

# Instability of $\mathcal{N} = 2$ gauge theory in compact space with an isospin chemical potential

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**ABSTRACT:** We investigate  $\mathcal{N} = 2$  super-Yang-Mills theory on a three sphere in the presence of an isospin chemical potential at strong coupling using the AdS/CFT correspondence. This system exhibits an instability for sufficiently large values of the chemical potential. In contrast to other related models, the first excitation to condense is not a vector meson but rather a scalar charged under the global  $SO(4)$  symmetry group. Furthermore, the spectrum of fluctuations exhibits an interesting cross-over behaviour as a function of the dimensionless temperature. We construct the new ground state of the non-linear theory.

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## 1 Introduction and summary

Various methods and models indicate that in QCD, the presence of a sufficiently large isospin chemical potential leads to condensation of mesonic particles. Such new ground states may have interesting properties, such as non-isotropy or even non-homogeneity. Intuitively, condensation will take place when the chemical potential is of the order of the mass of the lightest meson in the system. In QCD, this would thus imply that a modest chemical potential leads to the formation of a pion condensate, and increasing the potential further is then expected to trigger rho meson condensation and so forth. While this intuitive picture is simple, computing such condensation effects from first principles is hard, though is indeed suggested by phenomenological models [1–3] as well as studies that make use of the string/gauge theory correspondence [4–6] (though not all of these contain the analogue of a pion sector). Obtaining these results from the lattice still remains a challenge.

When additional scales are present in the problem, it is conceivable that more interesting or non-intuitive things happen. One particular situation that we will be interested in here is the effect of *finite volume*. In the context of the string/gauge

theory correspondence, there is a canonical example involving gauge theory in a space with finite volume, the dual description of which involves string theory on  $\text{AdS}_5 \times S^5$  in global coordinates. According to the conjecture, this describes  $\mathcal{N} = 4$  super-Yang-Mills theory on the compact manifold living on the boundary three-sphere (plus time). In its original form [7], the string/gauge theory correspondence links string theory on the Poincaré patch of  $\text{AdS}_5 \times S^5$  with  $\mathcal{N} = 4$  super-Yang-Mills theory on the non-compact  $\mathbb{R}^{3,1}$ . The Poincaré patch of the string theory background arises as the near-horizon limit of the supergravity solution for  $N_c \rightarrow \infty$  coincident D3-branes. This brane construction serves as the starting point for the “derivation” of the correspondence. In contrast, global  $\text{AdS}_5 \times S^5$  space does not arise as the near horizon limit of any D-brane configuration. However, it is strongly believed that the correspondence holds here as well, in the manner described above. Generalisation to finite temperature is straightforward and amounts to compactifying the time circle both on the gauge and gravity sides, with the size of the thermal circle  $S^1$  being inversely proportional to the temperature of the system. When the temperature is high enough, the system undergoes a Hawking-Page phase transition and empty thermal AdS space is replaced with a Schwarzschild black hole with a spherical horizon [8]. The existence of this phase transition is possible because of the presence of a dimensionless parameter (the ratio of the radii of the thermal circle and of the three-sphere), which is not present at infinite volume.

In order to study mesonic excitations in this model, one has to add matter fields to it, or in holographic language, to add probe branes. In order to stay as close as possible to the  $\mathcal{N} = 4$  model, one can add D7-branes, which was done in global AdS in [9]. This modification of the system breaks supersymmetry down to  $\mathcal{N} = 2$  and incorporates quarks in the fundamental representation of the gauge group. If the bare quark mass is chosen to vanish, the conformal invariance of the system is preserved. The isometries of the internal three-sphere give rise to a global  $SO(4)$  R-symmetry, and for  $N_f$  branes, one has in addition an  $SU(N_f)$  flavour symmetry. Chemical potentials can be introduced for all of these, but in the present paper we will study the effect of a chemical potential associated to the latter, and will call it an ‘isospin potential’ (see [10] for an analysis of some of the effects of an R-symmetry potential).

In this compact gauge theory, we find two features which to our knowledge have not been observed in the literature before. The first is that the particles which condense first are *not* vector excitations (‘rho mesons’), but rather scalar particles charged under the global symmetry group.<sup>1</sup> This is an effect which in fact survives

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<sup>1</sup>This could perhaps have been expected from the analysis of the meson spectrum of [11], which finds that these  $SO(4)$  charged scalar mesons are the lightest, but their work does not discuss the effect of a chemical potential, and the qualitative argument based on only a comparison of the mass may in any case not be the complete story.

the large radius limit at fixed temperature, i.e. the limit towards a black hole in the Poincaré patch. The anisotropy of the condensate is thus in the direction of the internal manifold, in contrast to what has been assumed in earlier literature for this model [5, 6, 12], and in contrast to results in the Sakai-Sugimoto model [4].

The second feature is that the thermal pole masses of some of the mesons can cross as the dimensionless ratio is varied.<sup>2</sup> This suggests that, as the chemical potential is increased from zero, the particles which condense first to form a new ground state may not always be the same, but depend on this parameter. We indeed confirm this crossing behaviour also for the condensate formation, by explicit computation of the fluctuation spectrum, construction of the condensate solutions and a computation of their free energies. In our model the crossing does not involve the lightest particle, so the ground state remains a condensate of  $SO(4)$  charged scalars, but in other models the situation may not be so simple, and it would be interesting to investigate this further.

## 2 Holography with a dual $S^3$

### 2.1 Brane embeddings in global $AdS_5 \times S^5$ and $AdS_5$ -Schwarzschild

In this section we will briefly review some properties of the global  $AdS_5 \times S^5$  and  $AdS_5$ -Schwarzschild spaces as well as various embeddings of D7-probe branes in these geometries [9]. String theory in global  $AdS_5 \times S^5$  space is believed to be dual to the  $\mathcal{N} = 4$  SYM theory at zero temperature, which lives at the boundary of this space, which is an  $S^3 \times \mathbb{R}$ . Turning on the temperature makes the time direction (both in the gauge and gravity sides) compact, with the radius being inversely proportional to the gauge theory temperature.

The metric of the global  $AdS_5$  spaces (both at zero and finite temperature) is given by

$$ds^2 = \left(1 + \frac{r^2}{R^2}\right) d\tau^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega_3^2 + R^2 d\Omega_5^2, \quad (2.1)$$

where the origin of the AdS space is at  $r = 0$ , the boundary is at  $r \rightarrow \infty$ , and  $R$  is the AdS radius. The Euclideanised time direction  $\tau$  is periodic with period  $R_\tau$ . Since the theory is conformal, it implies that only the ratio of the thermal circle  $\tau$  and the size of the boundary sphere on which dual theory is living,  $R/R_\tau$ , is physically observable, i.e. unchanged by the conformal symmetry of the theory.

As the temperature of the thermal AdS space is increased, a first order Hawking-Page phase transition takes place and global AdS-Schwarzschild space replaces the thermal AdS space as the proper ground state of the system [8]. The metric of the

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<sup>2</sup>This crossing behaviour is somewhat reminiscent of the mixing of states observed in [4].

global AdS-Schwarzschild black hole is given by

$$ds^2 = - \left( 1 + \frac{r^2}{R^2} - \frac{M^2}{r^2} \right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2} - \frac{M^2}{r^2}} + r^2 d\Omega_3^2 + R^2 d\Omega_5^2, \quad (2.2)$$

where  $M^2 = 8G_N m_{\text{bh}} / (3\pi)$  and  $m_{\text{bh}}$  is the mass of the black hole, while its temperature is given by

$$T = \frac{1}{4\pi} \left( \frac{2r_0}{R^2} + \frac{2M^2}{r_0^3} \right), \quad (2.3)$$

where

$$r_0^2 = R^2 \left( \frac{-1 + \sqrt{1 + 4M^2/R^2}}{2} \right). \quad (2.4)$$

The dimensionless ratio  $R/r_0$  is proportional to  $TR$ , and we will later on expand our results for large values of this parameter, interpreting this limit as one at finite temperature and large volume [9, 13], so that a comparison with results in the Poincaré patch can be made.

Introducing D7-probe branes in this geometry corresponds, in the holographic language, to adding flavour hypermultiplets to  $\mathcal{N} = 4$  SYM on the sphere  $S^3$ . A study of various D-brane probes, in particular D7-probe branes, in these geometries was performed in [9, 13]. It was found that at zero (and low) temperature, there are two possible D7-brane embeddings in this dual geometry. The first type of embeddings are those in which the D7-brane completely fills the  $\text{AdS}_5$  space and wraps the  $S^3 \in S^5$  which is equatorial (i.e it is a D-brane with vanishing extrinsic curvature). The second type of embeddings are those in which the D7-brane wraps a non-maximal  $S^3 \in S^5$  which shrinks along the radial direction of  $\text{AdS}_5$  and becomes zero before the brane reaches the origin of  $\text{AdS}_5$ . The first series is dual to the  $\mathcal{N} = 2$  SYM theory with massless hypermultiplets, while for the second class the hypermultiplet is massive, and its mass is related to the distance at which the D7-brane “stops” before the origin of the  $\text{AdS}_5$ . Interestingly, as the “quark” mass is varied a topology changing phase transition occurs [9]. It was further analysed in detail in [13] that this phase transition is actually third order, unlike most of the phase transition associated to probe branes in holographic duals, which are usually first order.

In the high temperature phase, the situation is similar to that in infinite volume. One finds that Lorentzian and black hole embeddings exhaust all possibilities. As for infinite volume, these correspond to D7-branes that stay outside the horizon or reach the black hole horizon respectively.

In this paper we will be studying various D-brane embeddings in the presence of an “isospin” chemical potential. For that purpose, instead of using the coordinates (2.1), it will be useful to use another set of coordinates given by

$$ds^2 = - \frac{u^2}{R^2} \left( 1 + \frac{R^2}{4u^2} \right)^2 dt^2 + u^2 \left( 1 - \frac{R^2}{4u^2} \right)^2 d\bar{\Omega}_3^2 + \frac{R^2}{u^2} (du^2 + u^2 d\Omega_5^2), \quad (2.5)$$

which is related to (2.1) via the coordinate change

$$u = \frac{1}{2}(r + \sqrt{r^2 + R^2}). \quad (2.6)$$

In these coordinates the origin of the AdS space is at  $u = R/2$ , while the boundary is an  $S^3$  at  $u \rightarrow \infty$ . We will also use

$$ds^2 = -\frac{1}{4z^2} (1 + z^2)^2 dt^2 + \frac{R^2}{4z^2} (1 - z^2)^2 d\bar{\Omega}_3^2 + R^2 \frac{dz^2}{z^2} + R^2 d\Omega_2^2, \quad (2.7)$$

which is related to the previous coordinates by  $z = R/(2u)$ . Let us also note that in the  $z$  coordinate the origin of AdS space is at  $z = 1$ , while the boundary is at  $z = 0$ .

Similarly for the system at finite temperature in addition to metric (2.2) we will also use  $(u, t)$  coordinates,

$$ds^2 = -\frac{2\rho_H^2}{uR^2} \frac{F(u)}{W(u)} dt^2 + 2\frac{\rho_H^2}{u} W(u) d\bar{\Omega}_3^2 + \frac{R^2}{4u^2 F(u)} du^2 + R^2 d\Omega_5^2, \quad (2.8)$$

where

$$F(u) = 1 - u^2, \quad \frac{\rho_H^4}{R^4} = \frac{1}{16} + \frac{M^2}{4R^2}, \quad W(u) = 1 - \frac{uR^2}{4\rho_H^2}. \quad (2.9)$$

Note here that the variable  $u$  is dimensionless and ranges from  $u \in [0, 1]$ , where the horizon is at  $u = 1$ , and the boundary is at  $u = 0$ . The Hawking temperature of this black hole is

$$T = \frac{\sqrt{2}\rho_H}{\pi R^2 \sqrt{1 - \frac{R^2}{4\rho_H^2}}}. \quad (2.10)$$

For numerical investigation it turns out that another form of the metric at finite  $T$  will also be useful.<sup>3</sup> If we change coordinates as

$$v = \frac{1 - u}{1 - \frac{uR^2}{4\rho_H^2}} \quad \Lambda = \frac{8\rho_H^2}{4\rho_H^2 + R^2}, \quad (2.11)$$

then the metric becomes

$$ds^2 = -\frac{v(\Lambda - v)}{(1 - v)(2 - \Lambda)} dt^2 + \frac{\Lambda - 1}{(2 - \Lambda)(1 - v)} R^2 d\bar{\Omega}_3^2 + \frac{\Lambda - 1}{4(1 - v)^2 v(\Lambda - v)} R^2 dv^2 + R^2 d\Omega_5^2, \quad (2.12)$$

where

$$d\Omega_3^2 = \frac{1}{4}(d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta d\alpha d\gamma). \quad (2.13)$$

In these coordinates  $v = 0$  is the horizon while  $v = 1$  is the boundary, and  $\Lambda$  is function of the temperature given by

$$\Lambda = \frac{4(\pi TR)^2}{3(\pi TR)^2 - \sqrt{(\pi TR)^2(-2 + (\pi TR)^2)}}. \quad (2.14)$$

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<sup>3</sup>This is related to the fact that the solutions at vanishing bare quark mass can be related to Heun functions in these coordinates.

## 2.2 Chemical potentials and homogeneous solutions

In this section we will construct solutions which correspond to adding a chemical potential associated to the global  $SU(2)$  flavour symmetry. This symmetry originates from the fact that we are considering two coinciding D7-probe branes. In addition to the  $SU(2)$  symmetry, our systems also exhibits another global  $SO(4)$  symmetry associated to the residual global isometry of the system of probes. In principle one could also consider switching on a chemical potential which is associated to this symmetry group, as it was done in e.g. [10]. However, our prime interest will be the physically more relevant  $SU(2)$  group, which has direct analogue with the  $SU(2)$  flavour symmetry group in the Sakai-Sugimoto model [4].

In order to turn on a chemical potential corresponding to this global  $SU(2)$  “isospin” symmetry, let us consider two coincident D7-branes, which for simplicity have equatorial embedding  $\theta = 0$ , so that the induced metric on the D7-branes is

$$ds^2 = -\frac{u^2}{R^2} \left(1 + \frac{R^2}{4u^2}\right)^2 dt^2 + \frac{R^2}{u^2} du^2 + u^2 \left(1 - \frac{R^2}{4u^2}\right)^2 d\bar{\Omega}_3^2 + R^2 d\Omega_3^2. \quad (2.15)$$

In other words, the D7-probes fill out the full  $\text{AdS}_5$  space, as well as a maximal  $S^3 \in S^5$ . We should note that there are two  $S^3$  factors present on the world-volume of the brane, one  $\bar{\Omega}_3$  which is dual to the boundary  $S^3$  and another one  $S^3$  in  $S^5$ , which is part of the global symmetry group.

As explained in the previous section, in order to turn on a chemical potential we need to turn on the  $A_0$  component of the gauge field, such that it satisfies the boundary condition

$$A_0(x, z \rightarrow 0) \rightarrow \mu_I \tau^3. \quad (2.16)$$

As a first guess for finding the ground state of the system in the presence of this chemical potential, we consider the homogeneous ansatz

$$A = A_0^{(3)}(u) \tau^3 dt, \quad (2.17)$$

so that the DBI action becomes

$$S = -T_{D7} 8\pi^4 R^3 \int dz \sqrt{-g(z)} \left[ 1 + \frac{\pi^2 R^4}{2\lambda} \left( \partial_z A_0^{(3)}(z) \right)^2 g^{tt}(z) g^{zz}(z) \right]^{1/2}. \quad (2.18)$$

The equation of motion for the field  $A_0(z)$  can be integrated once, yielding

$$\partial_z (A_0^{(3)}(z) R) = \frac{4cz(1+z^2)}{\sqrt{(1-z^2)^6 + 32 \frac{c^2 \pi^2}{\lambda} z^6}}, \quad (2.19)$$

where  $c$  is an integration constant. We are looking for a physical configuration which is smooth and differentiable everywhere. Specifically, we require that the field  $A_0$  and its derivatives are smooth at the origin of AdS space. However, by expanding the

right hand side of the above equation near the origin of AdS space one sees that the radial derivative of the  $A_0$  field is non-vanishing. In other words, it is not possible to obtain any nontrivial (different from a constant solution) homogeneous solution which is smooth at the origin. This observation persists also for the Yang-Mills truncation of the DBI action, and it also holds in the Poincaré limit (see [14]). Hence as the starting configuration for our fluctuation analysis we will use a homogeneous solution with non-zero isospin potential, given by

$$A_0 = a_0 \tau^3, \quad a_0 = \text{const.} \quad (2.20)$$

This solution implies that at zero temperature, the homogeneous background does not lead to generation of an isospin density.

Above the Hawking-Page transition, the situation is similar. The homogeneous ansatz yields a first order differential equation,

$$\partial_u A_0^{(3)}(u) \left(1 - \frac{uR^2}{4\rho_H^2}\right)^2 = a_0, \quad (2.21)$$

where  $a_0$  is an integration constant. This is solved by

$$A_0^{(3)}(u) = \frac{\mu_I(1-u)}{1 - \frac{uR^2}{4\rho_H^2}}, \quad (2.22)$$

where we have imposed the boundary condition that  $A_0(u \rightarrow 0) = \mu_I$  at the boundary, and also required vanishing of  $A_0$  at the horizon of the black hole in AdS space. In contrast to the low temperature situation, we see that there is now a non-vanishing isospin density present, even for this homogeneous system.

### 3 Perturbative analysis of the homogeneous vacuum at $T = 0$

While the homogeneous and isotropic solution which was discussed in the previous section is a legitimate solution to the equations of motion, we expect that for large enough values of the chemical potential this configuration will become unstable and “decay” into another, presumably non-homogeneous or non-isotropic ground state. Our expectations are based on a similar analysis which was previously performed for the Sakai-Sugimoto model in [4]. A major difference, however, with respect to the analysis of that paper is that we are now dealing with a field theory on a *compact* space. In the present section we will discuss the perturbative stability analysis at  $T = 0$ , which from a technical perspective largely follows the meson spectrum analysis of [11]; we recall some elements of that construction for completeness and in order to be able to compare with the finite temperature analysis which is to follow in section 4.



### 3.1 Scalar fluctuations at zero temperature

We will start the perturbative analysis of the homogeneous solution (2.20) by considering scalar perturbations. By scalars we here mean scalars in the dual theory that are also scalars from the point of view of the D7-probe, i.e. gauge theory scalar fields which are uncharged under the  $SO(4)$ . Some of the scalars in the dual gauge theory originate from the components of the gauge field on the D7-probe and will be analysed in the next section. Our starting point is the flat (i.e. maximal) D7-probe embedding with the world-volume field (2.20) turned on. The induced metric on the world-volume was written in (2.15). Since the D7-probe brane is filling out the full  $AdS_5$  space and wrapping a maximal  $S^3$  in  $S^5$ , there are only two transverse scalars to the brane world-volume, and they are within the  $S^5$ . To see which scalars these are, let us write the metric on  $S^5$  as

$$ds_5^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\Omega_3^2). \quad (3.1)$$

Instead of using the  $(\theta, \phi)$  coordinates it will be more convenient to introduce coordinates

$$w_1 = R \sin\theta \cos\phi \quad w_2 = R \sin\theta \sin\phi, \quad (3.2)$$

so that the metric on  $S^5$  becomes

$$ds_5^2 = \left(1 - \frac{w_1^2 + w_2^2}{R^2}\right) d\Omega_3^2 + dW^2(w_1, w_2). \quad (3.3)$$

Here  $dW^2$  is a complicated expression in terms of  $w_1, w_2$ , which however significantly simplifies for  $w_1 = 0, w_2 = w$ , as it then becomes

$$dW^2(w_1, w_2) \rightarrow \frac{dw^2}{1 - \frac{w^2}{R^2}}. \quad (3.4)$$

It can easily be checked from the equations of motion that it is indeed consistent to set one of the  $w_1, w_2$  to zero.<sup>4</sup> We expect that the first instability will appear already for the lowest lying (S-wave) mode on the dual gauge theory sphere  $\bar{S}^3$ , which is also a singlet on  $S^3 \in S^5$ . Therefore, when looking for the instabilities in the system, we will look at the fluctuation which is a function of only the time and  $u$ -coordinates. Let us define the fluctuation variable as

$$\Psi(t, u) = \delta w_2(t, u). \quad (3.5)$$

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<sup>4</sup>The fluctuation in the other direction leads to the same spectrum, so we will not comment on it any further (though we should emphasise that this is a property of the equatorial embedding not shared by non-zero bare quark mass embeddings).

The induced metric on the D7-brane becomes

$$ds^2 = -\frac{u^2}{R^2} \left(1 + \frac{R^2}{4u^2}\right)^2 dt^2 + u^2 \left(1 - \frac{R^2}{4u^2}\right)^2 d\bar{\Omega}_3^2 + \frac{R^2}{u^2} du^2 + R^2 \left(1 - \frac{\Psi(t, u)^2}{R^2}\right) d\Omega_3^2 + \left(\frac{\partial\Psi(t, u)}{\partial t} dt + \frac{\partial\Psi(t, u)}{\partial u} du\right)^2. \quad (3.6)$$

We next need to write down the action for the scalar fluctuation to leading order in  $\alpha'$ . A subtle point here is that all scalars are in the adjoint representation of the  $SU(2)$  group on the world-volume of two D7-branes. The approach we adopt here to write the action, is to first treat all scalars as *abelian* and derive the action for the fluctuation by linearising the DBI action. In the last step we then promote all fields to be in the adjoint representation by introducing an overall trace in front of the action (for more on this and other approaches, see e.g. [6, 15]).

Following these steps we end up with the action governing the scalar fluctuations,

$$S = -T_{D7} \frac{4\pi^6 R^4}{\lambda} \int du dt u^3 \left(1 + \frac{R^2}{4u^2}\right) \left(1 - \frac{R^2}{4u^2}\right)^3 \times \left[ -\frac{R^2}{2} \left(\frac{4u}{4u^2 + R^2}\right)^2 D_t \Psi^{(a)} D_t \Psi^{(a)} + \frac{u^2}{2R^2} \partial_u \Psi^{(a)} \partial_u \Psi^{(a)} - \frac{3}{2} \frac{\Psi^{(a)} \Psi^{(a)}}{R^2} \right]. \quad (3.7)$$

Since we expect that an instability will appear in the gauge direction orthogonal to the background field  $A_0$ , we make the ansatz for the fluctuation field to be

$$\Psi(t, u) = e^{-i\omega t} (\Psi_\omega^{(1)}(u)\tau^1 + \Psi_\omega^{(2)}(u)\tau^2), \quad (3.8)$$

where we have focused on one Fourier mode.

The equations of motion for the components  $\Psi_\omega^{(1)}$  and  $\Psi_\omega^{(2)}$  are coupled, but can be decoupled by changing variables as

$$\Psi_\omega^{(\pm)}(u) = \Psi_\omega^{(1)}(u) \pm i\Psi_\omega^{(2)}(u). \quad (3.9)$$

The equations of motion for the components  $\Psi^{(\pm)}$  are

$$\frac{\partial_u(\sqrt{-g}g^{uu}\partial_u\Psi_\omega^{(\pm)})}{\sqrt{-g}g^{uu}} - \frac{g^{tt}}{g^{uu}} (\omega \pm A_0^{(3)})^2 \Psi_\omega^{(\pm)} + \frac{3}{R^2 g^{uu}} \Psi_\omega^{(\pm)} = 0, \quad (3.10)$$

which for the specific metric on the D7-brane world-volume become

$$\partial_u^2 \Psi_\omega^{(\pm)} + \frac{\partial_u \left( u^5 \left(1 + \frac{R^2}{4u^2}\right) \left(1 - \frac{R^2}{4u^2}\right)^3 \right)}{u^5 \left(1 + \frac{R^2}{4u^2}\right) \left(1 - \frac{R^2}{4u^2}\right)^3} \partial_u \Psi_\omega^{(\pm)} + \left[ \frac{R^4}{u^4 \left(1 + \frac{R^2}{4u^2}\right)^2} (\omega \pm \mu)^2 + \frac{3}{u^2} \right] \Psi_\omega^{(\pm)} = 0. \quad (3.11)$$

These equations can be solved by reducing them to Schrödinger form. A very similar equation has been analysed in [11] for the determination of the mesonic spectrum on the world-volume of a probe  $D7$  brane at  $T = 0$ , in global AdS space. Following steps similar to those in [11], and focusing on modes which are constant both on the  $S^3 \in S^5$  and on the gauge theory  $\bar{S}^3$ , we obtain for the spectrum of fluctuations

$$(\mu + \omega)R = \pm(3 + 2n) \quad n = 0, 1, 2, \dots \quad (3.12)$$

Here  $n$  is the main quantum number. We see that the key effect of the non-vanishing chemical potential is to shift the frequency  $\omega \rightarrow \omega + \mu$ . Because of this we see that for large enough chemical potential  $\mu > \mu_{\text{crit}} = 3/R$ , the frequency of the lowest lying mode becomes zero, signalling that the homogeneous solution potentially becomes unstable at this value of the chemical potential, and a condensate of the scalar might form.

### 3.2 Vector fluctuations at zero temperature

Following the perturbative analysis in the scalar sector, we now turn our attention to vectors. We again expect unstable modes, but would like to know whether or not they occur before the instability of the scalar sector. An analysis of the vector mode spectrum was performed in infinite volume limit (on the Poincaré patch) in [16] and then later extended to non-zero chemical potential in [5, 6, 12]. At finite volume and vanishing temperature and chemical potential the spectrum can be found in [11]. The upshot of the analysis of [16] is that the lowest lying supermultiplet consists of two transverse scalars describing the transverse fluctuations of the D7-brane in the  $S^5$ , one scalar which originates from the vector component in the internal  $S^3 \in S^5$  wrapped by the D7-brane, and gauge components in the non-compact directions of  $\text{AdS}_5$ . As one moves to the compact case, i.e. global AdS space [11], the states from this supermultiplet get reorganised (split) so that the lightest state in the compact space is the scalar which originates from the component of the gauge field in the direction of the internal  $S^3 \in S^5$ . The vector components in the direction of the dual sphere as well as the transverse scalars both have larger masses. We now want to see how the fluctuations from the vector sector are shifted upon introducing a chemical potential.

#### 3.2.1 The gauge theory vector fluctuations

Let us start with the vector components in the direction of the sphere  $\bar{S}^3$  of the dual gauge theory. These fluctuations are dual to the vector excitations in the gauge theory. Similarly to what we did for scalars, we start by writing the fluctuations as

$$A = A_0^{(3)}(u)\tau^3 dt + Ra_i^{(1)}(t, u, \bar{\Omega}_3)\tau^1 d\bar{\theta}^i + Ra_i^{(2)}(t, u, \bar{\Omega}_3)\tau^2 d\bar{\theta}^i, \quad (3.13)$$

where  $i = (1, 2, 3)$  are indices on the dual  $\bar{S}^3$ , and as for scalars we fluctuate in the gauge directions orthogonal to  $A_0^{(3)}$ . Since we know that the lowest lying vector is a

singlet on  $S^3$  we do not have to consider excitations which depend on the coordinates of the internal sphere.<sup>5</sup>

Next, we linearise the Yang-Mills action on the world-volume of the D7-probe,

$$S = -T_{D7}\pi^2 \frac{R^4}{2\lambda} \int d^8\xi \sqrt{-g} \left[ (\partial_u A_0^{(3)})^2 g^{tt} g^{uu} + R^2 \left( ((D_t a_i)^{(a)})^2 g^{tt} g^{ii} + (\partial_u a_i^{(a)})^2 g^{uu} g^{ii} \right) + R^2 (f_{ij}^a)^2 g^{ii} g^{jj} \right], \quad (3.14)$$

where

$$(D_t a_i)^{(a)} = \partial_t a_i^{(a)} - \epsilon^{abc} A_0^{(b)} a_i^{(c)}, \quad f_{ij}^a = (\partial_i a_j^{(a)} - \partial_j a_i^{(a)}). \quad (3.15)$$

Here,  $g_{ij}$  is the metric on  $\bar{S}^3$ . The equations of motion for the fluctuations  $a_i^{(a)}$  are given by

$$\begin{aligned} \sqrt{-g} \epsilon^{abc} A_0^{(c)} D_t a_i^{(b)} g^{tt} g^{ii} + \sqrt{-g} \partial_t \left( D_t a_i^{(a)} \right) g^{tt} g^{ii} + \partial_u \left( \sqrt{-g} \partial_u a_i^{(a)} g^{uu} g^{ii} \right) \\ + \sum_j \left( \sqrt{-g} \left( \partial_j a_i^{(a)} - \partial_i a_j^{(a)} \right) g^{jj} g^{ii} \right) = 0, \end{aligned} \quad (3.16)$$

In order to solve this equation let us Fourier transform in the time direction. In order to decouple the equations for the fluctuations  $a_i^{(1)}$  and  $a_i^{(2)}$  we introduce a new pair of variables

$$\bar{X}_i^{(\pm)}(u, \omega, \bar{\Omega}_3) = e^{-i\omega t} \left( a_i^{(1)}(u, \omega, \bar{\Omega}_3) \pm i a_i^{(2)}(u, \omega, \bar{\Omega}_3) \right). \quad (3.17)$$

This finally yields the fluctuation equations

$$\begin{aligned} \partial_u \left( \sqrt{-g} g^{uu} g^{ii} \partial_u \bar{X}_i^{(\pm)} \right) + \sum_j \partial_j \left( \sqrt{-g} g^{jj} g^{ii} \left( \partial_j \bar{X}_i^{(\pm)} - \partial_i \bar{X}_j^{(\pm)} \right) \right) \\ - \sqrt{-g} g^{tt} g^{ii} \left( \omega \pm A_0^{(3)} \right)^2 \bar{X}_i^{(\pm)} = 0, \end{aligned} \quad (3.18)$$

which are equivalent to

$$D^u \left( \partial_u \bar{X}_i^{(\pm)} \right) + \nabla^j \left( \partial_j \bar{X}_i^{(\pm)} - \partial_i \bar{X}_j^{(\pm)} \right) - g^{tt} \left( \omega \pm A_0^{(3)} \right)^2 \bar{X}_i^{(\pm)} = 0. \quad (3.19)$$

In order to solve these equations, we make a factorised ansatz for  $\bar{X}_i$ , as a product of radial and angular functions, and expand the angular part  $\bar{X}_i^{\bar{\Omega}_3}(\bar{\Omega}_3)$  on  $\bar{S}^3$  in terms of vector spherical harmonics. In general, the fluctuations could also depend on a direction on internal sphere  $S^3$ . However, as mentioned before, we will focus only

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<sup>5</sup>In the language of [11] we consider type II fluctuations, with type I fluctuations to be considered in the next subsection. Looking ahead, it turns out that type III fluctuations condense after type I fluctuations and we will not consider them in detail here.

on singlets under the global  $SO(4)$  symmetry group. Also, while there exists three type of vector spherical harmonics on  $\bar{S}^3$ , it has been argued in [11, 16], that for fluctuations of the vector field which are taking place in  $\bar{S}^3 \in \text{AdS}_5$ , only  $Y_i^{l,\pm}$  which transform in  $((\frac{l\pm 1}{2}, \frac{l\pm 1}{2}))$ ,  $l \geq 1$  irreducible representations of  $SO(4)$  are relevant. Hence we expand the fluctuations as

$$\bar{X}_i^{(\pm)}(u, \omega, \bar{\Omega}_3) = \sum_{\bar{l}, s=\pm} \bar{\Phi}_{\omega, \bar{l}, s}^{(\pm)}(u) Y_i^{\bar{l}, s}, \quad (3.20)$$

where the index  $(\pm)$  refers to the two linear combinations of modes as defined in (3.17), and the  $\pm$  index refers to the value of the index  $s$  labelling the vector spherical harmonics.

The spherical harmonics satisfy the identities

$$\begin{aligned} \nabla_i \nabla^i Y_j^{l,\pm} - R_j^k Y_k^{l,\pm} &= -(l+1)^2 Y_j^{l,\pm}, \\ \epsilon^{ijk} \nabla_j Y_k^{l,\pm} &= \pm(l+1) Y_{l,\pm}^i, \\ \nabla^i Y_i^{l,\pm} &= 0, \end{aligned} \quad (3.21)$$

Using these identities the equation for the vector fluctuations can be rewritten as

$$\begin{aligned} \bar{\Phi}_{\omega, \bar{l}, s}^{(\pm)''}(u) + \frac{\partial_u \left( \sqrt{-g(u)} g^{uu}(u) P(u) \right)}{\left( \sqrt{-g(u)} g^{uu}(u) P(u) \right)} \bar{\Phi}_{\omega, \bar{l}, s}^{(\pm)'}(u) \\ - \frac{1}{g^{uu}(u)} \left[ g^{tt}(u) \left( \omega \pm A_0^{(3)}(u) \right)^2 + (\bar{l}+1)^2 P(u) \right] \bar{\Phi}_{\omega, \bar{l}, s}^{(\pm)}(u) = 0, \end{aligned} \quad (3.22)$$

where the  $\pm$  sign in front of  $A_0$  in the equation is correlated with the  $(\pm)$  sign on the  $\bar{\Phi}^{(\pm)}$  and  $P(u)$  is the inverse of the  $u$ -dependent part of the the metric factor in front of  $d\bar{\Omega}_3^2$ . We should note that this equation is independent of the quantum number  $s = \pm 1$ , which will be different when we start looking at the vector fluctuations in the direction of the internal sphere. In the case of zero temperature the fluctuation equation becomes (see also [11])

$$\begin{aligned} \bar{\Phi}_{\omega, \bar{l}}^{(\pm)''}(u) + \frac{\partial_u \left( u^3 \left( 1 + \frac{R^2}{4u^2} \right) \left( 1 - \frac{R^2}{4u^2} \right) \right)}{u^3 \left( 1 + \frac{R^2}{4u^2} \right) \left( 1 - \frac{R^2}{4u^2} \right)} \bar{\Phi}_{\omega, \bar{l}}^{(\pm)'}(u) \\ + \left[ \frac{R^4}{u^4 \left( 1 + \frac{R^2}{4u^2} \right)^2} (\omega \pm \mu)^2 - \frac{R^2}{u^4 \left( 1 - \frac{R^2}{4u^2} \right)^2} (\bar{l}+1)^2 \right] \bar{\Phi}_{\omega, \bar{l}}^{(\pm)}(u) = 0. \end{aligned} \quad (3.23)$$

This equation is very similar to (3.11), and can again be cast in Schrödinger form. We then find that the spectrum of vector fluctuations for the  $\bar{l}$ -th spherical harmonics is given by

$$(\omega \pm \mu)R = (3 + \bar{l} + 2n) \quad n = 0, 1, 2, 3, \dots \quad \bar{l} = 1, 2, 3, \dots, \quad (3.24)$$

Here  $n$  is again the main quantum number, and  $\bar{l}$  is an  $SO(4)$  quantum number corresponding to the sphere  $\bar{S}^3 \in \text{AdS}_5$ . We see that the result is again the same as the one in [11] if we consider  $SO(4)$  singlets (i.e. set  $l = 0$ ), except that the chemical potential shifts the frequency  $\omega \rightarrow \omega \pm \mu$ .

From equation (3.24) we see that for a critical value of the chemical potential given by  $\mu_{\text{crit}} = 4/R$ , the lowest lying mode  $\bar{l} = 1$  will become massless. Therefore, we expect that when the chemical potential is larger than this value, the system potentially becomes unstable.

### 3.2.2 The charged scalar fluctuations

Let us now consider fluctuations of the vector field in the direction of the internal  $S^3 \in S^5$ , which are dual to an  $SO(4)$  charged scalar field in the gauge theory. Since the WZW term in the action is now non-zero, when considering fluctuations, we have to modify the action from (3.14) by adding the term

$$S_{WZW} = \frac{T_{D7}\pi^2 R^4}{\lambda} \int \text{Tr}(C \wedge F \wedge F), \quad dC = \frac{4}{R} \text{Vol}(\text{AdS}_5). \quad (3.25)$$

Similarly as before, we make an ansatz as in (3.13) except that the index  $i$  is now taking values in the internal  $S^3$ . In addition, we will also allow the fluctuations to depend on both  $S^3$  and  $\bar{S}^3$  variables. In order to decouple the equations of motion we redefine variables as in (3.17) and make a factorised ansatz

$$X_i^{(\pm)}(u, \omega, \Omega_3, \bar{\Omega}_3) = e^{-i\omega t} \Phi^{(\pm)}(u) \bar{Y}^{\bar{l}}(\bar{\Omega}_3) Y_i^{l,s}(\Omega_3), \quad (3.26)$$

where the index  $(\pm)$  refers to the sign in the linear combination (3.17),  $i$  denotes the index in the direction of the internal  $S^3$ , and the index  $s = \pm 1$ . Also to shorten the notation we have suppressed indices on the functions  $\Phi^{(\pm)}$ , which should really also carry indices  $(\omega, s, \bar{l}, l)$ .

Following the same procedure as for scalars and the other gauge components we arrive at the equation for the fluctuations (see also [11])

$$\begin{aligned} \partial_u^2 \Phi^{(\pm)} + \frac{\partial_u \left( u^5 \left( 1 + \frac{R^2}{4u^2} \right) \left( 1 - \frac{R^2}{4u^2} \right)^3 \right)}{u^5 \left( 1 + \frac{R^2}{4u^2} \right) \left( 1 - \frac{R^2}{4u^2} \right)^3} \partial_u \Phi^{(\pm)} + \frac{R^2}{u^4 \left( 1 + \frac{R^2}{4u^2} \right)^2} (\omega \pm \mu)^2 \Phi^{(\pm)} \\ - \left[ (l+1)^2 + \bar{l}(\bar{l}+2) \frac{R^2}{u^2 \left( 1 - \frac{R^2}{4u^2} \right)^2} + 4s(l+1) \right] \frac{1}{u^2} \Phi^{(\pm)} = 0, \end{aligned} \quad (3.27)$$

where the sign in the  $(\omega \pm \mu)$  is the same as for  $\Phi^{(\pm)}$ . We should note that this equation explicitly depends on the quantum number  $s$ , which is labelling the vector harmonics on  $S^3$ . This is in contrast to the previous case for the equation for vector fluctuations on  $\bar{S}^3$ .

Putting equation (3.27) in Schrödinger form, like we did for the other fluctuations, we obtain for the spectrum

$$(\omega \pm \mu)R = 3 + 2s + l + 2n + \bar{l} \quad \text{where}$$

$$\bar{l} = 0, 1, 2, \dots \quad l = 1, 2, 3, \dots \quad n = 0, 1, 2, \dots \quad s = \pm 1. \quad (3.28)$$

This is the same as (4.31) of [11] except for the shift  $\omega \rightarrow \omega + \mu$ . Let us also note that the lowest lying excitation carries quantum numbers ( $\bar{l} = 0, l = 1, s = -1, n = 0$ ), and this mode will reach zero frequency when  $\mu > \mu_{\text{crit}} = 2/R$ .<sup>6</sup>

## 4 Perturbative analysis of the homogeneous vacuum at $T \neq 0$

In the previous section we have observed that the homogeneous isotropic ground state at non-zero chemical potential and zero temperature was unstable under both scalar and vector fluctuations. We would now like to see how is this modified once the temperature is turned on, paying particular attention to the order in which the instabilities set in as a function of temperature.

We should emphasise that in contrast to the zero temperature case, where all fluctuations have real frequency  $\omega$  (corresponding to stable mesonic scalar and vector particles), at finite temperature (above the Hawking-Page transition), even in the absence of chemical potential all fluctuation frequencies have a non-vanishing imaginary part. When the chemical potential is zero the imaginary part of these frequencies are negative, corresponding to the fact that these excitations are decaying in time, i.e. that they describe *quasi-stable* particles. However, as the chemical potential is turned on, if there is indeed an instability present, we expect that the negative imaginary part of the frequencies will become positive, i.e. that a decaying excitation would become an exponentially growing mode, which signals an instability. In what follows, we will therefore focus on studying the imaginary part of the quasi-normal modes of the system.

We start our analysis by looking at the scalar fluctuations, and repeat the procedure similar to that at zero temperature. Again, we use the metric on the  $S^5$  as in (3.3), keeping only the transverse scalar  $\Psi(t, u)$  nonzero (see equation (3.5)) and making it depend only on time and the radial direction  $u$ . As argued before, such excitation is consistent with the full equations of motion, and should correspond to the lowest energy mode, an S-wave on  $\bar{S}^3$ . The induced metric on the D7-brane

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<sup>6</sup>The fact that this excitation has the lowest mass at  $\mu = 0$  was also observed in [11], but no attempt was made to study its condensation under the influence of a chemical potential.

world-volume is then

$$ds^2 = -\frac{2\rho_H^2 F(u)}{uR^2 W(u)} dt^2 + 2\frac{\rho_H^2}{u} W(u) d\bar{\Omega}_3^2 + \frac{R^2}{4u^2 F(u)} du^2 + R^2 \left( 1 - \frac{\Psi(t, u)^2}{R^2} \right) d\Omega_3^2 + \left( \frac{\partial \Psi(t, u)}{\partial t} dt + \frac{\partial \Psi(t, u)}{\partial u} du \right)^2, \quad (4.1)$$

where

$$F(u) = 1 - u^2, \quad W(u) = 1 - \frac{uR^2}{4\rho_H^2}. \quad (4.2)$$

As before, we Fourier transform the scalar  $\Psi(t, u)$  and make the ansatz that it is pointing in the direction orthogonal to the  $A_0^{(3)}\tau^3$  in colour space,

$$\Psi(t, u) = \int \frac{d\omega}{2\pi} e^{-i\omega t} (\Psi_\omega^{(1)}(u)\tau^1 + \Psi_\omega^{(2)}(u)\tau^2). \quad (4.3)$$

We then again change variables as

$$\Psi_\omega^{(\pm)}(u) = \Psi_\omega^{(1)}(u) \pm i\Psi_\omega^{(2)}(u), \quad (4.4)$$

so that the equations for the  $\Psi^{(\pm)}$  fluctuations decouple and are given by<sup>7</sup>

$$\frac{\partial_u \left( \frac{W(u)F(u)}{u} \partial_u \Psi_\omega^{(\pm)}(u) \right)}{\frac{W(u)F(u)}{u}} + \frac{R^4 W(u)}{8u\rho_H^2 F(u)^2} \left( \omega \pm A_0^{(3)}(u) \right)^2 \Psi_\omega^{(\pm)}(u) + \frac{3}{4u^2 F(u)} \Psi_\omega^{(\pm)}(u) = 0. \quad (4.5)$$

We solve the fluctuation equation by imposing that the modes satisfy an incoming boundary condition at horizon,

$$\Psi_\omega^{(\pm)}(u) \Big|_{u \approx 1} = (1 - u)^{-i\omega/(4\pi T)} (1 + O(1 - u)). \quad (4.6)$$

The equations for the vector fluctuations are derived in a similar way. Let us first consider the vector fluctuations which are dual to vectors. We take them to be singlets under the global  $SO(4)$  ( $l = 0$ ), and orthogonal to the isospin chemical potential in the gauge group as we did at zero temperature, see (3.13). Following steps similar to those at zero temperature, instead of  $a_i^{(1)}, a_i^{(2)}$ , we introduce a new

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<sup>7</sup>Note that at  $T = 0$ , the equations only depend on  $\omega - \mu$ , and hence the critical chemical potential coincides with the frequency of the lightest mode. At  $T > 0$  the  $A_0^{(3)}$  component is no longer a constant, and obtaining the critical chemical potential is more complicated (physical states are no longer straight lines in the  $\omega, \mu$  plane).



pair of variables  $X_i^{(\pm)}$ , as in (3.17) and Fourier expand it in spherical harmonics as in (3.20). Hence, we arrive at the equations of motion for these fluctuations

$$\partial_u^2 \bar{\Phi}_\omega^{(\pm)s, \bar{l}} + \frac{\partial_u F(u) \partial_u \bar{\Phi}_\omega^{(\pm)s, \bar{l}}}{F(u)} - (\bar{l} + 1)^2 \frac{R^2}{8\rho_H^2 W(u) F(u) u} \bar{\Phi}_\omega^{(\pm)s, \bar{l}} + \frac{R^4 W(u)}{8u\rho_H^2 F(u)^2} (\omega \pm A_0^{(3)})^2 \bar{\Phi}_\omega^{(\pm)s, \bar{l}} = 0. \quad (4.7)$$

As for scalars, we impose incoming boundary conditions at the black hole horizon

$$\bar{\Phi}_\omega^{(\pm)s, \bar{l}} \Big|_{u \approx 1} = (1 - u)^{-i\omega/(4\pi T)} (1 + O(1 - u)). \quad (4.8)$$

Finally, we turn to the vector fluctuations dual to the charged scalars. Following similar steps as we did at zero temperature we arrive at the equations governing these fluctuations (see equation (3.27))

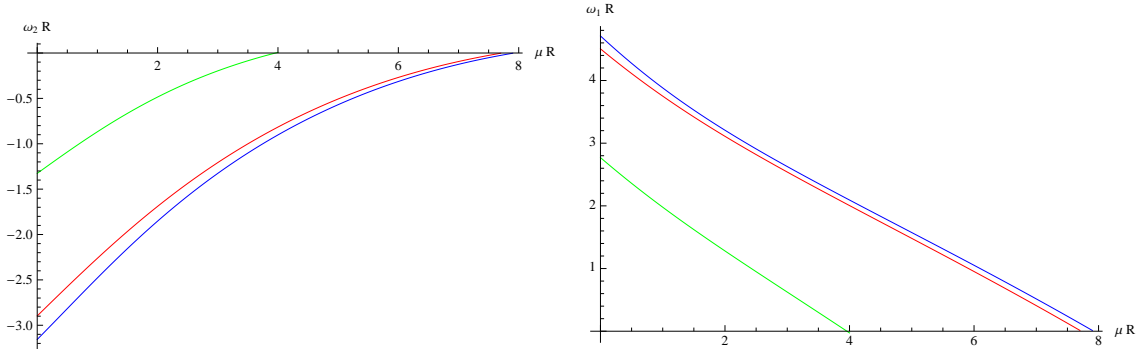
$$\frac{\partial_u \left( \frac{W(u)F(u)}{u} \partial_u \Phi^{(\pm)} \right)}{\frac{W(u)F(u)}{u}} - (l + 1)^2 \frac{1}{4u^2 F(u)} \Phi^{(\pm)} - \bar{l}(\bar{l} + 2) \frac{R^2}{8\rho_H^2 W(u) F(u) u} \Phi^{(\pm)} - \frac{s(l + 1)}{u^2 F(u)} \Phi^{(\pm)} + \frac{R^4 W(u)}{8u\rho_H^2 F(u)^2} (\omega \pm A_0^{(3)})^2 \Phi^{(\pm)} = 0. \quad (4.9)$$

where we have again suppressed the indices  $(\omega, \bar{l}, l, s)$  on the functions  $\Phi$  and we impose incoming boundary conditions at the horizon of the black hole

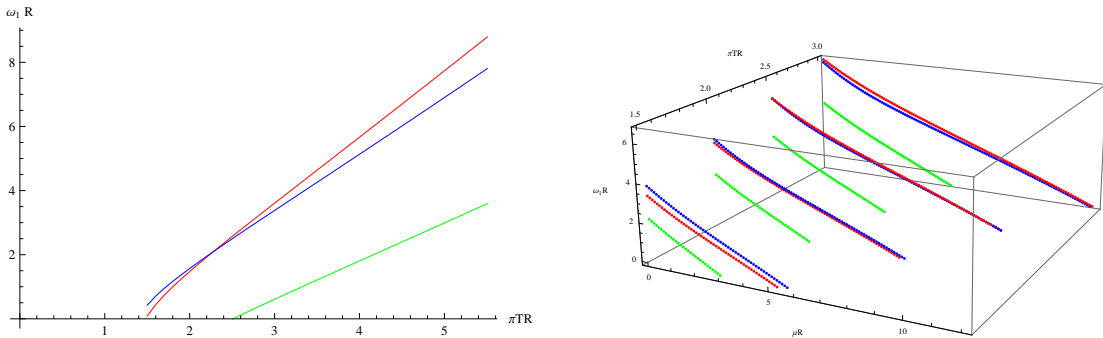
$$\Phi^{(\pm)} \Big|_{u \approx 1} = (1 - u)^{-i\omega/(4\pi T)} (1 + O(1 - u)). \quad (4.10)$$

In order to solve the fluctuation equations (4.5), (4.7) and (4.9), we use a shooting technique, in which we start from the horizon and look for modes that decay at infinity i.e. we look for the modes that describe normalisable excitations. These boundary conditions will be satisfied only for a discrete set of frequencies. We plot the imaginary parts of those frequencies for the scalar and two vectors, for fixed temperature and various values of the chemical potential  $\mu$ , in figure 1.

We see that as the value of the chemical potential is increased, the imaginary parts of the frequencies, which were initially all negative, become less and less negative and approach zero. When the chemical potential exceeds a critical value, the imaginary parts become positive one by one, signalling the presence of unstable modes in the system. Similarly, the real parts of the frequencies are decreasing to zero as the chemical potential grows, signalling again the onset of an instability. For a particular value of the temperature presented on the left plot in the figure 1, we see that the vector dual to the gauge theory charged scalar remains the lightest in



**Figure 1.** Plots of the imaginary (left) and real (right) parts of the frequencies for the lowest lying uncharged scalar fluctuation (red), the vector fluctuation (blue) and the charged scalar fluctuation (green), at fixed temperature  $\pi TR = 2$ , as a function of the chemical potential.

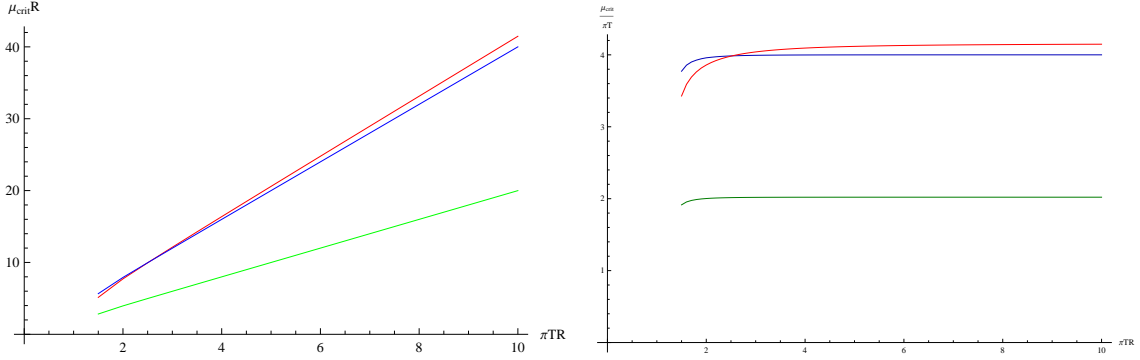


**Figure 2.** Left is a plot of the real parts of the frequencies for the various modes (colours as in figure 1) as functions of the temperature at fixed value  $\mu R = 5$  of the chemical potential. The plot on the right shows the real parts of the frequencies as functions of both temperature and chemical potential.

the spectrum and condenses first, followed by the transverse scalar and finally the vector.

In general, one would expect that particles condense roughly when the chemical potential become of the order of their mass. It is thus of interest to look at the behaviour of the masses<sup>8</sup> as a function of temperature. Figure 2 shows the result of this analysis. We here observe another interesting phenomenon, namely that there is a crossover point at some critical value of the temperature, above which the lightest vector becomes lighter than the transverse scalar. This suggests that above the crossover temperature, the lightest vector would condense before the lightest

<sup>8</sup>We use pole masses here for convenience as they are easy to obtain from the quasi-normal mode analysis, and the intuition we want to verify is anyhow qualitative.



**Figure 3.** Critical chemical potential as function of temperature, in two different dimensionless combinations. The figure on the right shows more clearly what happens in the  $TR \rightarrow \infty$  limit, which can be interpreted as the large radius limit at fixed temperature.

transverse scalar, if it had not been for the  $SO(4)$  charged scalar that condenses even earlier. One can indeed see that the corresponding imaginary parts cross as well, approximately at this point, see figure 3.

For large  $TR$ , the results read

$$\begin{aligned}
 \text{charged scalar :} \quad & \mu_{\text{crit}} R \approx 2.00\pi TR - \frac{2.00 \times 0.05}{4\pi TR} + \dots, \\
 \text{vector :} \quad & \mu_{\text{crit}} R \approx 4.00\pi TR - \frac{4.00 \times 0.05}{4\pi TR} + \dots, \\
 \text{uncharged scalar :} \quad & \mu_{\text{crit}} R \approx 4.16\pi TR - \frac{4.16 \times 1.00}{4\pi TR} + \dots.
 \end{aligned} \tag{4.11}$$

The leading order terms should agree with those obtained in the Poincaré patch, though to our knowledge only the one for the vector has been computed in the literature [6, 12, 17]. The result for the critical chemical potential of [6, 12] (when extrapolated to zero bare quark mass) seems to be somewhat larger than ours, which may be due to the fact that we have used a Yang-Mills truncation rather than the full DBI action.

## 5 The new ground states at zero and finite temperature

In the previous two sections we have seen that for large enough chemical potential the homogeneous ground state on  $\bar{S}^3 \in \text{AdS}_5$  becomes unstable under both scalar and various components of vector fluctuations. This is happening both at zero and non-zero temperature. In particular we observe that, at zero temperature, vector fluctuations in the direction of the internal  $S^3 \in S^5$  are the first to become unstable. As the temperature is increased, these vector components remain the first to become unstable. On the other hand, the ordering in which the other components of the

vector fluctuations and the scalar fluctuations become unstable is dependent on the temperature, as there is a ‘crossover’ temperature above which all vector components first become unstable.

Our previous analysis was done in perturbation theory, i.e. at the linearised level. So we would now like to see if the instabilities which we have found are present in the full non-linear theory, and to explore the new ground state in which the system settles for large enough values of the chemical potential.

### 5.1 The new ground state at zero temperature

In section 3.2.2 we have observed that as the chemical potential is turned on, when its value reaches  $\mu \geq 2/R$ , the lowest lying mode of the vector component in the internal  $S^3$ , becomes massless, signalling the onset of possible instability in the system. We have also seen that for even larger values of the chemical potential, the scalar becomes massless at  $\mu \geq 3/R$  and the other components of the vector develop an instability for  $\mu \geq 4/R$ .

To see whether the appearance of these massless modes indeed signals a real instability, we will now turn to the full non-linear theory and try to explicitly construct the new ground state to which the system would evolve as a consequence of the instability. As the perturbative analysis suggests that vector components in the internal  $S^3$  direction are first to condense, we will start the analysis of the new ground state by turning on only those components. We will later, for comparison, also analyse possible ground states due to condensation of the other fluctuations, and verify that those always have higher energy than the scalar condensate.

When writing down an ansatz for the scalar condensate ground state, we will use the fact that in perturbation theory, the first unstable mode is an  $\bar{l} = 0, l = 1, n = 0, s = -1$  wave (where  $\bar{l}$  labels modes in the  $\bar{S}^3$  and  $l$  labels modes in the  $S^3 \in S^5$ ). As far as the  $A_0$  component is concerned, at linearised level one cannot see the back-reaction of the scalar on the background value of this field, so in principle one cannot say if in the new ground state the  $A_0$  component will start to depend on the angular coordinates or not.

As a simplest attempt we take  $A_0$  to remain homogeneous, i.e. independent of the  $\bar{S}^3$  angular coordinates. With this ansatz, potential problems in the equations of motion could originate from expression of the form “ $A_\alpha A_\beta g_{S^3}^{\alpha\beta}$ ”, which are now turned on due to the non-vanishing vector field in the direction of the internal sphere  $S^3$ . Since these terms will typically produce spherical harmonics of higher  $l$ -number we would need to balance them in the equations of motion. However, we expect that the ground state would originate from condensation of only the lowest harmonic, so that higher  $l$ -harmonics are not needed. It is possible to reconcile these two observations if the “ $A_\alpha A_\beta g_{S^3}^{\alpha\beta}$ ” expression is independent of the angular coordinates. This can indeed be achieved for a particular linear combination of spherical harmonics given

by

$$Y_\alpha = \frac{ik_0}{K} Y_\alpha^{1,0,0,-1} + \frac{(k_1 + ik_2)}{K} Y_\alpha^{1,0,-1,-1} + \frac{(k_1 - ik_2)}{K} Y_\alpha^{1,0,1,-1}$$

where  $K \equiv \sqrt{k_0^2 + 2(k_1^2 + k_2^2)}$ , (5.1)

and  $k_0, k_1, k_2$  are three arbitrary real numbers which are not simultaneously vanishing, and  $(l, m_1, m_2, s)$  are the quantum numbers of the spherical harmonics. We should note here that value of these quantum numbers will be taken to be the same as those of the lowest lying excitation we have previously found in the perturbative analysis. Explicitly, the spherical harmonics are given by

$$\begin{aligned} Y^{1,0,0,-1} &= \frac{i}{2} d\alpha + \frac{i}{2} \cos \beta d\gamma \\ Y^{1,0,1,-1} &= -\frac{1}{2\sqrt{2}} e^{-i\alpha} d\beta - \frac{i}{2\sqrt{2}} \sin \beta e^{-i\alpha} d\gamma \\ Y^{1,0,-1,-1} &= -\frac{1}{2\sqrt{2}} e^{i\alpha} d\beta + \frac{i}{2\sqrt{2}} \sin \beta e^{i\alpha} d\gamma, \end{aligned} \quad (5.2)$$

where  $\alpha, \beta, \gamma$  are Euler coordinates on  $S^3 \in S^5$ ,

$$ds_{S^3}^2 = \frac{1}{4} (d\alpha^2 + d\beta^2 + d\gamma^2 + 2 \cos \beta d\alpha d\gamma). \quad (5.3)$$

It is also useful to keep in mind that

$$(Y_i^{l,m_1,m_2,s})^* = -(-1)^{m_1-m_2} Y_i^{l,-m_1,-m_2,s}. \quad (5.4)$$

Our ansatz for the new ground state is

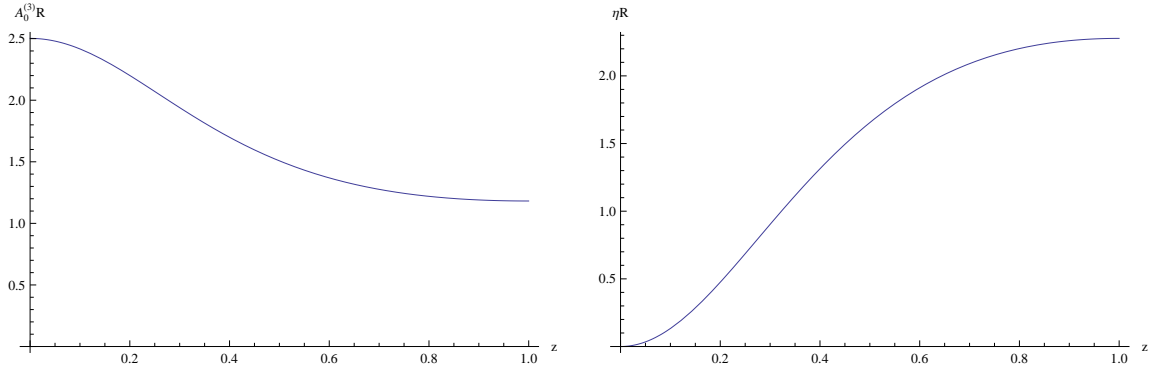
$$A_0 = A_0^{(3)}(u) \tau^3, \quad A_\alpha = R \eta(u) Y_\alpha(\Omega_3) \tau^1 \quad (5.5)$$

Plugging this into equations of motion, and using identities (3.21) we get an equation for  $A_0(u)$

$$\frac{\partial_u \left( (\partial_u A_0^{(3)}) u^3 \left(1 - \frac{R^2}{4u^2}\right)^3 / \left(1 + \frac{R^2}{4u^2}\right) \right)}{u^3 \left(1 - \frac{R^2}{4u^2}\right)^3 / \left(1 + \frac{R^2}{4u^2}\right)} - \frac{R^2}{u^2} (\eta(u))^2 A_0^{(3)}(u) = 0, \quad (5.6)$$

and an equation for the function  $\eta(u)$

$$\frac{\partial_u \left( \partial_u \eta(u) u^5 \left(1 + \frac{R^2}{4u^2}\right) \left(1 - \frac{R^2}{4u^2}\right)^3 \right)}{u^5 \left(1 + \frac{R^2}{4u^2}\right) \left(1 - \frac{R^2}{4u^2}\right)^3} + \frac{R^4}{u^4 \left(1 + \frac{R^2}{4u^2}\right)^2} (A_0^{(3)}(u))^2 \eta(u) + \frac{4}{u^2} \eta(u) = 0. \quad (5.7)$$



**Figure 4.** Profile of the fields  $A_0$  (left) and  $\eta$  (right) of the charged scalar condensate, evaluated at  $\mu R = 2.5$ . The boundary is at  $z = 0$  and the AdS centre is at  $z = 1$ .

Note that these are independent of the parameter  $K$ , which only appears in the angular part of the equations of motion, which is automatically satisfied for our ansatz.

Equations (5.6), and (5.7) are written in the non-compact coordinate  $u$  for which the AdS centre is at  $u = R/2$  and the boundary is at  $u = \infty$ . However, for numerical considerations it is more convenient to perform a coordinate change to compact coordinates  $z = R/2u$ , so that the AdS origin is at  $z = 1$ , and the boundary at  $z = 0$ . The equations of motion then are given by

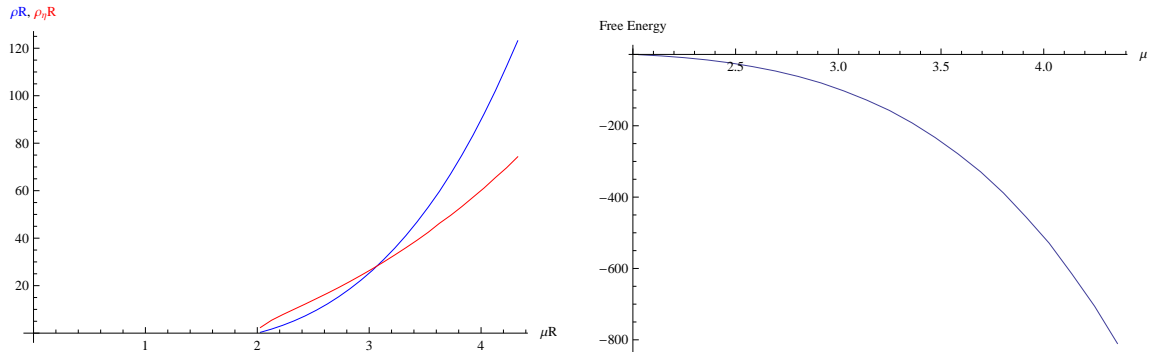
$$\begin{aligned} \partial_z^2 A_0^{(3)}(z) + \frac{1 + 8z^2 + 3z^4}{z^5 - z} \partial_z A_0^{(3)}(z) - \frac{R^2}{z^2} \eta(z)^2 A_0^{(3)}(z) &= 0, \\ \partial_z^2 \eta(z) + \frac{3 + 4z^2 + 5z^4}{z^5 - z} \partial_z \eta(z) + \left( \frac{4}{z^2} + \frac{4R^2}{(1 + z^2)^2} A_0^{(3)}(z)^2 \right) \eta(z) &= 0. \end{aligned} \quad (5.8)$$

We are interested in the solutions of these equations that are regular everywhere, and in particular at the origin of AdS space. This removes half of the solutions, as can be seen by looking at the  $z \rightarrow 1$  limit of the above equations. Namely, assuming that  $A_0$  and  $\eta$  are regular at the AdS origin, it is easy to see that the above equations reduce to the conditions that the first derivatives of  $A_0$  and  $\eta$  are vanishing at the origin. Hence, the general regular solution will be parametrised by two parameters  $a$ ,  $b$ . We then solve the equations of motion by shooting from the AdS origin, and look for the solutions at the boundary such that  $\eta$  is normalisable, while  $A_0$  is not. This normalisability condition further reduces the number of parameters by one. Hence in the expansion near infinity

$$A_0^{(3)} = \mu - \rho z^2 + \dots, \quad A_\alpha^{(1)} = R \rho_\eta Y_\alpha z^2 + \dots. \quad (5.9)$$

both the densities  $\rho$  and  $\rho_\eta$  are functions of the chemical potential  $\mu$ .

We plot the radial profile of the functions  $A_0(z)$  and  $\eta(z)$  for one particular solution in figure 4. As required, we see that the solution is regular everywhere, and



**Figure 5.** The plot on the left shows the isospin density  $\rho R$  (blue) and scalar density  $\rho_\eta R$  (red) of the charged scalar condensate as functions of the isospin chemical potential. The plot on the right shows the scaled free energy as function of chemical potential.

approaches the origin of AdS with vanishing derivative, so that no cusp is present. We also study various solutions for different values of chemical potential, see figure 5. The shooting procedure shows that there is a critical value of the chemical potential  $\mu_{\text{crit}} \sim 2/R$  below which there is no nontrivial solution present. Above  $\mu = \mu_{\text{crit}}$  a nontrivial condensate of scalar particles forms, and in the neighbourhood of  $\mu_{\text{crit}}$ , this condensate is to a good approximation given by

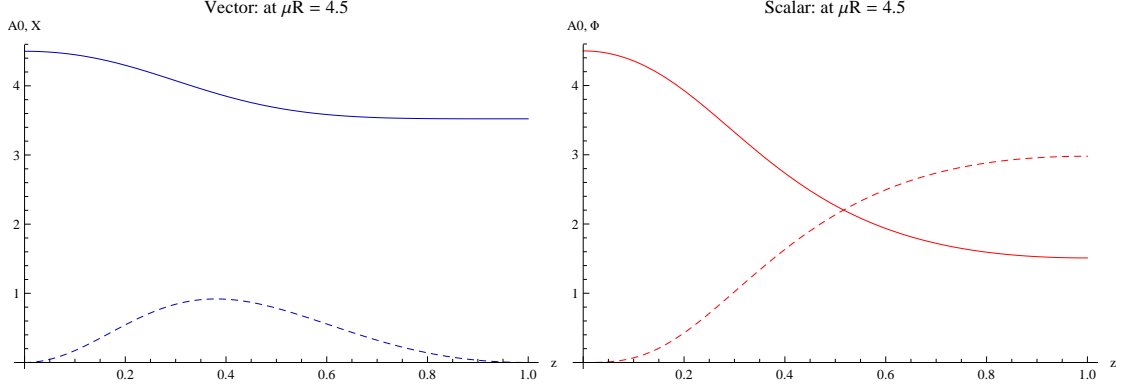
$$\rho_\eta = \begin{cases} 0 & \text{for } \mu < \mu_{\text{crit}} \\ \sqrt{\mu - \mu_{\text{crit}}} & \text{for } \mu > \mu_{\text{crit}}. \end{cases} \quad (5.10)$$

We have also evaluated the free energy for various values of the chemical potential (see figure 5), and observed that it is less than the (vanishing) free energy of the trivial configuration, which is in agreement with the statement that this is the ground state.

In summary, our analysis shows that for large enough value of the chemical potential, this system undergoes a second order phase transition in which the homogeneous isotropic solution is replaced with a non-isotropic one. The order parameter in this transition is the density  $\rho_\eta$ , and the critical exponent is the same as in the Landau-Ginsburg theory with positive quartic potential.

In order to complete the picture, and to show that (as expected from the perturbative analysis) the charged scalar condensate is always the one with the lowest energy, we will now construct condensates of the transverse scalar and the vector, and show that their energies are always higher than the one of the charged scalar. When constructing the transverse scalar ground state we recall that perturbative analysis suggested that the s-wave is the first excitation of the scalar which becomes massless. Hence we make a homogeneous (i.e. only  $u$ -dependent) ansatz as follows

$$A = A_0^{(3)}(u)\tau^3 dt, \quad \Phi = \Phi^{(1)}(u)\tau^1, \quad (5.11)$$



**Figure 6.** Left: a plot of the functions  $A_0(z)$  (solid curve) and  $\psi(z)$  (dashed curve) for the vector solution. Right: plots of the functions  $A_0(z)$  (solid) and  $\Phi$  (dashed) for the scalar configuration, both evaluated at a fixed value of chemical potential  $\mu R = 4.5$ .

where the vector  $A_0$  is present to account for the non-vanishing chemical potential, while all other vector components are zero. The equations of motion for the fields  $(A_0, \Phi)$  are given by

$$\begin{aligned} \partial_z^2 A_0^{(3)}(z) + \frac{1 + 8z^2 + 3z^4}{z^5 - z} \partial_z A_0^{(3)}(z) - \frac{R^2}{z^2} \Phi(z)^2 A_0^{(3)}(z) &= 0, \\ \partial_z^2 \Phi(z) + \frac{3 + 4z^2 + 5z^4}{z^5 - z} \partial_z \Phi(z) + \left( \frac{3}{z^2} + \frac{4R^2}{(1 + z^2)^2} A_0^{(3)}(z)^2 \right) \Phi(z) &= 0. \end{aligned} \quad (5.12)$$

Similarly, when constructing the ground state originating from vector condensation, we start with an ansatz which is similar to that of the vector component dual to a charged scalar, i.e. we write

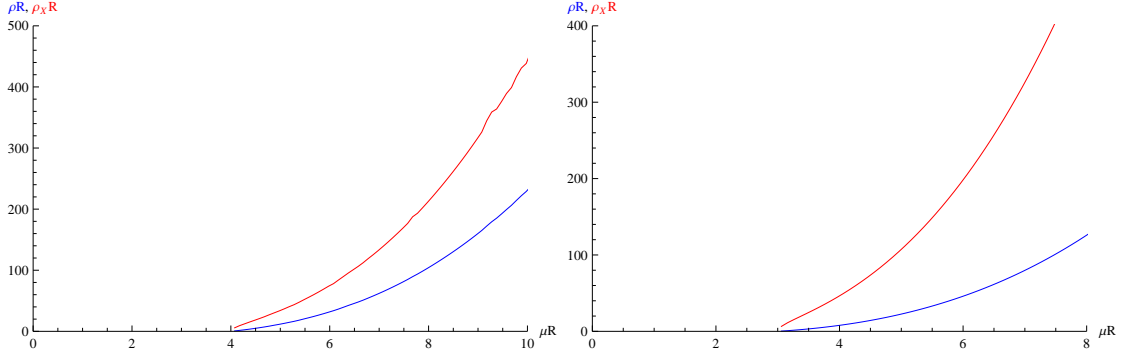
$$A_0 = A_0^{(3)}(u) \tau^3, \quad A_{\bar{\alpha}} = \psi(u) Y_{\bar{\alpha}}(\bar{\Omega}_3) \tau^1, \quad (5.13)$$

where  $Y_{\bar{\alpha}}$  is as in (5.5), except that the index  $\bar{\alpha} = 1, 2, 3$  now refers to the  $\bar{S}^3 \in \text{AdS}_5$ . The equations of motion then become

$$\begin{aligned} \partial_z^2 A_0^{(3)}(z) + \frac{1 + 8z^2 + 3z^4}{z^5 - z} \partial_z A_0^{(3)}(z) - \frac{4}{(1 - z^2)^2} \psi(z)^2 A_0^{(3)}(z) &= 0, \\ \partial_z^2 \psi(z) + \frac{1 + 3z^4}{z^5 - z} \partial_z \psi(z) + \left( \frac{4R^2}{(1 + z^2)^2} A_0^{(3)}(z)^2 - \frac{16}{(1 - z^2)^2} \right) \psi(z) &= 0. \end{aligned} \quad (5.14)$$

We should emphasise here that this equation is derived from an ansatz which uses spherical harmonics with  $\bar{l} = 1, s = -1$ , similar to the ansatz we used when we constructed the state for the vector dual to a charged scalar. However, we have also seen in the perturbative analysis that vector fluctuations in the direction of  $\bar{S}^3 \in \text{AdS}_5$  are insensitive to the quantum number  $s$ , unlike the fluctuation in direction





**Figure 7.** Plots of the densities  $\rho$  (blue) and  $\rho_\eta$  (red), as a function of the chemical potential  $\mu$ , at zero temperature, for the vector condensate (left) and scalar condensate (right).

of  $S^3 \in S^5$ . Therefore, it should also be possible to construct an alternative state with spherical harmonics with  $\bar{l} = 1, s = 1$ . This is indeed the case, and the free energy of this state is the same as for the state with  $\bar{l} = 1, s = -1$ .

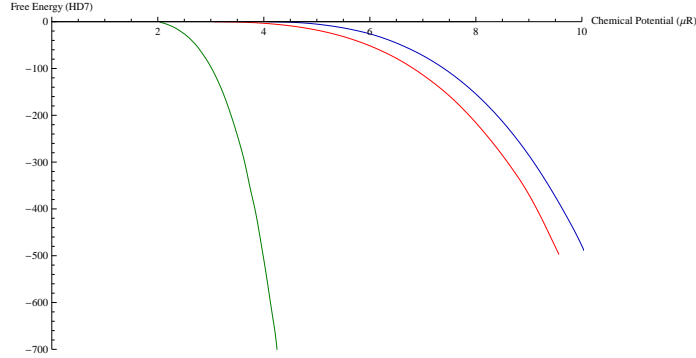
Equations (5.12) and (5.14) are solved in the same fashion as equation (5.8), that is by the shooting method and imposing that the solution is regular everywhere and in particular at the origin of  $\text{AdS}_5$ . The solutions for the radial functions  $(A_0, \Phi)$  for the scalar configuration and  $(A_0, \psi)$  for the vector are plotted in figure 6. We also plot the densities for both configurations (defined analogous to (5.9)) as functions of the chemical potential, see figure 7.

In order to compare various configurations we plot the free energies for all three states, see figure 8. As expected from the perturbative analysis, we see that the state which originates from a condensation of the vector components which are dual to a charged scalar has the lowest free energy. We also see that as the chemical potential is increased, the difference between the free energies of the other two states and the true ground state becomes larger. It is, however, likely that new instabilities will kick in at some point. Investigating that in detail would require at the least a perturbative analysis around this new ground state, which we will not attempt here.

## 5.2 The new ground state at finite temperature

So far we have seen that at zero temperature, the ground state originates from the condensation of vector components which are dual to a charged scalar, exactly as perturbation theory suggested. We now want to see what is happening with this new ground state as the temperature is turned on. We start by making the same ansatz as at zero temperature, see (5.5). The equations of motion in the coordinates (2.12) are given by

$$\partial_v^2 A_0^{(3)}(v) - \frac{\Lambda - 1}{4(\Lambda - v)(1 - v)^2 v} R^2 \eta(v)^2 A_0^{(3)}(v) = 0, \quad (5.15)$$



**Figure 8.** Scaled free energy of the zero-temperature condensates as a function of the dimensionless chemical potential. The vector is plotted in blue, the scalar in red, the charged scalar green. The black line along the  $x$ -axis denotes the old ground state.

together with

$$\begin{aligned} \partial_v^2 \eta(v) + \left( \frac{1}{1-v} + \frac{1}{v} - \frac{1}{\Lambda-v} \right) \partial_v \eta(v) \\ + \frac{\Lambda-1}{(\Lambda-v)(1-v)^2 v} \left( 1 + \frac{(2-\Lambda)(1-v)R^2}{4(\Lambda-v)v} A_0^{(3)}(v)^2 \right) \eta(v) = 0. \end{aligned} \quad (5.16)$$

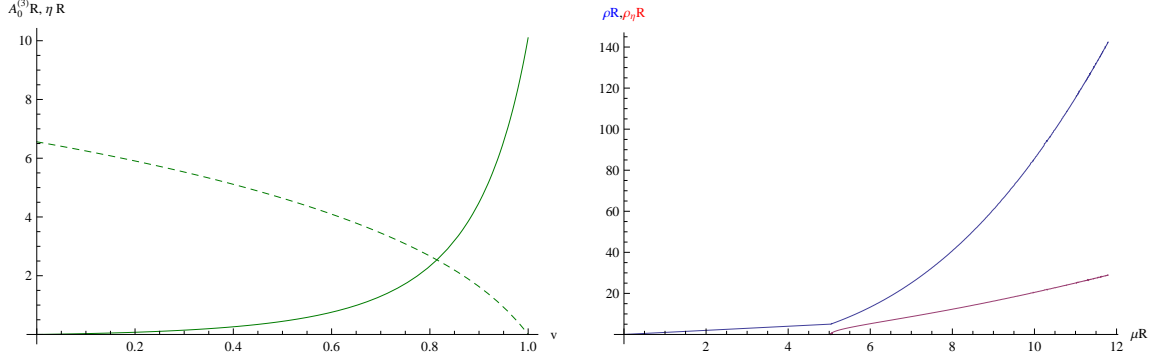
We are interested in finding regular solutions to these equations. It is easy to see that the solutions which are regular are parametrised by two free parameters. A general, perturbative expansion of the solution near the black hole horizon which is regular is given by

$$\begin{aligned} A_0^{(3)}(v) &= av + \frac{ab^2(\Lambda-1)}{8\Lambda R^2} v^2 + O(v^3), \\ \eta(v) &= b - \frac{(\Lambda-1)b}{\Lambda} v - \frac{12b(\Lambda-1) + (2-\Lambda)(\Lambda-1)a^2 R}{16\Lambda^2} v^2 + O(v^3), \end{aligned} \quad (5.17)$$

i.e. a regular solution is parametrised by two real numbers  $a, b$ . We also see that the general regular solution for  $A_0$  vanishes at the horizon, as required by global regularity. We solve this system of equations again using a shooting method with two free parameters. As at zero temperature, we require in addition that the solution for  $\eta$  is normalisable at infinity, or explicitly

$$A_0^{(3)}(v) = \mu - \rho(1-v) + O((1-v)^2), \quad \eta(v) = \rho_\eta(1-v) + O((1-v)^2). \quad (5.18)$$

This is possible only for a particular pair of parameters  $a, b$ , or in other words both densities  $\rho, \rho_\eta$  are functions of the chemical potential. An example of the radial profiles for a regular solution is plotted in figure 9. We also plot both densities as a function of chemical potential (see right plot on figure 9). We observe that, just as at zero temperature, the densities increase as the chemical potential is increased.



**Figure 9.** Left are plots of profile of the fields  $A_0$  (solid) and  $\eta$  (dashed) for the charged scalar, evaluated at  $\pi TR = 2.5$  and  $\mu R = 10.1$ . The boundary is at  $v = 1$  and horizon at  $v = 0$ . Right plot is for densities  $\rho$  (blue) and  $\rho_\eta$  (red), as function of chemical potential  $\mu$ , at fixed temperature  $\pi TR = 2.5$ .

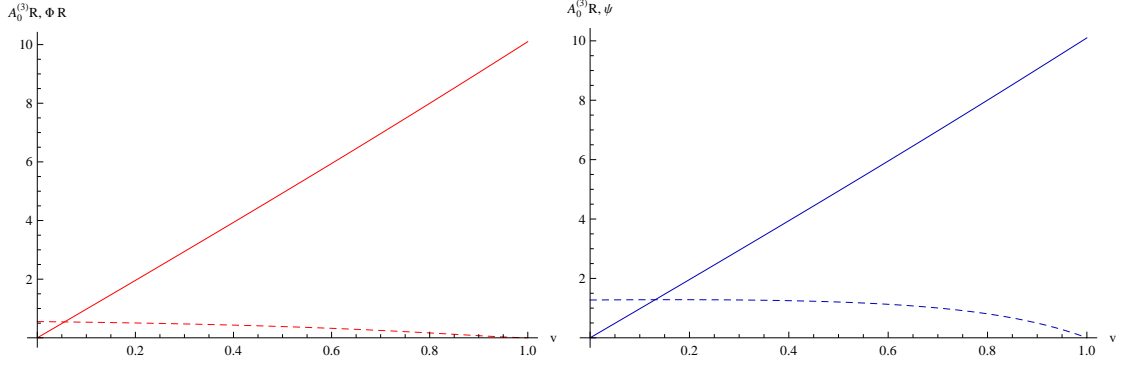
As for zero temperature, we should make sure that possible alternative states which appear due to condensation of other unstable particles have a larger free energy (as suggested by perturbation theory). We start with the scalar ground state. We make the same ansatz for the ground state, as we did at the zero temperature, see (5.11). The equations of motion in the coordinates (2.12) are given by

$$\begin{aligned} \partial_v^2 A_0^{(3)}(v) - \frac{\Lambda - 1}{4(\Lambda - v)(1 - v)^2 v} \chi(v)^2 R^2 A_0^{(3)}(v) &= 0, \\ \partial_v^2 \chi(v) + \left( \frac{1}{1 - v} + \frac{1}{v} - \frac{1}{\Lambda - v} \right) \partial_v \chi(v) & \\ + \frac{\Lambda - 1}{(\Lambda - v)(1 - v)^2 v} \left( 1 + \frac{(2 - \Lambda)(1 - v)R^2}{4(\Lambda - v)v} A_0^{(3)}(v)^2 \right) \chi(v) &= 0. \end{aligned} \quad (5.19)$$

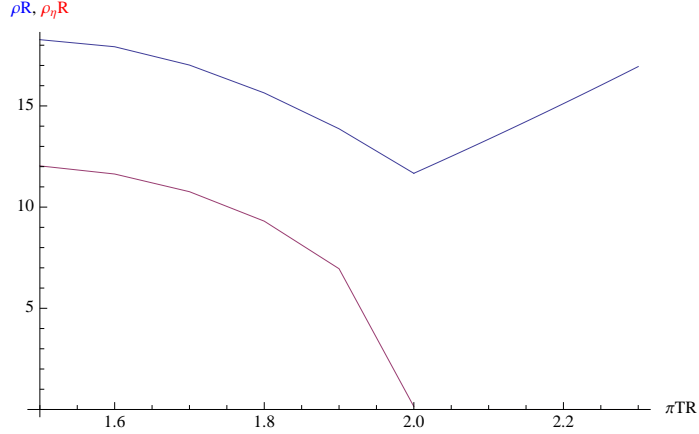
Similarly, the ground state originating from the vectors is derived starting with the ansatz (5.13). The equations of motion are given by

$$\begin{aligned} \partial_v^2 A_0^{(3)}(v) - \frac{2 - \Lambda}{4(\Lambda - v)(1 - v)v} \psi(v)^2 A_0^{(3)}(v) &= 0, \\ \partial_v^2 \psi(v) + \left( \frac{1}{v} - \frac{1}{\Lambda - v} \right) \partial_v \psi(v) & \\ - \frac{2 - \Lambda}{(\Lambda - v)(1 - v)v} \left( 1 - \frac{(\Lambda - 1)R^2}{4(\Lambda - v)v} A_0^{(3)}(v)^2 \right) \psi(v) &= 0. \end{aligned} \quad (5.20)$$

As before we consider only regular solutions to the equations (5.19) and (5.20) and require that the solutions are normalisable. Sample solutions for rather arbitrary values of the temperature and chemical potential are plotted in figure 10. We have



**Figure 10.** Plots of solutions  $(A_0(z), \Phi(z))$ , and  $(A_0(z), \psi(z))$  for the scalar and vector condensates respectively, at fixed temperature  $\pi TR = 2.5$  and chemical potential  $\mu R = 4.5$  (the curve for  $A_0$  is rather straight only because the plot is made for a chemical potential only slightly above the critical value).



**Figure 11.** Dependence of the charged scalar ground state densities on  $TR$  at fixed chemical potential  $\mu R = 4.005$ .

also evaluated the densities for these solutions, and find a qualitatively similar dependence on the chemical potential as before. We have also verified that indeed the charge scalar always has a lower free energy than the condensate of the other particles, as predicted by perturbation theory.

Finally, we present in figure 11 the dependence of the charged scalar condensate densities on the temperature, for fixed chemical potential. This shows how increasing the temperature ‘melts’ the condensate.

## 6 Discussion and outlook

In this paper we have analysed the stability of the homogeneous isotropic phase of conformal  $\mathcal{N} = 2$  super-Yang-Mills theory on a three sphere in the presence of an isospin chemical potential. We have found that for sufficiently large chemical potential, the theory exhibits unstable vector and scalar modes, as well an unstable vector mode which is dual to an  $SO(4)$  charged scalar in the dual gauge theory. The latter modes turn out to condense first. We have verified this explicitly by constructing the condensate and showing that it has the lowest energy. The new ground state is anisotropic in the directions of the internal three sphere within the five sphere, but isotropic on the gauge theory sphere. Therefore, the new ground state breaks the global  $SO(4)$  symmetry.

Since the anisotropy of the system originates from the compactness of the internal three sphere, this anisotropy persists if we take the limit towards a non-compact system, i.e. the limit in which the radius of the sphere on which the dual gauge theory lives is taken to be very large (at fixed temperature).

We have observed that the spectrum of fluctuations of the more massive vector and scalar mesons crosses as a function of  $TR$ . This does not influence the formation of the dual charged scalar condensate, but it conceivably plays a role for larger values of the chemical potential. In particular, by doing a fluctuation analysis around the dual charged scalar condensate, one expects that at some point new instabilities will set in, corresponding to the more massive particles condensing as well. An analysis of this type has to our knowledge not appeared for any string/gauge theory model, but our results indicate that the structure of condensate formation may be more intricate than previously suspected.

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