are equal (see also [8]). For our particles, the inertial, passive and active gravitational mass are therefore equal to m_p where by Eq. (5):

$$m_p = 4\pi \int_0^{R_p} r^2 \mathrm{d}r \left[\frac{\kappa\rho + \Delta V}{\kappa f'^2(\Phi)}\right]. \tag{11}$$

This equivalence holds for particles both in GR and in Palatini $f(\mathcal{R}, R_{\mu\nu}R^{\mu\nu})$ theories [4]. It follows that the motion of isolated particles (situated in a vacuum) in Palatini theories is exactly the same as it is in GR with $\Lambda = \Lambda_{\mathrm{eff}}$. This has a very simple physical explanation. In all modified gravity theories one may think of the modifications as being due to some new, effective force. In Palatini theories, there is no extra dynamical degree of freedom, so this 'new' force is also non-dynamical, *i.e.*, does not propagate and only acts at points. The closest analogue to this in particle physics would be Fermi's original proposal for a theory of the weak interaction. The effective new force is entirely *local* and depends on gradients in the $\mathcal{T}$ or $\mathcal{T}^{\mu}_{\nu}$. There is therefore no way for one isolated particle to influence the motion of another, and the force simply vanishes in a vacuum. The effective force will alter the internal configuration of the particle nonetheless, but this will not effect the motion of the particle as a whole.

Note that the above discussion relies on the specific model for the matter distribution in a system. In most systems it is realistic to take the matter as to be distributed in localized clumps (what we refer to as particles) which only interact with each other gravitationally (or through extra forces due to the modification of gravity, which as we said above is of zero range and thus completely irrelevant for classical particles). For the extreme environments like a neutron star, however, the energy density is of order nuclear density, and there is essentially no space between different particles so that the above matter model breaks down: in this case we expect the naïve averaging procedure to be applicable. For cosmology and most other astrophysical systems our averaging is nonetheless more realistic. We emphasize that the absence of WEP violation does not require the internal configuration of the particles is static (as we have assumed). Rather, it is a result of the fact that the extra force has zero range so that there is no composition-dependent extra force between different particles which do not overlap with each other. Additionally Palatini $f(R)$ theories are equivalent to a generalized Brans-Dicke, and, in common with other Brans-Dicke theories, the extra force has no composition dependence.

We now apply our results to a cosmological setting. At late times and on a microscopic level, most matter in the Universe is made up of small particles and a bit of radiation. In $f(\mathcal{R})$ theories, $\mathcal{T} = 0$ for radiation and so the presence of radiation does not alter the relation between Φ and $\mathcal{T}$. Our analysis is therefore directly applicable to this setting. The Universe must therefore evolve precisely as it would in GR with a cosmological constant $\Lambda_{\rm eff}$. This can be seen in an alternative manner: averaging Eq. (10) over volume $\mathcal{V}$ containing N particles with mass m_p and using Eq. (3) we find

$$\kappa\rho^{\rm eff}_{\rm matter} = -\left\langle \kappa T^{0}_{{\rm eff}\,0}\right\rangle = \kappa m_p \frac{N}{\mathcal{V}}.$$

Using Eq. (7) we find that in the rest frame of the particle $\left\langle \kappa T^{i}_{{\rm eff}\,j}\right\rangle = 0$. More generally then $\left\langle \kappa T^{i}_{{\rm eff}\,j}\right\rangle \sim \mathcal{O}(\kappa\rho^{\rm eff}_{\rm matter}\delta v^i \delta v_j)$ where δv^i is the relative particle velocity. Thus, when correctly coarse-grained over cosmological scales, $T^{\mu}_{{\rm eff}\,\nu}$ describes a collisionless dust; we have assumed that the peculiar velocities, δv, of the particles are small and dropped terms of $\mathcal{O}(\delta v^2)$. The cosmological evolution of the Universe in such theories is therefore precisely the same as it is in GR with a cosmological constant $\Lambda_{\rm eff}$ and a dust with energy density $\rho^{\rm eff}_{\rm matter}$. This argument also holds for most astrophysical systems. In particular, solar system tests are evaded and the Parametrized Post-Newtonian parameters are indistinguishable from those of GR.

Many of the predictions made in the literature (*e.g.* [3]) do not, therefore, follow from a Palatini $f(\mathcal{R})$ model, but may still be correct for some other modified gravity theory. Any such theory would, however, likely be subject to additional constraints from local tests of gravity.

Despite of their similar behaviours, Palatini $f(\mathcal{R})$ gravities and GR are not equivalent. Even though motions of isolated bodies are the same in both theories, the internal structure and dynamics of them are generally different [2]. This is because, for the body to be stable under gravity one must require that Eq. (7) holds, and the appearance of ΔV in this equation clearly indicates an alteration to GR. As we mentioned above, the pressure gradients inside the particle must be modified with respect to those in GR in order to exactly cancel the gradients in Φ, and so we expect the distributions of matter inside this particle, that is, the internal structures, according to these two theories to be different.

As another example, the electron energy levels in atoms are altered in Palatini $f(\mathcal{R})$ theories (see also [11] for a recent work). Such alterations occur, because in these theories the effective electron mass depends on the electron density, and so is different for different energy levels. In the absence of any Palatini modification, we define the energy and number density of an electron with total angular momentum j, in energy level n, to be: $\bar{E}_{nj} = -\alpha^2 m_e \mathcal{E}_{nj}$ and $\bar{n}_e^{(nj)}$ respectively where α is the fine structure constant, and m_e the electron mass. In Palatini $f(\mathcal{R})$ theories, the electron mass depends on Φ and hence on the electron density which is different for different values of n and j. We assume that $f(\Phi) \approx b\Phi(1+\varepsilon(\Phi))$ so that the density dependence of the electron mass is slight. The total energy of an electron (up to an overall constant) is then [4]:

$$E_{nj} = \alpha^2 m_{nj}^{\rm eff} \mathcal{E}_{nj}$$

where

$$m_{nj}^{\rm eff} = m_e(1 + \Delta_{nj}/2\alpha^2 \mathcal{E}_{nj})$$

for some m_e and to leading order in α^2:

$$\Delta_{nj} = -\left\langle \varepsilon(\Phi) \right\rangle_{nj} \equiv -\int \mathrm{d}^3 x\, \bar{n}_e^{(nj)}(x) \varepsilon(\bar{\Phi}_{nj}(\mathbf{x})) \qquad (12)$$

where $\bar{\Phi}_{nj}(\mathbf{x})/b = \kappa m_e n_e^{(nj)}(\mathbf{x})$. Thus, in Palatini theories determinations of the electron mass from transitions between different energy levels using the standard formula for E_{nl} would find different answers for each transition unless $\Delta_{nl} = $ const; this possibility is however very strongly constrained. So far both ε and m_e have only been defined up to an overall constant *i.e.* $\varepsilon \to \varepsilon + \delta_0$, $m_e \to m_e(1+\delta_0)$ for some constant $\delta_0 \ll 1$ is allowed. We fix the definition of m_e, and hence also ε, so m_e is the effective electron mass for the ground state *i.e.* $\left\langle \varepsilon \right\rangle_{10} = 0$. Using measurements of the electron mass from the transitions 1S-2S and 2S-8D [12] we have

$$\left| \left\langle \varepsilon(\Phi) \right\rangle_{20} - \frac{16}{21} \left\langle \varepsilon(\Phi) \right\rangle_{83} \right| < 8 \times 10^{-16}. \qquad (13)$$

Respectively for (10), (20) and (83) states the values of $\rho_e = m_e n_e$ near the expected electron radius are: $3 \times 10^{-5}\,\mathrm{g\,cm^{-3}}$, $10^{-6}\,\mathrm{g\,cm^{-3}}$ and $10^{-8}\,\mathrm{g\,cm^{-3}}$. If, for instance, $\varepsilon(\Phi) \approx \mathrm{const} + \epsilon_0 \Phi / b H_0^2$, where H_0^2 is the value of the cosmological constant today, Eq. (13) gives the very strong constraint:

$$|\epsilon_0| \approx |f''(\Phi) H_0^2 / f'(\Phi)| \lesssim 4 \times 10^{-40}$$

In summary, much of our intuition about how the microscopic behaviour of gravity affects physics on large scales is based upon Einstein's general relativity. In this *Letter* we show that such an intuition *cannot* simply be generalized to modified gravity theories without a detailed analysis of the energy-momentum microstructure. Indeed, naïvely averaging over the microscopic structure will generally lead to incorrect predictions, and inaccurate conclusions as to the validity of the theory. In particular, the naïve averaging procedure is *invalid* in Palatini theories. A correct averaging procedure shows that the cosmology of Palatini $f(R)$ models is identical to that of GR and fine tuning problems associated with the cosmological constant are neither alleviated nor, it should be said, worsened. Furthermore, the relative motion of particles in Palatini theories is indistinguishable from that predicted by GR. Interestingly, although Palatini $f(R)$ theories were designed to modify gravity on large scales, they actually modify physics on the smallest scales (*e.g.* the energy levels of electrons) leaving the larger scales practically unaltered. In general, before considering any astrophysical consequences of a modified gravity theory, it is important then to check that it does not make unrealistic predictions for atomic physics.

One may wonder whether similar problems arise in the metric $f(R)$ gravity theories. In this paper, we have been concerned with averaging over very small scales (e.g. atomic scales). In the metric $f(R)$ theories, the extra scalar degree of freedom is *dynamical* and so it can and does propagate. This means that its dependence on the distribution of matter is not so rigid as in Palatini theories. Over the very small scales of particles, although the energy density of matter might change rapidly (from nuclear density inside the particles to zero outside), the extra scalar degree of freedom Φ is not required to change so abruptly. In metric $f(R)$ theories, the field equation for Φ has the form: $\nabla^2\Phi + V'(\Phi) \propto (\rho - 3p)$, where p is the pressure of matter and ρ its energy density. Over very small scales, the kinetic term ($\nabla^2\Phi$) dominates over the potential term (V') and so reduces to $\nabla^2\Phi \propto (\rho - 3p)$. This means that over small scales the leading order deviation from GR, determined by Φ, obeys a linear second order differential equation. In Palatini theories, Φ is related algebraically to $\rho - 3p$. If Φ depends *algebraically* and *linearly* on $\rho - 3p$, we reduce to GR. It follows that in Palatini theories the leading order deviation from GR is necessarily determined by a non-linear algebraic relation to $\rho - 3p$.

One may think of metric $f(R)$ theories as being scalar-tensor theories with a gravitational strength coupling to matter, and Palatini theories as being ones with an essentially infinite strength coupling to matter (in this sense they also have an infinite mass, but the ratio of the mass to the coupling is finite). If one considers a general scalar-tensor theory with arbitrary strength coupling (but defined so that the mass divided by the coupling is fixed) then one would expect to see a cross-over, for some coupling strength, from a behaviour where the macroscopic dynamics are determined, to leading order, by the microscopic field equations (and averaging works as one might expect), to a behaviour where the average macroscopic dynamics are *not* described by the microscopic field equations. This is precisely the behaviour that was found in the study conducted in [13]. For a relatively weak (e.g. gravitational strength) coupling, such as in metric $f(R)$ theories, averaging over small scales works as one would generally expect (at least to determine the leading order deviation from GR). It should be stressed though that these problems with averaging may re-emerge in metric $f(R)$ on larger scales (e.g. those of large scale structure in the Universe). A discussion of this scenario is beyond the scope of this article.

Another point worthy of discussion is that the conclusions of this paper concerning the macroscopic behaviour of Palatini $f(R)$ theories applies generally well to much wider class of theories. Specifically, for a given theory on can always write (for a some choice of conformal frame) the modified Einstein equations as:

$$R^{\mu}{}_{\nu} - \frac{1}{2}R\delta^{\mu}{}_{\nu} + \Lambda\delta^{\mu}{}_{\nu} = \kappa T^{\mu}_{\mathrm{m}\,\nu} + t^{\mu}{}_{\nu}, \tag{14}$$

where Λ is some cosmological constant term, $T^{\mu}_{\mathrm{m}\,\nu}$ is the energy momentum tensor of matter and $t^{\mu}{}_{\nu}$ represents all of the modifications from the standard Einstein equation. All of our conclusions of the macroscopic behaviour of Palatini $f(R)$ theories, will then apply if, for some choice of Λ, both terms on the right hand side of Eq. (14) vanish outside particles. One could then replace $\kappa T^{\mu}_{\mathrm{m}\,\nu} + t^{\mu}{}_{\nu}$ by $\kappa \tilde{T}^{\mu}{}_{\nu}$, and this new energy momentum tensor would then only have support where $T^{\mu}{}_{\mathrm{m}\,\nu}$ i.e. if $T^{\mu}{}_{\mathrm{m}\,\nu}$ describes a system of particles separated by vacuum then so does $\tilde{T}^{\mu}{}_{\nu}$. In any such theory, one could, as is done in Ref. [4], deduce the dynamics of particles, and hence the averaged dynamics of a set of particles, simply by considering surface integrals which depend only on the form of the metric outside the particles. Outside the particles, the field equations in this modified theory reduce to $G^{\mu}{}_{\nu} + \Lambda\delta^{\mu}{}_{\nu}$ i.e. just vacuum GR. In this modified theory, particles in a vacuum would therefore move precisely like particles in a vacuum in GR. Since the motion of such particles in the later does not dependent on the precise composition of the particles, it is essentially irrelevant that we have replaced the original matter energy momentum tensor with a modified one. It is straightforward to see that if the motion of particles is equivalent to that in GR for one choice of conformal frame, it is equivalent for all choices of conformal frame.

Acknowledgements

The authors thank G. Olmo, E. Flanagan and N. Kaloper for discussions. BL, DFM and DJS acknowledge the Overseas Research Studentship Award, Cambridge Overseas Trust, the Humboldt Foundation and STFC respectively.

References

[1] S. M. Carroll *et al.*, Phys. Rev. D **70** 043528 (2004); T. P. Sotiriou and V. Faraoni, arXiv:0805.1726 [gr-qc]; S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002); D. F. Mota and J. D. Barrow, Mon. Not. Roy. Astron. Soc. **349** (2004) 291; A. W. Brookfield et al., Phys. Rev. D **73**, 083515 (2006) [Erratum-ibid. D **76**, 049901 (2007)]; E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006); M. Manera and D. F. Mota, Mon. Not. Roy. Astron. Soc. **371** (2006) 1373 ; N. J. Nunes and D. F. Mota, Mon. Not. Roy. Astron. Soc. **368** (2006) 751 ; J. D. Bekenstein, Phys. Rev. D **70**, 083509 (2004) [Erratum-ibid. D **71**, 069901 (2005)]; C. Skordis et al., Phys. Rev. Lett. **96**, 011301 (2006); T. Koivisto and D. F. Mota, Phys. Rev. D **75**, 023518 (2007); T. Clifton, D. F. Mota and J. D. Barrow, Mon. Not. Roy. Astron. Soc. **358**, 601 (2005); F. Bourliot et al., Phys. Rev. D **75**, 063508 (2007); B. Li, J. D. Barrow and D. F. Mota, Phys. Rev. D **76**, 044027 (2007).
[2] E. E. Flanagan, Phys. Rev. Lett. **92**, 071101 (2004) and Class. Quant. Grav. **21**, 3817 (2004).
[3] S. Fay *et al.*, Phys. Rev. D **75**, 063509 (2007); M. Borunda, B. Janssen and M. Bastero-Gil, arXiv:0804.4440 [hep-th]; F. Bauer and D. A. Demir, arXiv:0803.2664 [hep-ph]; G. J. Olmo, Phys. Rev. D **77**, 084021 (2008); E. Barausse, T. P. Sotiriou and J. C. Miller, arXiv:0801.4852 [gr-qc]; S. Lee, arXiv:0801.4606 [gr-qc]; T. Koivisto, Phys. Rev. D **73** 083517 (2006); M. Amarzguioui et al., Astron. Astrophys. **454**, 707 (2006); B. Li, J. D. Barrow and D. F. Mota, Phys. Rev. D **76**, 104047 (2007); B. Li, D. Fonseca Mota and J. D. Barrow, Phys. Rev. D **77** (2008) 024032; G. J. Olmo, Phys. Rev. D **72**, 083505 (2005).
[4] B. Li, D. F. Mota and D. J. Shaw, Phys. Rev. D **78**, 064018 (2008).
[5] G. J. Olmo *Phys. Rev. Lett.* **98**, 061101 (2007).
[6] A. Dominguez, D. Barraco, Phys. Rev. D **70**, 043505 (2004)
[7] A. Krasinski 1996, *Inhomogeneous Cosmological Models*, (CambridgeUP, Cambridge); T. Buchert, arXiv:0707.2153; A.A. Coley et al, Phys. Rev. Lett. 95 (2005) 151102; S. Rasanen, JCAP 0402 (2004) 003; J. Behrend, I. A. Brown and G. Robbers, JCAP **0801**, 013 (2008).
[8] E. Racine & E. Flanagan *Phys. Rev. D* **71** 044010, (2005).
[9] R. C. Tolman (1934) *Relativity, Thermodynamics & Cosmology*, (Clarendon Press, Oxford).
[10] E. Barausse *et al.*, *Class. Quant. Grav.* **25**, 062001 (2008).
[11] G. J. Olmo *Phys. Rev. D***77**, 084021 (2008).
[12] Th. Udem *et al. Phys. Rev. Lett.* **79**, 2646 (1997); B. de Beauvoir *et al.* (1997) *Phys Rev Lett.* **78**, 440 (1997).
[13] D. F. Mota and D. J. Shaw *Phys. Rev. D***75**, 063501 (2007); D. F. Mota and D. J. Shaw Phys. Rev. Lett. **97**, 151102 (2006).