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Finite entropy characterizes topological rigidity on connected groups

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Abstract. Let X_1, X_2 be mixing connected algebraic dynamical systems with the descending chain condition. We show that every equivariant continuous map $\mathsf{X}_1 \to \mathsf{X}_2$ is affine (that is, X_2 is *topologically rigid*) if and only if the system X_2 has finite topological entropy.

1. *Introduction*

An *algebraic $\mathbb{Z}^d$-action* α on a compact abelian group X is a homomorphism $\alpha : \mathbf{n} \mapsto \alpha(\mathbf{n})$ from $\mathbb{Z}^d$ to the group $\mathrm{Aut}(X)$ of continuous automorphisms of X. Compact groups are assumed to be metrizable throughout and are written multiplicatively; e is used to denote the identity element of any group. Write $\mathsf{X} = (X, \alpha)$ for such an algebraic dynamical system, and call the system X connected, mixing and so on if X is connected, α is mixing, and so on.

Any algebraic system X preserves λ_X, the Haar measure on X. The system X is *mixing* if

$$\lim_{\mathbf{n}\to\infty} \lambda_X(A_1 \cap \alpha(\mathbf{n})(A_2)) = \lambda_X(A_1) \cdot \lambda_X(A_2)$$

for all measurable sets $A_1, A_2 \subset X$.

A map $\phi : \mathsf{X}_1 \to \mathsf{X}_2$ between algebraic dynamical systems is *equivariant* if $\phi \circ \alpha_1(\mathbf{n}) = \alpha_2(\mathbf{n}) \circ \phi$ for all $\mathbf{n} \in \mathbb{Z}^d$, and is *affine* if there is a continuous group homomorphism $\psi : X_1 \to X_2$ and an element $y \in X_2$ with $\phi(x) = \psi(x) \cdot y$.

Topological (respectively, measurable) rigidity is a property of the systems X_1 and X_2 that forces an equivariant continuous (respectively measurable) map to coincide everywhere (respectively almost everywhere) with an affine map. We fix X_1 throughout to be a mixing, connected algebraic $\mathbb{Z}^d$-action, so will speak loosely of rigidity as a property of the target system X_2.

For $d \geq 1$, denote by $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ the ring of Laurent polynomials with integral coefficients in d commuting variables $u_1, \ldots, u_d$. An element f of R_d is written

$$f(\mathbf{u}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{u}^{\mathbf{n}}$$

with $\mathbf{u}^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$, $f_{\mathbf{n}} \in \mathbb{Z}$ for all $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, and $f_{\mathbf{n}} = 0$ for all but finitely many $\mathbf{n} \in \mathbb{Z}^d$.

If $\mathsf{X} = (X, \alpha)$ is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the countable dual group $M = \widehat{X}$ is a module over the ring R_d under the operation

$$f \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \widehat{\alpha}(\mathbf{n})(a)$$

for $f \in R_d$ and $a \in M$. The module M is called the *dual module* of X. Conversely, a countable module M over R_d determines an algebraic $\mathbb{Z}^d$-action $\mathsf{X}_M = (X_M, \alpha_M)$ by setting

$$\widehat{\alpha_M}(\mathbf{n})(a) = \mathbf{u}^{\mathbf{n}} \cdot a$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in M$.

An algebraic $\mathbb{Z}^d$-action X is *Noetherian* if the dual module is Noetherian. The following properties are equivalent:

- X_M is Noetherian;
- M is finitely generated over R_d (this is equivalent to M being Noetherian since R_d is itself Noetherian);
- any descending chain of closed α_M-invariant subgroups of X_M stabilizes (the descending chain condition, see [**9**]).

The topological entropy of the system X_M is defined and computed in terms of the module M in [**11**].

Rigidity properties of algebraic $\mathbb{Z}^d$-actions have been studied by several authors. Measurable equivariant maps between mixing zero-entropy algebraic $\mathbb{Z}^d$-actions exhibit strong regularity properties (see [**4, 5, 8, 10**]). For a $\mathbb{Z}$-action generated by an automorphism θ on a connected finite-dimensional compact abelian group, it is known that the topological centralizer of the action admits non-affine maps if and only if θ is not ergodic (cf. [**1, 2, 16**]). Ergodic automorphisms of infinite-dimensional groups may have non-affine maps in their centralizers (see Example 1.3). In [**3**] it is shown that for any expansive connected algebraic $\mathbb{Z}^d$-action X, the topological centralizer of α consists of affine maps (expansiveness is a condition that implies the descending chain condition; for $d = 1$ it forces the compact group X to be finite-dimensional).

In this paper we prove the following result, which characterizes a form of topological rigidity in terms of topological entropy.

THEOREM 1.1. *Let X_1, X_2 be connected mixing Noetherian algebraic $\mathbb{Z}^d$-actions. Then the following properties are equivalent:*

(1) *every equivariant continuous map $\mathsf{X}_1 \to \mathsf{X}_2$ is an affine map;*
(2) *the system X_2 has finite topological entropy.*

For $d \geq 2$, this result applies to situations where the underlying group X_2 is infinite-dimensional, so the lifting techniques of [**1**] and [**16**] cannot be applied directly. Write $\mathbb{T} \subset \mathbb{C}$ for the multiplicative unit circle.

Example 1.2. To illustrate Theorem 1.1, consider the $\mathbb{Z}^2$-action X where $X \subset \mathbb{T}^{\mathbb{Z}^2}$ is the closed subgroup consisting of all $x \in \mathbb{T}^{\mathbb{Z}^2}$ with

$$x(m+1,n) \cdot x(m,n) \cdot x(m,n+1) = 1 \quad \text{for all } m,n \in \mathbb{Z},$$

and α is the shift action of $\mathbb{Z}^2$ on X. The system X is mixing and has finite entropy. It follows that every continuous equivariant map from X to itself is an affine map. In contrast, the *measurable* centralizer of X contains many non-affine maps, since X is measurably isomorphic to a $\mathbb{Z}^2$ Bernoulli shift (see [**13, 17**]).

Example 1.3. For the case of a single automorphism, the compact group being finite-dimensional forces the entropy to be finite. Ergodic automorphisms of infinite-dimensional groups are not topologically rigid in general.

For example, the shift automorphism of $X = \mathbb{T}^{\mathbb{Z}}$ defines an ergodic $\mathbb{Z}$-action of infinite entropy that is not topologically rigid: if $f : \mathbb{T} \to \mathbb{T}$ is any map, then the shift map commutes with the map $\phi : X \to X$ defined by $(\phi(x))_k = f(x_k)$. The module corresponding to this action is a Noetherian R_1-module.

On the other hand, an ergodic automorphism of $\mathbb{T}^{\mathbb{Z}}$ that splits into a direct product of automorphisms of finite-dimensional tori *is* topologically rigid. The module corresponding to this action is not Noetherian. It is not known whether such an action can have finite topological entropy (see [**12**]).

The next example shows that Theorem 1.1 does not hold for non-Noetherian actions.

Example 1.4. Let F_d denote the field of fractions of R_d, considered as a R_d-module. Let X_1 denote the algebraic $\mathbb{Z}^d$-action corresponding to F_d. Notice that F_d is torsion-free as an R_d-module, and X_1 has infinite entropy. For any $\mathbf{n} \in \mathbb{Z}^d$, multiplication by $\mathbf{u}^{\mathbf{n}} - 1$ is an automorphism of F_d. By duality, the map $x \mapsto \alpha_1(\mathbf{n})(x) - x$ is a continuous automorphism of X_1 for any $\mathbf{n} \in \mathbb{Z}^d$. In particular, X_1 does not have any non-trivial periodic orbits. Now let X_2 be any mixing connected algebraic $\mathbb{Z}^d$-action with a dense set of periodic orbits (any Noetherian system has this property). Since continuous equivariant maps take periodic orbits to periodic orbits, it follows that any continuous equivariant map from X_2 to X_1 is trivial.

Example 1.4 is similar in spirit to a remark of Comfort (see [**7**]): there are no non-trivial homomorphisms $\mathbb{T} \to \widehat{\mathbb{Q}}$ since torsion elements are dense in $\mathbb{T}$ but absent in $\widehat{\mathbb{Q}}$.

2. *Algebraic $\mathbb{Z}^d$-actions*

In this section basic results and terminology on algebraic $\mathbb{Z}^d$-actions are collected. A prime ideal $\mathfrak{p} \subset R_d$ is *associated with* the R_d-module M if there exists $m \in M$ with $\mathfrak{p} = \{f \in R_d \mid f \cdot m = 0\}$. The set of prime ideals associated with M is denoted $\mathrm{Asc}(M)$. If M is Noetherian, then $\mathrm{Asc}(M)$ is finite. The torsion submodule of M is defined by

$$\mathrm{Tor}(M) = \{m \in M \mid r \cdot m = 0 \text{ for some non-zero } r \in R_d\}.$$

A module M is said to be a *torsion module* if $\mathrm{Tor}(M) = M$.

The following result taken from [**14**, Theorem 6.5] characterizes mixing in algebraic terms.

LEMMA 2.1. *The algebraic $\mathbb{Z}^d$-action X_M is mixing if and only if for every $\mathfrak{p} \in \mathrm{Asc}(M)$ and for every non-zero $\mathbf{n} \in \mathbb{Z}^d$, the polynomial $\mathbf{u}^{\mathbf{n}} - 1$ does not lie in $\mathfrak{p}$.*

An algebraic $\mathbb{Z}^d$-action X_2 is an *algebraic factor* of X_1 if there is a surjective continuous equivariant homomorphism $\phi : X_1 \to X_2$.

The next lemma shows that if the module corresponding to an algebraic $\mathbb{Z}^d$-action is Noetherian, then infinite topological entropy can only be created by the presence of (a factor of) a full shift with infinite alphabet. This result is easily obtained from [**11**] or [**14**, Proposition 19.4]; a proof is included here to make the paper relatively self-contained and to show how the algebraic properties of the module M interact with the dynamical properties of the system X_M.

LEMMA 2.2. *For a Noetherian system X_M the following conditions are equivalent:*
(1) *X_M does not admit a non-trivial closed α_M-invariant subgroup H with the property that the restriction of α_M to H is an algebraic factor of the shift action of $\mathbb{Z}^d$ on $(\mathbb{T}^n)^{\mathbb{Z}^d}$ for some $n > 0$;*
(2) *M is a torsion module;*
(3) *X_M has finite topological entropy.*

Proof. (1) $\implies$ (2). Suppose that M is not a torsion module and $N = M/\mathrm{Tor}(M)$. Then N is a non-zero torsion-free R_d-module.

We claim that N is isomorphic to a submodule of the free module R_d^n of rank n for some $n \geq 1$. Let N_0 denote the localization of N at the prime ideal $\{0\}$. Since N is Noetherian, N_0 is a finite-dimensional vector space over $F = \mathbb{Z}(u_1^{\pm 1}, \dots, u_d^{\pm 1})$, the quotient field of R_d. Let $B = \{b_1, \dots, b_n\}$ be any F-basis of N_0. The map $m \mapsto m/1$ embeds N as a submodule of N_0. Choose a finite R_d-generating set A of N, and an element $p \in R_d$ with the property that $p \cdot a$ lies in the R_d-submodule generated by B for all $a \in A$. The submodule generated by $\{b_1/p, \dots, b_n/p\}$ contains N, and is a free R_d-module of rank n. This proves the claim.

The system X_N is therefore an algebraic factor of the shift action on $(\mathbb{T}^n)^{\mathbb{Z}^d}$. Since N is a quotient of M, by duality there exists a closed α-invariant subgroup $H \subset X$ such that the restriction of α to H is conjugate to X_N.

(2) $\implies$ (1). Let $H \subset X$ be a closed α-invariant subgroup such that the restriction of α to H is an algebraic factor of the shift action on $(\mathbb{T}^n)^{\mathbb{Z}^d}$ for some $n > 0$. The dual module of the shift action on $(\mathbb{T}^n)^{\mathbb{Z}^d}$ is isomorphic to the direct sum of n copies of R_d, which implies that $\widehat{H}$ is a torsion-free R_d-module. On the other hand, $\widehat{H}$ is a quotient of the R_d-module M, which is a torsion module by assumption (2). Hence $\widehat{H} = \{0\}$, so that H is trivial.

(2) $\implies$ (3). Let $\{m_1, \dots, m_k\}$ generate M as an R_d-module. For $j = 1, \dots, k$ let $I_j \subset R_d$ denote the ideal defined by

$$I_j = \{p \in R_d \mid p \cdot m_j = 0\}.$$

Since M is a torsion module, each I_j is non-zero. For $j = 1, \dots, k$ let M_j denote the R_d-module R_d/I_j. Since each I_j is non-zero, X_{M_j} has finite entropy by [**11**, Theorem 3.1]. Let

$$M' = M_1 \oplus \cdots \oplus M_k.$$

Since each X_{M_j} has finite entropy, $\mathsf{X}_{M'}$ also has finite entropy. The map $(r_1, \dots, r_k) \mapsto r_1 m_1 + \cdots + r_k m_k$ expresses M as a quotient of M'. The dual of this map embeds X_M as a sub-action of $\mathsf{X}_{M'}$, so in particular X_M has finite entropy.

(3) $\implies$ (2). If M is not a torsion module, then it contains R_d as a submodule. By duality, the shift action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$ is therefore an algebraic factor of X_M. Since the former action has infinite entropy, X_M has infinite topological entropy. □

3. *van Kampen's theorem*

In the proof of Theorem 1.1 the following structure theorem of van Kampen [**15**] will be used in place of the lifting of toral maps. This result splits continuous maps into a 'linear' part (a character) and a 'nonlinear' part in a unique way. It is also used in this connection by Walters [**16**].

THEOREM 3.1. *Let X be a compact connected abelian group and let $f : X \to \mathbb{T}$ be a continuous map with $f(e) = 1$. Then there exist a character $\phi \in \widehat{X}$ and a continuous map $S(f) : X \to \mathbb{R}$ such that*

$$S(f)(e) = 0, \quad f(x) = \phi(x) \cdot e^{2\pi i S(f)(x)} \quad \textit{for all } x \in X. \tag{1}$$

Moreover, ϕ and $S(f)$ are uniquely defined by (1).

4. *Rigidity of equivariant maps*

For any algebraic $\mathbb{Z}^d$-action $\mathsf{X} = (X, \alpha)$ and for any locally compact abelian group A, denote by A^X the group of all continuous maps

$$h : X \to A, \quad h(e) = e,$$

equipped with point-wise multiplication. The action α induces the structure of an R_d-module on A^X by defining

$$p \cdot h(x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} p(\mathbf{n}) \cdot h \circ \alpha(\mathbf{n})(x).$$

A key observation is that $\widehat{X}$ can be regarded as a submodule of $\mathbb{T}^X$ with this structure.

PROPOSITION 4.1. *Let $\mathsf{X} = (X, \alpha)$ be a connected $\mathbb{Z}^d$-action. Then $\mathbb{R}^X$ and $\mathbb{T}^X/\widehat{X}$ are isomorphic as R_d-modules.*

Proof. The correspondence $f \mapsto S(f)$ from Theorem 3.1 induces a map S from $\mathbb{T}^X$ to $\mathbb{R}^X$. If f_1, f_2 are elements of $\mathbb{T}^X$ then, by the uniqueness part of Theorem 3.1, $S(f_1\overline{f_2}) = S(f_1) - S(f_2)$, so that S is a group homomorphism. Similarly, if θ is a continuous endomorphism of G, then $S(f \circ \theta) = S(f) \circ \theta$. Hence, $S : \mathbb{T}^X \to \mathbb{R}^X$ is an R_d-module homomorphism. For any f in $\mathbb{R}^X$, $S(e^{2\pi i f}) = f$, so that the map S is surjective. Since $\ker(S) = \widehat{X}$, the statement follows. □

If f and g are functions from $\mathbb{Z}^d$ to $\mathbb{C}$ and g has finite support, the convolution $f * g : \mathbb{Z}^d \to \mathbb{C}$ is given by

$$f * g(\mathbf{i}) = \sum_{\mathbf{j} \in \mathbb{Z}^d} f(\mathbf{i} - \mathbf{j}) \cdot g(\mathbf{j}).$$

Write $L^2(\mathbb{Z}^d)$ for the set of all square-integrable functions $\mathbb{Z}^d \to \mathbb{C}$ (with respect to the counting measure on $\mathbb{Z}^d$).

In addition to van Kampen's theorem, a simple version of the L^2 zero-divisor problem is needed (see [**6**] for an overview).

PROPOSITION 4.2. *If $f \in L^2(\mathbb{Z}^d)$ has $f * g = 0$ for some non-zero function $g : \mathbb{Z}^d \to \mathbb{C}$ with finite support, then f is identically zero.*

Proof. Since the support of g is finite, there exists $\mathbf{n} \in \mathbb{Z}^d$ such that the support of $g * \delta_{\mathbf{n}}$ is contained in $\mathbb{N}^d$. Replacing g by $g * \delta_{\mathbf{n}}$ if necessary, we may assume that the support of g is contained in $\mathbb{N}^d$. Let $\widehat{f}, \widehat{g} \in L^2(\mathbb{T}^d)$ denote the Fourier transforms of f and g respectively. By the choice of g, $\widehat{g} = p|_{\mathbb{T}^d}$ for some non-zero polynomial $p(\mathbf{z}) \in \mathbb{C}[z_1, \ldots, z_d]$. Define $V(p) \subset \mathbb{T}^d$ by

$$V(p) = \{x \in \mathbb{T}^d \mid p(x) = 0\}.$$

We claim that $\lambda_d(V(p)) = 0$ for any non-zero p, where λ_d is the Haar measure on $\mathbb{T}^d$. This may be proved by induction on d. If $d = 1$, then $V(p) \subset \mathbb{T}$ is finite since every non-zero polynomial has only finitely many roots. If $d > 1$, choose polynomials $p_0, \ldots, p_k$ in $\mathbb{C}[z_1, \ldots, z_{d-1}]$ such that

$$p(z_1, \ldots, z_d) = \sum_{i=0}^{k} p_i(z_1, \ldots, z_{d-1}) z_d^i.$$

Since p is non-zero, p_i is non-zero for some i. By the inductive hypothesis, $\lambda_{d-1}(V(p_i)) = 0$. If $(z_1, \ldots, z_{d-1})$ lies in $\mathbb{T}^{d-1} \backslash V(p_i)$, then the map $z \mapsto p(z_1, \ldots, z_{d-1}, z)$ is a non-zero polynomial in $\mathbb{C}[z]$. This implies that for any $(z_1, \ldots, z_{d-1}) \in \mathbb{T}^{d-1} \backslash V(p_i)$, the set

$$\{z \in \mathbb{T} \mid (z_1, \ldots, z_{d-1}, z) \in V(p)\}$$

is finite. By Fubini's theorem,

$$\lambda_d(V(p)) = \int_{\mathbb{T}^{d-1}} \int_{\mathbb{T}} \mathbb{I}_{V(p)} \, d\lambda_1 \, d\lambda_{d-1} = 0,$$

where $\mathbb{I}$ is the indicator function, which proves the claim. Since $\widehat{g} = p|_{\mathbb{T}^d}$ and $\widehat{f} \cdot \widehat{g} = \widehat{f * g} = 0$, it follows that $\widehat{f} = 0$ almost everywhere, so that $f = 0$. □

LEMMA 4.3. *If $\mathsf{X} = (X, \alpha)$ is a mixing connected algebraic $\mathbb{Z}^d$-action, then* $\mathrm{Tor}(\mathbb{T}^X) \subset \widehat{X}$.

Proof. Let M denote the set of all square-integrable functions

$$h : \widehat{X} \to \mathbb{C}, \quad h(e) = 0.$$

Defining

$$p \cdot h(\phi) = \sum_{\mathbf{i} \in \mathbb{Z}^d} p(\mathbf{i}) h(\phi \circ \alpha(\mathbf{i}))$$

for $p \in R_d$ gives M the structure of an R_d-module.

We claim first that M is torsion-free. Let h be an element of $\mathrm{Tor}(M)$; for any non-trivial $\chi \in \widehat{X}$ define a function $h_\chi : \mathbb{Z}^d \to \mathbb{C}$ by

$$h_\chi(\mathbf{i}) = h(\chi \circ \alpha(\mathbf{i})).$$

Since α is mixing, the map $\mathbf{i} \mapsto \chi \circ \alpha(\mathbf{i})$ is one-to-one. Hence,

$$\sum_{\mathbf{i} \in \mathbb{Z}^d} |h_\chi(\mathbf{i})|^2 \leq \sum_{\chi \in \widehat{X}} |h(\chi)|^2 < \infty.$$

This shows that $h_\chi \in L^2(\mathbb{Z}^d)$ for all $\chi \in \widehat{X}$. Note that $L^2(\mathbb{Z}^d)$ itself is an R_d-module with respect to the multiplication $p \cdot h = p * h$. Furthermore, the map $h \mapsto h_\chi$ is an R_d-module homomorphism from M to $L^2(\mathbb{Z}^d)$. Since $\mathrm{Tor}(L^2(\mathbb{Z}^d)) = \{0\}$ by Proposition 4.2, we conclude that $h_\chi = 0$ for all χ, so that $h = 0$. This proves that M is torsion-free.

For f in $\mathbb{R}^X$, let $\widehat{f} \in M$ denote the Fourier transform of f. Since $\widehat{f \circ \theta}(\phi) = \widehat{f}(\phi \circ \theta)$ for any continuous endomorphism θ of X, the map $f \mapsto \widehat{f}$ is an R_d-module homomorphism from $\mathbb{R}^X$ to M. By the Fourier inversion theorem this map is injective. Since $\mathrm{Tor}(M) = \{0\}$, this implies that $\mathrm{Tor}(\mathbb{R}^X) = \{0\}$. Proposition 4.1 then shows that $\mathrm{Tor}(\mathbb{T}^X) \subset \widehat{X}$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that X_2 has finite entropy, and let f be an equivariant continuous map $\mathsf{X}_1 \to \mathsf{X}_2$. Define $f_0 : X_1 \to X_2$ by

$$f_0(x) = f(x) - f(e). \tag{2}$$

Since f is equivariant, so is f_0.

Fix an arbitrary character $\phi \in \widehat{X_2}$. By Lemma 2.2, $\widehat{X_2}$ is a torsion module, so ϕ lies in the torsion submodule of $\mathbb{T}^{X_2}$. Since f_0 is equivariant and $f_0(e) = 1$, the map $h \mapsto h \circ f_0$ is an R_d-module homomorphism $\mathbb{T}^{X_2} \to \mathbb{T}^{X_1}$. Hence $\phi \circ f_0$ is an element of the torsion submodule of $\mathbb{T}^{X_1}$. By Lemma 4.3, $\phi \circ f_0$ lies in $\widehat{X_1}$. Since the initial choice of ϕ was arbitrary, this shows that $\phi \mapsto \phi \circ f_0$ is a group homomorphism from $\widehat{X_2}$ to $\widehat{X_1}$. By duality, there exists a continuous homomorphism $\theta : X_1 \to X_2$ such that $\phi \circ f_0 = \phi \circ \theta$ for all $\phi \in \widehat{X_2}$. Since characters separate points, this implies that $f_0 = \theta$. Hence $f = f(e) + f_0$ is an affine map.

If X_2 has infinite entropy, then by Lemma 2.2 there exists a non-trivial closed α_2-invariant subgroup $H \subset X_2$ with the property that the restriction of α_2 to H is an algebraic factor of the shift action on $(\mathbb{T}^n)^{\mathbb{Z}^d}$ for some $n > 0$. Let $K \subset (\mathbb{T}^n)^{\mathbb{Z}^d}$ be a proper, closed shift-invariant subgroup such that the restriction of α_2 to H is algebraically conjugate to the shift action of $\mathbb{Z}^d$ on $(\mathbb{T}^n)^{\mathbb{Z}^d}/K$. Since any non-affine equivariant map from X_1 to H gives rise to a non-affine equivariant map $\mathsf{X}_1 \to \mathsf{X}_2$, without loss of generality we may assume that $X_2 = (\mathbb{T}^n)^{\mathbb{Z}^d}/K$, and α_2 is the shift action.

For any continuous map $q : X_1 \to \mathbb{R}^n$ define a map

$$\sigma(q) : X_1 \to (\mathbb{T}^n)^{\mathbb{Z}^d}$$

by

$$\sigma(q)(x)(\mathbf{n}) = \exp \circ\, q \circ \alpha_1(\mathbf{n})(x).$$

Let $\pi : (\mathbb{T}^n)^{\mathbb{Z}^d} \to (\mathbb{T}^n)^{\mathbb{Z}^d}/K$ denote the projection map. For any

$$q : X_1 \to \mathbb{R}^n,$$

$\pi \circ \sigma(q)$ is a continuous equivariant map from X_1 to X_2. We claim that there exists a non-zero continuous equivariant map from X_1 to X_2 of the form $\pi \circ \sigma(q)$ for some continuous map $q : X_1 \to \mathbb{R}^n$ with $q(e) = 0$.

For any finite set $F \subset \mathbb{Z}^d$, let Π_F denote the projection map

$$\Pi_F : (\mathbb{T}^n)^{\mathbb{Z}^d} \to (\mathbb{T}^n)^F.$$

Since K is a proper closed subgroup of $(\mathbb{T}^n)^{\mathbb{Z}^d}$, there exists a finite set $F \subset \mathbb{Z}^d$, and a point $\mathbf{x} \in (\mathbb{T}^n)^F$, such that $\mathbf{x}$ does not lie in the image of Π_F. Since X_1 is mixing, for any $\mathbf{i} \neq \mathbf{j} \in \mathbb{Z}^d$, the kernel of $\alpha_1(\mathbf{i}) - \alpha_1(\mathbf{j})$ is a proper closed subgroup of X_1. In particular, there exists $y \in X_1$ such that $y \neq e$, and

$$\alpha_1(\mathbf{i})(y) \neq \alpha_1(\mathbf{j})(y) \quad \text{for any } \mathbf{i}, \mathbf{j} \in F.$$

Choose $\mathbf{z} \in (\mathbb{R}^n)^F$ such that $\exp(\mathbf{z}(\mathbf{i})) = \mathbf{x}(\mathbf{i})$ for all $\mathbf{i} \in F$. Let

$$q : X_1 \to \mathbb{R}^n$$

be any continuous map with $q(e) = 0$ and $q \circ \alpha_1(\mathbf{i})(y) = \mathbf{z}(\mathbf{i})$ for all $\mathbf{i} \in F$. Since $\pi \circ \sigma(q)(y)$ does not lie in K, this proves the claim.

Now let $q : X_1 \to \mathbb{R}^n$ be any continuous map such that $q(e) = 0$, and $\pi \circ \sigma(q) : X_1 \to X_2$ is a non-zero map. For any $t \in [0, 1]$, define maps $q_t : X_1 \to \mathbb{R}^n$ and $h_t : X_1 \to X_2$ by $q_t(x) = tq(x)$, $h_t(x) = \pi \circ \sigma(q_t)$. For any $t \in [0, 1]$, h_t is a continuous equivariant map from X_1 to X_2, and $h_t(e) = e$. We claim that h_t is non-affine for some $t \in (0, 1]$.

Suppose this is not the case. Then for each $t \in [0, 1]$, h_t is a continuous homomorphism from X_1 to X_2. Let Y denote the set of all continuous maps from X_1 to X_2. Choose any metric ρ on X_2 that gives the topology, and define a metric ρ_0 on Y by

$$\rho_0(h_1, h_2) = \sup\{\rho(h_1(x), h_2(x)) \mid x \in X_1\}.$$

The map $t \mapsto h_t$ is continuous with respect to ρ_0 and the set of all continuous homomorphisms from X_1 to X_2 forms a discrete subset of Y. Hence $t \mapsto h_t$ is constant, which contradicts the fact that $h_0 = 0$ and $h_1 \neq 0$. This proves that some h_t is not an affine map. Since h_t is a continuous equivariant map from X_1 to X_2 for any $t \in [0, 1]$, Theorem 1.1 follows. $\square$

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