Asymptotic theory for the multidimensional random on-line nearest-neighbour graph

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Abstract

The on-line nearest-neighbour graph on a sequence of n uniform random points in $(0,1)^d$ ($d \in \mathbb{N}$) joins each point after the first to its nearest neighbour amongst its predecessors. For the total power-weighted edge-length of this graph, with weight exponent $\alpha \in (0, d/2]$, we prove $O(\max\{n^{1-(2\alpha/d)}, \log n\})$ upper bounds on the variance. On the other hand, we give an $n \to \infty$ large-sample convergence result for the total power-weighted edge-length when $\alpha > d/2$. We prove corresponding results when the underlying point set is a Poisson process of intensity n.

Key words and phrases: Random spatial graphs; network evolution; variance asymptotics; martingale differences.

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1 Introduction

The (random) on-line nearest-neighbour graph, which we describe in detail below, is one of the simplest models of the evolution of (random) spatial networks. Graphs with an 'on-line' construction, whereby vertices are added one by one and connected to existing vertices according to some rule, have recently been the subject of considerable study in relation to the modelling of real-world networks. Examples of modelling applications include the internet, social networks, and communications networks in general. The literature is extensive (see e.g. [6, 11] for surveys), but mostly non-rigorous; rigorous mathematical results are fewer in

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number, even for simple models, and existing results concentrate on graph-theoretic rather than geometrical properties (see e.g. [4, 5]).

In recent years, much progress has been made in obtaining large-sample limit theorems for functionals defined on graphs in geometric probability, see e.g. [2, 3, 9, 12–14, 17, 18]. The graphs in question are locally determined in a certain sense. A natural functional of interest is the total (Euclidean) edge length of the graph, or, more generally, the total power-weighted edge-length, i.e. the sum of the α-powers of each edge length for a fixed weight exponent $\alpha > 0$. The on-line nearest-neighbour graph (ONG) is of particular theoretical interest since its total power-weighted length functional has both normal and non-normal limiting regimes, depending on the exponent α. (Another example of such a graph was given in [15], but there spatial boundary effects were crucial.) Moreover, the complete central limit theorem for the ONG seems just beyond reach of existing general results such as those of [3, 13, 14, 17] which employ various concepts of 'stabilization'.

The ONG is constructed on points arriving sequentially in $\mathbb{R}^d$ by connecting each point (vertex) after the first to its nearest (in the Euclidean sense) predecessor. Many real-world networks have certain characteristics in common, including spatial structure, localization (connections tend to join nearby nodes), and sequential growth (the network evolves over time by the addition of new nodes). The ONG is one of the simplest models of spatial network evolution that captures these features.

The ONG appeared in [4] as a growth model of the world wide web graph (for $d = 2$), as a simplified version of the so-called FKP network model [7]. [4] studied, amongst other things, the vertex-degree distribution of the ONG. Here we are concerned with geometrical properties: in particular, the large-sample asymptotic behaviour of the total power-weighted edge length of the ONG on uniform random points in the unit cube $(0, 1)^d$, $d \in \mathbb{N} := \{1, 2, 3, \ldots\}$.

In the present paper, we add to previous work on the ONG. In [19], explicit laws of large numbers were given for the total power-weighted length of the random ONG in $(0, 1)^d$, via an application of general results from [18]. [13, 16] gave partial classification of the distributional limits of the power-weighted length of the ONG on uniform random points in $(0, 1)^d$. In particular, when $d = 1$, for exponent $\alpha > 1/2$, [16] showed, by a 'divide-and-conquer' approach (and the 'contraction method' [10]), that the limiting distribution of the centred total power-weighted length of the ONG is described in terms of a distributional fixed-point equation. In particular, these distributional limits are not Gaussian.

It is natural to look for central limit theorems (CLTs), i.e. proving that, for general dimensions $d \in \mathbb{N}$, for suitable values of α, the total weight, centred and appropriately scaled, converges in distribution to a Gaussian limit. Penrose [13] gave such a CLT for $d \in \mathbb{N}$ and $\alpha \in (0, d/4)$: see Section 2 below. As stated in [13, 16], it is suspected that a CLT holds throughout $\alpha \in (0, d/2]$. One contribution of the present paper is to give variance upper bounds for the total power-weighted edge length of the ONG for $\alpha \in (0, d/2]$. These upper bounds are believed to be tight, and are consistent with the conjectured central limit theory. Our methods for estimating variances are based on a martingale difference approach, and delicate estimates of changes in the power-weighted length of the ONG on re-sampling a particular vertex.

We also give a convergence in distribution result for the total power-weighted length of the ONG, centred as necessary, for $\alpha > d/2$. This improves on an earlier result from [16], where such a result was given for $\alpha > d$. We prove this result via a refinement of the martingale difference technique that yields the variance bounds.

Intuition behind the $\alpha = d/2$ phase transition in the limiting behaviour is provided by the fact that increasing the weight exponent α increases the relative importance of longer edges; for large enough α this amplifies the inhomogeneities in the structure of the ONG ('old' edges tend to be much longer) and so destroys the Gaussian behaviour.

In the next section we give a formal definition of the model and state our main results.

2 Definitions and results

Let $d \in \mathbb{N}$. Let $(\mathbf{X}_1, \mathbf{X}_2, \ldots)$ be a sequence of points in $(0,1)^d$. For $n \in \mathbb{N}$, let $\mathcal{X}_n$ denote the finite sequence $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$. The on-line nearest-neighbour graph (ONG) on vertex set $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ is constructed by joining each point of $\mathcal{X}_n$ after the first by an edge to its nearest neighbour amongst those points that precede it in the sequence. That is, for $i = 2, \ldots, n$ we join $\mathbf{X}_i$ by a directed edge $(\mathbf{X}_i, \mathbf{X}_j)$ to $\mathbf{X}_j$, $1 \le j < i$, satisfying

$$\|\mathbf{X}_j - \mathbf{X}_i\| = \min_{1 \le k < i} \|\mathbf{X}_k - \mathbf{X}_i\|,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. We use lexicographic order on $\mathbb{R}^d$ to break any ties. The resulting directed graph is the ONG on $\mathcal{X}_n$, denoted $\mathrm{ONG}(\mathcal{X}_n)$.

It is sometimes more convenient to view the ONG as an undirected graph, by ignoring the directedness of the edges. From this perspective $\mathrm{ONG}(\mathcal{X}_n)$ is a tree; in view of the directed graph picture, it can be seen as rooted at $\mathbf{X}_1$.

From now on we take the points $\mathbf{X}_1, \mathbf{X}_2, \ldots$ to be random. On an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(\mathbf{U}_1, \mathbf{U}_2, \ldots)$ be a sequence of independent uniformly distributed random vectors in $(0,1)^d$. For $n \in \mathbb{N}$, let $\mathcal{U}_n := (\mathbf{U}_1, \ldots, \mathbf{U}_n)$. The points $\{\mathbf{U}_1, \ldots, \mathbf{U}_n\}$ of the sequence $\mathcal{U}_n$ then constitute a binomial point process consisting of n independent uniform random vectors in $(0,1)^d$.

For $\mathbf{x} \in \mathbb{R}^d$ and $\mathcal{X} \subset \mathbb{R}^d$, let $d(\mathbf{x}; \mathcal{X}) := \inf_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \|\mathbf{x} - \mathbf{y}\|$ denote the distance from $\mathbf{x}$ to its Euclidean nearest neighbour in $\mathcal{X} \setminus \{\mathbf{x}\}$. For $d \in \mathbb{N}$ and $\alpha > 0$, define the total power-weighted edge length of $\mathrm{ONG}(\mathcal{U}_n)$ by $\mathcal{O}^{d,\alpha}(\mathcal{U}_1) := 0$ and for $n \ge 2$

$$\mathcal{O}^{d,\alpha}(\mathcal{U}_n) := \sum_{i=2}^{n} (d(\mathbf{U}_i; \mathcal{U}_{i-1}))^\alpha.$$

Also, define the centred version $\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n) := \mathcal{O}^{d,\alpha}(\mathcal{U}_n) - \mathbb{E}[\mathcal{O}^{d,\alpha}(\mathcal{U}_n)]$. We are interested in the behaviour of $\mathcal{O}^{d,\alpha}(\mathcal{U}_n)$ as $n \to \infty$.

We also consider the ONG defined on a Poisson number of points. Let $(N(t); t \ge 0)$ be the counting process of a homogeneous Poisson process of unit rate in $(0, \infty)$, independent of $(\mathbf{U}_1, \mathbf{U}_2, \ldots)$. Thus for $\lambda > 0$, $N(\lambda)$ is a Poisson random variable with mean λ. With $\mathcal{U}_n$ as defined above, for $\lambda > 0$ set $\mathcal{P}_\lambda := \mathcal{U}_{N(\lambda)}$. In the Poisson case, we again use the notation $\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{P}_\lambda) = \mathcal{O}^{d,\alpha}(\mathcal{P}_\lambda) - \mathbb{E}[\mathcal{O}^{d,\alpha}(\mathcal{P}_\lambda)]$ for the (deterministically) centred version. Note that the points of the sequence $\mathcal{P}_\lambda$ constitute a homogeneous (marked) Poisson point process of intensity λ on $(0,1)^d$. In this 'Poissonized' version of the ONG, we are again interested in the large-sample asymptotics, i.e. the limit $\lambda \to \infty$.

For $d \in \mathbb{N}$ let v_d denote the volume of the unit-radius Euclidean d-ball, i.e.

$$v_d := \pi^{d/2} \left[\Gamma \left(1 + (d/2) \right) \right]^{-1};$$

see e.g. equation (6.50) of [8]. The following result summarizes previous work (see Theorem 4 of [19] and Theorem 2.1 of [16]) on the first-order behaviour of $\mathcal{O}^{d,\alpha}(\mathcal{U}_n)$. Here and subsequently '$\stackrel{L^p}{\longrightarrow}$' denotes convergence in L^p-norm, $p \geq 1$.

Proposition 2.1 *[16, 19] Let $d \in \mathbb{N}$. For $\alpha \in (0, d)$, as $n \to \infty$*

$$n^{(\alpha-d)/d} \mathcal{O}^{d,\alpha}(\mathcal{U}_n) \stackrel{L^1}{\longrightarrow} \frac{d}{d-\alpha} v_d^{-\alpha/d} \Gamma(1 + (\alpha/d)).$$

For $\alpha = d$, as $n \to \infty$

$$\mathbb{E}[\mathcal{O}^{d,d}(\mathcal{U}_n)] \sim v_d^{-1} \log n.$$

For $\alpha > d$, there exists $\mu(d, \alpha) \in (0, \infty)$ such that as $n \to \infty$

$$\mathbb{E}[\mathcal{O}^{d,\alpha}(\mathcal{U}_n)] \to \mu(d, \alpha).$$

Remarks. (a) In the particular case $d = 1$, Proposition 2.1 of [16] gives

$$\mu(1, \alpha) = \frac{2}{\alpha(\alpha+1)} \left(1 + \frac{2^{-\alpha}}{\alpha - 1} \right), \quad (\alpha > 1).$$

(b) These results carry over to the Poisson point process case with $\mathcal{O}^{d,\alpha}(\mathcal{P}_n)$: this observation follows from now well-known 'Poissonization' methods.

Second-order (i.e. convergence in distribution) results for $\mathcal{O}^{d,\alpha}(\mathcal{U}_n)$ and $\mathcal{O}^{d,\alpha}(\mathcal{P}_\lambda)$ were given in [13, 16]. Specifically, Theorem 3.6 of Penrose [13] gives a CLT for $\alpha \in (0, d/4)$ and Theorem 2.1(ii) of [16] gives convergence to a non-Gaussian limit for $\alpha > d$. We summarize these results in Proposition 2.2 below. Denote by $\mathcal{N}(0, \sigma^2)$ the normal distribution with mean 0 and variance $\sigma^2 \geq 0$; this includes the degenerate case $\mathcal{N}(0, 0) \equiv 0$. Here and subsequently '$\stackrel{d}{\longrightarrow}$' denotes convergence in distribution.

Proposition 2.2 *Suppose $d \in \mathbb{N}$.*

(i) Suppose $\alpha \in (0, d/4)$. Then [13] there exist constants $\sigma^2_{d,\alpha} \in [0, \infty)$ and $\delta^2_{d,\alpha} \in [0, \sigma^2_{d,\alpha}]$ such that

$$\lim_{\lambda \to \infty} \lambda^{(2\alpha-d)/d} \mathbb{V}\mathrm{ar}[\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{P}_\lambda)] = \sigma^2_{d,\alpha}, \quad \lim_{n \to \infty} n^{(2\alpha-d)/d} \mathbb{V}\mathrm{ar}[\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n)] = \sigma^2_{d,\alpha} - \delta^2_{d,\alpha}, \quad (1)$$

and as $\lambda, n \to \infty$

$$\lambda^{(2\alpha-d)/(2d)} \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{P}_\lambda) \stackrel{d}{\longrightarrow} \mathcal{N}(0, \sigma^2_{d,\alpha}), \quad n^{(2\alpha-d)/(2d)} \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n) \stackrel{d}{\longrightarrow} \mathcal{N}(0, \sigma^2_{d,\alpha} - \delta^2_{d,\alpha}). \quad (2)$$

(ii) Suppose $\alpha > d$. Then [16] there exists a mean-zero non-Gaussian random variable $Q(d, \alpha)$ such that as $n \to \infty$

$$\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n) \longrightarrow Q(d, \alpha), \quad (3)$$

where the convergence is almost sure and in L^p, for any $p \geq 1$.

It is conjectured (see [13, 16]) that the CLTs of Proposition 2.2(i) are in fact valid for all $\alpha \in (0, d/2)$:

Conjecture 2.1 *[13, 16] Suppose $d \in \mathbb{N}$. The limit theorems (1) and (2) are also valid for $\alpha \in [d/4, d/2)$.*

In ongoing work, we have made some progress towards Conjecture 2.1, but do not yet have a proof.

The first main result of the present paper, Theorem 2.1 below, provides a version of the variance upper bounds in (1) for all $\alpha \in (0, d/2]$. Theorem 2.1 is thus consistent with Conjecture 2.1, and the bounds in Theorem 2.1 are believed to be sharp (up to a constant factor).

Theorem 2.1 *Suppose $d \in \mathbb{N}$.*

(i) For $\alpha \in (0, d/2)$, there is a constant $C \in (0, \infty)$ such that for all $n \in \mathbb{N}$, $\lambda \geq 1$

$$\mathbb{V}\mathrm{ar}[\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n)] \leq C n^{1-(2\alpha/d)}, \quad \mathbb{V}\mathrm{ar}[\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{P}_\lambda)] \leq C \lambda^{1-(2\alpha/d)}. \tag{4}$$

(ii) There is a constant $C \in (0, \infty)$ such that for all $n \in \mathbb{N}$, $\lambda \geq 1$

$$\mathbb{V}\mathrm{ar}[\tilde{\mathcal{O}}^{d,d/2}(\mathcal{U}_n)] \leq C \log(1+n), \quad \mathbb{V}\mathrm{ar}[\tilde{\mathcal{O}}^{d,d/2}(\mathcal{P}_\lambda)] \leq C \log(1+\lambda). \tag{5}$$

Our second main result extends (3) to all $\alpha > d/2$ and also to the Poisson case.

Theorem 2.2 *Suppose $d \in \mathbb{N}$ and $\alpha > d/2$. Then there exists a mean-zero random variable $Q(d, \alpha)$ (which is non-Gaussian for $\alpha > d$) such that:*

(i) as $n \to \infty$

$$\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n) \xrightarrow{L^2} Q(d, \alpha); \tag{6}$$

(ii) and, with the coupling of $\mathcal{U}_n$ and $\mathcal{P}_n$ given by $\mathcal{P}_n := \mathcal{U}_{N(n)}$,

$$\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{P}_n) \xrightarrow{L^2} Q(d, \alpha). \tag{7}$$

Remarks. (a) The fact that for $\alpha > d$ the random variables $Q(d, \alpha)$ in (3) and Theorem 2.2 are not normal follows since convergence also holds without any centring; see Theorem 2.1(ii) of [16]. In the special case $d = 1$, a weaker version of (6), with convergence in distribution only, was given for $\alpha > 1/2$ in Theorem 2.2 of [16]. In the $d = 1$ case, more information can be obtained about the distribution of $Q(1, \alpha)$ using a 'divide-and-conquer' technique; see [16], in particular Theorem 2.2, where the distribution of $Q(1, \alpha)$, $\alpha > 1/2$ is given (in the binomial setting, and the result carries over to the Poisson setting by Theorem 2.2 here). Indeed, $Q(1, \alpha)$, $\alpha > 1/2$, is given by the unique solution to a distributional fixed-point equation, and in particular is not Gaussian; see [16] for details. We suspect that $Q(d, \alpha)$ is non-Gaussian for $\alpha \in (d/2, d]$ also for $d \geq 2$.

(b) A closely related 'directed' version of the one-dimensional ONG is the 'directed linear tree' introduced in [15], in which each point in a sequence of points in $(0,1)$ is joined to its nearest predecessor to the *left*. Following the methods of the present paper, one can obtain results for that model analogous to the $d = 1$ cases of all those in this section.

Theorem 2.1(ii) suggests that the case $\alpha = d/2$ is of a special nature. Moreover, the case $d = 2$, $\alpha = 1$ is of natural interest, where we have the total Euclidean length of the ONG on random points in $(0,1)^2$. We conjecture the following.

Conjecture 2.2 *Let $d \in \mathbb{N}$. There exists a constant $\sigma^2_{d,d/2} \in (0,\infty)$ such that*

$$(\log n)^{-1/2} \tilde{\mathcal{O}}^{d,d/2}(\mathcal{U}_n) \xrightarrow{d} \mathcal{N}(0, \sigma^2_{d,d/2}), \text{ as } n \to \infty.$$

The proof (or refutation) of Conjecture 2.2 seems to be a challenging open problem.

The structure of the remainder of the paper is as follows. In Section 3 we give some preparatory results on the properties of the ONG. In Section 4 we use a martingale difference technique to prove Theorem 2.1. In Section 5 we refine the martingale difference technique to give a proof of Theorem 2.2.

3 Preliminaries

First we introduce some more notation. Let $\mathrm{card}(\mathcal{X})$ denote the cardinality (number of elements) of a finite set $\mathcal{X}$, and let $\mathbf{0}$ be the origin of $\mathbb{R}^d$ $(d \in \mathbb{N})$. For measurable $R \subset \mathbb{R}^d$, let $|R|$ denote the d-dimensional Lebesgue measure of R. Let $\mathrm{diam}(R) = \sup_{\mathbf{x},\mathbf{y} \in R} \|\mathbf{x} - \mathbf{y}\|$ denote the (Euclidean) diameter of a bounded set $R \subset \mathbb{R}^d$. Let $B(\mathbf{x}; r)$ be the (closed) Euclidean d-ball with centre $\mathbf{x} \in \mathbb{R}^d$ and radius $r > 0$.

In the analysis in Sections 4 and 5 below, we will need detailed properties of the change in total weight of the ONG on $\mathcal{U}_n$ when the point $\mathbf{U}_i$, $i \in \{1,\ldots,n\}$, is independently re-sampled, i.e., replaced by an independent copy $\mathbf{U}_i'$. The changes due to edges *incident* to $\mathbf{U}_i, \mathbf{U}_i'$ require most work to deal with. To study these, we make use of the fact that an edge from $\mathbf{U}_j$ with $j > i$ can only be incident to $\mathbf{U}_i$ if $\mathbf{U}_j$ falls in the Voronoi cell of $\mathbf{U}_i$ with respect to $\{\mathbf{U}_1,\ldots,\mathbf{U}_i\}$. Hence the preliminary results in this section begin with an analysis of such Voronoi cells.

The next lemma gives bounds on the expected diameter of Voronoi cells in $(0,1)^d$ with respect to $\mathcal{U}_n$. For $n \in \mathbb{N}$, let $V_n(\mathbf{x})$ be the Voronoi cell of $\mathbf{x} \in (0,1)^d$ with respect to $\{\mathbf{x}, \mathbf{U}_1, \ldots, \mathbf{U}_n\}$:

$$V_n(\mathbf{x}) := \left\{ \mathbf{y} \in (0,1)^d : \|\mathbf{x} - \mathbf{y}\| \leq \min_{1 \leq i \leq n} \|\mathbf{y} - \mathbf{U}_i\| \right\} \subseteq (0,1)^d. \tag{8}$$

Lemma 3.1 *Let $d \in \mathbb{N}$, $\beta > 0$. Then there exists $C \in (0,\infty)$ such that for all $n \in \mathbb{N}$*

$$\sup_{\mathbf{x} \in (0,1)^d} \mathbb{E}[(\mathrm{diam}(V_n(\mathbf{x})))^\beta] \leq C n^{-\beta/d}.$$

We will prove Lemma 3.1 using a construction of overlapping and nested cones from p. 1027 of [14]. The argument works for an arbitrary convex set, not just $(0,1)^d$, but here we only need the latter.

For $d \in \{2,3,\ldots\}$, we can (and do) choose $I \in \mathbb{N}$ and construct C_i, $1 \le i \le I$ a finite collection of infinite closed cones in $\mathbb{R}^d$ with angular radius $\pi/12$ and apex at $\mathbf{0}$, with $\cup_{i=1}^I C_i = \mathbb{R}^d$. Let $C_i(\mathbf{x})$ be the translate of C_i with apex at $\mathbf{x} \in \mathbb{R}^d$. Let $C_i^+(\mathbf{x})$ be the closed cone with apex and principal axis coincident with those of $C_i(\mathbf{x})$ but with angular radius $\pi/6$. When $d = 1$, we take $I = 2$ and let $C_1 = [0,\infty)$, $C_2 = (-\infty, 0]$, and for $\mathbf{x} \in \mathbb{R}$ set $C_1(\mathbf{x}) = C_1^+(\mathbf{x}) = [\mathbf{x}, \infty)$ and $C_2(\mathbf{x}) = C_2^+(\mathbf{x}) = (-\infty, \mathbf{x}]$.

Let $d \in \mathbb{N}$. For $\mathbf{x} \in \mathbb{R}^d$ and $r > 0$, let $C_i(\mathbf{x};r) := C_i(\mathbf{x}) \cap B(\mathbf{x};r)$ and $C_i^+(\mathbf{x};r) := C_i^+(\mathbf{x}) \cap B(\mathbf{x};r)$. For $n \in \mathbb{N}$, define the event

$$E_n(\mathbf{x};r) := \bigcap_{i:1\le i\le I, \mathrm{diam}(C_i(\mathbf{x};r)\cap(0,1)^d)=r} \{\mathcal{U}_n \cap C_i^+(\mathbf{x};r) \neq \emptyset\},$$

with the convention that an empty intersection is Ω. Then $E_n(\mathbf{x};r) \subseteq E_{n+1}(\mathbf{x};r)$, and for $s \ge r$, $E_n(\mathbf{x};r) \subseteq E_n(\mathbf{x};s)$. For $\mathbf{x} \in (0,1)^d$, set

$$R_n(\mathbf{x}) := \inf\{r > 0 : E_n(\mathbf{x};r) \text{ occurs}\}. \tag{9}$$

Note that a.s., $R_n(\mathbf{x}) \le d^{1/2}$. The next lemma is the main step in the proof of Lemma 3.1.

Lemma 3.2 *Suppose $d \in \mathbb{N}$. For $\beta > 0$ there exists $C \in (0,\infty)$ such that for all $n \in \mathbb{N}$*

$$\sup_{\mathbf{x}\in(0,1)^d} \mathbb{E}[R_n(\mathbf{x})^\beta] \le Cn^{-\beta/d}.$$

Proof. For $\mathbf{x} \in (0,1)^d$ and $r > 0$, $\mathbb{P}(R_n(\mathbf{x}) \ge r) \le \mathbb{P}(E_n(\mathbf{x};r)^c)$, so that

$$\mathbb{P}(R_n(\mathbf{x}) > r) \le \mathbb{P}\left(\bigcup_{i:1\le i\le I, \mathrm{diam}(C_i(\mathbf{x})\cap(0,1)^d)\ge r} \{\mathcal{U}_n \cap C_i^+(\mathbf{x};r) = \emptyset\} \right), \tag{10}$$

with the convention that an empty union is empty. Suppose $d \in (0, d^{1/2}]$. For any i with $\mathrm{diam}(C_i(\mathbf{x}) \cap (0,1)^d) \ge r$, we can by convexity choose a (non-random) $\mathbf{z} \in C_i(\mathbf{x}) \cap (0,1)^d$ at distance $r/2$ from $\mathbf{x}$. Then (since $\frac{1}{4} < \frac{1}{2}\sin\frac{\pi}{12}$) we have that $B(\mathbf{z};r/4) \cap (0,1)^d$ is contained in $C_i^+(\mathbf{x};r)$ and, since $r \le d^{1/2}$, has $|B(\mathbf{z};r/4) \cap (0,1)^d| \ge Cr^d$ for some $C \in (0,\infty)$ depending only on d. Hence for any i with $\mathrm{diam}(C_i(\mathbf{x}) \cap (0,1)^d) \ge r$,

$$\mathbb{P}(\mathcal{U}_n \cap C_i^+(\mathbf{x};r) = \emptyset) \le \mathbb{P}(\mathcal{U}_n \cap B(\mathbf{z};r/4) \cap (0,1)^d = \emptyset) \le (1 - Cr^d)^n, \tag{11}$$

for some $C \in (0,\infty)$ depending only on d. Applying Boole's inequality in (10), using (11), and noting that $1 - x \le \mathrm{e}^{-x}$ for any $x \ge 0$, we have that there are constants $C, C' \in (0,\infty)$, depending only on d, such that for all $r > 0$ and $n \in \mathbb{N}$

$$\sup_{\mathbf{x}\in(0,1)^d} \mathbb{P}(R_n(\mathbf{x}) > r) \le C' \exp(-Cnr^d).$$

Hence for $\beta > 0$ and $n \in \mathbb{N}$, setting $s = Cnr^{d/\beta}$,

$$\sup_{\mathbf{x}\in(0,1)^d} \mathbb{E}[R_n(\mathbf{x})^\beta] = \sup_{\mathbf{x}\in(0,1)^d} \int_0^\infty \mathbb{P}(R_n(\mathbf{x}) > r^{1/\beta})\mathrm{d}r \le C' \int_0^\infty \exp(-Cnr^{d/\beta})\mathrm{d}r$$
$$\le C'n^{-\beta/d} \int_0^\infty s^{(\beta/d)-1} \exp(-s)\mathrm{d}s = C'n^{-\beta/d}\Gamma(\beta/d),$$

using Euler's Gamma integral (see e.g. 6.1.1 in [1]) for the last equality. □

Now we can complete the proof of Lemma 3.1.

Proof of Lemma 3.1. With $R_n(\mathbf{x})$ as defined at (9), we claim that

$$\mathrm{diam}(V_n(\mathbf{x})) \le 2R_n(\mathbf{x}) \tag{12}$$

for all $\mathbf{x} \in (0,1)^d$ and all $n \in \mathbb{N}$. Thus for $\beta > 0$, $\mathbb{E}[(\mathrm{diam}(V_n(\mathbf{x})))^\beta] \le C\mathbb{E}[(R_n(\mathbf{x}))^\beta] \le C'n^{-\beta/d}$, by Lemma 3.2, proving Lemma 3.1.

To verify the claim (12), suppose that $\mathbf{y} \in (0,1)^d$ lies at distance $s > r = R_n(\mathbf{x})$ from $\mathbf{x}$. Then we can choose i such that $\mathbf{y} \in C_i(\mathbf{x})$, so clearly $\mathrm{diam}(C_i(\mathbf{x}) \cap (0,1)^d) > r$ and $\mathrm{diam}(C_i(\mathbf{x}; r) \cap (0,1)^d) = r$. By definition of $R_n(\mathbf{x})$ we must have some point of $\mathcal{U}_n \cap C_i^+(\mathbf{x}; r)$; but then this point lies closer to $\mathbf{x}$ than $\mathbf{y}$ does, so $\mathbf{y}$ is not in the Voronoi cell $V_n(\mathbf{x})$. Thus $\sup_{\mathbf{y}\in V_n(\mathbf{x})} \|\mathbf{x}-\mathbf{y}\| \le R_n(\mathbf{x})$. Then the triangle inequality implies the result. □

Next we establish the results that we will need later to control the changes in the ONG on re-sampling a vertex. Let $D \subset \mathbb{R}^d$ be a measurable, non-null convex region and let $\mathbf{x} \in D$. Let $(\mathbf{X}_1, \mathbf{X}_2, \ldots)$ be a sequence of independent uniform random points on D. We use the notation $\mathcal{X}_n = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$, and for $\mathbf{x} \in D$ set $\mathcal{X}_n^{\mathbf{x}} := (\mathbf{x}, \mathbf{X}_1, \ldots, \mathbf{X}_n)$. For a finite sequence $\mathcal{X}$ of points in $\mathbb{R}^d$ and two points $\mathbf{x}, \mathbf{y}$ of $\mathcal{X}$, let $E(\mathbf{x}, \mathbf{y}; \mathcal{X})$ denote the event that $(\mathbf{x}, \mathbf{y})$ is an edge in the ONG on $\mathcal{X}$. Let $\mathcal{O}_{\mathbf{x}}^{d,\alpha}(D; n)$ denote the total power-weighted length, with weight exponent $\alpha > 0$, of edges incident to $\mathbf{x}$ in the ONG on sequence $\mathcal{X}_n^{\mathbf{x}}$, i.e.

$$\mathcal{O}_{\mathbf{x}}^{d,\alpha}(D; n) := \sum_{i=1}^n \mathbf{1}_{E(\mathbf{X}_i, \mathbf{x}; \mathcal{X}_i^{\mathbf{x}})} \|\mathbf{X}_i - \mathbf{x}\|^\alpha. \tag{13}$$

In the special case $D = (0,1)^d$, we will write $\mathbf{U}_i$ for $\mathbf{X}_i$, $\mathcal{U}_n$ for $\mathcal{X}_n$ and $\mathcal{U}_n^{\mathbf{x}}$ for $\mathcal{X}_n^{\mathbf{x}}$, and we abbreviate notation to

$$\mathcal{O}_{\mathbf{x}}^{d,\alpha}(n) := \mathcal{O}_{\mathbf{x}}^{d,\alpha}((0,1)^d; n).$$

Lemma 3.3 *Let $d \in \mathbb{N}$. Suppose $\alpha > 0$. There exists $C \in (0,\infty)$ such that*

$$\sup_{n\in\mathbb{N}} \sup_{\mathbf{x}\in(0,1)^d} \mathbb{E}[\mathcal{O}_{\mathbf{x}}^{d,\alpha}(n)] \le C. \tag{14}$$

Moreover there exists $C \in (0,\infty)$ such that for any m, n with $0 \le m < n$

$$\sup_{\mathbf{x}\in(0,1)^d} \mathbb{E}[\mathcal{O}_{\mathbf{x}}^{d,\alpha}(n) - \mathcal{O}_{\mathbf{x}}^{d,\alpha}(m)] \le C(m+1)^{-\alpha/d}. \tag{15}$$

Proof. Fix $d \in \mathbb{N}$. For $i \in \mathbb{N}$ and $\mathbf{x} \in (0,1)^d$, set

$$W_i := \mathcal{O}_{\mathbf{x}}^{d,1}(i) - \mathcal{O}_{\mathbf{x}}^{d,1}(i-1) = \mathbf{1}_{E(\mathbf{U}_i,\mathbf{x};\mathcal{U}_i^{\mathbf{x}})} \|\mathbf{U}_i - \mathbf{x}\|,$$

with the convention $\mathcal{O}_{\mathbf{x}}^{d,1}(0) := 0$. Thus W_i is the length of the edge from $\mathbf{U}_i$ to $\mathbf{x}$ in the ONG on $\mathcal{U}_n^{\mathbf{x}}$, if such an edge exists, or zero otherwise. Then for $n \in \mathbb{N}$

$$\mathcal{O}_{\mathbf{x}}^{d,\alpha}(n) = \sum_{i=1}^{n} W_i^{\alpha}. \tag{16}$$

Let $i \geq 2$. Given $\{\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}\}$, $W_i > 0$ only if $\mathbf{U}_i$ falls inside the Voronoi cell of $\mathbf{x}$ with respect to $\{\mathbf{x}, \mathbf{U}_1, \ldots, \mathbf{U}_{i-1}\}$, that is $V_{i-1}(\mathbf{x})$ as defined at (8). In addition, given that $\mathbf{U}_i \in V_{i-1}(\mathbf{x})$ (an event of probability $|V_{i-1}(\mathbf{x})|$), we have $W_i \leq \mathrm{diam}(V_{i-1}(\mathbf{x}))$. So for $i \geq 2$

$$\begin{aligned} \mathbb{E}[W_i^{\alpha} \mid \mathbf{U}_1, \ldots, \mathbf{U}_{i-1}] &= \mathbb{E}[W_i^{\alpha} \mathbf{1}_{\{\mathbf{U}_i \in V_{i-1}(\mathbf{x})\}} \mid \mathbf{U}_1, \ldots, \mathbf{U}_{i-1}] \\ &\leq |V_{i-1}(\mathbf{x})| (\mathrm{diam}(V_{i-1}(\mathbf{x})))^{\alpha} \leq (\mathrm{diam}(V_{i-1}(\mathbf{x})))^{d+\alpha}. \end{aligned} \tag{17}$$

Then taking expectations in (17) we obtain

$$\sup_{\mathbf{x} \in (0,1)^d} \mathbb{E}[W_i^{\alpha}] \leq \sup_{\mathbf{x} \in (0,1)^d} \mathbb{E}[(\mathrm{diam}(V_{i-1}(\mathbf{x})))^{d+\alpha}] \leq C(i+1)^{-1-(\alpha/d)}, \tag{18}$$

for some $C \in (0, \infty)$ and all $i \in \mathbb{N}$, by Lemma 3.1. Then we obtain (14) by taking expectations in (16) and using (18). Similarly we obtain (15), this time using the fact that for $1 \leq m < n$

$$\mathbb{E}[\mathcal{O}_{\mathbf{x}}^{d,\alpha}(n) - \mathcal{O}_{\mathbf{x}}^{d,\alpha}(m)] = \sum_{i=m+1}^{n} \mathbb{E}[W_i^{\alpha}] \leq C \sum_{i=m+1}^{\infty} (i+1)^{-1-(\alpha/d)},$$

by (18). This completes the proof. □

In addition to $\mathcal{O}_{\mathbf{x}}^{d,\alpha}(D; n)$, we consider the related quantity

$$\hat{\mathcal{O}}_{\mathbf{x}}^{d,\alpha}(D; n) := \sum_{i=2}^{n} \mathbf{1}_{E(\mathbf{X}_i,\mathbf{x};\mathcal{X}_i^{\mathbf{x}})} (d(\mathbf{X}_i; \{\mathbf{X}_1, \ldots, \mathbf{X}_{i-1}\}))^{\alpha};$$

that is, the total weight of the edges in the ONG on $\mathcal{X}_n$ from those points that would be joined to $\mathbf{x}$ in the ONG on $\mathcal{X}_n^{\mathbf{x}}$. In the case $D = (0,1)^d$, we use the abbreviation

$$\hat{\mathcal{O}}_{\mathbf{x}}^{d,\alpha}(n) := \hat{\mathcal{O}}_{\mathbf{x}}^{d,\alpha}((0,1)^d; n).$$

The following result is analogous to Lemma 3.3.

Lemma 3.4 *Let $d \in \mathbb{N}$. Suppose $\alpha > 0$. There exists $C \in (0, \infty)$ such that*

$$\sup_{n \in \mathbb{N}} \sup_{\mathbf{x} \in (0,1)^d} \mathbb{E}[\hat{\mathcal{O}}_{\mathbf{x}}^{d,\alpha}(n)] \leq C. \tag{19}$$

Moreover there exists $C \in (0, \infty)$ such that for any m, n with $0 \leq m < n$

$$\sup_{\mathbf{x} \in (0,1)^d} \mathbb{E}[\hat{\mathcal{O}}_{\mathbf{x}}^{d,\alpha}(n) - \hat{\mathcal{O}}_{\mathbf{x}}^{d,\alpha}(m)] \leq C(m+1)^{-\alpha/d}. \tag{20}$$

Proof. The proof is similar to that of Lemma 3.3. For $i \in \{2, 3, 4, \ldots\}$ set

$$\hat{W}_i = \hat{\mathcal{O}}^{d,\alpha}_{\mathbf{x}}(i) - \hat{\mathcal{O}}^{d,\alpha}_{\mathbf{x}}(i-1) = \mathbf{1}_{E(\mathbf{U}_i, \mathbf{x}; \mathcal{U}_i^{\mathbf{x}})} d(\mathbf{U}_i; \{\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}\}),$$

where we take $\hat{\mathcal{O}}^{d,\alpha}_{\mathbf{x}}(1) := 0$. Then $\hat{W}_i > 0$ only if $\mathbf{U}_i \in V_{i-1}(\mathbf{x})$. Given that $\mathbf{U}_i \in V_{i-1}(\mathbf{x})$, it follows from the triangle inequality that

$$\hat{W}_i \leq d(\mathbf{U}_i; \{\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}\}) \leq d(\mathbf{U}_i; \{\mathbf{x}\}) + d(\mathbf{x}; \{\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}\}) \leq C \mathrm{diam}(V_{i-1}(\mathbf{x})),$$

for some $C \in (0, \infty)$ depending only on d. It follows that there exists $C \in (0, \infty)$ such that for all $i \in \mathbb{N}$

$$\mathbb{E}[\hat{W}_i^\alpha \mid \mathbf{U}_1, \ldots, \mathbf{U}_{i-1}] \leq C |V_{i-1}(\mathbf{x})| (\mathrm{diam}(V_{i-1}(\mathbf{x})))^\alpha \leq C (\mathrm{diam}(V_{i-1}(\mathbf{x})))^{d+\alpha}.$$

Thus by Lemma 3.1, for some $C \in (0, \infty)$ and all $i \in \mathbb{N}$,

$$\sup_{\mathbf{x} \in (0,1)^d} \mathbb{E}[\hat{W}_i^\alpha] \leq C (i+1)^{-1-(\alpha/d)},$$

and the lemma follows. □

The remaining results of this section will be used later to convert between Poisson and binomial results. The first is a technical lemma.

Lemma 3.5 *Suppose $\beta \geq 1$ and $x > 0$. Then,*

$$-\frac{1}{\beta} x^{1-\beta} \exp(-x^\beta) \leq \int_0^x \exp(-t^\beta) \mathrm{d}t - \Gamma(1 + (1/\beta)) \leq 0. \tag{21}$$

Proof. Suppose $\beta \geq 1$ and $x > 0$. We have

$$\int_0^x \exp(-t^\beta) \mathrm{d}t = \int_0^\infty \exp(-t^\beta) \mathrm{d}t - \int_x^\infty \exp(-t^\beta) \mathrm{d}t. \tag{22}$$

We deal with each integral on the right-hand side of (22) separately, using the change of variable $y = t^\beta$. By Euler's Gamma integral (see e.g. 6.1.1 in [1]) we have

$$\int_0^\infty \mathrm{e}^{-t^\beta} \mathrm{d}t = \frac{1}{\beta} \int_0^\infty y^{(1/\beta)-1} \mathrm{e}^{-y} \mathrm{d}y = \frac{1}{\beta} \Gamma(1/\beta) = \Gamma(1 + (1/\beta)). \tag{23}$$

For the second integral on the right-hand side of (22) we have

$$0 \leq \int_x^\infty \mathrm{e}^{-t^\beta} \mathrm{d}t = \frac{1}{\beta} \int_{x^\beta}^\infty y^{(1/\beta)-1} \mathrm{e}^{-y} \mathrm{d}y \leq \frac{1}{\beta} (x^\beta)^{(1/\beta)-1} \int_{x^\beta}^\infty \mathrm{e}^{-y} \mathrm{d}y = \frac{1}{\beta} x^{1-(1/\beta)} \mathrm{e}^{-x^\beta}. \tag{24}$$

Then from (22) with (23) and (24) we obtain (21). □

To deduce the Poisson parts of Theorems 2.1 and 2.2 we will need some estimates of incremental expectations, improving upon those in Section 3 of [16]. For $n \in \mathbb{N}$ set

$$Z_n := \mathcal{O}^{d,1}(\mathcal{U}_n) - \mathcal{O}^{d,1}(\mathcal{U}_{n-1}),$$

taking $\mathcal{O}^{d,1}(\mathcal{U}_0) := 0$. Thus Z_n is the gain in length on addition of the nth point in the ONG on $(\mathbf{U}_1, \mathbf{U}_2, \ldots)$. Then for $n \in \mathbb{N}$

$$\mathcal{O}^{d,\alpha}(\mathcal{U}_n) = \sum_{i=1}^{n} Z_i^\alpha. \tag{25}$$

Note that (25) with (27) below implies that for $\alpha \in (0, d)$

$$\mathbb{E}[\mathcal{O}^{d,\alpha}(\mathcal{U}_n)] = \frac{d}{d-\alpha} v_d^{-\alpha/d} \Gamma(1 + (\alpha/d)) n^{1-(\alpha/d)} + O(\max\{1, n^{1-(\alpha/d)-(1/d)+\varepsilon}\}), \tag{26}$$

for any $\varepsilon > 0$, which improves upon the $o(n^{1-(\alpha/d)})$ error term implicit in Theorem 2.1(i) of [16].

Lemma 3.6 *Suppose $d \in \mathbb{N}$ and $\alpha \in (0, d]$. Then for $n \in \mathbb{N}$*

$$\mathbb{E}[Z_n^\alpha] = v_d^{-\alpha/d} \Gamma(1 + (\alpha/d)) n^{-\alpha/d} + h(n), \tag{27}$$

where $h(n) = O(n^{-(\alpha/d)-(1/d)+\varepsilon})$ as $n \to \infty$, for any $\varepsilon > 0$.

Proof. Let $d \in \mathbb{N}$. For $r > 0$ and $\mathbf{x} \in (0,1)^d$, set $A(\mathbf{x}; r) := |(0,1)^d \cap B(\mathbf{x}; r)|$. For $n \geq 2$,

$$\mathbb{P}(Z_n^\alpha > z \mid \mathbf{U}_n) = \mathbb{P}(\{\mathbf{U}_1, \ldots, \mathbf{U}_{n-1}\} \cap B(\mathbf{U}_n; z^{1/\alpha}) = \emptyset \mid \mathbf{U}_n) = (1 - A(\mathbf{U}_n; z^{1/\alpha}))^{n-1}.$$

For $r > d^{1/2}$, $A(\mathbf{x}; r) = 1$ for all $\mathbf{x} \in (0,1)^d$. Then for $\mathbf{U}_n \in (0,1)^d$,

$$\mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n] = \int_0^\infty \mathbb{P}(Z_n > z^{1/\alpha} \mid \mathbf{U}_n) \mathrm{d}z = \int_0^{d^{\alpha/2}} (1 - A(\mathbf{U}_n; z^{1/\alpha}))^{n-1} \mathrm{d}z. \tag{28}$$

Fix $\varepsilon \in (0, 1/d)$ small. For all n large enough so that $n^{\varepsilon-(1/d)} < 1/2$, let S_n denote the region $[n^{\varepsilon-(1/d)}, 1 - n^{\varepsilon-(1/d)}]^d$. For $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in (0,1)^d$ let $m(\mathbf{x}) := \min\{x_1, \ldots, x_d, 1 - x_1, \ldots, 1 - x_d\}$, i.e. the shortest distance from $\mathbf{x}$ to the boundary of $(0,1)^d$. Consider $\mathbf{x} \in S_n$. For $0 < r \leq m(\mathbf{x})$, $A(\mathbf{x}; r) = v_d r^d$, and for $r < d^{1/2}$, $C r^d \leq A(\mathbf{x}; r) \leq v_d r^d$ for some $C \in (0, v_d)$ depending only on d. Thus from (28)

$$\mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n \in S_n] \geq \int_0^{m(\mathbf{U}_n)^\alpha} (1 - v_d z^{d/\alpha})^{n-1} \mathrm{d}z \geq \int_0^{n^{\varepsilon\alpha-(\alpha/d)}} (1 - v_d z^{d/\alpha})^{n-1} \mathrm{d}z, \tag{29}$$

since $m(\mathbf{U}_n) \geq n^{\varepsilon-(1/d)}$ for $\mathbf{U}_n \in S_n$. For $x > 0$ Taylor's Theorem with Lagrange remainder implies that $e^{-x} = 1 - x + Cx^2$ where $C \in [0, 1/2]$, so for $z < n^{\varepsilon\alpha-(\alpha/d)}$ and n large enough, we have that

$$\begin{aligned}
\left(1 - v_d z^{d/\alpha}\right)^{n-1} &\geq \left(\exp(-v_d z^{d/\alpha}) - \frac{1}{2} v_d^2 z^{2d/\alpha}\right)^n \\
&= \exp(-v_d n z^{d/\alpha}) \left(1 - \frac{1}{2} v_d^2 z^{2d/\alpha} \exp(v_d z^{d/\alpha})\right)^n \\
&\geq \exp(-v_d n z^{d/\alpha}) \left(1 + O\left(n^{2d\varepsilon-2} \exp(v_d n^{d\varepsilon-1})\right)\right)^n
\end{aligned}$$

$$= \exp(-v_d n z^{d/\alpha})(1 + O(n^{2d\varepsilon-1})),$$

as $n \to \infty$, since $\varepsilon < 1/d$. So from (29) we have that

$$\mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n \in S_n] \geq (1 + O(n^{2d\varepsilon-1})) \int_0^{n^{\varepsilon\alpha-(\alpha/d)}} \exp(-v_d n z^{d/\alpha}) \mathrm{d}z. \tag{30}$$

Now, setting $s = (v_d n)^{\alpha/d} z$, for $\alpha \in (0, d]$

$$\begin{aligned} \int_0^{n^{\varepsilon\alpha-(\alpha/d)}} \exp(-v_d n z^{d/\alpha}) \mathrm{d}z &= (n v_d)^{-\alpha/d} \int_0^{v_d^{\alpha/d} n^{\varepsilon\alpha}} \exp(-s^{d/\alpha}) \mathrm{d}s \\ &= (n v_d)^{-\alpha/d} \Gamma(1 + (\alpha/d)) + O(\exp(-v_d n^{\varepsilon d})), \end{aligned} \tag{31}$$

using (21) for the final equality. So we obtain from (30) and (31) that for $\varepsilon > 0$

$$\mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n \in S_n] \geq (n v_d)^{-\alpha/d} \Gamma(1 + (\alpha/d)) + O(n^{2d\varepsilon-1-(\alpha/d)}).$$

For the upper bound, using the fact that $1 - x \leq \mathrm{e}^{-x}$ for $x \in (0, 1)$ we have from (28)

$$\begin{aligned} \mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n \in S_n] &= \int_0^{d^{\alpha/2}} (1 - A(\mathbf{U}_n; z^{1/\alpha}))^{n-1} \mathrm{d}z \\ \leq \int_0^{n^{\varepsilon\alpha-(\alpha/d)}} \exp(-v_d (n-1) z^{d/\alpha}) \mathrm{d}z &+ \int_{n^{\varepsilon\alpha-(\alpha/d)}}^{\infty} \exp(-C(n-1) z^{d/\alpha}) \mathrm{d}z. \end{aligned} \tag{32}$$

For $\alpha \in (0, d]$, the second term on the right-hand side of (32) is $O(\exp(-C n^{\varepsilon d}))$, using (24) with $t = (C(n-1))^{\alpha/d} z$, $\beta = d/\alpha$, and $x = (C(n-1))^{\alpha/d} n^{\varepsilon\alpha-(\alpha/d)}$. Also, the first term on the right-hand side of (32) is bounded by

$$\exp(v_d n^{\varepsilon d-1}) \int_0^{n^{\varepsilon\alpha-(\alpha/d)}} \exp(-v_d n z^{d/\alpha}) \mathrm{d}z = (n v_d)^{-\alpha/d} \Gamma(1 + (\alpha/d)) + O(n^{\varepsilon d-1-(\alpha/d)}),$$

by (31). So from (32), for the upper bound we obtain

$$\mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n \in S_n] \leq (n v_d)^{-\alpha/d} \Gamma(1 + (\alpha/d)) + O(n^{d\varepsilon-1-(\alpha/d)}).$$

Combining the upper and lower bounds we have

$$\mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n \in S_n] = (n v_d)^{-\alpha/d} \Gamma(1 + (\alpha/d)) + O(n^{d\varepsilon-1-(\alpha/d)}), \tag{33}$$

for $\alpha \in (0, d]$ and ε small enough. Now consider $\mathbf{x} \in (0, 1)^d \setminus S_n$. Here $C r^d \leq A(\mathbf{x}; r) \leq v_d r^d$ for $r < d^{1/2}$, and by similar arguments to above, we obtain

$$\mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n \notin S_n] = O(n^{-\alpha/d}). \tag{34}$$

Since $\mathbb{P}(\mathbf{U}_n \notin S_n) = O(n^{\varepsilon-(1/d)})$, we obtain from (33) and (34) that for any $\varepsilon > 0$

$$\begin{aligned} \mathbb{E}[Z_n^\alpha] &= \mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n \in S_n] \mathbb{P}(\mathbf{U}_n \in S_n) + \mathbb{E}[Z_n^\alpha \mid \mathbf{U}_n \notin S_n] \mathbb{P}(\mathbf{U}_n \notin S_n) \\ &= (n v_d)^{-\alpha/d} \Gamma(1 + (\alpha/d)) + O(n^{\varepsilon-(\alpha/d)-(1/d)}), \end{aligned}$$

and so we have (27). □

4 Proof of Theorem 2.1

The aim of this section is to prove the upper bounds on variances for $\mathcal{O}^{d,\alpha}(\mathcal{U}_n)$ and $\mathcal{O}^{d,\alpha}(\mathcal{P}_\lambda)$ given in Theorem 2.1. The following martingale-difference result is the key to the proof of the binomial parts of Theorem 2.1. Some extra work is then needed to derive the 'Poissonized' version of the result.

Lemma 4.1 *Let $d \in \mathbb{N}$ and $\alpha > 0$. For each $n \in \mathbb{N}$, there exist mean-zero random variables $D_i^{(n)}$, $i = 1, 2, \ldots, n$, such that:*

(i) $\sum_{i=1}^n D_i^{(n)} = \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n)$;

(ii) $\mathbb{E}[D_i^{(n)} D_j^{(n)}] = 0$ *for* $i \neq j$;

(iii) there exists $C \in (0, \infty)$ *such that* $\mathbb{E}[(D_i^{(n)})^2] \leq C i^{-2\alpha/d}$ *for all* n, i.

Before proving the lemma, we introduce some more notation. For $n \in \mathbb{N}$, let $\mathcal{F}_n$ denote the σ-field generated by $\mathcal{U}_n$. Let $\mathcal{F}_0$ denote the trivial σ-field. For ease of notation during this proof, set $Y_n = \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n)$. Then we can write for $n \in \mathbb{N}$

$$Y_n = \sum_{i=1}^n D_i^{(n)},$$

where for $i \in \{1, 2, \ldots, n\}$

$$D_i^{(n)} = \mathbb{E}[Y_n \mid \mathcal{F}_i] - \mathbb{E}[Y_n \mid \mathcal{F}_{i-1}], \tag{35}$$

and for fixed n the $D_i^{(n)}$, $i = 1, \ldots, n$ are martingale differences, and hence orthogonal (see e.g. Chapter 12 of [20]). This establishes parts (i) and (ii) of the lemma. It remains to estimate $\mathbb{E}[(D_i^{(n)})^2]$. Given $\mathcal{U}_n = (\mathbf{U}_1, \ldots, \mathbf{U}_n)$, for $i \in \{1, \ldots, n\}$ let $\mathbf{U}_i'$ be an independent copy of $\mathbf{U}_i$ (independent of $\mathbf{U}_1, \mathbf{U}_2, \ldots$) and set

$$\mathcal{U}_n^i := (\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_i', \mathbf{U}_{i+1}, \ldots, \mathbf{U}_n),$$

so $\mathcal{U}_n^i$ is $\mathcal{U}_n$ with the ith member of the sequence independently re-sampled. Define

$$\Delta_i^{(n)} := \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n^i) - \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n) = \mathcal{O}^{d,\alpha}(\mathcal{U}_n^i) - \mathcal{O}^{d,\alpha}(\mathcal{U}_n),$$

the change in Y_n on re-sampling the point $\mathbf{U}_i$. Then it is the case that

$$D_i^{(n)} = -\mathbb{E}[\Delta_i^{(n)} \mid \mathcal{F}_i].$$

We split $\Delta_i^{(n)}$ into six components as follows. Let $\Delta_{i,1}^{(n)}$ be the weight of the edge from $\mathbf{U}_i$ in the ONG on $\mathcal{U}_n$, and let $\Delta_{i,2}^{(n)}$ be the weight of the edge from $\mathbf{U}_i'$ in the ONG on $\mathcal{U}_n^i$. Let $\Delta_{i,3}^{(n)}$ be the total weight of the edges incident to $\mathbf{U}_i$ in the ONG on $\mathcal{U}_n$, and let $\Delta_{i,4}^{(n)}$ be the total weight of the edges incident to $\mathbf{U}_i'$ in the ONG on $\mathcal{U}_n^i$. Let $\Delta_{i,5}^{(n)}$ be the total weight of edges in the ONG on $(\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_{i+1}, \ldots, \mathbf{U}_n)$ from points in $\mathcal{U}_n$ that are joined to $\mathbf{U}_i'$ in the ONG on $\mathcal{U}_n^i$. Let $\Delta_{i,6}^{(n)}$ be the total weight of edges in the ONG on $(\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_{i+1}, \ldots, \mathbf{U}_n)$ from points in $\mathcal{U}_n$ that are joined to $\mathbf{U}_i$ in the ONG on $\mathcal{U}_n$. Then

$$\Delta_i^{(n)} = \Delta_{i,2}^{(n)} + \Delta_{i,4}^{(n)} + \Delta_{i,6}^{(n)} - \Delta_{i,1}^{(n)} - \Delta_{i,3}^{(n)} - \Delta_{i,5}^{(n)}.$$

The next result will be crucial for the proof of Lemma 4.1.

Lemma 4.2 *For any $\alpha > 0$ there exists $C \in (0, \infty)$ such that for all $\ell \in \{1, \ldots, 6\}$*

$$\mathbb{E}[(\mathbb{E}[\Delta_{i,\ell}^{(n)} \mid \mathcal{F}_i])^2] \leq C i^{-2\alpha/d}, \tag{36}$$

for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$.

Proof. First consider $\ell \in \{1, 2\}$. By the conditional Jensen's inequality,

$$\mathbb{E}[(\mathbb{E}[\Delta_{i,\ell}^{(n)} \mid \mathcal{F}_i])^2] \leq \mathbb{E}[\mathbb{E}[(\Delta_{i,\ell}^{(n)})^2 \mid \mathcal{F}_i]] = \mathbb{E}[(\Delta_{i,\ell}^{(n)})^2].$$

For $\ell \in \{1, 2\}$, we have from Lemma 3.1 in [16] (cf (27) above) that for $\alpha > 0$, $\mathbb{E}[(\Delta_{i,\ell}^{(n)})^2] = \mathbb{E}[Z_i^{2\alpha}] \leq C i^{-2\alpha/d}$ for all i, n. Thus for $\ell \in \{1, 2\}$ there is a constant $C \in (0, \infty)$ such that (36) holds for all i and n.

Now consider $\ell \in \{3, 4\}$. For $i \in \mathbb{N}$, let $V_i := V_{i-1}(\mathbf{U}_i)$ be the Voronoi cell of $\mathbf{U}_i$ with respect to $\{\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_i\}$. Similarly, let $V_i' := V_{i-1}(\mathbf{U}_i')$ be the Voronoi cell of $\mathbf{U}_i'$ with respect to $\{\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_i'\}$.

By convexity, there exists a d-cube of side length at most $2\mathrm{diam}(V_i)$ which contains V_i and also lies inside $(0,1)^d$. Let B_i denote a minimal-volume such cube.

Points of $\{\mathbf{U}_{i+1}, \ldots, \mathbf{U}_n\}$ that fall outside of V_i can never be joined to $\mathbf{U}_i$ and can only serve to decrease the total weight incident to $\mathbf{U}_i$ (by shrinking the subsequent Voronoi cells). Hence removing any point of $\{\mathbf{U}_{i+1}, \ldots, \mathbf{U}_n\}$ that falls outside V_i (and in particular any that falls outside B_i) can only increase the total weight of edges incident to $\mathbf{U}_i$. Moreover, $\{\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}\}$ necessarily lie outside V_i and their removal can only increase the total weight incident to $\mathbf{U}_i$. In other words, for any $j \geq i+1$ and any subsequence $\mathcal{U}_j'$ of $\mathcal{U}_j$ containing $\mathbf{U}_i$ and $\mathbf{U}_j$, we have $E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_j) \subseteq E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_j')$, and $\mathbb{P}(E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_j)) = 0$ for any $\mathbf{U}_j \notin V_i$ and in particular any $\mathbf{U}_j \notin B_i$.

It follows that

$$\Delta_{i,3}^{(n)} = \sum_{j=i+1}^{n} \mathbf{1}_{E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_j)} \|\mathbf{U}_j - \mathbf{U}_i\|^\alpha \leq \sum_{j: i+1 \leq j \leq n, \mathbf{U}_j \in B_i} \mathbf{1}_{E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_{j,i})} \|\mathbf{U}_j - \mathbf{U}_i\|^\alpha,$$

where $\mathcal{U}_{j,i}$ is the subsequence of $(\mathbf{U}_i, \ldots, \mathbf{U}_j)$ consisting only of those points in B_i. So in particular, given $\mathcal{F}_i$, $\Delta_{i,3}^{(n)}$ is stochastically dominated by $\mathcal{O}_{\mathbf{U}_i}^{d,\alpha}(B_i; N)$ where $N \sim \mathrm{Bin}(n - i, |B_i|)$ is the number of points of $\{\mathbf{U}_{i+1}, \ldots, \mathbf{U}_n\}$ that fall in B_i. (Recall the definition of $\mathcal{O}_{\mathbf{x}}^{d,\alpha}(D; n)$ from (13).) We thus have that, given $\mathcal{F}_i$, $\Delta_{i,3}^{(n)}$ is stochastically dominated by

$$\mathcal{O}_{\mathbf{U}_i}^{d,\alpha}(B_i; n) \stackrel{\mathrm{d}}{=} |B_i|^{\alpha/d} \mathcal{O}_{\mathbf{x}}^{d,\alpha}(n),$$

by scaling, for some $\mathbf{x} \in (0,1)^d$. Since $|B_i| \le C(\mathrm{diam}(V_i))^d$, we have in particular that for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$

$$\mathbb{E}[\Delta_{i,3}^{(n)} \mid \mathcal{F}_i] \le C(\mathrm{diam}(V_i))^\alpha \sup_{\mathbf{x} \in (0,1)^d} \mathbb{E}[\mathcal{O}_{\mathbf{x}}^{d,\alpha}(n)] \le C(\mathrm{diam}(V_i))^\alpha,$$

by (14). Thus by Lemma 3.1, for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$,

$$\mathbb{E}[(\mathbb{E}[\Delta_{i,3}^{(n)} \mid \mathcal{F}_i])^2] \le C\mathbb{E}[(\mathrm{diam}(V_i))^{2\alpha}] \le Ci^{-2\alpha/d}. \tag{37}$$

Similarly, $\mathbb{E}[\Delta_{i,4}^{(n)} \mid \mathcal{F}_i] \le C\mathbb{E}[(\mathrm{diam}(V_i'))^\alpha \mid \mathcal{F}_i]$ so that, by the conditional Jensen's inequality,

$$\mathbb{E}[(\mathbb{E}[\Delta_{i,4}^{(n)} \mid \mathcal{F}_i])^2] \le C\mathbb{E}[(\mathrm{diam}(V_i'))^{2\alpha}] \le Ci^{-2\alpha/d}, \tag{38}$$

for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$ by Lemma 3.1 once more, since $V_i' \stackrel{\mathrm{d}}{=} V_i$. Thus from (37) and (38) we verify the $\ell \in \{3, 4\}$ cases of (36).

Finally consider $\ell \in \{5, 6\}$. Recall that $\mathcal{U}_{j,i}$ is the subsequence of $(\mathbf{U}_i, \ldots, \mathbf{U}_j)$ consisting only of those points in B_i. By the argument above for $\Delta_{i,3}^{(n)}$, we have that

$$\begin{aligned}\Delta_{i,6}^{(n)} &= \sum_{j=i+1}^{n} \mathbf{1}_{E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_j)} \left(d(\mathbf{U}_j; \{\mathbf{U}_1, \ldots, \mathbf{U}_{j-1}\} \setminus \{\mathbf{U}_i\})\right)^\alpha \\ &\le \sum_{j: i+1 \le j \le n, \mathbf{U}_j \in B_i} \mathbf{1}_{E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_{j,i})} \left(d(\mathbf{U}_j; \{\mathbf{U}_1, \ldots, \mathbf{U}_{j-1}\} \setminus \{\mathbf{U}_i\})\right)^\alpha. \end{aligned} \tag{39}$$

List the points of $\mathcal{U}_{j,i}$ in order of increasing mark (index) as $(\mathbf{U}_i, \mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_s})$. For $j \ge j_1 + 1$, observe that removing points outside B_i can only increase the distance from $\mathbf{U}_j$ to its nearest neighbour amongst $\{\mathbf{U}_1, \ldots, \mathbf{U}_{j-1}\} \setminus \{\mathbf{U}_i\}$, since we know $\mathbf{U}_{j_1} \in B_i$. Thus we have that for $j \ge j_1 + 1$

$$d(\mathbf{U}_j; \{\mathbf{U}_1, \ldots, \mathbf{U}_{j-1}\} \setminus \{\mathbf{U}_i\}) \le d(\mathbf{U}_j; (\{\mathbf{U}_1, \ldots, \mathbf{U}_{j-1}\} \setminus \{\mathbf{U}_i\}) \cap B_i). \tag{40}$$

Then from (39) and (40) we obtain

$$\begin{aligned}\Delta_{i,6}^{(n)} \le\ & \left(d(\mathbf{U}_{j_1}; \{\mathbf{U}_1, \ldots, \mathbf{U}_{j_1-1}\} \setminus \{\mathbf{U}_i\})\right)^\alpha \\ &+ \sum_{j: i+1 \le j \le n, \mathbf{U}_j \in B_i} \mathbf{1}_{E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_{j,i})} \left(d(\mathbf{U}_j; (\{\mathbf{U}_1, \ldots, \mathbf{U}_{j-1}\} \setminus \{\mathbf{U}_i\}) \cap B_i)\right)^\alpha. \end{aligned} \tag{41}$$

To bound the length of the edge from $\mathbf{U}_{j_1}$, we note that any point $\mathbf{y} \in V_n(\mathbf{x})$ has $d(\mathbf{y}; \{\mathbf{U}_1, \ldots, \mathbf{U}_n\}) \le 2\mathrm{diam}(V_n(\mathbf{x}))$. Hence

$$d(\mathbf{U}_{j_1}; \{\mathbf{U}_1, \ldots, \mathbf{U}_{j_1-1}\} \setminus \{\mathbf{U}_i\}) \le C\mathrm{diam}(V_i). \tag{42}$$

Recall the definition of $\hat{\mathcal{O}}_{\mathbf{x}}^{d,\alpha}(D; n)$ from just above Lemma 3.4. Then from (41) with (42), we have that, given $\mathcal{F}_i$, $\Delta_{i,6}^{(n)}$ is stochastically dominated by

$$\hat{\mathcal{O}}_{\mathbf{U}_i}^{d,\alpha}(B_i; n) + C(\mathrm{diam}(V_i))^\alpha \stackrel{\mathrm{d}}{=} |B_i|^{\alpha/d} \hat{\mathcal{O}}_{\mathbf{x}}^{d,\alpha}(n) + C(\mathrm{diam}(V_i))^\alpha,$$

for some $\mathbf{x} \in (0,1)^d$. Taking expectations, we obtain from Lemma 3.4 that

$$\mathbb{E}[\Delta_{i,6}^{(n)} \mid \mathcal{F}_i] \leq C|B_i|^{\alpha/d} + C(\mathrm{diam}(V_i))^\alpha \leq C'(\mathrm{diam}(V_i))^\alpha.$$

Then by Lemmas 3.1 we obtain

$$\mathbb{E}[(\mathbb{E}[\Delta_{i,6}^{(n)} \mid \mathcal{F}_i])^2] \leq C\mathbb{E}[(\mathrm{diam}(V_i))^{2\alpha}] \leq C' i^{-2\alpha/d},$$

for all n, i. A similar argument holds for $\Delta_{i,5}^{(n)}$, and thus verifies the $\ell \in \{5,6\}$ cases of (36). This completes the proof of the lemma. □

Proof of Lemma 4.1. With $D_i^{(n)}$ given by (35), parts (i) and (ii) of the lemma are immediate, as described above. The Cauchy–Schwarz inequality and (36) imply

$$\mathbb{E}[(D_i^{(n)})^2] = \mathbb{E}[(\mathbb{E}[\Delta_i^{(n)} \mid \mathcal{F}_i])^2] = \mathbb{E}\left[\left(\sum_{\ell=1}^{6}(-1)^\ell \mathbb{E}[\Delta_{i,\ell}^{(n)} \mid \mathcal{F}_i]\right)^2\right] \leq C i^{-2\alpha/d},$$

for all n, i. This yields part (iii) of the lemma. □

To deduce the Poisson version of Theorem 2.1, and later Theorem 2.2, we prove the following series of lemmas.

Lemma 4.3 *Let $N(n)$ be a Poisson random variable with mean $n \geq 1$. For $\beta \in [0,1)$,*

$$\mathbb{V}\mathrm{ar}[N(n)^{1-\beta}] \leq C n^{1-2\beta}; \quad (43)$$

$$\mathbb{E}[(N(n)^{1-\beta} - n^{1-\beta})^2] \leq C n^{1-2\beta}; \quad (44)$$

$$\text{and} \quad \mathbb{E}[(\log(1+N(n)) - \log(1+n))^2] \leq C n^{-1}; \quad (45)$$

for some $C \in (0,\infty)$ and all $n \geq 1$.

Proof. Let $n \geq 1$. First we prove (43), (44). Let $\beta \in [0,1)$. Set $K_n := N(n) - n$. Then

$$N(n)^{1-\beta} = n^{1-\beta}(1 + n^{-1}K_n)^{1-\beta}, \quad (46)$$

where by the Intermediate Value Theorem we have that $(1+n^{-1}K_n)^{1-\beta} = 1 + (1-\beta)(1+H_n)^{-\beta} n^{-1} K_n$ for some H_n with $|H_n| \leq n^{-1}|K_n|$. Hence

$$N(n)^{1-\beta} - n^{1-\beta} = n^{-\beta}(1-\beta)(1+H_n)^{-\beta} K_n, \quad (47)$$

so that for $C \in (0,\infty)$

$$\mathbb{E}[(N(n)^{1-\beta} - n^{1-\beta})^2] = C n^{-2\beta}\mathbb{E}[(1+H_n)^{-2\beta} K_n^2]. \quad (48)$$

Let A_n denote the event $\{|K_n| < n^{3/4}\}$. Then

$$\mathbb{E}[(1+H_n)^{-2\beta} K_n^2] = \mathbb{E}[(1+H_n)^{-2\beta} K_n^2 \mathbf{1}_{A_n}] + \mathbb{E}[(1+H_n)^{-2\beta} K_n^2 \mathbf{1}_{A_n^c}].$$

Here, by Cauchy–Schwarz,

$$\mathbb{E}[(1+H_n)^{-2\beta}K_n^2\mathbf{1}_{A_n^c}] \leq (\mathbb{E}[(1+H_n)^{-4\beta}K_n^4])^{1/2}(\mathbb{P}(A_n^c))^{1/2}.$$

But by (47), for $C \in (0,\infty)$, $\mathbb{E}[(1+H_n)^{-4\beta}K_n^4] = Cn^{4\beta}\mathbb{E}[|N(n)^{1-\beta} - n^{1-\beta}|^4]$, so that

$$\mathbb{E}[(1+H_n)^{-2\beta}K_n^2\mathbf{1}_{A_n^c}] \leq Cn^{2\beta}(\mathbb{E}[|N(n)^{1-\beta} - n^{1-\beta}|^4])^{1/2}(\mathbb{P}(A_n^c))^{1/2}, \quad (49)$$

which tends to zero as $n \to \infty$, by standard Chernoff-type Poisson tail bounds (see e.g. Lemma 1.2 in [12]). Also, given A_n, $|H_n| \leq n^{-1}|K_n| \leq n^{-1/4}$, so that

$$\mathbb{E}[(1+H_n)^{-2\beta}K_n^2\mathbf{1}_{A_n}] \leq C\mathbb{E}[K_n^2 \mid A_n] \leq C\mathbb{E}[K_n^2]\mathbb{P}(A_n)^{-1} = Cn\mathbb{P}(A_n)^{-1} \sim Cn, \quad (50)$$

as $n \to \infty$, by standard Poisson tail bounds. So from (48), (49) and (50) we obtain (44).

Now from (47) we have

$$\mathbb{V}\mathrm{ar}[N(n)^{1-\beta}] = Cn^{-2\beta}\mathbb{V}\mathrm{ar}[(1+H_n)^{-\beta}K_n] \leq Cn^{-2\beta}\mathbb{E}[(1+H_n)^{-2\beta}K_n^2].$$

Then from (49) and (50) we obtain (43).

Finally, the Intermediate Value Theorem implies that

$$\log(1+N(n)) - \log(1+n) = \log(1+(1+n)^{-1}K_n) = (1+n)^{-1}(1+H_n)^{-1}K_n,$$

where, as before, $|H_n| \leq n^{-1}|K_n|$. Hence

$$\mathbb{E}[(\log(1+N(n)) - \log(1+n))^2] = (1+n)^{-2}\mathbb{E}[(1+H_n)^{-2}K_n^2].$$

Now (50) still holds with $\beta = 1$, while instead of (49) in this case we have

$$\mathbb{E}[(1+H_n)^{-2}K_n^2\mathbf{1}_{A_n^c}] \leq (1+n)^2(\mathbb{E}[(\log(1+N(n)) - \log(1+n))^4])^{1/2}(\mathbb{P}(A_n^c))^{1/2},$$

by Cauchy–Schwarz, which again tends to zero as $n \to \infty$. Thus we obtain (45). □

Lemma 4.4 *Let $d \in \mathbb{N}$ and $\alpha > 0$. Let $N(n)$ be a Poisson random variable with mean $n \geq 1$. Then there exists $C \in (0,\infty)$ such that for all $n \geq 1$*

$$\mathbb{E}[\mathbb{V}\mathrm{ar}[\mathcal{O}^{d,\alpha}(\mathcal{U}_{N(n)}) \mid N(n)]] \leq C + \sup_{1\leq m\leq 2n} \mathbb{V}\mathrm{ar}[\mathcal{O}^{d,\alpha}(\mathcal{U}_m)]. \quad (51)$$

Proof. We have that

$$\mathbb{E}[\mathbb{V}\mathrm{ar}[\mathcal{O}^{d,\alpha}(\mathcal{U}_{N(n)}) \mid N(n)]] \leq \sup_{1\leq m\leq 2n} \mathbb{V}\mathrm{ar}[\mathcal{O}^{d,\alpha}(\mathcal{U}_m)] + C\mathbb{E}[(N(n))^2\mathbf{1}_{\{N(n)>2n\}}],$$

using the trivial bound that $\mathcal{O}^{d,\alpha}(\mathcal{U}_{N(n)}) \leq CN(n)$. By Cauchy–Schwarz, the last term in the above display is bounded by a constant times

$$(\mathbb{E}[(N(n))^4])^{1/2}(\mathbb{P}(N(n) > 2n))^{1/2},$$

which tends to 0 as $n \to \infty$ by standard Poisson tail bounds. So we obtain (51). □

Lemma 4.5 *Let $d \in \mathbb{N}$ and $\alpha \in (0, d]$. For $n \in \mathbb{N}$, let $\mu_n := \mathbb{E}[\mathcal{O}^{d,\alpha}(\mathcal{U}_n)]$. Let $N(n)$ be a Poisson random variable with mean n. There exists $C \in (0, \infty)$ such that for all $n \geq 1$*

$$\mathbb{E}[(\mu_{N(n)} - \mu_{\lfloor n \rfloor})^2] \leq C n^{1-(2\alpha/d)}.$$

Proof. Taking expectations in (25), we have that for $n \in \mathbb{N}$

$$\mu_n = \sum_{i=1}^{n} \mathbb{E}[Z_i^\alpha].$$

First suppose that $\alpha \in (0, d)$. By (27) we have that, for integers ℓ, m with $1 \leq \ell < m$,

$$\mu_m - \mu_\ell = \sum_{i=\ell+1}^{m} \mathbb{E}[Z_i^\alpha] = \frac{d}{d-\alpha} v_d^{-\alpha/d} \Gamma(1 + (\alpha/d))(m^{1-(\alpha/d)} - \ell^{1-(\alpha/d)}) + \sum_{i=\ell+1}^{m} h(i) + O(m^{-\alpha/d}) + O(\ell^{-\alpha/d}). \tag{52}$$

In particular, for $n \geq 1$

$$|\mu_{N(n)} - \mu_{\lfloor n \rfloor}| = C|N(n)^{1-(\alpha/d)} - n^{1-(\alpha/d)}| + \delta(n), \tag{53}$$

where from (52) the random variable $\delta(n)$ satisfies

$$|\delta(n)| \leq \sum_{i=\min\{N(n), \lfloor n \rfloor\}}^{\max\{N(n), \lfloor n \rfloor\}} |h(i)| + O(\min\{n, 1 + N(n)\}^{-\alpha/d}). \tag{54}$$

On the other hand, for $\alpha = d$, this time (27) implies that for $1 \leq \ell < m$,

$$\mu_m - \mu_\ell = v_d^{-1}(\log(1+m) - \log(1+\ell)) + \sum_{i=\ell+1}^{m} h(i) + O(m^{-1}) + O(\ell^{-1}). \tag{55}$$

In particular, for $n \geq 1$, (55) gives

$$|\mu_{N(n)} - \mu_{\lfloor n \rfloor}| = C|\log(1 + N(n)) - \log(1+n)| + \delta(n), \tag{56}$$

where again $\delta(n)$ satisfies (54), now with $\alpha = d$.

We now claim that for all $\alpha \in (0, d]$, $\delta(n)$ as defined by (53) or (56) satisfies

$$\mathbb{E}[\delta(n)^2] = o(n^{1-(2\alpha/d)}), \text{ as } n \to \infty. \tag{57}$$

Then in the case $\alpha \in (0, d)$, (57) with (53), (44) and Cauchy–Schwarz yields the lemma. In the case $\alpha = d$, the result follows from (57) with (56), (45) and Cauchy–Schwarz again.

It remains to prove the claim (57). We start from the fact that for $\alpha \in (0, d]$, $\delta(n)$ satisfies (54). Note that there exists $C \in (0, \infty)$ such that for $n \geq 1$

$$\mathbb{E}[\min\{n, 1 + N(n)\}^{-2\alpha/d}] \leq n^{-2\alpha/d} + \mathbb{E}[(1 + N(n))^{-2\alpha/d}] \leq C n^{-2\alpha/d}, \tag{58}$$

as can be proved by standard Poisson tail estimates as used elsewhere in the present paper (cf Lemma 5.1 for an analogous binomial result).

Now we deal with the main term in (54). We have that for $n \geq 1$, $\alpha \in (0,d]$ and any $\eta \in (0,1/2)$,

$$\sup_{m \in \mathbb{N}: |m - \lfloor n \rfloor| \leq n^{(1/2)+\eta}} \sum_{i=\min\{m,\lfloor n \rfloor\}}^{\max\{m,\lfloor n \rfloor\}} |h(i)| \leq (2n^{(1/2)+\eta}+1) \sup_{m \in \mathbb{N}: |m - \lfloor n \rfloor| \leq n^{(1/2)+\eta}} |h(m)|; \qquad (59)$$

it follows from (59) and Lemma 3.6 that for any $\eta \in (0,1/2)$, $\varepsilon > 0$, there exists $C \in (0,\infty)$ such that for all $n \geq 1$

$$\sup_{m \in \mathbb{N}: |m - \lfloor n \rfloor| \leq n^{(1/2)+\eta}} \sum_{i=\min\{m,\lfloor n \rfloor\}}^{\max\{m,\lfloor n \rfloor\}} |h(i)| \leq C n^{(1/2)+\eta-(1/d)-(\alpha/d)+\varepsilon}. \qquad (60)$$

In particular, this is $o(n^{(1/2)-(\alpha/d)})$ for sufficiently small ε, η. Now we have, with $\eta > 0$ as above, for $n \geq 1$

$$\mathbb{E}\left[\left(\sum_{i=\min\{N(n),\lfloor n \rfloor\}}^{\max\{N(n),\lfloor n \rfloor\}} |h(i)|\right)^2\right] \leq \left(\sup_{m \in \mathbb{N}: |m - \lfloor n \rfloor| \leq n^{(1/2)+\eta}} \sum_{i=\min\{m,\lfloor n \rfloor\}}^{\max\{m,\lfloor n \rfloor\}} |h(i)|\right)^2$$
$$+ C\mathbb{E}[|N(n) - \lfloor n \rfloor|^2 \mathbf{1}_{\{|N(n)-\lfloor n \rfloor| > n^{(1/2)+\eta}\}}], \qquad (61)$$

and by Cauchy–Schwarz

$$\mathbb{E}[|N(n) - \lfloor n \rfloor|^2 \mathbf{1}_{\{|N(n)-\lfloor n \rfloor| > n^{(1/2)+\eta}\}}] \leq (\mathbb{E}[|N(n) - \lfloor n \rfloor|^4] \mathbb{P}(|N(n) - \lfloor n \rfloor| > n^{(1/2)+\eta}))^{1/2},$$

which is $o(n^{1-(2\alpha/d)})$ as $n \to \infty$, by standard Poisson tail bounds. Thus from (61), (60), (58), and Cauchy–Schwarz, we verify (57). □

Proof of Theorem 2.1. First we prove the binomial parts of (4) and (5). By part (i) of Lemma 4.1, we have that $\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n) = \sum_{i=1}^n D_i^{(n)}$ for each $n \in \mathbb{N}$. By the orthogonality of the $D_i^{(n)}$ (part (ii) of Lemma 4.1) we have that for $n \in \mathbb{N}$

$$\mathbb{V}\mathrm{ar}[\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n)] = \sum_{i=1}^n \mathbb{E}[(D_i^{(n)})^2],$$

which by part (iii) of Lemma 4.1 yields the upper bounds as claimed.

We now deduce the Poisson parts of (4) and (5). For ease of notation, let $X_n := \mathcal{O}^{d,\alpha}(\mathcal{U}_n)$ and $\mu_n := \mathbb{E}[X_n]$. Then if $N(n)$ is Poisson with mean $n \geq 1$, $\mathcal{O}^{d,\alpha}(\mathcal{P}_n)$ has the distribution of $X_{N(n)}$ and its expectation is $\mathbb{E}[\mu_{N(n)}] =: a_n$. Write

$$\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{P}_n) = X_{N(n)} - a_n = (X_{N(n)} - \mu_{N(n)}) + (\mu_{\lfloor n \rfloor} - a_n) + (\mu_{N(n)} - \mu_{\lfloor n \rfloor}). \qquad (62)$$

Then $\mathbb{V}\mathrm{ar}[\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{P}_n)] = \mathbb{V}\mathrm{ar}[(X_{N(n)} - \mu_{N(n)}) + (\mu_{N(n)} - \mu_{\lfloor n \rfloor})]$. We have

$$\mathbb{V}\mathrm{ar}[X_{N(n)} - \mu_{N(n)}] = \mathbb{E}[\mathbb{V}\mathrm{ar}[X_{N(n)} - \mu_{N(n)} \mid N(n)]] = \mathbb{E}[\mathbb{V}\mathrm{ar}[\mathcal{O}^{d,\alpha}(\mathcal{U}_{N(n)}) \mid N(n)]].$$

By (51) this is bounded by a constant times $\sup_{m\le 2n}\mathbb{V}\mathrm{ar}[\mathcal{O}^{d,\alpha}(\mathcal{U}_m)]$, which, using the binomial parts of (4) and (5), is bounded by a constant times $n^{1-(2\alpha/d)}$ for $\alpha\in(0,d/2)$ and by a constant times $\log(1+n)$ for $\alpha=d/2$. So we have for $C\in(0,\infty)$ and $n\ge 1$

$$\mathbb{V}\mathrm{ar}[X_{N(n)}-\mu_{N(n)}]\le\begin{cases} Cn^{1-(2\alpha/d)} & \text{if } \alpha\in(0,d/2);\\ C\log(1+n) & \text{if } \alpha=d/2.\end{cases}\tag{63}$$

The final term on the right-hand side of (62) satisfies Lemma 4.5. So by (62) with Lemma 4.5, (63), and Cauchy–Schwarz, we obtain the Poisson parts of (4) and (5). □

5 Proof of Theorem 2.2

By Lemma 4.1 we have that for $\alpha>d/2$, for all $n\in\mathbb{N}$

$$\mathbb{V}\mathrm{ar}[\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n)]=\sum_{i=1}^n\mathbb{E}[(D_i^{(n)})^2]\le C\sum_{i=1}^n i^{-2\alpha/d}\le C'<\infty.$$

In order to show that $\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n)$ in fact converges, we employ a refinement of the martingale difference technique of Section 4. First we need two more lemmas.

Lemma 5.1 *Suppose $X\sim\mathrm{Bin}(n,p)$ for $n\in\mathbb{N}$ and $p\in(0,1)$. Then for any $\beta>0$ there exists $C\in(0,\infty)$ such that for all $n\in\mathbb{N}$ and all $p\in(0,1)$*

$$\mathbb{E}[(1+X)^{-\beta}]\le C(np)^{-\beta}.$$

Proof. We have that

$$\begin{aligned}\mathbb{E}[(1+X)^{-\beta}]&\le(1+(np/2))^{-\beta}+\mathbb{E}[(1+X)^{-\beta}\mathbf{1}_{\{X<np/2\}}]\\&\le C(np)^{-\beta}+(\mathbb{E}[(1+X)^{-2\beta}])^{1/2}(\mathbb{P}(X<np/2))^{1/2},\end{aligned}$$

for some $C\in(0,\infty)$ and all $n\in\mathbb{N}$, $p\in(0,1)$, using Cauchy–Schwarz. But for $\beta>0$, $(1+X)^{-2\beta}\le 1$ a.s., so $\mathbb{E}[(1+X)^{-2\beta}]\le 1$. Also, by standard binomial tail bounds (see e.g. Lemma 1.1 in [12]), $\mathbb{P}(X<np/2)=O(\exp(-Cnp))$ for all n,p. □

Lemma 5.2 *Suppose $d\in\mathbb{N}$ and $\alpha>d/2$. For $\varepsilon>0$ sufficiently small, we have that*

$$\lim_{n\to\infty}\sup_{m:|n-m|\le n^{(1/2)+\varepsilon}}\left|\mathbb{E}[\mathcal{O}^{d,\alpha}(\mathcal{U}_n)]-\mathbb{E}[\mathcal{O}^{d,\alpha}(\mathcal{U}_m)]\right|=0.\tag{64}$$

Proof. For ease of notation, let $\mu_n:=\mathbb{E}[\mathcal{O}^{d,\alpha}(\mathcal{U}_n)]$. By monotonicity of μ_n,

$$\sup_{m:|n-m|\le n^{(1/2)+\varepsilon}}|\mu_n-\mu_m|\le\max\{\mu_{n+\lceil n^{(1/2)+\varepsilon}\rceil}-\mu_n,\mu_n-\mu_{n-\lceil n^{(1/2)+\varepsilon}\rceil}\},$$

so it suffices to show that both terms in the maximum tend to zero as $n\to\infty$. Consider the $\alpha\in(d/2,d)$ case of (52). Now by Lemma 3.6 we have, for small enough $\varepsilon>0$,

$$\sum_{i=n+1}^{n+\lceil n^{(1/2)+\varepsilon}\rceil}h(i)\le Cn^{(1/2)+\varepsilon}\sup_{i:n\le i\le n+\lceil n^{(1/2)+\varepsilon}\rceil}h(i)\le Cn^{(1/2)-(\alpha/d)-(1/d)+2\varepsilon}=o(n^{(1/2)-(\alpha/d)}),$$

which tends to 0 as $n \to \infty$, given that $\alpha > d/2$. Thus by (52), as $n \to \infty$,

$$\mu_{n+\lceil n^{(1/2)+\varepsilon}\rceil} - \mu_n = Cn^{1-(\alpha/d)}((1+n^{-(1/2)+\varepsilon})^{1-(\alpha/d)} - 1) + o(1),$$

for some $C \in (0, \infty)$. But this is $O(n^{(1/2)-(\alpha/d)+\varepsilon})$, which tends to zero for $\alpha > d/2$ and ε small enough. Similarly for $\mu_n - \mu_{n-\lceil n^{(1/2)+\varepsilon}\rceil}$. Thus we obtain (64) for $\alpha \in (d/2, d)$.

Now suppose that $\alpha = d$. This time we have (55); by Lemma 3.6 the sum in (55) tends to 0 as $m, \ell \to \infty$. Thus for $\varepsilon > 0$ small enough

$$\mu_{n+\lceil n^{(1/2)+\varepsilon}\rceil} - \mu_n = v_d^{-1} \log\left(\frac{n + \lceil n^{(1/2)+\varepsilon}\rceil}{n}\right) + o(1) = O(n^{\varepsilon-(1/2)}) + o(1) \to 0,$$

and similarly for $\mu_n - \mu_{n-\lceil n^{(1/2)+\varepsilon}\rceil}$. Thus we get (64) for $\alpha = d$. The case $\alpha > d$ is straightforward, since there (see Proposition 2.1) $\mu_n \to \mu(d, \alpha) \in (0, \infty)$ as $n \to \infty$. $\square$

To prepare for the proof of Theorem 2.2, we modify the technique used in the proof of Lemma 4.1 above. For $n, m \in \mathbb{N}$ with $m < n$ set $Y_n^{(m)} := \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n) - \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_m)$, i.e. $Y_n^{(m)}$ is the centred total weight of edges in the ONG on $\mathcal{U}_n$ counting only edges from points after the first m in the sequence. With $\mathcal{F}_i$ the σ-field generated by $(\mathbf{U}_1, \ldots, \mathbf{U}_i)$, set

$$D_i^{(n,m)} := \mathbb{E}[Y_n^{(m)} \mid \mathcal{F}_i] - \mathbb{E}[Y_n^{(m)} \mid \mathcal{F}_{i-1}],$$

so that for fixed n, m the $D_i^{(n,m)}$ are martingale differences and

$$Y_n^{(m)} = \sum_{i=1}^{n} D_i^{(n,m)}.$$

As in Section 4, for $i \in \mathbb{N}$ let $\mathbf{U}_i'$ be an independent copy of $\mathbf{U}_i$. For $i \leq n$ let $\mathcal{U}_n^i$ be the sequence $\mathcal{U}_n$ but with $\mathbf{U}_i$ replaced by $\mathbf{U}_i'$. If $i > n$, we take $\mathcal{U}_n^i = \mathcal{U}_n$. Define

$$\begin{aligned}\Delta_i^{(n,m)} &:= [\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n^i) - \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_m^i)] - [\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n) - \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_m)] \\ &= [\mathcal{O}^{d,\alpha}(\mathcal{U}_n^i) - \mathcal{O}^{d,\alpha}(\mathcal{U}_m^i)] - [\mathcal{O}^{d,\alpha}(\mathcal{U}_n) - \mathcal{O}^{d,\alpha}(\mathcal{U}_m)].\end{aligned}$$

Then, similarly to before,

$$D_i^{(n,m)} = -\mathbb{E}[\Delta_i^{(n,m)} \mid \mathcal{F}_i].$$

Analogously to before, we decompose $\Delta_i^{(n,m)}$ into six parts. For $i > m$, let $\Delta_{i,1}^{(n,m)}$ be the weight of the edge from $\mathbf{U}_i$, and $\Delta_{i,2}^{(n,m)}$ be the weight of the edge from $\mathbf{U}_i'$. For $i \leq m$, set $\Delta_{i,1}^{(n,m)} = \Delta_{i,2}^{(n,m)} = 0$. For all i, let $\Delta_{i,\ell}^{(n,m)}$ for $\ell = 3, 4$ be the total weight of edges incident to $\mathbf{U}_i$, $\mathbf{U}_i'$ respectively from $\{\mathbf{U}_{m+1}, \mathbf{U}_{m+2}, \ldots, \mathbf{U}_n\}$. Let $\Delta_{i,5}^{(n,m)}$ be the total weight of edges in the ONG on $(\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_{i+1}, \ldots, \mathbf{U}_n)$ from points in $\{\mathbf{U}_{m+1}, \ldots, \mathbf{U}_n\}$ that are joined to $\mathbf{U}_i'$ in the ONG on $\mathcal{U}_n^i$. Let $\Delta_{i,6}^{(n,m)}$ be the total weight of edges in the ONG on $(\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_{i+1}, \ldots, \mathbf{U}_n)$ from points in $\{\mathbf{U}_{m+1}, \ldots, \mathbf{U}_n\}$ that are joined to $\mathbf{U}_i$ in the ONG on $\mathcal{U}_n$. Then we have

$$\Delta_i^{(n,m)} = \Delta_{i,2}^{(n,m)} + \Delta_{i,4}^{(n,m)} + \Delta_{i,6}^{(n,m)} - \Delta_{i,1}^{(n,m)} - \Delta_{i,3}^{(n,m)} - \Delta_{i,5}^{(n,m)}.$$

Note that $\Delta_{i,\ell}^{(n,m)} \geq \Delta_{i,\ell}^{(n,m+1)}$ and $\Delta_{i,\ell}^{(n,1)} = \Delta_{i,\ell}^{(n)}$ as defined in Section 4. Analogously to Lemma 4.2 above, we have the following.

Lemma 5.3 *For any $\alpha > 0$ there exists $C \in (0, \infty)$ such that for all $\ell \in \{1, \ldots, 6\}$*

$$\mathbb{E}[(\mathbb{E}[\Delta_{i,\ell}^{(n,m)} \mid \mathcal{F}_i])^2] \leq C i^{-2\alpha/d}, \tag{65}$$

for $m \leq i \leq n$, and, for $i < m \leq n$,

$$\mathbb{E}[(\mathbb{E}[\Delta_{i,\ell}^{(n,m)} \mid \mathcal{F}_i])^2] \leq C(\max\{m-i, i\})^{-2\alpha/d}. \tag{66}$$

Proof. The argument in Lemma 4.2 carries through, so that (65) holds for all i. Indeed, $\Delta_{i,\ell}^{(n,m)} \leq \Delta_{i,\ell}^{(n)}$ and so Lemma 4.2 implies (65) for all i, ℓ. Thus to obtain (66) we need to show that there exists $C \in (0, \infty)$ such that for all ℓ and all $i < m \leq n$

$$\mathbb{E}[(\mathbb{E}[\Delta_{i,\ell}^{(n,m)} \mid \mathcal{F}_i])^2] \leq C(m-i)^{-2\alpha/d}. \tag{67}$$

Thus suppose $i < m$. In this case, we need only consider $\Delta_{i,\ell}^{(n,m)}$ for $\ell \geq 3$, since $\Delta_{i,\ell}^{(n,m)} = 0$ for $\ell \in \{1, 2\}$. First take $\ell = 3$, dealing with the edges incident to $\mathbf{U}_i$. There are $m - i$ points of $\mathcal{U}_n$ with mark (index) greater than i but not more than m, and edges from these points to $\mathbf{U}_i$ are not counted in $\Delta_{i,3}^{(n,m)}$. Recall that V_i, V_i' is the Voronoi cell of $\mathbf{U}_i, \mathbf{U}_i'$ respectively with respect to itself and $\{\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}\}$, and B_i is a minimal-volume d-cube with $V_i \subseteq B_i \subseteq (0, 1)^d$.

By an argument analogous to that in the proof of Lemma 4.2, discarding points of $\{\mathbf{U}_{i+1}, \ldots, \mathbf{U}_m\}$ that fall outside B_i can only increase $\Delta_{i,3}^{(n,m)}$. It follows that, with the same notation as in that proof,

$$\Delta_{i,3}^{(n,m)} = \sum_{j=m+1}^{n} \mathbf{1}_{E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_j)} \|\mathbf{U}_j - \mathbf{U}_i\|^\alpha \leq \sum_{j: m+1 \leq j \leq n, \mathbf{U}_j \in B_i} \mathbf{1}_{E(\mathbf{U}_j, \mathbf{U}_i; \mathcal{U}_{j,i})} \|\mathbf{U}_j - \mathbf{U}_i\|^\alpha.$$

Let $M \sim \text{Bin}(m - i, |B_i|)$ be the number of points of $\{\mathbf{U}_{i+1}, \ldots, \mathbf{U}_m\}$ that fall in B_i. Thus $\Delta_{i,3}^{(n,m)}$ is stochastically dominated by

$$\mathcal{O}_{\mathbf{U}_i}^{d,\alpha}(B_i; n) - \mathcal{O}_{\mathbf{U}_i}^{d,\alpha}(B_i; M) \stackrel{\text{d}}{=} |B_i|^{\alpha/d} [\mathcal{O}_{\mathbf{x}}^{d,\alpha}(n) - \mathcal{O}_{\mathbf{x}}^{d,\alpha}(M)],$$

for some $\mathbf{x} \in (0, 1)^d$, by scaling. Hence for some $C \in (0, \infty)$

$$\mathbb{E}[\Delta_{i,3}^{(n,m)} \mid \mathcal{F}_i] \leq C |B_i|^{\alpha/d} \mathbb{E}[(M+1)^{-\alpha/d} \mid \mathcal{F}_i],$$

by (15). By Lemma 5.1, $\mathbb{E}[(M+1)^{-\alpha/d} \mid \mathcal{F}_i] \leq C|B_i|^{-\alpha/d}(m-i)^{-\alpha/d}$, so that for $i < m$

$$\mathbb{E}[\Delta_{i,3}^{(n,m)} \mid \mathcal{F}_i] \leq C(m-i)^{-\alpha/d}.$$

For $\ell = 4$ a similar argument (with V_i replaced by V_i') holds. Thus we obtain (67) for $\ell \in \{3, 4\}$. For $\ell \in \{5, 6\}$ a similar argument applies, using (20) instead of (15) this time. □

Proof of Theorem 2.2. By Lemma 5.3 and Cauchy–Schwarz we have that for $i < m$,

$$\mathbb{E}[(D_i^{(n,m)})^2] \leq C(\max\{m-i, i\})^{-2\alpha/d},$$

while for $i \geq m$, $\mathbb{E}[(D_i^{(n,m)})^2] \leq C i^{-2\alpha/d}$. Thus for $\alpha > 0$, for $m < n$

$$\mathbb{E}[|\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n) - \tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_m)|^2] = \mathbb{E}[(Y_n^{(m)})^2] = \sum_{i=1}^{n} \mathbb{E}[(D_i^{(n,m)})^2]$$
$$\leq C \sum_{i=1}^{\lceil m/2 \rceil} (m-i)^{-2\alpha/d} + C \sum_{i=\lfloor m/2 \rfloor}^{m} i^{-2\alpha/d} + C \sum_{i=m+1}^{n} i^{-2\alpha/d}. \tag{68}$$

In particular, for $\alpha > d/2$ the right-hand side of (68) is bounded by a constant times $m^{1-(2\alpha/d)}$, which tends to 0 as n, m tend to infinity. Thus for $\alpha > d/2$, $\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n)$ is a Cauchy sequence in L^2, and hence as $n \to \infty$ it converges in L^2 to some limit random variable $Q(d,\alpha)$, with $\mathbb{E}[Q(d,\alpha)] = \lim_{n\to\infty} \mathbb{E}[\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{U}_n)] = 0$. Thus we obtain (6).

Finally, we prove the Poisson part (7). As before, let $X_n := \mathcal{O}^{d,\alpha}(\mathcal{U}_n)$ and $\mu_n := \mathbb{E}[X_n]$. For $N(n)$ Poisson with mean $n > 0$, $\mathcal{O}^{d,\alpha}(\mathcal{P}_n)$ has the distribution of $X_{N(n)}$ and expectation $\mathbb{E}[\mu_{N(n)}] =: a_n$. Consider, for $n > 0$

$$\mathbb{E}[|(X_{N(n)} - \mu_{N(n)}) - Q(d,\alpha)|^2] \leq \sup_{m \geq n/2} \mathbb{E}[|(X_m - \mu_m) - Q(d,\alpha)|^2]$$
$$+ \mathbb{E}[|(X_{N(n)} - \mu_{N(n)}) - Q(d,\alpha)|^2 \mathbf{1}_{\{N(n) < n/2\}}]. \tag{69}$$

For $\alpha > d/2$, the L^2 convergence of $X_n - \mu_n$ to $Q(d,\alpha)$ (from (6)) implies that the first term on the right-hand side of (69) tends to zero, and that the second term is bounded by a constant times $\mathbb{P}(N(n) < n/2)$, which tends to zero as $n \to \infty$. So, for $\alpha > d/2$,

$$X_{N(n)} - \mu_{N(n)} \xrightarrow{L^2} Q(d,\alpha), \text{ as } n \to \infty. \tag{70}$$

First suppose $\alpha > d$. Here (see Proposition 2.1) $\mu_n \to \mu := \mu(d,\alpha) \in (0,\infty)$ as $n \to \infty$. It follows, by a similar argument to (69), that $\mu_{N(n)}$ converges to μ in L^2 and $a_n = \mathbb{E}[\mu_{N(n)}] \to \mu$ also. Thus, with (70), as $n \to \infty$

$$\tilde{\mathcal{O}}^{d,\alpha}(\mathcal{P}_n) = X_{N(n)} - a_n = (X_{N(n)} - \mu_{N(n)}) + (\mu_{N(n)} - \mu) + (\mu - a_n) \xrightarrow{L^2} Q(d,\alpha).$$

For $\alpha \in (d/2, d]$, $\mu_n \to \infty$ as $n \to \infty$. Recall (62). With $p_m(n) = \mathbb{P}(N(n) = m)$, the middle bracket in (62) satisfies, for $\varepsilon > 0$,

$$|a_n - \mu_{\lfloor n \rfloor}| = \sum_{m \in \mathbb{N}: |m-n| < n^{(1/2)+\varepsilon}} |\mu_m - \mu_{\lfloor n \rfloor}| p_m(n) + \sum_{m \in \mathbb{N}: |m-n| \geq n^{(1/2)+\varepsilon}} |\mu_m - \mu_{\lfloor n \rfloor}| p_m(n). \tag{71}$$

Using the trivial bound $\mu_m \leq Cm$, the second sum in (71) is bounded by a constant times

$$\sum_{m \in \mathbb{N}: |m-n| \geq n^{(1/2)+\varepsilon}} (m+n) p_m(n) \leq \mathbb{E}[(N(n)+n) \mathbf{1}_{\{|N(n)-n| \geq n^{(1/2)+\varepsilon}\}}],$$

which by Cauchy–Schwarz is bounded by

$$(\mathbb{E}[(N(n)+n)^2])^{1/2} (\mathbb{P}(|N(n)-n| \geq n^{(1/2)+\varepsilon}))^{1/2} \to 0,$$

as $n \to \infty$, by standard Poisson tail bounds. The first sum in (71) satisfies

$$\sum_{m \in \mathbb{N}: |m-n| < n^{(1/2)+\varepsilon}} |\mu_m - \mu_{\lfloor n \rfloor}| p_m(n) \leq \sup_{m \in \mathbb{N}: |m-n| < n^{(1/2)+\varepsilon}} |\mu_m - \mu_{\lfloor n \rfloor}|,$$

which tends to zero as $n \to \infty$ by (64). Thus for $\alpha \in (d/2, d]$, as $n \to \infty$,

$$|a_n - \mu_{\lfloor n \rfloor}| \to 0. \tag{72}$$

Also, from Lemma 4.5 we have that, for $\alpha \in (d/2, d]$, $\mathbb{E}[|\mu_{N(n)} - \mu_{\lfloor n \rfloor}|^2] \to 0$, so that

$$\mu_{N(n)} - \mu_{\lfloor n \rfloor} \xrightarrow{L^2} 0, \text{ as } n \to \infty. \tag{73}$$

Thus from (62) with (70), (72) and (73) we obtain the result for $\alpha \in (d/2, d]$ also. □

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