Choosability on H-Free Graphs*

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Abstract. A graph is H-free if it has no induced subgraph isomorphic to H. We determine the computational complexity of the CHOOSABILITY problem restricted to H-free graphs for all but four graphs H. We also show that if H is a linear forest, then the problem is fixed-parameter tractable when parameterized by k.

Keywords. Choosability, H-free graphs, linear forest.

1 Introduction

Graph coloring is without doubt one of the most fundamental concepts in both structural and algorithmic graph theory. The well-known COLORING problem, which takes as input a graph G and an integer k, asks whether G admits a k-*coloring*, i.e., a mapping $c : V(G) \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$. The corresponding problem where k is fixed, i.e., not part of the input, is denoted by k-COLORING. Since graph coloring naturally arises in a vast number of theoretical and practical applications, it is not surprising that many variants and generalizations of the COLORING problem have been studied over the years. There are some very good surveys [1,28] and a book [22] on the subject.

One well-known generalization of graph coloring, called LIST COLORING, was introduced by Vizing [29] and Erdős, Rubin and Taylor [7]. This problem takes as input a graph G and a *list assignment* $\mathcal{L} = \{L(u) \mid u \in V(G)\}$ of admissible colors for its vertices, and asks whether G admits a coloring c such that $c(u) \in L(u)$ for each $u \in V(G)$; such a coloring c is said to *respect* the list assignment $\mathcal{L}$. We say that $\mathcal{L}$ is a k-*list assignment* if each of the lists $L(u)$ in $\mathcal{L}$ contains k colors. A graph G is k-*choosable* if for every k-list assignment $\mathcal{L}$ of G there exists a coloring that respects $\mathcal{L}$. The corresponding decision problem is called the CHOOSABILITY problem. If k is fixed, we denote this problem by k-CHOOSABILITY.

The CHOOSABILITY problem has received increasing attention since the beginning of the 90s. The problem is known to be very difficult: Gutner and Tarsi

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[16] showed that k-Choosability is Π_2^p-complete on bipartite graphs for any fixed $k \geq 3$, where Π_2^p is a complexity class in the polynomial hierarchy containing both NP and coNP (we refer to [9] for the exact definition of this complexity class). On the positive side, 2-Choosability can be solved in polynomial time on all graphs [7]. The problems 3-Choosability and 4-Choosability remain Π_2^p-complete on planar graphs [15], whereas every planar graph is 5-choosable [27].

We study the Choosability problem on graphs that are characterized by a forbidden induced subgraph H. A graph G is *H-free* if none of its induced subgraphs is isomorphic to H. We write $F_1 + F_2$ to denote the disjoint union of two graphs F_1 and F_2, and rF to denote the disjoint union of r copies of a graph F. The path on ℓ vertices is denoted by P_ℓ, and $K_{1,3}$ denotes the claw, i.e., the star with four vertices.

In Section 2, we determine the computational complexity of the Choosability problem restricted to H-free graphs for all but four graphs H. More precisely, we show that the problem is NP-hard for every graph H that does not belong to $\{K_{1,3}, P_1, 2P_1, 3P_1, P_1 + P_2, P_1 + P_3, P_2, P_3, P_4\}$. On the positive side, we show that the problem can be solved in polynomial time if $H \in \{P_1, 2P_1, 3P_1, P_2, P_3\}$. This leaves open the complexity of the problem when $H \in \{K_{1,3}, P_1+P_2, P_1+P_3, P_4\}$. Let us briefly discuss these four remaining cases. If $H \in \{P_1+P_2, P_1+P_3, P_4\}$, then the class of H-free graphs contains the class of complete bipartite graphs as a subclass. Any attempt to show polynomial-time solvability of Choosability on H-free graphs for $H \in \{P_1 + P_2, P_1 + P_3, P_4\}$ can therefore initially be restricted to complete bipartite graphs only. Although several results on choosability of complete bipartite graphs exist in the literature [20,25,26], the computational complexity of the Choosability problem on complete bipartite graphs remains open. The last case, $H = K_{1,3}$, is also well-studied in the literature [12,13,14,17]. The class of $K_{1,3}$-free graphs, better known as claw-free graphs, contains the class of line graphs as a subclass. The well-known and long-standing List Coloring Conjecture, usually attributed to Vizing (cf. [18]), states that every line graph is k-choosable if and only if it is k-colorable. It is known that the k-Coloring problem is NP-complete on line graphs for all $k \geq 3$ [21,24]. This implies that k-Choosability is NP-complete on line graphs for every $k \geq 3$, unless the List Coloring Conjecture is false. Consequently, assuming the List Coloring Conjecture is true, k-Choosability is NP-hard on $K_{1,3}$-free graphs for every $k \geq 3$, implying that Choosability is NP-hard on this graph class as well.

In Section 3, we consider the parameterized complexity of the Choosability problem on H-free graphs where H is a *linear forest*, i.e., a disjoint union of one or more paths. Golovach and Heggernes [10] proved that Choosability on P_5-free graphs is fixed-parameter tractable with parameter k, i.e., can be solved in time $f(k) \cdot |V(G)|^{O(1)}$ for some function f that only depends on k. We generalize this result by showing that Choosability on H-free graphs is fixed-parameter tractable with parameter k for *every* linear forest H. Our result is obtained by combining results of Alon [1] and Fellows et al. [8] together with a very recent result of Atminas, Lozin and Razgon [2] on graphs without long induced paths. We point out that extending our result to fixed-parameter tractability for non-linear forests H is not possible, assuming that the List Coloring Conjecture is true and $\mathsf{P} \neq \mathsf{NP}$. The reason is that every non-linear forest H contains a vertex of degree at least 3, implying that every H-free graph is $K_{1,3}$-free for such graphs H. As we noticed above, k-CHOOSABILITY is NP-hard on $K_{1,3}$-free graphs for every $k \geq 3$ if the List Coloring Conjecture is true.

The fact that CHOOSABILITY on H-free graphs turns out to be fixed-parameter tractable with parameter k for every linear forest H is somewhat surprising, given the fact that a similar result cannot be obtained for the COLORING problem unless $\mathsf{P} = \mathsf{NP}$. The latter follows from the fact that there exist combinations of integers k and linear forests H for which the k-COLORING problem for H-free graphs is NP-complete; for example, it is known that 4-COLORING is NP-complete for P_8-free graphs [5] and that 6-COLORING is NP-complete for P_7-free graphs [4]. Only very few parameterized results for COLORING on H-free graphs are known. It is known that this problem is fixed-parameter tractable with parameter k when $H = rP_1 + P_2$ for any fixed integer r [6], but it is not known whether COLORING is fixed-parameter tractable with parameter k when restricted to $2P_2$-free graphs. Although Hoáng et al. [19] proved that k-COLORING can be solved in polynomial time on P_5-free graphs for every integer k, they posed fixed-parameter tractability of COLORING with parameter k for this graph class as an open question.

2 Polynomial-time Solvable and NP-Hard Cases

In this section we determine the computational complexity of the CHOOSABILITY problem on H-free graphs for all but four graphs H. We will use the following dichotomy result of Král', Kratochvíl, Tuza, and Woeginger [23].

Theorem 1 ([23]). *Let H be a fixed graph. The* COLORING *problem restricted to H-free graphs is polynomial-time solvable if H is an induced subgraph of P_4 or of $P_1 + P_3$, and is* NP*-complete otherwise.*

Theorem 1 provides a complete complexity classification of the COLORING problem on H-free graphs. Similar classifications exist for two well-known generalizations of COLORING, namely LIST COLORING and PRECOLORING EXTENSION [11]. The latter problem takes as input a graph G, an integer k and a precoloring of G, i.e., a mapping $c_W : W \to \{1, 2, \ldots, k\}$ for some subset $W \subseteq V(G)$, and asks whether c_W can be extended to a k-coloring of G. The following result is a significant step towards a complete complexity classification for the CHOOSABILITY problem restricted to H-free graphs.

Theorem 2. *Let H be a fixed graph. The* CHOOSABILITY *problem restricted to H-free graphs is* NP*-hard if $H \notin \{K_{1,3}, P_1, 2P_1, 3P_1, P_1+P_2, P_1+P_3, P_2, P_3, P_4\}$, and is polynomial-time solvable if $H \in \{P_1, 2P_1, 3P_1, P_2, P_3\}$.*

Proof. We start by considering the cases for which we must prove polynomial-time solvability. Because an H'-free graph is H-free whenever H' is an induced subgraph of a graph H, we may restrict ourselves to the cases $H = 3P_1$ and $H = P_3$. Let $H = 3P_1$. It is known that a $3P_1$-free graph is k-choosable for some integer k if and only if it is k-colorable [13]. Since $3P_1$ is an induced subgraph of P_1+P_3, the result follows from Theorem 1. Let $H = P_3$. Because every connected component of a P_3-free graph G is a complete graph, and therefore $3P_1$-free, we obtain polynomial-time solvability by applying the previous argument on each connected component of G.

Now consider the cases $H \notin \{K_{1,3}, P_1, 2P_1, 3P_1, P_1 + P_2, P_1 + P_3, P_2, P_3, P_4\}$, i.e., we let H be a graph that is not an induced subgraph of $K_{1,3}$, $P_1 + P_3$ or P_4. We distinguish two cases, depending on whether or not H has a *dominating vertex*, i.e., a vertex that is adjacent to all other vertices.

Case 1. H has no dominating vertex.

We make a reduction from the Coloring problem, which remains NP-complete on H-free graphs according to Theorem 1. Let G be an H-free graph and k an integer. We denote the degree of a vertex $u \in V(G)$ by $d_G(u)$. Then we may assume without loss of generality that $d_G(u) \geq k$ for all $u \in V(G)$, as otherwise we perform the following well-known procedure (see e.g. [5]). We repeatedly delete a vertex with degree at most $k-1$ from G until no such vertex remains. The resulting graph G_k (which may be empty) is k-colorable if and only if G is k-colorable. Moreover, G_k is H-free and can be obtained in polynomial time.

We define $k^* = k + \sum_{u \in V(G)} (d_G(u) - k + 1)$ and construct a graph G^* by adding a set of $k^* - k$ vertices $T = \{t_1, \ldots, t_{k^*-k}\}$ that are adjacent to each other and to every vertex of G. Because H has no dominating vertex and G is H-free, we find that G^* is H-free. Hence, it remains to show that G is k-colorable if and only if G^* is k^*-choosable.

First suppose that G^* is k^*-choosable. Then G^* has a coloring c that respects the list assignment $\mathcal{L}^* = \{L^*(u) \mid u \in V(G^*)\}$ with $L^*(u) = \{1, \ldots, k^*\}$ for all $u \in V(G^*)$. Because the $k^* - k$ vertices in T are mutually adjacent, they are all colored differently by c. Hence, by taking the restriction of c to $V(G)$, we find that G is k-colorable.

Now suppose that G is k-colorable. We prove that G^* is k^*-choosable. In order to do this, let $\mathcal{L}^* = \{L^*(u) \mid u \in V(G^*)\}$ be an arbitrary k^*-list assignment of G^*. We will construct a coloring of G^* that respects $\mathcal{L}^*$. We start by coloring the vertices of T and, if possible, reducing G^* by applying the following procedure:

– As long as there is an uncolored vertex t_j such that $L^*(t_j)$ contains an unused color x and there is a vertex $u \in V(G)$ with $x \notin L^*(u)$ do as follows: give t_j color x and delete all vertices $u \in V(G)$ for which at least $d_G(u) - k + 1$ used colors are not in $L^*(u)$. Afterwards, we consider the vertices of the remaining set $T' \subseteq T$ one by one and give them any unused color from their list.

It is possible to color all vertices of T by this procedure, because $|L^*(t_j)| = k^*$ for $j = 1, \ldots, k^* - k$ and $|T| = k^* - k \leq k^*$. The procedure is correct due to the following reason. Let $u \in V(G)$. After coloring all vertices of T we can partition T into two sets A_u and B_u, where A_u consists of those vertices of T that received a color not in $L^*(u)$ and $B_u = T \setminus A_u$ consists of those vertices of T that received a color from $L^*(u)$. Then the number of available colors for u is $k^* - |B_u| = k^* - (|T| - |A_u|) = k^* - (k^* - k - |A_u|) = k + |A_u|$, whereas u still has $d_G(u)$ uncolored neighbors in G^*. If $k + |A_u| \geq d_G(u) + 1$, or equivalently, if $|A_u| \geq d_G(u) - k + 1$, then we may delete u; after coloring all vertices of $V(G^*) \setminus \{u\}$, we are guaranteed that there exists at least one color in $L^*(u)$ that is not used on the neighborhood of u in G^*, and we can give u this color.

After coloring the vertices in T as described above, we let U denote the subset of vertices of $V(G)$ that were not deleted while coloring T. Recall the set T' defined in the procedure. We distinguish two cases.

First suppose $T' = \emptyset$. Then every $t \in T$ received a color that does not appear in the list $L^*(u)$ for at least one vertex $u \in V(G)$ that was not yet deleted from the graph at the moment t was colored. Consequently, the size of some set A_u increases by 1 whenever a vertex of T receives a color. Recall that a vertex $u \in U$ is deleted from the graph as soon as the size of A_u reaches $d_G(u) - k + 1$. Since $|T| = k^* - k = \sum_{u \in V(G)} (d_G(u) - k + 1)$, every vertex of $V(G)$ is deleted from the graph at some point during the procedure. Hence $U = \emptyset$, implying that G^* is k^*-choosable due to the correctness of our procedure.

Now suppose $T' \neq \emptyset$ and let $t' \in T'$. Because $|L^*(t')| = k^*$ and $|T| = k^* - k$, the list $L^*(t')$ contains a set D of k colors that are not used as a color for any vertex in T (including t' itself). We will show that $D \subseteq L^*(u)$ for every $u \in U$. For contradiction, suppose there exists a color $y \in D$ and a vertex $w \in U$ such that $y \notin L^*(w)$. By the definition of T', vertex t' received a color z that appears in the list $L^*(u)$ for every $u \in U$. But according to our procedure, we would not have colored t' with color z if color y was also available; note that y is not used to color any vertex in $T \setminus \{t'\}$ by the definition of D. This yields the desired contradiction, implying that $D \subseteq L^*(u)$ for every $u \in U$. By symmetry of the colors, we may assume that $D = \{1, \ldots, k\}$. We assumed that G is k-colorable, so G has a coloring $c : V(G) \to \{1, \ldots, k\}$, and we can safely assign color $c(u)$ to each $u \in U$. Due to this and the correctness of our procedure, we conclude that G^* is also k-choosable when $T' \neq \emptyset$.

Case 2. H has a dominating vertex.

First suppose that H is K_3-free, where K_3 is the complete graph on three vertices. Because $H \notin \{P_1, P_2, P_3, K_{1,3}\}$, H has a dominating vertex, and H is K_3-free, H is isomorphic to the star $K_{1,s}$ with $s+1$ vertices for some $s \geq 4$. Consequently, H has exactly one dominating vertex. Let $H' = sP_1$ be the graph obtained from H by removing this dominating vertex. As shown in Case 1, the Choosability problem is NP-hard for H'-free graphs, and hence for H-free graphs as well.

Now suppose that H is not K_3-free. Recall that for any fixed integer $k \geq 3$ the k-Choosability problem is Π_2^p-complete, and hence NP-hard, for bipartite graphs [16]. Since K_3 is an induced subgraph of H, the class of H-free graphs is a superclass of the class of K_3-free graphs, which in turn is a superclass of the class of bipartite graphs. This completes the proof of Theorem 2. $\square$

3 Fixed-Parameter Tractability

Our next result implies that, for every linear forest H, the Choosability problem for H-free graphs is fixed-parameter tractable when parameterized by k.

Theorem 3. *For any linear forest H, the k-Choosability problem restricted to H-free graphs can be solved in linear time for every fixed integer k.*

Proof. Let H be a linear forest with connected components $H_1, \ldots, H_p$. By definition, each H_i is a path on $|V(H_i)|$ vertices. We define $\ell = |V(H_1)| + \ldots + |V(H_p)| + p - 1$. Note that the class of H-free graphs is a subclass of the class of P_ℓ-free graphs. We prove that Choosability can be solved in linear time on P_ℓ-free graphs for any fixed integer k, implying Theorem 3.

Let G be a P_ℓ-free graph. Atminas, Lozin and Razgon [2] showed that for any two integers s and r, there exists an integer $t(s,r)$ such that any graph of treewidth at least $t(s,r)$ contains the path P_s as an induced subgraph or the complete bipartite graph $K_{r,r}$ as a subgraph. In order to use this result, we set $s = \ell$ and $r = r^* = \lceil 4\binom{k^4}{k}\log(2\binom{k^4}{k})\rceil$ (which we explain later). Using Bodlaender's algorithm [3] we can test in linear time whether the treewidth of G is at most $t(\ell, r^*) - 1$.

If the treewidth of G is at most $t(\ell, r^*) - 1$, then we use a result due to Fellows et al. [8], stating that the Choosability problem is linear-time solvable for graphs of bounded treewidth. Suppose that G has treewidth at least $t(\ell, r^*)$. Then due to the aforementioned result of Atminas, Lozin and Razgon [2] and our assumption that G is P_ℓ-free, we find that G contains a graph F isomorphic to K_{r^*,r^*} as a subgraph. The average degree of a graph with n vertices and m edges is equal to $2m/n$. Alon [1] proved that any graph with average degree at least $\lceil 4\binom{k^4}{k}\log(2\binom{k^4}{k})\rceil$ is not k-choosable. Because F has average degree $r^* = \lceil 4\binom{k^4}{k}\log(2\binom{k^4}{k})\rceil$, this means that F is not k-choosable. Since G is a supergraph of F, we conclude that G is not k-choosable either in this case. $\square$

4 Conclusions

We finish this note with the following remark. On general graphs, Choosability is provably harder than List Coloring under the assumption that $\mathsf{NP} \neq \mathsf{coNP}$, as the former problem is Π_2^p-complete [7] whereas the latter problem is NP-complete. However, Theorem 2 shows that Choosability becomes easier than List Coloring when restricted to $3P_1$-free graphs. After all, Choosability is polynomial-time solvable on H-free graphs if $H = 3P_1$ by Theorem 2, while List Coloring is NP-complete even on $(3P_1, P_1 + P_2)$-free graphs, which are exactly those graphs that can be obtained from complete graphs by removing a number of matching edges [11].

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