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# A DIRECTIONAL UNIFORMITY OF PERIODIC POINT DISTRIBUTION AND MIXING

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Abstract. For mixing $\mathbb{Z}^d$-actions generated by commuting automorphisms of a compact abelian group, we investigate the directional uniformity of the rate of periodic point distribution and mixing. When each of these automorphisms has finite entropy, it is shown that directional mixing and directional convergence of the uniform measure supported on periodic points to Haar measure occurs at a uniform rate independent of the direction.

## 1. Introduction.

It is well-known that, under mild hypotheses, sufficiently smooth functions mix at an exponential rate, and periodic point measures become uniformly distributed on smooth functions at an exponential rate, for dynamical systems with hyperbolic behaviour or comparable regularity properties. For example, Bowen [2, 1.26] shows an 'exponential cluster property', that Anosov diffeomorphisms preserving a smooth measure mix Lipschitz functions exponentially fast, and Lind [9] shows similar properties for Hölder functions on quasihyperbolic toral automorphisms. On compact groups, smoothness conditions can be phrased in terms of how well a function can be approximated by a function with finitely supported Fourier transform (that is, by trigonometric polynomials). Thus for a group automorphism $\alpha : X \to X$ of a compact metric abelian group $X$, and an exhaustive increasing sequence $H_1 \subset H_2 \subset \cdots$ of finite subsets of the character group $\widehat{X}$, the rate of mixing and the rate of uniform distribution of periodic points amount to the existence of functions $\phi$ and $\psi$, with $\phi(k) \to \infty$ and $\psi(k) \to \infty$ as $k \to \infty$, such that

1. $H_k \cap \widehat{\alpha}^n H_k = \{0\}$ for $|n| > \phi(k)$ (a rate of mixing), and
2. $H_k \cap (\widehat{\alpha}^n - 1)H_k = \{0\}$ for $|n| > \psi(k)$ (a rate of equidistribution of periodic points).

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Statement (i) gives a class of functions $\mathcal{C}(X)$ with prescribed decay of Fourier coefficients, a rate function $\phi' = o(1)$, and $C = C(f,g)$ for which

$$f,g \in \mathcal{C}(X) \implies \left|\int f(x)g(\alpha^n x)\mathrm{d}\mu(x) - \int f\mathrm{d}\mu \int g\mathrm{d}\mu\right| < C\phi'(n), \tag{1}$$

where $\mu$ denotes Haar measure on $X$. Statement (ii) gives a rate function $\psi' = o(1)$ and a constant $C = C(f)$ for which

$$f \in \mathcal{C}(X) \implies \left|\int f\mathrm{d}\mu_n - \int f\mathrm{d}\mu\right| < C\psi'(n), \tag{2}$$

where $\mu_n$ denotes Haar measure on the subgroup of points fixed by $\alpha^n$. For a given group $X$, the class of test functions on which the mixing and uniform distribution may be seen depends, via the exhaustive sequence, on the functions $\phi$ and $\psi$. For explicit calculations of this sort when $X = \mathbb{T}^d$ is the torus, and $\mathcal{C}(X)$ is a class of Hölder functions, see Lind [9, Sect. 4].

Our interest here is in commuting automorphisms with finite entropy, together defining an algebraic $\mathbb{Z}^d$-action $\alpha$ (an *entropy rank one action* in the sense of Einsiedler and Lind [5]); this means that $\alpha$ is a homomorphism from $\mathbb{Z}^d$ to the group of continuous automorphisms of $X$. We write $\alpha^{\mathbf{n}}$ for the automorphism $\alpha(\mathbf{n})$. Examples include commuting toral automorphisms, Ledrappier's example [8], the (invertible extension of the) $\times 2, \times 3$ system, and many others (Schmidt's monograph [13] describes many dynamical properties of these systems).

A non-uniform rate of mixing, or a non-uniform rate of convergence of periodic point measures, for a $\mathbb{Z}^d$-action $\alpha$, corresponds to the statements (1) and (2) respectively for each of the maps $\alpha^{\mathbf{n}}$ with $\mathbf{n} \neq 0$. The uniformity of the title amounts to asking if, having fixed an appropriate exhaustion $H_1 \subset H_2 \subset \cdots$ of the character group $\widehat{X}$, there is a uniform way to choose the functions $\phi$ and $\psi$ witnessing a directional uniformity in mixing and in the distribution of periodic points. We show that the decay functions $\phi'$ and $\psi'$ can be chosen so that they depend only on the distance from the origin in $\mathbb{Z}^d$, by proving the following theorem.

**Theorem 1.1.** *Let $(X,\alpha)$ be a mixing entropy rank one $\mathbb{Z}^d$-action by automorphisms of a compact abelian group $X$ satisfying the descending chain condition on closed $\alpha$-invariant subgroups. Write $\mu$ for Haar measure on $X$ and $\mu_{\mathbf{n}}$ for Haar measure on the subgroup of points fixed by the automorphism $\alpha^{\mathbf{n}}$. Then there is a class of smooth functions $\mathcal{C}(X)$ strictly containing the trigonometric polynomials, and rate functions $\phi' = o(1)$, $\psi' = o(1)$, such that, for any $f,g \in \mathcal{C}(X)$,*

$$\left|\int f(x)g(\alpha^{\mathbf{n}}x)\mathrm{d}\mu(x) - \int f\mathrm{d}\mu \int g\mathrm{d}\mu\right| < C(f,g)\phi'(\|\mathbf{n}\|),$$

*and*

$$\left|\int f\mathrm{d}\mu_{\mathbf{n}} - \int f\mathrm{d}\mu\right| < C(f)\psi'(\|\mathbf{n}\|).$$

In addition to the motivation already given, this question arose as a result of our paper [12], where it is shown that for a large class of such systems there is a uniform lower bound to the exponential rate of growth in the number of periodic points, and the papers [10] and [11], in which more subtle directionally uniform bounds and counts for periodic points are found. Because of the diversity of underlying compact groups, and our main interest in uniformity, we have not delved into the articulation between the growth in the exhaustive sequence (measuring the smoothness of the function class) and the permitted growth in the control functions $\phi$ and $\psi$ (measuring the rate of mixing or of equidistribution of periodic points).

Our methods combine the formalism introduced by Kitchens and Schmidt in [7], Diophantine arguments, and Einsiedler and Lind's adelic Lyapunov vectors for entropy rank one actions [5].

2. **Algebraic $\mathbb{Z}^d$-actions.** Following Kitchens and Schmidt, we exploit the correspondence between an algebraic $\mathbb{Z}^d$-action $\alpha$ by automorphisms of a compact abelian metric group $X$ and a module over the ring of Laurent polynomials

$$R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}].$$

This is achieved by identifying each dual automorphism $\widehat{\alpha}^{\mathbf{n}}$ with multiplication by $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$, then extending this in a natural way to polynomials. As a result, attention may be restricted to a fixed $R_d$-module $M$ with dynamical properties of $\alpha$ translated into algebraic properties of $M$. For example, the descending chain condition on closed $\alpha$-invariant subgroups of $X$ corresponds to $M$ being Noetherian. The mixing property for $\alpha$ translates to multiplication by $u^{k\mathbf{n}}$ being injective on $M$ for all $k \in \mathbb{N}$ and all $\mathbf{n} \neq 0$. These two properties will be assumed throughout. A full introduction to algebraic $\mathbb{Z}^d$-actions and the correspondence just described is given in Schmidt's monograph [13].

Some especially useful algebraic machinery is available when $\alpha$ has entropy rank one (that is, when each element of the action has finite topological entropy). Since $M$ is assumed to be Noetherian, it has a finite set of associated prime ideals $\operatorname{Ass}(M) \subset \operatorname{Spec}(R_d)$. Furthermore, since $\alpha$ is mixing and of entropy rank one, for each $\mathfrak{p} \in \operatorname{Ass}(M)$, the module $R_d/\mathfrak{p}$ has Krull dimension one, written

$$\operatorname{kdim}(R_d/\mathfrak{p}) = 1$$

(see [5, Prop. 6.1] and [10, Lem. 2.3]). Therefore, the field of fractions of $R_d/\mathfrak{p}$ is a global field that we denote by $\mathbb{K}(\mathfrak{p})$. Let $\mathcal{P}(\mathbb{K}(\mathfrak{p}))$ denote the set of places of $\mathbb{K}(\mathfrak{p})$, $|\cdot|_v$ the absolute value corresponding to the place $v$, and set

$$S(\mathfrak{p}) = \{v \in \mathcal{P}(\mathbb{K}(\mathfrak{p})) \mid |R_d/\mathfrak{p}|_v \text{ is an unbounded subset of } \mathbb{R}\},$$

which is a finite set because $R_d/\mathfrak{p}$ is finitely generated. Following Einsiedler and Lind, associate to $\mathfrak{p}$ the list of *Lyapunov vectors*

$$\mathcal{L}(\mathfrak{p}) = \{\boldsymbol{\ell}_v = (\log|\pi(u_1)|_v, \dots, \log|\pi(u_d)|_v) \mid v \in S(\mathfrak{p})\},$$

where $\pi : R_d \to R_d/\mathfrak{p}$ denotes the usual quotient map.

In what follows, our approach is to prove an algebraic version of Theorem 1.1 for a module of the form $R_d/\mathfrak{p}$, and then build up to a general Noetherian module $M$ using standard methods (see [14] for example).

3. **Two uniformities.** Let $M$ be an $R_d$-module, and suppose that $\left(H_k^M\right)_{k\geqslant 1}$ is an increasing sequence of finite subsets of $M$ with $\bigcup_{k=1}^{\infty} H_k^M = M$ (that is, an exhaustive sequence). Let $\phi_M$ and $\psi_M$ be functions (to be chosen later) with

$$\phi_M(k), \psi_M(k) \to \infty$$

as $k \to \infty$. We are interested in the following two properties of $M$.

I: $H_k^M \cap \mathbf{u}^{\mathbf{n}} H_k^M = \{0\}$ for all $\mathbf{n} \in \mathbb{Z}^d$ with $\|\mathbf{n}\| > \phi_M(k)$ (directional uniformity of mixing),

II: $H_k^M \cap (\mathbf{u}^{\mathbf{n}} - 1) H_k^M = \{0\}$ for all $\mathbf{n} \in \mathbb{Z}^d$ with $\|\mathbf{n}\| > \psi_M(k)$ (directional uniformity of distribution of periodic points).

**Theorem 3.1.** *Let $\mathfrak{p} \subset R_d$ be a prime ideal with kdim$(R_d/\mathfrak{p}) = 1$, and suppose that $\theta(k) \nearrow \infty$ as $k \to \infty$. If the module $M = R_d/\mathfrak{p}$ corresponds to a mixing action, then there exists an exhaustive increasing sequence $\left(H_k^M\right)_{k\geqslant 1}$ of finite subsets of $M$, and a constant $B > 0$, such that Property* I *is satisfied for*

$$\phi_M(k) = B \log \theta(k).$$

The proof of Theorem 3.1 is facilitated by the following result, which is adapted from [12].

**Lemma 3.2.** *If the prime ideal $\mathfrak{p} \subset R_d$ has kdim$(R_d/\mathfrak{p}) = 1$, and the module $R_d/\mathfrak{p}$ corresponds to a mixing algebraic $\mathbb{Z}^d$-action, then the set of Lyapunov vectors $\mathcal{L}(\mathfrak{p})$ spans $\mathbb{R}^d$.*

*Proof.* This can be seen using properties of the directional entropy function

$$h : \mathbb{R}^d \to \mathbb{R}$$

(see [3] and [6]). The proof of [12, Th. 1.1] shows that $h$ is bounded away from zero when the action is mixing. If $\mathcal{L}(\mathfrak{p})$ does not span $\mathbb{R}^d$, then there exists

$$\mathbf{w} \in \mathsf{S}_{d-1} = \{\mathbf{z} \in \mathbb{R}^d \mid \|\mathbf{z}\| = 1\}$$

such that

$$h(\mathbf{w}) = \sum_{v\in V} \max\{\boldsymbol{\ell}_v \cdot \mathbf{w}, 0\} = 0,$$

giving an immediate contradiction. □

*Proof of Theorem 3.1.* Write $\widehat{\mathbf{n}} = \mathbf{n}/\|\mathbf{n}\|$ for any non-zero integer vector $\mathbf{n}$. We claim that there is a constant $C > 0$ such that given any non-zero vector $\mathbf{n} \in \mathbb{Z}^d$, there exists $w \in S(\mathfrak{p})$ such that $|\boldsymbol{\ell}_w \cdot \widehat{\mathbf{n}}| > C$. If this were not the case, then compactness of $\mathsf{S}_{d-1}$ would give a point $\mathbf{z}$ in $\mathsf{S}_{d-1}$ such that $\sum_{v\in S(\mathfrak{p})} |\boldsymbol{\ell}_v \cdot \mathbf{z}| = 0$ (since $\mathbf{w} \mapsto \sum_{v\in S(\mathfrak{p})} |\boldsymbol{\ell}_v \cdot \mathbf{w}|$ is continuous on $\mathsf{S}_{d-1}$), meaning that $\mathcal{L}(\mathfrak{p})$ would lie in the subspace orthogonal to $\mathbf{z}$, contradicting Lemma 3.2.

Set

$$H_k^M = \{a \in M \mid \theta(k)^{-1} \leqslant |a|_v \leqslant \theta(k) \text{ for all } v \in S(\mathfrak{p})\} \cup \{0\}.$$

and note that $\left(H_k^M\right)_{k\geqslant 1}$ is an increasing exhaustive sequence of finite subsets of $M$ since $\theta(k) \nearrow \infty$ as $k \to \infty$.

Given $\mathbf{n} \in \mathbb{Z}^d$ with $\|\mathbf{n}\| > \phi_M(k)$, there exists $v \in S(\mathfrak{p})$ such that $|\boldsymbol{\ell}_v \cdot \widehat{\mathbf{n}}| > C$. Let $a \in H_k^M$ be non-zero. If $\boldsymbol{\ell}_v \cdot \widehat{\mathbf{n}} < 0$, then

$$\begin{aligned} |\pi(\mathbf{u}^{\mathbf{n}})a|_v &= \exp(\|\mathbf{n}\|\boldsymbol{\ell}_v \cdot \widehat{\mathbf{n}})|a|_v \\ &< \exp\left(\frac{-C\log\theta(k)}{C}\right)\theta(k)^{1/2} \\ &= \theta(k)^{-1/2}. \end{aligned}$$

On the other hand, if $\boldsymbol{\ell}_v \cdot \widehat{\mathbf{n}} > 0$, then

$$\begin{aligned} |\pi(\mathbf{u}^{\mathbf{n}})a|_v &= \exp(\|\mathbf{n}\|\boldsymbol{\ell}_v \cdot \widehat{\mathbf{n}})|a|_v \\ &> \exp\left(\frac{C\log\theta(k)}{C}\right)\theta(k)^{-1/2} \\ &= \theta(k)^{1/2}. \end{aligned}$$

Hence, $u^{\mathbf{n}}a \notin H_k^M$, and the statement of the theorem follows by setting $B = \frac{1}{C}$. □

We now turn our attention to Property II. For a mixing action arising from a Noetherian module $M$, an essential consequence of the entropy rank one assumption is that for each $\mathbf{n} \in \mathbb{Z}^d$, the set of points fixed by $\alpha^{\mathbf{n}}$ is finite, and the cardinality of this set is equal to $|M/(\mathbf{u}^{\mathbf{n}} - 1)M|$ by duality.

**Theorem 3.3.** *Let $\mathfrak{p} \subset R_d$ be a prime ideal with $kdim(R_d/\mathfrak{p}) = 1$, and suppose that $\theta(k) \nearrow \infty$ as $k \to \infty$. If the module $M = R_d/\mathfrak{p}$ corresponds to a mixing action, then there exists an increasing exhaustive sequence $H = \left(H_k^M\right)_{k\geqslant 1}$ of finite subsets of $M$, and constants $\sigma, A, C_1, C_2 > 0$ such that Property* II *is satisfied for*

$$\psi_M(k) = \max\left\{C_1, \frac{\sigma+1}{C_2}\log\left(\frac{\theta(k)}{(A^\sigma/2)^{1/(\sigma+1)}}\right)\right\}.$$

*Proof.* Just as in the proof of Theorem 3.1, there is a constant $C > 0$ such that given any non-zero $\mathbf{n} \in \mathbb{Z}^d$, there exists $w \in S(\mathfrak{p})$ such that

$$|\boldsymbol{\ell}_w \cdot \widehat{\mathbf{n}}| > C.$$

Without loss of generality, we may always choose $w$ such that

$$\boldsymbol{\ell}_w \cdot \widehat{\mathbf{n}} > C.$$

For, given $w \in S(\mathfrak{p})$ such that $\boldsymbol{\ell}_w \cdot \widehat{\mathbf{n}} < -C$, we can consider $\prod_{v\in S(\mathfrak{p})} |\pi(\mathbf{u}^{\mathbf{n}})|_v$ as follows: Since $\pi(\mathbf{u}^{\mathbf{n}})$ is a unit in $M$, the product formula implies that

$$\prod_{v\in S(\mathfrak{p})\setminus\{w\}} |\pi(\mathbf{u}^{\mathbf{n}})|_v = |\pi(\mathbf{u}^{\mathbf{n}})|_w^{-1}.$$

Hence, for some $v \in S(\mathfrak{p}) \setminus \{w\}$ it follows that

$$|\pi(\mathbf{u}^{\mathbf{n}})|_v > |\pi(\mathbf{u}^{\mathbf{n}})|_w^{-1/\sigma},$$

where $\sigma = |S(\mathfrak{p})| - 1$. Therefore,

$$\exp(\|\mathbf{n}\|\boldsymbol{\ell}_v \cdot \widehat{\mathbf{n}}) > \exp(-\|\mathbf{n}\|\boldsymbol{\ell}_w \cdot \widehat{\mathbf{n}}/\sigma).$$

Hence $\boldsymbol{\ell}_v \cdot \widehat{\mathbf{n}} > C/\sigma$, and we simply need to replace $C$ by $C/\sigma$.

As before, set

$$H_k^M = \{a \in M \mid \theta(k)^{-1} \leqslant |a|_v \leqslant \theta(k) \text{ for all } v \in S(\mathfrak{p})\} \cup \{0\},$$

which defines an increasing exhaustive sequence since $\theta(k) \nearrow \infty$ as $k \to \infty$.

Fix $\epsilon > 0$ and let $\mathbf{n} \in \mathbb{Z}^d$ satisfy $\|\mathbf{n}\| > \psi_M(k)$, where in the definition of $\psi_M$ we make the choices $\sigma = |S(\mathfrak{p})| - 1$, $C_2 = C - \epsilon$, and the constants $A$ and $C_1$ are to be specified later. Let $a \in H_k^M$ be non-zero and suppose that $(\mathbf{u}^{\mathbf{n}} - 1)a \in H_k^M$. This implies that $|\pi(\mathbf{u}^{\mathbf{n}} - 1)a|_w < \theta(k)$, so

$$|a|_w < \frac{\theta(k)}{|\pi(\mathbf{u}^{\mathbf{n}}) - 1|_w} < \frac{2\theta(k)}{|\pi(\mathbf{u}^{\mathbf{n}})|_w}. \tag{3}$$

By the product formula $\prod_{v\in\mathcal{P}(\mathbb{K}(\mathfrak{p}))} |a|_v = 1$, so

$$\prod_{v\in S(\mathfrak{p})\setminus\{w\}} |a|_v = |a|_w^{-1} \prod_{v\in\mathcal{P}(\mathbb{K}(\mathfrak{p}))\setminus S(\mathfrak{p})} |a|_v^{-1} \geqslant |a|_w^{-1},$$

as $|a|_v \leqslant 1$ for all $v \in \mathcal{P}(\mathbb{K}(\mathfrak{p})) \setminus S(\mathfrak{p})$. Thus at least one $v \in S(\mathfrak{p}) \setminus \{w\}$ satisfies

$$|a|_v \geqslant |a|_w^{-1/\sigma} > \left(\frac{|\pi(\mathbf{u}^{\mathbf{n}})|_w}{2\theta(k)}\right)^{1/\sigma}, \tag{4}$$

by (3).

If $v$ is archimedean, then by Baker's Theorem [1], there exist constants $A, B > 0$ such that

$$|\pi(\mathbf{u}^{\mathbf{n}}) - 1|_v \geqslant \frac{A}{\max_{1\leqslant i\leqslant d}\{n_i\}^B}.$$

If $v$ is non-archimedean, a similar bound holds by Yu's Theorem [15]. In both the archimedean and non-archimedean cases, given the ideal $\mathfrak{p}$, the constants arising can (in principle) be computed explicitly. Combining these bounds with (4) gives

$$\begin{aligned}
|\pi(\mathbf{u}^{\mathbf{n}} - 1)a|_v = |\pi(\mathbf{u}^{\mathbf{n}} - 1)|_v|a|_v &\geqslant \frac{A|\pi(\mathbf{u}^{\mathbf{n}})|_w^{1/\sigma}}{\max_{1\leqslant i\leqslant d}\{n_i\}^B(2\theta(k))^{1/\sigma}}\\
&= \frac{A\exp(\|\mathbf{n}\|\ell_w\cdot\widehat{\mathbf{n}}/\sigma)}{\max_{1\leqslant i\leqslant d}\{n_i\}^B(2\theta(k))^{1/\sigma}}\\
&\geqslant \frac{A\exp(\|\mathbf{n}\|C/\sigma)}{\max_{1\leqslant i\leqslant d}\{n_i\}^B(2\theta(k))^{1/\sigma}}\\
&\geqslant \frac{A}{(2\theta(k))^{1/\sigma}}\exp\left(\frac{(C-\epsilon)\|\mathbf{n}\|}{\sigma}\right), \qquad (5)
\end{aligned}$$

provided that $\|\mathbf{n}\|$ is large enough to ensure that

$$\max_{1\leqslant i\leqslant d}\{n_i\}^B \leqslant \exp(\|\mathbf{n}\|\epsilon/\sigma).$$

We may ensure this by a suitable choice of $C_1 = C_1(\epsilon)$ in the definition of $\psi_M(k)$, since $\|\mathbf{n}\| > \psi_M(k)$. Furthermore, since $\|\mathbf{n}\| > \psi_M(k)$, the right-hand side of (5) is strictly greater than

$$\frac{A}{(2\theta(k))^{1/\sigma}}\left(\frac{\theta(k)}{(A^\sigma/2)^{1/(\sigma+1)}}\right)^{1+1/\sigma} = \theta(k),$$

so $(\mathbf{u}^{\mathbf{n}} - 1)a \notin H_k^M$, disagreeing with our contrary assumption which therefore must have been false. □

Theorems 3.1 and 3.3 describe (in an opaque form) uniformity in rate of mixing and in the distribution of periodic points respectively for cyclic systems – those corresponding to cyclic $R_d$-modules. As usual, we need arguments from commutative algebra to build up to a more general picture. The next lemma allows the two properties to be inherited by a suitable extension of one action by another.

**Lemma 3.4.** *Let $L, M$ be $R_d$-modules with $L \subset M$, and suppose that both $L$ and $M/L$ are mixing.*

1. *If Property* I *holds for $L$ and for $M/L$, then there is an appropriate increasing exhaustive sequence $\left(H_k^M\right)_{k\geqslant 1}$ and function*

$$\phi_M(k) = \max\{\phi_L(k), \phi_{M/L}(k)\},$$

*such that it also holds for $M$.*

2. *If Property* II *holds for $L$ and for $M/L$, then there is an appropriate increasing exhaustive sequence $\left(H_k^M\right)_{k\geqslant 1}$ and function*

$$\psi_M(k) = \max\{\psi_L(k), \psi_{M/L}(k)\},$$

*such that it also holds for $M$.*

*Proof.* For each $k \in \mathbb{N}$, let

$$H_k^M = \bigcup_{x \in K \cap \pi^{-1}(H_k^{M/L})} x + H_k^L,$$

where $\pi : M \to M/L$ is the natural quotient map of $R_d$-modules, and $K$ is a set of coset representatives containing 0. By construction, each $H_M^k$ is finite and $\bigcup_{k=1}^{\infty} H_k^M = M$.

(i) Suppose that Property I is violated for some $\mathbf{n}$ with $\|\mathbf{n}\| > \phi_M(k)$. Then there exist $w, x \in K \cap \pi^{-1}(H_k^{M/L})$ and $g, h \in H_k^L$ such that

$$\mathbf{u}^{\mathbf{n}}(x+h) = w + g \neq 0. \tag{6}$$

In particular, $u^{\mathbf{n}}\pi(x) = \pi(w)$, which means that $\pi(w) = 0$ since Property I holds for $M/L$ by hypothesis. Therefore, $w = 0$ by our choice of $K$, and $\pi(x) = 0$ as multiplication by $\mathbf{u}^{\mathbf{n}}$ is an automorphism of $M/L$. It follows that $x = 0$ by our choice of $K$ and so (6) implies that $\mathbf{u}^{\mathbf{n}}h = g$ meaning that $g = 0$, as Property I holds for $L$. So, $w + g = 0$, contradicting (6).

(ii) Suppose that Property II is violated for some $\mathbf{n}$ with $\|\mathbf{n}\| > \psi_M(k)$. Then there exist $x \in K$, $w \in K \cap \pi^{-1}(H_k^{M/L})$, $g \in H_k^L$ and $h \in L$ such that

$$(\mathbf{u}^{\mathbf{n}} - 1)(x+h) = w + g \neq 0. \tag{7}$$

Thus $(\mathbf{u}^{\mathbf{n}} - 1)\pi(x) = \pi(w)$, which means that $\pi(w) = 0$ since Property II holds for $M/L$. This forces $\pi(x) = 0$ as multiplication by $(\mathbf{u}^{\mathbf{n}} - 1)$ is injective on $M/L$ (since $M/L$ corresponds to a mixing system). Therefore $x = 0$ by our choice of $K$, and so (7) implies that $(\mathbf{u}^{\mathbf{n}} - 1)h = g$ meaning $g = 0$, as Property II holds for $L$. So $w + g = 0$, contradicting (7). □

The next lemma shows that both properties are inherited when passing to factors of systems (factors of algebraic $\mathbb{Z}^d$-actions correspond to submodules under duality).

**Lemma 3.5.** *Let $L, M$ be $R_d$-modules with $L \subset M$ and suppose that $M$ is mixing.*

1. *If $M$ has Property* I*, then there is an appropriate increasing sequence $\left(H_k^L\right)_{k \geqslant 1}$ and function $\phi_L(k) = \phi_M(k)$ such that it also holds for $L$.*
2. *If $M$ has Property* II*, then there is an appropriate increasing exhaustive sequence $\left(H_k^L\right)_{k \geqslant 1}$ and function $\psi_L(k) = \psi_M(k)$ such that it also holds for $L$.*

*Proof.* For each $k \in \mathbb{N}$ let $H_k^L = H_k^M \cap L$. Then $\bigcup_{k=1}^{\infty} H_k^L = L$ and each $H_k^L$ is finite.

(i) For $\mathbf{n} \in \mathbb{Z}^d$ with $\|\mathbf{n}\| > \phi_L(k)$,

$$\begin{aligned} H_k^L \cap \mathbf{u}^{\mathbf{n}} H_k^L &= H_k^M \cap L \cap \mathbf{u}^{\mathbf{n}}(H_k^M \cap L) \\ &= H_k^M \cap L \cap \mathbf{u}^{\mathbf{n}} H_k^M \cap \mathbf{u}^{\mathbf{n}} L, \end{aligned}$$

since multiplication by $u^{\mathbf{n}}$ is injective. Furthermore, the right-hand side is $\{0\}$, as Property I holds for $M$.

(ii) Similarly, for $\mathbf{n} \in \mathbb{Z}^d$ with $\|\mathbf{n}\| > \psi_L(k)$,

$$\begin{aligned} H_k^L \cap (\mathbf{u}^{\mathbf{n}} - 1) H_k^L &= H_k^M \cap L \cap (\mathbf{u}^{\mathbf{n}} - 1)(H_k^M \cap L) \\ &= H_k^M \cap L \cap (\mathbf{u}^{\mathbf{n}} - 1) H_k^M \cap (\mathbf{u}^{\mathbf{n}} - 1) L, \end{aligned}$$

as multiplication by $\mathbf{u}^{\mathbf{n}} - 1$ is injective by the mixing assumption. Furthermore, the right-hand side is $\{0\}$, as Property II holds for $M$. □

We are now ready to pass the two uniformity properties up from cyclic modules to Noetherian modules.

**Theorem 3.6.** *Let $M$ be a Noetherian $R_d$-module corresponding to a mixing algebraic $\mathbb{Z}^d$-action of entropy rank one, and suppose $\theta(k) \nearrow \infty$ as $k \to \infty$. Then there is an increasing exhaustive sequence $\left(H_k^M\right)_{k\geqslant 1}$ of finite subsets of $M$, and there are constants $B, C > 0$, such that Property* I *is satisfied for*

$$\phi_M(k) = B \log \theta(k),$$

*and Property* II *is satisfied for*

$$\psi_M(k) = C \log \theta(k).$$

*Proof.* Write $\mathrm{Ass}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ and note that

$$\mathrm{kdim}(R_d/\mathfrak{p}_i) = 1$$

for each $1 \leqslant i \leqslant r$ by the mixing and entropy rank one assumptions. By [13, Cor. 6.3] or [14, Sect. 4], $M$ embeds in a module of the form $M' = \bigoplus_{i=1}^r M(i)$, where each $R_d$-module $M(i)$ for $1 \leqslant i \leqslant r$, has a prime filtration of the form

$$\{0\} = N_0^{(i)} \subset N_1^{(i)} \subset \cdots \subset N_{s(i)}^{(i)} = M(i), \tag{8}$$

with $N_j^{(i)}/N_{j-1}^{(i)} \cong R_d/\mathfrak{p}_i$ for all $1 \leqslant j \leqslant s(i)$. Each of these modules is mixing by [13, Th. 6.5].

We first consider Property I. For each module $M(i)$, by Theorem 3.1, Lemma 3.4, and induction on the prime filtration (8), we may find an increasing exhaustive sequence $\left(H_k^{M(i)}\right)_{k\geqslant 1}$ of finite subsets of $M(i)$ such that Property I is satisfied for

$$\phi_{M(i)}(k) = B_i \log \theta(k),$$

where $B_i > 0$ is the constant appearing in Theorem 3.1 (which follows from the proof of Lemma 3.4). For each $k \in \mathbb{N}$, set

$$H_k^{M'} = \bigoplus_{i=1}^r H_k^{M(i)}$$

and

$$\phi_{M'}(k) = B \log \theta(k),$$

where $B = \max_{1\leqslant i\leqslant r}\{B_i\}$. Therefore, Property I holds for $M'$ and the required result follows by applying Lemma 3.5.

Property II is obtained in an analogous way, noting that $\psi_M(k)$ can be replaced by $\psi_M(k) = C \log \theta(k)$ for a suitably large choice of $C$ in Theorem 3.3. □

4. **Remarks.** (1) Theorem 3.6 gives Theorem 1.1 simply by translation: the class of functions $\mathcal{C}(X)$ is defined by choosing a rate of decay for the size of coefficients in the Fourier expansion outside $H_k^M$ so rapid that the sum over $M \setminus H_k$ is $o(1)$ in $k$, choosing $\theta$, and then computing $\phi'_M$ and $\psi'_M$.

(2) Throughout, the function $\theta$ can be chosen arbitrarily. We have left it in the statements to facilitate more explicit estimates for specific classes of compact groups.

(3) It seems possible that the uniformity in mixing seen here could be present in higher entropy ranks. We initially attempted to prove this using adelic amoebas [4] in place of Lyapunov vectors. For a mixing action of higher entropy rank, an unpublished argument due to Einsiedler enables one to see that the adelic amoeba

spans $\mathbb{R}^d$, just as the set of Lyapunov vectors does for an entropy rank one action. However, finding a suitable exhaustive sequence in the dual module based on this appears to be rather problematic. Notably, however, one only needs to consider an action corresponding to a cyclic module as the method of passing up to Noetherian modules used here (Lemmas 3.4 and 3.5) works for all entropy ranks.

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