REVERSIBLE COMPLEX HYPERBOLIC ISOMETRIES

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Abstract. Let $\mathrm{PU}(n,1)$ denote the isometry group of the n-dimensional complex hyperbolic space $\mathrm{H}^n_{\mathbb{C}}$. An isometry g is called *reversible* if g is conjugate to g^{-1} in $\mathrm{PU}(n,1)$. If g can be expressed as a product of two involutions, it is called *strongly reversible*. We classify reversible and strongly reversible elements in $\mathrm{PU}(n,1)$. We also investigate reversibility and strong reversibility in $\mathrm{SU}(n,1)$.

1. Introduction

An element g in a group G is called *reversible* if there exists $h \in G$ such that $g^{-1} = hgh^{-1}$. The terminology 'real' has also been used in the literature to refer to the reversible elements, for example, see [9, 25, 11]. If h is an involution, that is $h^{-1} = h$, then this equation becomes $g^{-1} = hgh$ or equivalently $(hg)^2 = hghg = e$, the identity element. In other words, g can be decomposed as the product of two involutions h and hg. In this case g is called *strongly reversible*.

Reversible group elements have been studied in several contexts, for example see [9, 19, 20, 25, 26, 27]. The strongly reversible elements are also studied in several contexts, for example see [3, 4, 5, 7, 6, 15, 16, 17, 21, 29]. Some of these authors have used the terminology 'strongly real' or 'bireflectional' to refer to strongly reversible elements. From a representation theoretic point of view, the terminology 'real' is motivated by a theorem of Frobenius and Schur (1906) which says that if G is finite, the number of real-valued complex irreducible characters of G equals the number of real conjugacy classes of G, cf. [14]. On the other hand from geometric point of view, the terminology 'reversible' is more commonly used, cf. [18, 22, 23, 24]. We will use the terminology 'reversible' and 'strongly reversible'.

Reversible elements in real hyperbolic geometry have been investigated in many contexts. Let $I(\mathrm{H}^n_{\mathbb{R}})$ denote the full isometry group of the n-dimensional real hyperbolic space and let $I_o(\mathrm{H}^n_{\mathbb{R}})$ denote the identity component, which is the group of orientation preserving isometries of $\mathrm{H}^n_{\mathbb{R}}$. When $n = 2$ it is well known that every element of $I(\mathrm{H}^2_{\mathbb{R}})$ is strongly reversible (and so also reversible) but that there are elements of $I_o(\mathrm{H}^2_{\mathbb{R}}) = \mathrm{PSL}(2,\mathbb{R})$ that are not reversible. For example $z \longmapsto z + 1$ is not conjugate in $\mathrm{PSL}(2,\mathbb{R})$ to its inverse, $z \longmapsto z - 1$. Things are slightly different for $n = 3$. On page 47 of [10] Fenchel shows that every element of the group $I_o(\mathrm{H}^3_{\mathbb{R}}) = \mathrm{PSL}(2,\mathbb{C})$ is strongly reversible. On page 51 of [10] he also shows that every element of $I(\mathrm{H}^3_{\mathbb{R}})$ is strongly reversible. In higher dimensions, it follows from [13, Theorem 1.2] that every element of $I(\mathrm{H}^n_{\mathbb{R}})$ is strongly reversible, also see

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[3, 15, 16, 21, 29]. The reversible elements in $I_o(\mathrm{H}^n_{\mathbb{R}})$ have been classified in [12, 24], also see [18]. In [12], the first author obtained a linear-algebraic classification by identifying the orientation-preserving isometry group with $\mathrm{SO}_o(n,1)$. In [24], a geometric classification of the reversible elements in $I_o(\mathrm{H}^n_{\mathbb{R}})$ was obtained using the ball model of the hyperbolic space.

Let $\mathrm{H}^n_{\mathbb{C}}$ denote the n-dimensional complex hyperbolic space. Let $I(\mathrm{H}^n_{\mathbb{C}})$ denote the full isometry group which consists of holomorphic, as well as anti-holomorphic isometries. The group of all holomorphic isometries can be identified with the projective unitary group $\mathrm{PU}(n,1)$ which is an index 2 subgroup of $I(\mathrm{H}^n_{\mathbb{C}})$. Falbel and Zocca [8] proved that every element in $\mathrm{PU}(2,1)$ can be expressed as a product of two anti-holomorphic involutions, and so is strongly reversible in $I(\mathrm{H}^2_{\mathbb{C}})$. Choi [2] extended this result to the isometries of $\mathrm{H}^n_{\mathbb{C}}$. It follows from these results that every holomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$ is reversible in $I(\mathrm{H}^n_{\mathbb{C}})$.

In this paper we restrict ourselves to the group $\mathrm{PU}(n,1)$ and ask for reversible and strongly reversible elements in $\mathrm{PU}(n,1)$. However, for convenience, we work with the linear group $\mathrm{U}(n,1)$. We also investigate reversibility and strong reversibility in $\mathrm{SU}(n,1)$. Earlier, strongly reversible and reversible elements in unitary groups over a field $\mathbb{F}$ have been investigated by Djokovich [3] and Singh-Thakur [25] respectively. It is desirable to have an explicit and actual classification, not just characterisation, of the reversible elements in unitary groups over the complex numbers. Such a classification is not known in general. However, for the groups $\mathrm{U}(n,1)$ and $\mathrm{SU}(n,1)$ which are of interest to complex hyperbolic geometry, we have a very satisfactory answer to the classification problem of reversible elements. In this paper we offer a complete classification of the reversible and strongly reversible elements in $\mathrm{U}(n,1)$, in $\mathrm{SU}(n,1)$ or in $\mathrm{PU}(n,1)$. Most of our results are linear algebraic in nature. So people who are not familiar with complex hyperbolic geometry should think of our results as being about unitary groups with respect to an indefinite Hermitian form. The main results of the paper are Theorem 4.1, Theorem 4.2 and Theorem 4.5 in section 4. As a consequence we have the following.

Theorem 1.1. *Let T be an element in* $\mathrm{SU}(n,1)$.

- (i) *Let T be hyperbolic. Then T is reversible in* $\mathrm{SU}(n,1)$ *if and only if the characteristic polynomial of T has real coefficients.*
- (ii) *Let T be elliptic. Then T is reversible in* $\mathrm{SU}(n,1)$ *if and only if the characteristic polynomial of T has real coefficients and the eigenvalue of negative or indefinite type of T is 1 or -1.*
- (iii) *Let $T = NA$ be parabolic. Then T is reversible in* $\mathrm{SU}(n,1)$ *if and only if the characteristic polynomial of T has real coefficients and the null eigenvalue of T is 1 or -1 and the minimal polynomial of N is $(x-1)^3$.*

Strong reversibility is very closely related to decomposable subgroups. Will [28] has investigated when a subgroup of $\mathrm{SU}(2,1)$ generated by two loxodromic maps can be decomposed as an index two subgroup of a group generated by three involutions. He says that such a group is $\mathbb{R}$-decomposable if all three involutions are antiholomorphic, that is they are in $I(\mathrm{H}^2_{\mathbb{C}})$ but not in $\mathrm{SU}(2,1)$, and $\mathbb{C}$-decomposable when all three involutions are in $\mathrm{SU}(2,1)$. Will's criteria to decide whether a group is $\mathbb{R}$ or $\mathbb{C}$-decomposable involve traces of certain group elements being real. As a consequence of Theorem 1.1, we relate real traces in $\mathrm{SU}(2,1)$ and $\mathrm{SU}(3,1)$ to reversibility. The following result should be compared with Theorem 1 of Will [28].

Corollary 1.2. *Let T be an element in* $\mathrm{SU}(k,1)$ *for* $k=2$ *or* 3.

(i) *Let T be hyperbolic. Then T is reversible in* $\mathrm{SU}(k,1)$ *if and only if the trace of T is real.*
(ii) *Let T be elliptic. Then T is reversible in* $\mathrm{SU}(n,1)$ *if and only if the trace of T is real and the eigenvalue of negative or indefinite type of T is* 1 *or* -1.
(iii) *Let $T=NA$ be parabolic. Then T is reversible in* $\mathrm{SU}(n,1)$ *if and only if the trace of T is real, the null eigenvalue T is* 1 *or* -1 *and the minimal polynomial of N is* $(x-1)^3$.

2. Preliminaries

All the assertions made in this section are borrowed essentially from [1].

Let $\mathbb{V}\approx\mathbb{C}^{n+1}$ be a vector space of dimension $(n+1)$ over $\mathbb{C}$ equipped with the complex Hermitian form of *signature* $(n,1)$,

$$\langle z,w\rangle=\overline{w}^tJz=-z_0\overline{w}_0+z_1\overline{w}_1+\cdots+z_n\overline{w}_n,$$

where z and w are the column vectors in $\mathbb{V}$ with entries $z_0,\cdots,z_n$ and $w_0,\cdots,w_n$ respectively and J is the diagonal matrix $J=diag(-1,1,\cdots,1)$ representing the Hermitian form. Define

$$\mathbb{V}_0=\{z\in\mathbb{V}\mid\langle z,z\rangle=0\},\ \mathbb{V}_+=\{z\in\mathbb{V}\mid\langle z,z\rangle>0\},\ \mathbb{V}_-=\{z\in\mathbb{V}\mid\langle z,z\rangle<0\}.$$

A vector v is called *time-like*, *space-like* or *light-like* according as v is an element in $\mathbb{V}_-$, $\mathbb{V}_+$ or $\mathbb{V}_0$. Let $\mathbb{P}(\mathbb{V})$ be the projective space obtained from $\mathbb{V}$, i.e, $\mathbb{P}(\mathbb{V})=\mathbb{V}-\{0\}/\sim$, where $u\sim v$ if there exists λ in $\mathbb{C}^*$ such that $u=v\lambda$, and $\mathbb{P}(\mathbb{V})$ is equipped with the quotient topology. Let $\pi:\mathbb{V}-\{0\}\to\mathbb{P}(\mathbb{V})$ denote the projection map. We define $\mathrm{H}^n_{\mathbb{C}}=\pi(\mathbb{V}_-)$. The boundary $\partial\mathrm{H}^n_{\mathbb{C}}$ in $\mathbb{P}(\mathbb{V})$ is $\pi(\mathbb{V}_0-\{0\})$. The unitary group $\mathrm{U}(n,1)$ of the Hermitian space $\mathbb{V}$ acts by the holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$. We will not deal with the anti-holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$ in this paper. All isometries will be assumed to be holomorphic unless specified otherwise.

A matrix A in $\mathrm{GL}(n+1,\mathbb{C})$ is unitary with respect to the Hermitian form $\langle z,w\rangle$ if $\langle Az,Aw\rangle=\langle z,w\rangle$ for all $z,\,w\in\mathbb{V}$. Let $\mathrm{U}(n,1)$ denote the group of all matrices that are unitary with respect to our Hermitian form of signature $(n,1)$. By letting z and w vary through a basis of $\mathbb{V}$ we can characterise $\mathrm{U}(n,1)$ by

$$\mathrm{U}(n,1)=\{A\in\mathrm{GL}(n+1,\mathbb{C})\ :\ \bar{A}^tJA=J\}$$

The actual group of the isometries of $\mathrm{H}^n_{\mathbb{C}}$ is $\mathrm{PU}(n,1)=\mathrm{U}(n,1)/Z(\mathrm{U}(n,1))$, where the centre $Z(\mathrm{U}(n,1))$ can be identified with the circle group $\mathbb{S}^1=\{\lambda I\mid|\lambda|=1\}$. Thus an isometry T of $\mathrm{H}^n_{\mathbb{C}}$ lifts to a unitary transformation $\tilde{T}$ in $\mathrm{U}(n,1)$ and the fixed points of T correspond to eigenvectors of $\tilde{T}$. For our purpose, it is convenient to deal with $\mathrm{U}(n,1)$ rather than $\mathrm{PU}(n,1)$. We shall regard $\mathrm{U}(n,1)$ as acting on $\mathrm{H}^n_{\mathbb{C}}$ as well as on $\mathbb{V}$.

A subspace $\mathbb{W}$ of $\mathbb{V}$ is called *space-like*, *light-like*, or *indefinite* if the Hermitian form restricted to $\mathbb{W}$ is positive-definite, degenerate, or non-degenerate but indefinite respectively. If $\mathbb{W}$ is an indefinite subspace of $\mathbb{V}$, then the orthogonal complement $\mathbb{W}^\perp$ is space-like.

Definition 2.1. An eigenvalue λ of $T\in\mathrm{U}(n,1)$ is said to be of *negative type*, of *positive type* if every eigenvector in $\mathbb{V}_\lambda$ is in $\mathbb{V}_-$ or $\mathbb{V}_+$ respectively. The eigenvalue λ is called *null* if the λ-eigenspace $\mathbb{V}_\lambda$ is light-like. The eigenvalue λ is said to be of *indefinite type* if $\mathbb{V}_\lambda$ contains vectors in $\mathbb{V}_-$ and vectors in $\mathbb{V}_+$. Moreover, for λ

of indefinite type, the restriction of the Hermitian form to $\mathbb{V}_\lambda$ has signature $(r,1)$, $1 \leq r \leq n$, where $\dim \mathbb{V}_\lambda = r+1$.

A second model of $\mathrm{H}^n_{\mathbb{C}}$ is obtained by taking the section of $\mathbb{V}$ defined by $z_0 = 1$ and considering $\pi(\mathbb{V}_-)$. Thus a point $\mathtt{z} = (z_1, \dots, z_n) \in \mathrm{H}^n_{\mathbb{C}}$ corresponds to $z = [(1, z_1, \dots, z_n)]$ in $\pi(\mathbb{V})$. The vector $(1, z_1, \dots, z_n)$ is the *standard lift* of $\mathtt{z} \in \mathrm{H}^n_{\mathbb{C}}$ to $\mathbb{V}_-$. Further we see that $\mathtt{z} \in \mathrm{H}^n_{\mathbb{C}}$ provided

$$\langle z, z \rangle = -1 + |z_1|^2 + \cdots + |z_n|^2 < 0,$$

i.e. $|z_1|^2 + \cdots + |z_n|^2 < 1$. Thus $\pi(\mathbb{V}_-)$ can be identified with the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ given by

$$\mathbb{B}^n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n \ : \ |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

This identifies the boundary $\partial\mathrm{H}^n_{\mathbb{C}}$ with the *complex unit sphere*

$$\mathbb{S}^{2n-1} = \{(z_1, \cdots, z_n) \in \mathbb{C}^n \ : \ |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

The *Bergman metric* of $\mathrm{H}^n_{\mathbb{C}}$ is the distance function ρ given by

$$\cosh\left(\frac{\rho(\mathtt{z}, \mathtt{w})}{2}\right) = \frac{\langle z, w\rangle\langle w, z\rangle}{\langle z, z\rangle\langle w, w\rangle},$$

where z, w are the standard lifts of $\mathtt{z}$, $\mathtt{w}$ in $\mathrm{H}^n_{\mathbb{C}}$.

In the ball model of the hyperbolic space, by Brouwer's fixed point theorem it follows that every isometry T has a fixed point on the closure $\overline{\mathrm{H}^n_{\mathbb{C}}}$. An isometry T is called *elliptic* if it has a fixed point in $\mathrm{H}^n_{\mathbb{C}}$; it is called *parabolic* if it fixes a single point and this point lies in $\partial\mathrm{H}^n_{\mathbb{C}}$; it is called *hyperbolic* (or *loxodromic*) if it fixes exactly two points and they both lie on $\partial\mathrm{H}^n_{\mathbb{C}}$. Any non-central element T of $\mathrm{U}(n,1)$ must be one of the above three types; see [1].

It follows from the conjugacy classification in $\mathrm{U}(n,1)$, see [1, Theorem 3.4.1], that the elliptic and hyperbolic elements are semisimple, i.e. their minimal polynomial is a product of linear factors. The parabolic elements are not semisimple. A parabolic transformation T has the unique Jordan decomposition $T = AN$, where A is elliptic, N is unipotent and $AN = NA$.

Let T be elliptic. From the conjugacy classification it follows that all eigenvalues of T except for one are of positive type and the remaining eigenvalue is either of negative type or of indefinite type. Moreover, all eigenvalues will have norm 1.

Suppose T is hyperbolic. Then it has a pair of null eigenvalues $re^{i\theta}$, $r^{-1}e^{i\theta}$, $r > 1$, and the eigenspace of each such eigenvalue has dimension one. The other eigenvalues are of positive type and they all have norm one.

Suppose T is parabolic. If T is unipotent, i.e. all the eigenvalues are 1, then it has minimal polynomial $(x-1)^2$, or $(x-1)^3$. If T is a non-unipotent parabolic, then it has the Jordan decomposition $T = AN$ as above. In this case T has a null eigenvalue λ and the minimal polynomial of T contains a factor of the form $(x-\lambda)^2$ or $(x-\lambda)^3$. This implies that $\mathbb{V}$ has a T-invariant orthogonal decomposition $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$, where $T|_{\mathbb{W}}$ is semisimple, $\mathbb{U}$ is indefinite, $\dim \mathbb{U} = k$ with $k = 2$ or 3 and $T|_{\mathbb{U}}$ has characteristic, as well as minimal polynomial $(x-\lambda)^k$.

3. Reversible and strongly reversible elements in U(n) and SU(n)

Let U(n) denote the isometry group of $\mathbb{V}_o \approx \mathbb{C}^n$ equipped with the positive-definite Hermitian form $\langle z, w\rangle_o = z_1\overline{w}_1 + ... + z_n\overline{w}_n$. In this section we assume the well known facts that every eigenvalue of an element of U(n) is a complex number of unit modulus and that every element of U(n) is diagonalizable.

A polynomial $f(x)$ over $\mathbb{C}$ is called *self-dual* if its set of roots is invariant under taking reciprocals. That is, if $\lambda \in \mathbb{C}$ is a root of $f(x)$ of multiplicity k, then so is λ^{-1}. Note that when $\lambda = \pm 1$ this statement is vacuous. For a linear transformation T, let $\chi_T(x)$ denote the characteristic polynomial of T.

Strongly reversible elements in U(n) were considered in the work of Ellers [6], also see [3, 25].

Proposition 3.1 (Theorem 8 of Ellers [6])**.** *A transformation T in* U(n) *is strongly reversible if and only if its characteristic polynomial is self-dual.*

Since strongly reversible elements are reversible and having a self-dual characteristic polynomial is necessary for being reversible (see below), we immediately have:

Corollary 3.2. *A transformation T in* U(n) *is reversible if and only if its characteristic polynomial is self-dual.*

In the case of SU(n) things become slightly more delicate.

Proposition 3.3. *A transformation T in* SU(n) *is reversible if and only if its characteristic polynomial is self-dual. However, for an element T in* SU(n) *with self-dual characteristic polynomial the following two conditions are equivalent:*

(a) T is reversible but not strongly reversible;
(b) $n = 4m + 2$ with $m \in \mathbb{Z}$ and ± 1 is not an eigenvalue of T.

Proof. Suppose T is a reversible or strongly reversible element of SU(n). Then we can find $S \in$ SU(n) so that $STS^{-1} = T^{-1}$ (if T is strongly reversible then $S = S^{-1}$). Let $\mathbb{V}_\lambda$ denote the eigenspace of T corresponding to the eigenvalue λ. For each eigenvalue λ of T, it is clear that S bijectively maps the λ-eigenspace $\mathbb{V}_\lambda$ to the λ^{-1}-eigenspace $\mathbb{V}_{\lambda^{-1}}$. Therefore $\mathbb{V}_\lambda$ and $\mathbb{V}_{\lambda^{-1}}$ have the same dimension. This implies λ and λ^{-1} are roots of the characteristic polynomial $\chi_T(x)$ with the same multiplicity. Hence $\chi_T(x)$ is self-dual.

Conversely, suppose $\chi_T(x)$ is self-dual. Let E denote the set of eigenvalues $\lambda \neq \pm 1$ such that λ^{-1} is also an eigenvalue with the same multiplicity. Then $\mathbb{V}$ has a T-invariant orthogonal decomposition into eigenspaces

$$\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_{-1} \oplus \mathbb{W},$$

where $\mathbb{W} = \oplus_{\lambda\in E}(\mathbb{V}_\lambda \oplus \mathbb{V}_{\lambda^{-1}})$ and $\dim \mathbb{V}_\lambda = \dim \mathbb{V}_{\lambda^{-1}}$. If $v \in \mathbb{V}_\lambda$ then $T^{-1}v = \lambda^{-1}v$ and so $\mathbb{V}_\lambda$ is the λ^{-1}-eigenspace of T^{-1}. Similarly, $\mathbb{V}_{\lambda^{-1}}$ is the λ-eigenspace, of T^{-1}. Since $\mathbb{V}_\lambda$ and $\mathbb{V}_{\lambda^{-1}}$ are non-empty, we can find orthonormal bases $\{e_1, ..., e_r\}$ and $\{f_1, ..., f_r\}$ of $\mathbb{V}_\lambda$ and $\mathbb{V}_{\lambda^{-1}}$ respectively. Let $\mathbb{W}_\lambda = \mathbb{V}_\lambda \oplus \mathbb{V}_{\lambda^{-1}}$. Define $S_\lambda : \mathbb{W}_\lambda \to \mathbb{W}_\lambda$ by $S_\lambda(e_i) = f_i$ and $S_\lambda(f_i) = -e_i$ for each $i = 1, \ldots, r$. Then $S_\lambda T|_{\mathbb{W}_\lambda} S_\lambda^{-1} = T^{-1}|_{\mathbb{W}_\lambda}$ and $\det(S_\lambda) = 1$. Note, however that $(S_\lambda)^2 = -I$ on $\mathbb{W}_\lambda$, so S_λ is not an involution. Define

$$S_{\mathbb{W}} = \oplus_{\lambda\in E} S_\lambda.$$

Let $\mathbb{W}_1 = \mathbb{V}_1 \oplus \mathbb{V}_{-1}$ and define $S_1 : \mathbb{W}_1 \to \mathbb{W}_1$ to be the identity (it may be that $\mathbb{V}_1$ or $\mathbb{V}_{-1}$ is empty). Let $S = S_1 \oplus S_{\mathbb{W}}$. Then $S \in \mathrm{SU}(n)$ and $STS^{-1} = T^{-1}$. Thus T is reversible. This proves the first part of the theorem.

For the second part of the proposition, suppose we want T to be strongly reversible. Then we must change the above construction to ensure that S_λ is an involution. In this case, we define $\widetilde{S}_\lambda(e_i) = f_i$ and $\widetilde{S}_\lambda(f_i) = e_i$. Then $\widetilde{S}_\lambda^2 = I$ and $\det(\widetilde{S}_\lambda) = (-1)^{\dim(\mathbb{V}_\lambda)}$. Define

$$\widetilde{S}_{\mathbb{W}} = \oplus_{\lambda \in E} \widetilde{S}_\lambda.$$

Then $\widetilde{S}_{\mathbb{W}}$ is an involution and $\det(\widetilde{S}_{\mathbb{W}}) = (-1)^{\frac{1}{2}\dim(\mathbb{W})}$. If T does not have eigenvalue 1 or -1, that is both $\mathbb{V}_1$ and $\mathbb{V}_{-1}$ are empty, then $\mathbb{W} = \mathbb{V}$ and n is even. We see that $\widetilde{S} = \widetilde{S}_{\mathbb{W}}$ is in $\mathrm{SU}(n)$ only when n is a multiple of 4. Suppose n is odd or $n = 4m + 2$ with ± 1 as an eigenvalue. Then either $\mathbb{V}_1$ or $\mathbb{V}_{-1}$ is non-empty. Choose v in $\mathbb{V}_1 \oplus \mathbb{V}_{-1} = \mathbb{W}_1$ and define $\widetilde{S}_1$ by $\widetilde{S}_1(v) = (-1)^{\frac{1}{2}\dim(\mathbb{W})} v$ and $\widetilde{S}_1$ is the identity on the orthogonal complement of v in $\mathbb{W}_1$. Let $\widetilde{S} = \widetilde{S}_1 \oplus \widetilde{S}_{\mathbb{W}}$. Then $\widetilde{S}$ is an involution in $\mathrm{SU}(n)$ and $\widetilde{S}T\widetilde{S}^{-1} = \widetilde{S}T\widetilde{S} = T^{-1}$. Thus T is strongly reversible. Hence it follows that if T is reversible, but not strongly reversible, then we must have $n = 4m + 2$ and ± 1 is not an eigenvalue of T.

Conversely, suppose $n = 4m + 2$ and ± 1 is not an eigenvalue of T. Since T is reversible, we have $S \in \mathrm{SU}(n)$ with $STS^{-1} = T^{-1}$. If possible suppose that T is strongly reversible. Then, S can be chosen to be an involution. Now, observe that we can decompose $\mathbb{V}$ as a direct sum $\mathbb{V} = \mathbb{W}_+ \oplus \mathbb{W}_-$ so that $S : \mathbb{W}_+ \to \mathbb{W}_-$ and $S : \mathbb{W}_- \to \mathbb{W}_+$. (For example we can take $\mathbb{W}_+$ to be the direct sum of the eigenspaces $\mathbb{V}_\lambda$ where $\Im(\lambda) > 0$ and $\mathbb{W}_-$ to be the direct sum of the eigenspaces $\mathbb{V}_{\lambda^{-1}}$ where $\Im(\lambda^{-1}) = -\Im(\lambda) < 0$.) Note that $\mathbb{W}_+$ and $\mathbb{W}_-$ both have dimension $2m + 1$. Let $\{e_1, \ldots, e_{2m+1}\}$ be an orthonormal basis for $\mathbb{W}_+$. Then $\{S(e_1), \ldots, S(e_{2m+1})\}$ is an orthonormal basis of $\mathbb{W}_-$. Hence, with respect to the basis $\{e_1, S(e_1), \ldots, e_{2m+1}, S(e_{2m+1})\}$, we can write S as a block diagonal matrix where each block is a 2×2 off-diagonal matrix with off-diagonal entries 1. It is clear that each block has determinant -1 and hence S has determinant $(-1)^{2m+1} = -1$. This is a contradiction to the fact that S belongs to $\mathrm{SU}(n)$.

This completes the proof. □

4. Reversible and strongly reversible elements in $\mathrm{U}(n,1)$ and $\mathrm{SU}(n,1)$

4.1. **Statement of main theorems.** We now turn our attention to $\mathrm{U}(n,1)$ and $\mathrm{SU}(n,1)$. In this case it is no longer true that eigenvalues have unit modulus or that transformations are diagonalizable. Suppose T is a reversible element in $\mathrm{U}(n,1)$ or $\mathrm{SU}(n,1)$. Then there exist S in $\mathrm{U}(n,1)$, or $\mathrm{SU}(n,1)$ respectively, so that $STS^{-1} = T^{-1}$. This implies that if λ is an eigenvalue of T with multiplicity m, then so is λ^{-1}. Hence $\chi_T(x)$ is self-dual. What is interesting is the converse.

Theorem 4.1. *Suppose T is an element of $\mathrm{U}(n,1)$ or $\mathrm{SU}(n,1)$ whose characteristic polynomial is self-dual.*

(i) *Let T be elliptic. Then T is reversible if and only if the eigenvalue of negative or indefinite type of T is 1 or -1.*
(ii) *Let T be unipotent with minimal polynomial $(x-1)^2$. Then T is not reversible.*

(iii) *Let T be unipotent with minimal polynomial $(x-1)^3$. Then T is reversible.*
(iv) *Let $T = NA$ be non-unipotent parabolic. Then T is reversible if and only if the null eigenvalue of T is 1 or -1 and the minimal polynomial of N is $(x-1)^3$.*
(v) *Let T be hyperbolic. Then T is reversible.*

Note that the statement of part (ii) does not agree with Lemma 3.4.3 of Chen and Greenberg [1]. In fact there is an error in their proof in the case where $\mathbb{F} = \mathbb{C}$. On line 4 of page 71, they state that if s and s' are two purely imaginary complex numbers (that is $\mathrm{Re}(s) = \mathrm{Re}(s') = 0$) then we can find $\lambda \in \mathbb{C}$ so that $s' = \lambda s \overline{\lambda} = |\lambda|^2 s$. This is clearly impossible if $s' = -s$.

Again, things become slightly more delicate for strongly reversible elements.

Theorem 4.2. (i) *Let T be an element of $\mathrm{U}(n,1)$. Then T is strongly reversible if and only if it is reversible.*
(ii) *Let T be an element of $\mathrm{SU}(n,1)$ whose characteristic polynomial is self-dual. Then the following conditions are equivalent*
(a) T is reversible but not strongly reversible.
(b) T is hyperbolic, $n = 4m+1$ for $m \in \mathbb{Z}$ and ± 1 is not an eigenvalue of T.

The following lemma is fundamental to the analysis which follows.

Lemma 4.3. Goldman [11, Lemma 6.2.5]. *Let T be a transformation in $\mathrm{U}(n,1)$. If λ is an eigenvalue of T, then $\overline{\lambda}^{-1}$ is also an eigenvalue with the same multiplicity as that of λ.*

Furthermore, it is not hard to show that if $|\lambda| \neq 1$ then λ and $\overline{\lambda}^{-1}$ are (distinct) null eigenvalues. Of course, when $|\lambda| = 1$ (as in the case of $\mathrm{U}(n)$) we have $\lambda = \overline{\lambda}^{-1}$ and so, although true, this lemma does not give us any useful information.

We conclude this section by discussing what happens in $\mathrm{PU}(n,1)$. Suppose that T is in $\mathrm{PU}(n,1)$. Let $\widetilde{T}$ be a lift of T to $\mathrm{U}(n,1)$ and note that $e^{i\theta}\widetilde{T}$ corresponds to the same element of $\mathrm{PU}(n,1)$ for all $\theta \in [0, 2\pi)$.

Lemma 4.4. *For every $T \in \mathrm{PU}(n,1)$, there exists a unique lift $\widehat{T}$ of T to $\mathrm{U}(n,1)$ so that, for each fixed point of T in $\mathrm{H}^n_{\mathbb{C}} \cup \partial\mathrm{H}^n_{\mathbb{C}}$, the associated eigenvalue of $\widehat{T}$ is a positive real number.*

This lemma enables us to state the following.

Theorem 4.5. *Let $T \in \mathrm{PU}(n,1)$. Then T is reversible, or strongly reversible, if and only if the lift $\widehat{T}$ of T to $\mathrm{U}(n,1)$ given by Lemma 4.4 is reversible, or strongly reversible respectively.*

In particular, T is reversible, or strongly reversible, if and only if the following conditions hold.

(i) The characteristic polynomial of $\widehat{T}$ is self dual and,
(ii) if T is parabolic, the minimal polynomial of the unipotent part of $\widehat{T}$ is $(x-1)^3$.

4.2. **Proof of Theorem 4.1.**

Proof. (i) Suppose T is elliptic. Let λ be the eigenvalue of T of negative or indefinite type. Then $\mathbb{V}$ has an orthogonal decomposition into T-invariant subspaces $\mathbb{V} = \mathbb{V}_\lambda \oplus \mathbb{W}$, where $\mathbb{V}_\lambda$ is the eigenspace of λ. The space $\mathbb{V}_\lambda$ is indefinite and $\mathbb{W}$ is the space-like orthogonal complement. Clearly, $\mathbb{V}_\lambda$ is the eigenspace of T^{-1} corresponding to the eigenvalue of indefinite or negative type λ^{-1}. Now T is conjugate to T^{-1} if and only if they have the same eigenvalue of negative or indefinite type and $T|_{\mathbb{W}}$ is conjugate to $T^{-1}|_{\mathbb{W}}$. Now, $\lambda = \lambda^{-1}$ if and only if $\lambda = \pm 1$. Further, $T|_{\mathbb{W}}$ is a transformation in $\mathrm{U}(n+1-m)$ where $m = \dim \mathbb{V}_\lambda$. Since the characteristic polynomial of $T|_{\mathbb{W}}$ is self-dual, it follows from Lemma 3.1 that $T|_{\mathbb{W}}$ is conjugate to its inverse. This establishes the assertion for the case where T in in $\mathrm{U}(n, 1)$.

When $T \in \mathrm{SU}(n, 1)$ we need to be slightly more careful. Let $S|_{\mathbb{W}}$ be such that $S|_{\mathbb{W}} T|_{\mathbb{W}} S|_{\mathbb{W}}{}^{-1} = T|_{\mathbb{W}}{}^{-1}$. By adjusting $S|_{\mathbb{V}_\lambda}$ as in Proposition 3.3 if necessary, we may ensure that $\det(S) = 1$. Then $S \in \mathrm{SU}(n, 1)$ and S conjugates T to T^{-1}. Thus (i) follows in this case too.

(ii) Let T be unipotent. Then T has a minimal polynomial $(x-1)^2$ or $(x-1)^3$.

First, consider the case where the minimal polynomial is $(x-1)^2$. Using the Jordan normal form for T, we can find vectors u and v so that

$$T(u) = u, \quad T(v) = v + u, \quad T^{-1}(u) = u, \quad T^{-1}(v) = v - u.$$

Further u and v generate a non-degenerate T-invariant subspace $\mathbb{W}$ so that the restriction of $\langle \cdot, \cdot \rangle$ to $\mathbb{W}$ has signature $(1, 1)$. As T preserves $\langle \cdot, \cdot \rangle$ we have

$$\langle u, v \rangle = \langle Tu, Tv \rangle = \langle u, v + u \rangle = \langle u, v \rangle + \langle u, u \rangle.$$

This implies

$$\langle u, u \rangle = 0 \tag{4.1}$$

Since the Hermitian form has signature $(1, 1)$ on $\mathbb{W}$, we must have $\langle u, v \rangle \neq 0$.

If S conjugates T to T^{-1} then S maps the span of u and v to itself. Furthermore, since $S(u)$ is also a light-like eigenvector with eigenvalue 1, the uniqueness of the fixed point of T implies that S must send u to a multiple of itself and v to a linear combination of u and v. Suppose

$$S(u) = au, \quad S(v) = bu + cv.$$

Since S preserves the Hermitian form then

$$\langle u, v \rangle = \langle S(u), S(v) \rangle = \langle au, bu + cv \rangle = a\bar{c}\langle u, v \rangle$$

where we have used (4.1) at the last stage. Hence $a\bar{c} = 1$ since $\langle u, v \rangle \neq 0$. If we have $STS^{-1} = T^{-1}$ then $ST = T^{-1}S$. The images of u and v under these maps are

$$\begin{aligned} ST(u) &= S(u) = au, & ST(v) &= S(v+u) = (a+b)u + cv, \\ T^{-1}S(u) &= T^{-1}(au) = au, & T^{-1}S(v) &= T^{-1}(bu + cv) = (b-c)u + cv. \end{aligned}$$

Hence $(a+b)u + cv = (b-c)u + cv$, and so $a = -c$. Together with $a\bar{c} = 1$, this implies $|a|^2 = |c|^2 = -1$, which is clearly impossible.

(iii) Now consider the case where the minimal polynomial is $(x-1)^3$. Using the Jordan normal form of T we see that there are vectors u, v and w so that

$$T(u) = u, \quad T(v) = v + u, \quad T(w) = w + v.$$

Let $\mathbb{W}$ be the span of u, v and w. As T preserves $\langle \cdot, \cdot \rangle$ we must have

$$0 = \langle u, u \rangle = \langle u, v \rangle = \langle v, v \rangle + \langle u, w \rangle = \langle w, v \rangle + \langle v, w \rangle + \langle v, v \rangle. \tag{4.2}$$

As the restriction of $\langle \cdot, \cdot \rangle$ to $\mathbb{W}$ is non-degenerate, we have $\langle v, v \rangle \neq 0$. Define k by

$$k = \frac{\langle v, w \rangle}{2\langle v, v \rangle}.$$

Note that the last identity in (4.2) implies $2k + 2\overline{k} = -1$ Define S on $\mathbb{W}$ by

$$S(u) = -u, \quad S(v) = v + 2ku, \quad S(w) = -w + 2\overline{k}v + 2|k|^2 u.$$

Then

$$ST(u) = -u, \quad ST(v) = v + (2k-1)u, \quad ST(w) = -w + (2\overline{k}+1)v + (2|k|^2 + 2k)u.$$

It is easy to check that S and ST are involutions. Finally, we can check that S and ST preserve the Hermitian form. For example:

$$\begin{aligned}
\langle S(w), S(v) \rangle &= \langle -w + 2\overline{k}v + 2|k|^2 u, v + 2ku \rangle \\
&= -\langle w, v \rangle - 2\overline{k}\langle w, u \rangle + 2\overline{k}\langle v, v \rangle \\
&= -\langle w, v \rangle + 4\overline{k}\langle v, v \rangle \\
&= -\langle w, v \rangle + 2\langle w, v \rangle \\
&= \langle w, v \rangle.
\end{aligned}$$

Finally note that on the space-like orthogonal complement of $\mathbb{W}$, T restricts to the identity map. Thus T is strongly reversible.

(iv) Suppose T is a non-unipotent parabolic. Let $T = AN$ be the Jordan decomposition of T, where A is semisimple, N is unipotent and $AN = NA$. We say that an eigenvalue μ of T is *pure* if the corresponding eigenspace $\{v \in \mathbb{V} \mid (T - \mu I)v = 0\}$ coincides with the generalised eigenspace $\{v \in \mathbb{V} \mid (T - \mu I)^{n+1} v = 0\}$. Otherwise μ is *mixed*. Since T is parabolic, the null eigenvalue λ of T must be mixed. However, for A, λ is the eigenvalue of indefinite type and the generalised eigenspace $\mathbb{V}_\lambda$ of T will be the usual λ-eigenspace of A.

Also it follows from the Jordan decomposition that T is reversible if and only if A and N are both reversible, cf. [1, Theorem 3.4.1 (c)]. The result now follows from (i), (ii) and (iii).

(v) Suppose T is hyperbolic. Let λ be the (null) eigenvalue of T with $|\lambda| > 1$. Then $\mathbb{V}$ has a decomposition into T-invariant orthogonal subspaces: $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$, where $\mathbb{U}$ is the direct sum of the one dimensional null eigenspaces $\mathbb{V}_\lambda$ and $\mathbb{V}_{\overline{\lambda}^{-1}}$ and $\mathbb{W}$ is the space-like orthogonal complement to $\mathbb{U}$. The Hermitian form restricted to $\mathbb{U}$ has signature $(1, 1)$, hence $T|_{\mathbb{U}}$ can be considered as a transformation in $\mathrm{U}(1, 1)$. Furthermore $\mathbb{V}_\lambda$ is the λ^{-1}-eigenspace of $T^{-1}|_{\mathbb{U}}$ and $\mathbb{V}_{\overline{\lambda}^{-1}}$ is the $\overline{\lambda}$-eigenspace of $T^{-1}|_{\mathbb{U}}$. Hence, it is easy to see that $T|_{\mathbb{U}}$ is reversible in $\mathrm{U}(1, 1)$ if and only if λ is real. Thus the characteristic polynomial of $T|_{\mathbb{U}}$ is self-dual with real roots λ and $\lambda^{-1} = \overline{\lambda}^{-1}$, and $T|_{\mathbb{U}}$ is in $\mathrm{SU}(1, 1)$. Since $\langle , \rangle|_{\mathbb{W}}$ is positive-definite, $T|_{\mathbb{W}}$ can be considered as a transformation in $\mathrm{U}(n-1)$ or $\mathrm{SU}(n-1)$. By Lemma 3.1, $T|_{\mathbb{W}}$ is reversible. Hence the assertion follows. □

4.3. **Proof of Theorem 4.2.**

Proof. Let T be an element of $\mathrm{U}(n,1)$ or $\mathrm{SU}(n,1)$. If T is strongly reversible then it is reversible.

Suppose that T is reversible. Note that if T is not semisimple then, since it is reversible, the null eigenvalue is 1 or -1. Moreover, in the proof of Theorem 4.1 (iii) we have shown that a reversible unipotent map is strongly reversible. Hence if $\lambda \neq \pm 1$ then the dimension of $\mathbb{V}_\lambda$ is the same as the multiplicity of λ as a root of $\chi_T(x)$.

Following the proof of Proposition 3.3, let E denote the set of eigenvalues $\lambda \neq \pm 1$ of T and $\mathbb{W} = \oplus_{\lambda \in E}(\mathbb{V}_\lambda \oplus \mathbb{V}_{\lambda^{-1}})$. (Note that if T is unipotent then $\mathbb{W}$ is empty.) Then we can construct $\widetilde{S}_\mathbb{W}$ as in the proof of Proposition 3.3 so that $\widetilde{S}_\mathbb{W} T|_\mathbb{W} \widetilde{S}_\mathbb{W}^{-1} = T^{-1}|_\mathbb{W}$ and $\widetilde{S}_\mathbb{W}^2 = I$. Note that $\det(S_\mathbb{W}) = (-1)^{\frac{1}{2}\dim(\mathbb{W})}$. Let $\mathbb{U}$ be the orthogonal complement of $\mathbb{W}$. Then $\mathbb{U}$ contains the eigenspaces of ± 1 if these are eigenvalues. Defining $\widetilde{S}_\mathbb{U}$ to be the identity and $\widetilde{S} = \widetilde{S}_\mathbb{U} \oplus \widetilde{S}_\mathbb{W}$ immediately demonstrates that T is strongly reversible in $\mathrm{U}(n,1)$. If 1 or -1 is an eigenvalue of T then we can adjust $\widetilde{S}_\mathbb{U}$ as in Proposition 3.3 so that $\det(S) = 1$ and so T is strongly reversible in $\mathrm{SU}(n,1)$.

If T is unipotent then, by definition, 1 is an eigenvalue of T. If T is elliptic or non-unipotent parabolic then, since T is reversible, by Theorem 4.1 it has eigenvalue ± 1. In each case, we see that T is strongly reversible in $\mathrm{SU}(n,1)$.

Suppose $T \in \mathrm{SU}(n,1)$ is hyperbolic and reversible and that ± 1 is not an eigenvalue of T. Then necessarily n is odd. Let λ be the eigenvalue with $|\lambda| > 1$ and let $\mathbb{U}$ and $\mathbb{W}$ be as in the proof of Theorem 4.1(iv). Define $\widetilde{S}_\mathbb{U}$ to be an involution in $\mathrm{U}(1,1)$ that swaps the eigenspaces of λ and λ^{-1}. Note that $\det(S_\mathbb{U}) = -1$. We know that $T|_\mathbb{W}$ can be considered to be in $\mathrm{SU}(n-1)$. If T is strongly reversible in $\mathrm{SU}(n,1)$ then $T|_\mathbb{W}$ needs to be strongly reversible by an element $S_\mathbb{W}$ with determinant -1. By adapting the Proposition 3.3 we see that this is the case if and only if $(-1)^{\frac{1}{2}(n-1)} = -1$ and so $n-1 = 4m+2$. Hence T is strongly reversible in $\mathrm{SU}(n,1)$ when $n = 4m+3$. This proves the result. □

4.4. **Proof of Theorem 4.5.** We begin by proving Lemma 4.4

Proof. (Lemma 4.4.) Observe that if T is elliptic or parabolic it fixes a connected subset of $\mathrm{H}^n_\mathbb{C} \cup \partial \mathrm{H}^n_\mathbb{C}$ and this subset corresponds to an $e^{i\theta}$-eigenspace $\mathbb{V}_{e^{i\theta}}$ for some lift $\widetilde{T}$. Then $\widehat{T} = e^{-i\theta}\widetilde{T}$ has the property we claimed. If T is hyperbolic then its fixed points on $\partial \mathrm{H}^n_\mathbb{C}$ correspond to eigenspaces $\mathbb{V}_\lambda$ and $\mathbb{V}_\mu$ of some lift $\widetilde{T}$ of T. Using Lemma 4.3 we see that $\mu = \overline{\lambda}^{-1}$. In other words, $\lambda = re^{i\theta}$ and $\mu = \overline{\lambda}^{-1} = r^{-1}e^{i\theta}$ for some $r > 1$. Then $\widehat{T} = e^{-i\theta}\widetilde{T}$ has the property we claimed. □

Proof. (Theorem 4.5.) Let $T \in \mathrm{PU}(n,1)$ and let $\widehat{T} \in \mathrm{U}(n,1)$ be the lift of T coming from Lemma 4.4.

First suppose that $\widehat{T}$ is reversible. Then we can find $\widehat{S} \in \mathrm{U}(n,1)$ so that $\widehat{S}\widehat{T}\widehat{S}^{-1} = \widehat{T}^{-1}$. Applying the canonical projection from $\mathrm{U}(n,1)$ to $\mathrm{PU}(n,1)$ gives $S \in \mathrm{PU}(n,1)$ satisfying $STS^{-1} = T^{-1}$ and so T is reversible. Moreover, if $\widehat{T}$ is strongly reversible then $\widehat{S}$ has order two. Hence S has order (at most) 2. Therefore T is strongly reversible.

Conversely, suppose that $T \in \mathrm{PU}(n,1)$ is reversible. Then there exists $S \in \mathrm{PU}(n,1)$ so that $STS^{-1} = T^{-1}$. Let $\widehat{S}$ be any lift of S to $\mathrm{U}(n,1)$. Note that the expression $\widehat{S}\widehat{T}\widehat{S}^{-1}$ is independent of which lift we choose. If S has order 2 then multiplying $\widehat{S}$ by a scalar if necessary, we may suppose that $\widehat{S}$ also has order 2. Since $STS^{-1} = T^{-1}$ we see that $\widehat{S}\widehat{T}\widehat{S}^{-1} = k\widehat{T}^{-1}$ for some $k \in \mathbb{C}$. Note that $|k| = 1$.

If $z \in \mathrm{H}^n_{\mathbb{C}} \cup \partial\mathrm{H}^n_{\mathbb{C}}$ is fixed by T then $S(z)$ is fixed by T^{-1}, and so also by T. By the definition of $\widehat{T}$, we know that z corresponds to an eigenvector v of $\widehat{T}$ with eigenvalue λ, which is real and positive. Now consider $\widehat{S}v$.

$$\widehat{T}^{-1}\widehat{S}v = k^{-1}(\widehat{S}\widehat{T}\widehat{S}^{-1})\widehat{S}v = k^{-1}\widehat{S}\widehat{T}v = \lambda k^{-1}\widehat{S}v.$$

Therefore $\widehat{S}v$ is an eigenvector of $\widehat{T}^{-1}$ with eigenvalue λk^{-1}. That is, $\widehat{S}v$ is an eigenvector of $\widehat{T}$ with eigenvalue $\lambda^{-1}k$. Now $\widehat{S}v$ corresponds to a fixed point of T in $\mathrm{H}^n_{\mathbb{C}} \cup \partial\mathrm{H}^n_{\mathbb{C}}$, namely $S(z)$. Therefore, by the construction of $\widehat{T}$ we know that $\lambda^{-1}k$ is real and positive. Since λ^{-1} is real and positive and $|k| = 1$, we must have $k = 1$. Hence $\widehat{S}\widehat{T}\widehat{S}^{-1} = \widehat{T}^{-1}$. Thus $\widehat{T}$ is reversible.

By construction, if T is elliptic or parabolic, the eigenvalue of $\widehat{T}$ of negative or indefinite type is 1. Hence, the last part follows by applying Theorem 4.1 to $\widehat{T}$ in the reversible case and Theorem 4.2(i) in the strongly reversible case. □

5. Proof of Theorem 1.1

When $T \in \mathrm{SU}(n,1)$, the following lemma provides a necessary and sufficient condition for $\chi_T(x)$ to be self-dual.

Lemma 5.1. *Let T be in $\mathrm{SU}(n,1)$. Then $\chi_T(x)$, the characteristic polynomial of T, is self-dual if and only if the coefficients of $\chi_T(x)$ are real. In particular, if $\chi_T(x)$ is self dual, then the trace of T is real.*

Proof. Let λ be an eigenvalue of T. Then $\overline{\lambda}^{-1}$ is an eigenvalue of T with the same multiplicity, using Lemma 4.3. Suppose T is self-dual. Then $(\overline{\lambda}^{-1})^{-1} = \overline{\lambda}$ is also an eigenvalue of T. Hence the set of eigenvalues is invariant under complex conjugation. Since the coefficients of the characteristic polynomial are symmetric polynomials in the eigenvalues, they must be real. Conversely, if the coefficients of $\chi_T(x)$ are real then its roots are real or come in complex conjugate pairs. Again using Lemma 4.3 we see that if λ is an eigenvalue then so is λ^{-1}, and hence $\chi_T(x)$ is self dual. □

5.1. **Proof of Theorem 1.1.** Combining Theorem 4.1 with the above lemma, Theorem 1.1 follows.

As a corollary to Theorem 1.1 we have the following.

Corollary 5.2. *Let T be an element in $\mathrm{SU}(n,1)$ such that T is reversible in $\mathrm{SU}(n,1)$. Then the trace of T is real.*

The converse to the above corollary, in general, is false. For example, consider the hyperbolic element g in $\mathrm{SU}(4,1)$ with eigenvalues

$$\frac{(3+\sqrt{5})}{2}e^{i\pi/5}, \quad \frac{(3-\sqrt{5})}{2}e^{i\pi/5}, \quad -e^{i\pi/5}, \quad -e^{i\pi/5}, \quad -e^{i\pi/5}.$$

Then g has trace zero, but two of the other coefficients in the characteristic polynomial of g are not real, and so g is not reversible in $\mathrm{U}(4,1)$. So, for $n \geq 4$ the converse of Corollary 5.2 is not true. However, for $n = 2,\ 3$, we have a better situation.

Lemma 5.3. *For $k = 2, 3$, let T in $\mathrm{SU}(k,1)$ be such that the trace of T is real. Then the characteristic polynomial of T is self-dual.*

Proof. We shall prove the lemma for $k = 3$. The case $k = 2$ follows similarly. Our argument is very similar to Goldman's argument on page 206 of [11].

Let T be in $\mathrm{SU}(3,1)$. Let λ_j for $j = 1, 2, 3, 4$ be the eigenvalues of T and write $\tau = \mathrm{tr}(T) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$. Since $\det(T) = 1$, we immediately have

$$\lambda_1\lambda_2\lambda_3\lambda_4 = 1. \tag{5.1}$$

Then the characteristic polynomial of T is of the form

$$\chi_T(x) = x^4 - a_3x^3 + a_2x^2 - a_1x + 1.$$

Now by the relationship between roots and the coefficients of a polynomial we have

$$\begin{aligned} a_3 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \tau, \\ a_1 &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 \\ &= \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} + \lambda_4^{-1}, \end{aligned}$$

where we used (5.1) on the last line. Using Lemma 4.3 we know that for each j, there exists k such that $\lambda_j^{-1} = \overline{\lambda_k}$. Therefore

$$a_1 = \overline{\lambda}_1 + \overline{\lambda}_2 + \overline{\lambda}_3 + \overline{\lambda}_4 = \overline{\tau}.$$

Hence we can write the characteristic polynomial of T as

$$\chi_T(x) = x^4 - \tau x^3 + \sigma x^2 - \overline{\tau}x + 1.$$

We claim that σ is real. Now

$$\begin{aligned} \sigma &= \lambda_1\lambda_2 + \lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_2\lambda_4 + \lambda_1\lambda_4 + \lambda_2\lambda_3 \\ &= \lambda_3^{-1}\lambda_4^{-1} + \lambda_1^{-1}\lambda_2^{-1} + \lambda_2^{-1}\lambda_4^{-1} + \lambda_1^{-1}\lambda_3^{-1} + \lambda_2^{-1}\lambda_3^{-1} + \lambda_1^{-1}\lambda_4^{-1}, \text{ using (5.1)} \\ &= \overline{\lambda}_1\overline{\lambda}_2 + \overline{\lambda}_3\overline{\lambda}_4 + \overline{\lambda}_1\overline{\lambda}_3 + \overline{\lambda}_2\overline{\lambda}_4 + \overline{\lambda}_1\overline{\lambda}_4 + \overline{\lambda}_2\overline{\lambda}_3 \text{ (after permuting terms)} \\ &= \overline{\sigma} \end{aligned}$$

Hence, if τ is real, then $\chi_T(x)$ has real coefficients, and so all solutions are either real or come in conjugate pairs. Together with Lemma 4.3, this implies that if λ is a root, then so is λ^{-1}. Hence $\chi_T(x)$ is self-dual. □

5.2. **Proof of Corollary 1.2.** Combining the above lemma with Theorem 1.1 we have Corollary 1.2.

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