Excursions and path functionals for stochastic processes with asymptotically zero drifts

Ostap Hryniv* Mikhail V. Menshikov* Andrew R. Wade*†

29th January 2013

Abstract

We study discrete-time stochastic processes (X_t) on $[0, \infty)$ with asymptotically zero mean drifts. Specifically, we consider the critical (Lamperti-type) situation in which the mean drift at x is about c/x. Our focus is the recurrent case (when c is not too large). We give sharp asymptotics for various functionals associated with the process and its excursions, including results on maxima and return times. These results include improvements on existing results in the literature in several respects, and also include new results on excursion sums and additive functionals of the form $\sum_{s \leq t} X_s^\alpha$, $\alpha > 0$. We make minimal moments assumptions on the increments of the process. Recently there has been renewed interest in Lamperti-type process in the context of random polymers and interfaces, particularly nearest-neighbour random walks on the integers; some of our results are new even in that setting. We give applications of our results to processes on the whole of $\mathbb{R}$ and to a class of multidimensional 'centrally biased' random walks on $\mathbb{R}^d$; we also apply our results to the simple harmonic urn, allowing us to sharpen existing results and to verify a conjecture of Crane *et al.*

Keywords: Path functional; excursion; maximum; passage-time; additive functional; path integral; Lamperti's problem; centre of mass; centrally biased random walk.
AMS 2010 Subject Classifications: 60G17, 60J10, 60J55, 60F15, 82D60

1 Introduction

The study of functionals defined on paths of stochastic processes is a topic with classical foundations and extensive applications in modern probability; such functionals give a quantitative encapsulation of both probabilistic information about the recurrence behaviour of the process and geometrical information about the way in which the process explores the state-space. A substantial body of work is devoted to *additive functionals* of the form $\sum_{s=1}^t \Phi(X_s)$, where $X_1, X_2, \ldots$ is a discrete-time stochastic process on $\mathbb{R}^d$ and $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a given measurable function. The most basic choice, in which $\Phi(x)$ is taken to be the indicator function $\mathbf{1}\{x \in A\}$ of a Borel set $A \subseteq \mathbb{R}^d$, leads to occupation time for A. In the most well-studied case, X_t is a sum of i.i.d. random variables; the monograph

*Department of Mathematical Sciences, University of Durham, South Road, Durham, DH1 3LE, UK.
†Corresponding author. E-mail: `andrew.wade@durham.ac.uk`. Tel: +44 (0)191 334 4151

by Borodin and Ibragimov [6] is devoted to limit theory in the i.i.d. setting. There is also much work devoted to the case in which X_t is an *ergodic* Markov process. Classical work goes back to Markov and Bernstein (cf. [28, p. 2299]). If Φ is integrable with respect to the stationary distribution of the process, then a large collection of 'ergodic theorems' and distributional limit theorems (after suitable scaling and under various conditions) are known, and represent an active area of research: see e.g. [10, 24, 27, 28, 33] and references therein for an indication of the extensive literature.

In the present paper, we study stochastic processes on $\mathbb{R}^d$ of a more general type (assuming a regenerative property only, rather than the Markov property) and in the *near-critical* situation from the point of view of the asymptotic behaviour of the process. Near-criticality entails that the one-step mean drift of X_t is *asymptotically zero* in a sense that we describe more precisely later on. Processes with asymptotically zero drifts are of interest in their own right for exploring phase transitions in asymptotic behaviour, as first described in the general setting in fundamental work of Lamperti [30, 31]. A consequence of near-criticality is that many important distributions associated with such processes display heavy-tailed behaviour; these include passage times [5, 31] and, if they exist, stationary distributions [35]. In many cases of interest, these and other quantities have natural scaling exponents that depend on the details of the process.

Moreover, such processes are important from the point of view of applications for two main reasons: first, they serve as prototypical near-critical stochastic systems and hence for the development of new techniques, and second, they can often be extracted from more complex near-critical systems via the method of Lyapunov functions, to powerful analytic effect. One classical but important illustration of the latter point is provided by the Lyapunov function approach to Pólya's theorem on the recurrence/transience of symmetric simple random walk S_t on $\mathbb{Z}^d$: Pólya's theorem can be understood in an entirely one-dimensional setting by taking $X_t = \|S_t\|$, in which case the process X_t has asymptotically zero drift in the sense that

$$\mathbb{E}[X_{t+1} - X_t \mid S_t = \mathbf{x}] = c_d \|\mathbf{x}\|^{-1} + O(\|\mathbf{x}\|^{-2}),$$

for some constant $c_d > 0$ that depends only on d. Lamperti's recurrence classification for such processes [30] implies Pólya's theorem. Importantly, the same technique works for a very large class of random walks; in particular, the Markov property is not essential.

For these asymptotically zero drift processes, we study a class of additive functionals (or *path integrals*) of the form $\sum_{s=1}^t X_s^\alpha$, $\alpha \geq 0$. Moreover, we study the maximum functional $\max_{1 \leq s \leq t} X_s$ (which corresponds in a certain sense to the $\alpha \to \infty$ limit of the additive functional). We are interested in the large-t asymptotics of such functionals, in the case where X_t is recurrent. Our primary interest is not in the case where the process has sufficient 'ergodicity' properties that $t^{-1} \sum_{s=1}^t X_s^\alpha$ converges, but rather in the case where $\sum_{s=1}^t X_s^\alpha$ grows faster than linearly, including the case where X_t is *null*-recurrent. This is rather different to the emphasis of the classical work cited above, and accords with our focus on systems that are *near-critical* in some sense.

We make some further remarks on applications and related results. Borovkov *et al.* [7] consider analogues for queueing models of the path integrals that we study. As far as the authors are aware, there has been little work specifically concerned with additive functionals of processes with asymptotically zero drift. For a particular (null-recurrent) example of a nearest-neighbour random walk on the nonnegative integers, Fal' [15] proved distributional limits for functionals such as $\sum_{s=1}^t (1 + X_s)^{-\gamma}$, $\gamma > 0$ sufficiently large. Our interest is in functionals of the opposite nature.

A (normalization of a) particularly important path integral is the *center-of-mass* process associated with X_t defined by $G_t = t^{-1} \sum_{s=1}^{t} X_s$. The behaviour of G_t for Markov processes is only partially understood beyond the case in which sufficient ergodicity ensures that G_t converges to a limit. For the centre-of-mass associated with simple random walk on $\mathbb{Z}^d$, Grill [21] proves the interesting result that (compact-set) recurrence is present if and only if $d = 1$. The desire to understand Grill's result more generally was one of the original motivations for the work of the present paper; by analogy with Lamperti [30, 31], it is natural to begin in the setting of processes with asymptotically zero drifts.

Our approach is via a detailed study of the *excursions* of the process X_t, in which, once again, the heavy-tailed nature of the characteristics of the processes becomes evident. Thus, if η denotes the duration of an excursion, we are led to the study of *excursion functionals* such as $\sum_{s=1}^{\eta} X_s^\alpha$ (including the special case $\alpha = 0$ of η itself) and $\max_{1 \leq s \leq \eta} X_s$. As well as being key ingredients in the proofs of our large-t asymptotics, these quantities are of interest in their own right in various theoretical and applied contexts. For example, to apply Theorem 2.1 of [10] one needs to understand tail properties of an analogue of $\sum_{s=1}^{\eta} X_s^\alpha$; sums over excursions for processes with asymptotically zero drift turn out to be central to the analysis of the 'simple harmonic urn' [9] (see also Section 3.3 below).

In the last decade or so, significant interest in processes with asymptotically zero drifts has come from a community of probabilists and statistical physicists from the point of view of modelling the configurations of polymers and interfaces. A now standard approach in this field is to take as an underlying model a nearest-neighbour random walk with an asymptotically zero drift: see for example [1, 12, 23]. Such nearest-neighbour models are amenable to explicit calculation, often via intricate algebraic methods such as Karlin–McGregor spectral theory and orthogonal polynomials [26]; other recent work on these models, not directly motivated by polymer models, includes for example [11, 18, 29, 40]. This continued interest in asymptotically zero drift processes in the nearest-neighbour case is another motivation for the present paper, in which we present related results for a much more general class of models. We discuss the relation of our results to some of this recent work in more detail in Section 3.4.

The outline of the remainder of the paper is as follows. In Section 2 we give a formal statement of our half-line model and state our results on excursions and functionals in a series of subsections. In Section 3 we give applications of our half-line results to processes on the whole line (Section 3.1) and to multidimensional processes including centrally biased random walks on $\mathbb{R}^d$ (Section 3.2) and the simple harmonic urn (Section 3.3). Also, in Section 3.4, we make some remarks on how our model and results complement recent results, restricted to nearest-neighbour random walks, in the context of models of random polymers and interfaces. The proofs of the results in Sections 2 and 3 are given in Sections 4 and 5 respectively.

2 Main results on path functionals

2.1 Description of the model

We formally describe our process $X := (X_t)_{t \in \mathbb{N}}$ ($\mathbb{N} := \{1, 2, \ldots\}$) and our structural assumptions on its state-space $\mathcal{S}$. Recall that a subset R of $\mathbb{R}^d$ is *locally finite* if $R \cap H$ is finite for all bounded $H \subset \mathbb{R}^d$. Our basic assumption is the following.

(A0) (a) Let $\mathcal{S}$ be a locally finite, unbounded subset of $[0, \infty)$ with $0 \in \mathcal{S}$.

(b) Suppose that $(X_t)_{t\in\mathbb{N}}$ is an $\mathcal{S}$-valued process adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$, and $\mathbb{P}[X_1 = 0] = 1$.

We also assume the following form of 'irreducibility'.

(A1) Suppose that for each $x, y \in \mathcal{S}$ there exist $m(x, y) \in \mathbb{N}$ and $\varphi(x, y) > 0$ such that

$$\mathbb{P}[X_{t+m(X_t,y)} = y \mid \mathcal{F}_t] \geq \varphi(X_t, y), \text{ a.s., for all } t \in \mathbb{N}. \tag{2.1}$$

If X is a time-homogeneous Markov process, (2.1) reduces to the usual sense of irreducibility that, for any $x, y \in \mathcal{S}$, there exists $m(x, y) \in \mathbb{N}$ such that $\mathbb{P}[X_{m(x,y)} = y \mid X_1 = x] > 0$. The assumption (2.1) allows us to work with more general processes, such as functions of Markov process: see the discussion at the end of this subsection. A consequence of (A0) and (A1) is that $\limsup_{t\to\infty} X_t = \infty$, a.s.; see Proposition 2.1 below.

We make some 'Lamperti-style' assumptions on the increments of X. Throughout we use the notation $\Delta_t := X_{t+1} - X_t$. We will typically need to assume that for some $p > 2$ (at least), some $\delta > 0$, and some constant $C \in (0, \infty)$, for all $t \in \mathbb{N}$,

$$\mathbb{E}[|\Delta_t|^p \mid \mathcal{F}_t] \leq C(1 + X_t)^{p-2-\delta}, \text{ a.s.} \tag{2.2}$$

Given that (2.2) holds for some $p > 2$, $\mathbb{E}[\Delta_t^k \mid \mathcal{F}_t]$ is a.s. finite for $k \in \{1, 2\}$. We make some further assumptions on the moments of the increments, as follows. For notational convenience, throughout the paper we write $\log^q x$ for $(\log x)^q$, $q \in \mathbb{R}$.

(A2) Suppose that for some $c \in \mathbb{R}$ and $s^2 \in (0, \infty)$, as $X_t \to \infty$,

$$\mathbb{E}[\Delta_t \mid \mathcal{F}_t] = cX_t^{-1} + o(X_t^{-1}\log^{-1} X_t), \text{ a.s.,} \tag{2.3}$$

$$\mathbb{E}[\Delta_t^2 \mid \mathcal{F}_t] = s^2 + o(\log^{-1} X_t), \text{ a.s.} \tag{2.4}$$

We make an important note on notation: our usage assumes that implicit constants in Landau $O(\,\cdot\,)$, $o(\,\cdot\,)$ symbols are *non-random* and independent of t, so that asymptotic expressions such as (2.3) and (2.4) are understood to hold uniformly in t and sample points ω (on a set of probability 1). So, for example, (2.4) means that for any $\varepsilon > 0$ we can choose $x < \infty$ so that $|\mathbb{E}[\Delta_t^2 \mid \mathcal{F}_t] - s^2| \leq \varepsilon / \log X_t$, a.s., on $\{X_t > x\}$, for any $t \in \mathbb{N}$.

We study X via its *excursions* from 0. Set $\tau_0 := 1$ and for $n \in \mathbb{N}$ define

$$\tau_n := \min\{t > \tau_{n-1} : X_t = 0\},$$

with the usual convention that $\min \emptyset = \infty$. That is, $\tau_0, \tau_1, \tau_2, \ldots$ are the successive times of visit to the origin by X; if X visits 0 only finitely often then $\tau_n = \infty$ for all n large enough. When $\tau_n < \infty$ we denote, for $n \in \mathbb{N}$, $\eta_n := \tau_n - \tau_{n-1}$, the duration of the nth excursion; also set $\eta_0 := 1$. Provided that $\tau_n < \infty$, we denote the nth *excursion* ($n \in \mathbb{N}$) by $\mathcal{E}_n := (X_t)_{\tau_{n-1} \leq t \leq \tau_n - 1}$. Let $N := \min\{n \in \mathbb{N} : \tau_n = \infty\}$.

(A3) (a) Suppose that, for all $n \in \mathbb{N}$, $\mathbb{P}[\eta_{n+1} < \infty \mid \tau_n < \infty] = \mathbb{P}[\eta_1 < \infty]$.

(b) Suppose that, on $\{N = \infty\}$, $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence.

Part (b) of (A3) assumes a full regenerative structure in the case of an infinite number of returns to 0. Part (a) makes a weaker assumption, needed to deal with the event $\{N < \infty\}$. A useful reference for regenerative processes is [2, Chapter VI].

Our irreducibility and regenerative assumptions have the following basic consequence.

Proposition 2.1. *Suppose that (A0) and (A1) hold. Then* $\limsup_{t\to\infty} X_t = \infty$*, a.s. If, in addition, part (a) of (A3) also holds, then either:*

(i) (transience) $\mathbb{P}[\eta_1 < \infty] < 1$ *and* $\lim_{t\to\infty} X_t = \infty$ *a.s.; or*

(ii) (recurrence) $\mathbb{P}[\eta_1 < \infty] = 1$ *and* $\liminf_{t\to\infty} X_t = 0$ *a.s.*

We give the proof of Proposition 2.1 in Section 4.3, along with the proofs of the other results that we state in the present section.

Before describing our main results, we indicate why we have chosen our particular assumptions. It is too restrictive for the applications that we have in mind to assume that X is itself a Markov process; our more general framework enables us to work with, for example, $X_t = \|Y_t\|$ where Y_t is a Markov process on $\mathbb{R}^d$. More generally, suppose that $(Y_t)_{t\in\mathbb{N}}$ is an irreducible time-homogeneous Markov process on an arbitrary countable set Σ, and let $f : \Sigma \to [0,\infty)$ be measurable such that $f^{-1}(x)$ is finite for each x. Let $\mathcal{F}_t = \sigma(Y_1,\dots,Y_t)$ and take $X_t = f(Y_t)$. Then X_t is $\mathcal{F}_t$-adapted and has the countable state-space $\mathcal{S} = f(\Sigma)$. Moreover, by irreducibility, given u and v such that $f(u) = x$ and $f(v) = y$, there exist $\varphi(x,y) > 0$ and $m(x,y) \in \mathbb{N}$ such that $\mathbb{P}[Y_{t+m(x,y)} = v \mid Y_t = u] \geq \varphi(x,y)$, using the fact that, by our assumption on Σ and f, there are only finitely many possible u, v pairs for a given x, y. Hence (2.1) holds. If $f^{-1}(0) = 0$ is unique, (A3) follows from the strong Markov property. This generality is very useful for applications: we describe one such example in detail in Section 3.2 below.

The remaining parts of this section are devoted to our results. Our excursion-based approach is only applicable in the recurrent case, so first we give a recurrence classification. Then we move on to detailed properties of excursions and the tails of associated random variables, and finally to $t \to \infty$ asymptotics of functionals defined on paths of the process up to some given time t. We exhibit various tail or scaling exponents for the quantities that we study.

2.2 Recurrence classification

We say X is *transient* if $\mathbb{P}[\eta_1 < \infty] < 1$; otherwise it is *recurrent*. If recurrent, we say that X is *positive*-recurrent if $\mathbb{E}[\eta_1] < \infty$ and *null*-recurrent if $\mathbb{E}[\eta_1] = \infty$. Under (A2), the quantity $-2c/s^2$ will play a central role in all that follows, and we introduce the notation

$$r := -2c/s^2. \tag{2.5}$$

The recurrence classification for X is as follows.

Theorem 2.2. *Suppose that (A0)–(A3) hold, and (2.2) holds with* $p > 2$*. Then* X *is*

(i) transient if $r < -1$*;*

(ii) null-recurrent if $-1 \leq r \leq 1$*;*

(iii) positive-recurrent if $r > 1$*.*

Theorem 2.2 is essentially due to Lamperti [30,31] (in the case $|r| \neq 1$) and Menshikov *et al.* [34] (in the case $|r| = 1$) under somewhat different conditions. We do not give a detailed proof of Theorem 2.2 here, but sketch in Section 4.3 how these existing results may be adapted to our setting.

2.3 The maximum of an excursion

Let $M_n := \max_{\tau_{n-1} \leq t < \tau_n} X_t$, the maximum attained by $\mathcal{E}_n$. The next result gives tail bounds, in the recurrent case, for the $(M_n)_{n\in\mathbb{N}}$, which are i.i.d. under our assumptions.

Theorem 2.3. *Suppose that (A0)–(A3) hold. Suppose that $r > -1$ and (2.2) holds with $p > \max\{2, 1+r\}$. Then for any $\varepsilon > 0$, for all x sufficiently large,*

$$x^{-1-r}(\log x)^{-\varepsilon} \leq \mathbb{P}[M_1 \geq x] \leq x^{-1-r}(\log x)^{1+\varepsilon}. \tag{2.6}$$

In particular, $\mathbb{E}\left[M_1^{1+r}\right] = \infty$ but, for any $\varepsilon > 0$, $\mathbb{E}\left[M_1^{1+r-\varepsilon}\right] < \infty$.

Remarks 2.1. (a) Symmetric simple random walk on the half-line with reflection at 0 (and, indeed, any of a host of more general zero-drift models) has $r = 0$, and is thus right on the boundary of having a finite expectation for M_1. (b) In the tail bound (2.6) and similar results in the sequel, the polynomial term is sharp but we do not necessarily strive for the best possible exponent for the logarithmic term. Our results are all sharp enough, however, to classify completely which moments do or do not exist for the random variable in question.

2.4 The duration of an excursion

The following result is a sharpening in our context of [5, Propositions 1 and 2], which themselves extended work of Lamperti [31].

Theorem 2.4. *Suppose that (A0)–(A3) hold. Suppose that $r > -1$ and (2.2) holds with $p > \max\{2, 1+r\}$. Then for any $\varepsilon > 0$, for all t sufficiently large,*

$$t^{-\frac{1+r}{2}}(\log t)^{-\varepsilon} \leq \mathbb{P}[\eta_1 \geq t] \leq t^{-\frac{1+r}{2}}(\log t)^{2+r+\varepsilon}. \tag{2.7}$$

In particular, $\mathbb{E}\big[\eta_1^{\frac{1+r}{2}}\big] = \infty$ but, for any $\varepsilon > 0$, $\mathbb{E}\big[\eta_1^{\frac{1+r}{2}-\varepsilon}\big] < \infty$.

The proof of the upper tail bound in (2.7) is based on general results of [4]. The lower bound in (2.7) is new in the generality given here; under more restrictive assumptions (including uniformly bounded increments for X_t) it can be derived from [3, Corollary 1]. Our proof of the lower bound is based on the intuitively appealing Lemma 4.11 below. Lamperti [31] was the first to systematically study the problem of the existence or non-existence of moments $\mathbb{E}[\eta_1^q]$: his results covered only integer q. Subsequently Aspandiiarov *et al.* extended Lamperti's results to all $q > 0$ (see the Appendix of [5]), but neither [31] nor the results of [5] determine whether the boundary case $\mathbb{E}[\eta_1^{(1+r)/2}]$ is finite or infinite; as mentioned above, results of [3] can be used to settle the boundary case, but under more restrictive conditions on the increments than we use in Theorem 2.4. (The results of [5, 31] related to Theorem 2.4 are stated in the Markovian setting, but their methods, similar to ours, work more generally.)

2.5 Number of excursions

Let N_t denote the number of excursions up until time t, i.e., $N_t := \max\{n \in \mathbb{N} : \tau_n \leq t\}$.

Theorem 2.5. *Suppose that (A0)–(A3) hold.*

(i) Suppose that $-1 < r \le 1$ and (2.2) holds with $p > 2$. Then for any $\varepsilon > 0$, a.s., for all but finitely many t,

$$t^{\frac{1+r}{2}} (\log t)^{-3-r-\varepsilon} \le N_t \le t^{\frac{1+r}{2}} (\log t)^{1+\varepsilon}. \tag{2.8}$$

(ii) Suppose that $r > 1$ and (2.2) holds with $p > 1 + r$. Then a.s., as $t \to \infty$, $t^{-1} N_t \to \frac{1}{\mathbb{E}[\eta_1]} \in (0, \infty)$.

2.6 Occupation times and stationary distribution

In this section $\mathbb{E}[\eta_1] < \infty$. Define for $t \in \mathbb{N}$ and $x \in \mathcal{S}$ the occupation times $L_t(x) := \sum_{s=1}^{t} \mathbf{1}\{X_s = x\}$. Also define the occupation times during the nth excursion by

$$\ell_n(x) := \sum_{t=\tau_{n-1}}^{\tau_n - 1} \mathbf{1}\{X_t = x\}. \tag{2.9}$$

The next result is essentially a consequence of 'ergodic theory' for regenerative processes. The limiting distribution π that appears in Theorem 2.6 is the usual (unique) stationary distribution if X is an irreducible positive-recurrent Markov process.

Theorem 2.6. *Suppose that (A0)–(A3) hold, $r > 1$, and (2.2) holds with $p > 1 + r$. Then setting*

$$\pi(x) := \frac{\mathbb{E}[\ell_1(x)]}{\mathbb{E}[\eta_1]}, \tag{2.10}$$

we have that $\pi(x) > 0$, $\sum_{x \in \mathcal{S}} \pi(x) = 1$, and, for any $x \in \mathcal{S}$, $t^{-1} L_t(x) \to \pi(x)$ a.s. and in L^q for any $q \ge 1$. Finally, if, in addition, the distribution of η_1 is not supported on $k\mathbb{N}$ for any $k > 1$, we have that, for any $x \in \mathcal{S}$, $\lim_{t \to \infty} \mathbb{P}[X_t = x] = \pi(x)$.

Remark 2.2. In the case of a Markov process with uniformly bounded increments, under assumptions otherwise similar to ours, results of Menshikov and Popov [35] show that, for $r > 1$, $\pi(x) = x^{-r+o(1)}$ as $x \to \infty$. The asymptotics of $\pi(x)$ are not of direct interest to the topic of the present paper, and so we do not discuss this further here, but our methods can be used to extend such results to the present more general setting.

2.7 Running maximum process

In this section we consider the process of maxima of X, i.e., $\max_{1 \le s \le t} X_s$.

Theorem 2.7. *Suppose that (A0)–(A3) hold.*

(i) Suppose that $-1 < r \le 1$ and (2.2) holds with $p > 2$. Then for any $\varepsilon > 0$, a.s., for all but finitely many t,

$$t^{\frac{1}{2}} (\log t)^{-\frac{4+r}{1+r} - \varepsilon} \le \max_{1 \le s \le t} X_s \le t^{\frac{1}{2}} (\log t)^{\frac{3}{1+r} + \varepsilon}.$$

(ii) Suppose that $r > 1$ and (2.2) holds with $p > 1 + r$. Then for any $\varepsilon > 0$, a.s., for all but finitely many t,

$$t^{\frac{1}{1+r}} (\log t)^{-\frac{1}{1+r} - \varepsilon} \le \max_{1 \le s \le t} X_s \le t^{\frac{1}{1+r}} (\log t)^{\frac{2}{1+r} + \varepsilon}.$$

Remarks 2.3. (a) Related bounds in a general Lamperti-type setting are given in [36, Section 4]; the excursion-based approach adopted here has both advantages and disadvantages compared to the method of [36]. The upper bounds in Section 4 of [36] essentially apply in the present setting (concretely, use [36, Theorem 3.2] with Lemma 4.1 here), and lead to slightly sharper upper bounds than those in our Theorem 2.7. (See also Section 6 of [8] for some variations on these upper bounds.) However, the lower bounds in [36] cannot readily be applied here, even assuming a uniform bound on the increments of X_t. Thus our lower bounds in Theorem 2.7 represent progress over previous results.

(b) Our excursion-based approach sheds no light on the transient case $r < -1$. For $r < -1$, under several additional assumptions, [36, Theorem 4.2] shows that there exists $D \in (0, \infty)$ such that a.s., for all but finitely many t, $X_t \geq t^{1/2}(\log t)^{-D}$. This result can be viewed as a generalization of the classical Dvoretzky–Erdős theorem on rate of escape of transient simple symmetric random walk in $\mathbb{Z}^d$ ($d \geq 3$) [13].

(c) In various special cases of certain nearest-neighbour random walks on $\mathbb{Z}^+ := \{0, 1, 2, \dots\}$, using methods restricted to the nearest-neighbour case, sharper versions of one or other of the bounds in Theorem 2.7(i) are given in [16, 18, 23, 37, 38, 40]; of these, only [23] also has a version of Theorem 2.7(ii).

2.8 Single-excursion sums

For $\alpha \geq 0$ and $n \in \mathbb{N}$ set

$$\xi_n^{(\alpha)} := \sum_{t=\tau_{n-1}}^{\tau_n - 1} X_t^\alpha = \sum_{x \in \mathcal{S}} x^\alpha \ell_n(x), \tag{2.11}$$

with the occupation time notation of (2.9); note that $\xi_n^{(0)} = \eta_1$. Our next result gives tail bounds for $\xi_1^{(\alpha)}$. Theorem 2.8 has applications in its own right: for example in [10, Theorem 2.1, p. 908] one is required to verify a condition similar to $\mathbb{E}[(\xi_1^{(\alpha)})^{2+\delta}] < \infty$.

Theorem 2.8. *Suppose that (A0)–(A3) hold. Suppose that $r > -1$ and (2.2) holds with $p > \max\{2, 1 + r\}$. Let $\alpha \geq 0$. Then for any $\varepsilon > 0$, for all x sufficiently large,*

$$x^{-\frac{1+r}{\alpha+2}}(\log x)^{-\varepsilon} \leq \mathbb{P}[\xi_1^{(\alpha)} \geq x] \leq x^{-\frac{1+r}{\alpha+2}}(\log x)^{\frac{2+2r}{\alpha+2}+1+\varepsilon}. \tag{2.12}$$

In particular, $\mathbb{E}\big[(\xi_1^{(\alpha)})^{\frac{1+r}{\alpha+2}}\big] = \infty$ *but, for any* $\varepsilon > 0$, $\mathbb{E}\big[(\xi_1^{(\alpha)})^{\frac{1+r}{\alpha+2}-\varepsilon}\big] < \infty$.

Remarks 2.4. (a) The $\alpha = 0$ case of Theorem 2.8 reduces to Theorem 2.4. Theorem 2.8 can also be seen as a generalization of Theorem 2.3, since here $\lim_{\alpha\to\infty}(\xi_1^{(\alpha)})^{1/\alpha} = M_1$, a.s., so for any x, $\mathbb{P}[\xi_1^{(\alpha)} \geq x^\alpha] \to \mathbb{P}[M_1 \geq x]$ as $\alpha \to \infty$.

(b) For simplicity we have stated our results for functionals based on $x \mapsto x^\alpha$, but our methods apply to any nonnegative nondecreasing function (cf Lemma 4.11 below).

2.9 Path integrals

Fix $\alpha \geq 0$ and define $S_t^{(\alpha)} := \sum_{s=1}^t X_s^\alpha$. We have the following asymptotic results on $S_t^{(\alpha)}$.

Theorem 2.9. *Suppose (A0)–(A3) hold, $r > -1$, and (2.2) holds with $p > \max\{2, 1+r\}$.*

(i) Suppose that $-1 < r \le 1$. Then for any $\varepsilon > 0$, a.s., for all but finitely many t,

$$t^{\frac{\alpha+2}{2}} (\log t)^{-\frac{(\alpha+2)(4+r)}{1+r} - \varepsilon} \le S_t^{(\alpha)} \le t^{\frac{\alpha+2}{2}} (\log t)^{\frac{3\alpha+6}{1+r} + 2 + \varepsilon}.$$

(ii) Suppose that $1 < r \le 1 + \alpha$. Then for any $\varepsilon > 0$, a.s., for all but finitely many t,

$$t^{\frac{\alpha+2}{1+r}} (\log t)^{-\frac{\alpha+2}{1+r} - \varepsilon} \le S_t^{(\alpha)} \le t^{\frac{\alpha+2}{1+r}} (\log t)^{\frac{2\alpha+4}{1+r} + 2 + \varepsilon}.$$

(iii) Suppose that $r > 1 + \alpha$. Then, with π as defined at (2.10), as $t \to \infty$, a.s.,

$$t^{-1} S_t^{(\alpha)} \to \frac{\mathbb{E}[\xi_1^{(\alpha)}]}{\mathbb{E}[\eta_1]} = \sum_{x \in \mathcal{S}} x^\alpha \pi(x) =: \nu_\alpha \in (0, \infty). \tag{2.13}$$

Theorem 2.9(iii) is essentially a consequence of 'ergodic theory' for regenerative processes (see e.g. [2, Theorem VI.3.1, p. 178]) but our proof of Theorem 2.9(i)–(ii) yields part (iii) at little additional effort, so we give the self-contained proof in Section 4.3.

A case of special interest is when $\alpha = 1$, in which case it is natural to study the normalized sum $t^{-1} S_t^{(1)}$ which is just the *centre of mass* of $(X_1, \ldots, X_t)$. Denote

$$G_t := t^{-1} S_t^{(1)} = t^{-1} \sum_{s=1}^{t} X_s. \tag{2.14}$$

Theorem 2.7 yields the following immediate corollary for G_t. For simplicity of presentation, we suppress the logarithmic factors in Theorem 2.7 by stating Corollary 2.10 parts (i) and (ii) on the logarithmic scale.

Corollary 2.10. *Suppose (A0)–(A3) hold, $r > -1$, and (2.2) holds with $p > \max\{2, 1 + r\}$.*

(i) Suppose that $-1 < r \le 1$. Then $\lim_{t\to\infty} \frac{\log G_t}{\log t} = \frac{1}{2}$, a.s.

(ii) Suppose that $1 < r \le 2$. Then $\lim_{t\to\infty} \frac{\log G_t}{\log t} = \frac{2-r}{1+r} \in [0, 1/2)$, a.s.

(iii) Suppose that $r > 2$. Then for $\nu_1 \in (0, \infty)$ given by (2.13), $\lim_{t\to\infty} G_t = \nu_1$, a.s.

Remark 2.5. Comparing the scaling exponents in Corollary 2.10 to those in Theorem 2.7, we see that they coincide (taking value $\frac{1}{2}$) in the null-recurrent case, but differ in the positive-recurrent case ($\frac{2-r}{1+r} < \frac{1}{1+r}$ for $r > 1$). The intuition here is that in the positive-recurrent case, the process rarely visits the scale of the maximum, so $G_t \ll \max_{1\le s\le t} X_s$.

3 Applications

3.1 Processes on the whole real line

In this section we give applications of our results from Section 2 on half-line processes to models defined on the whole line, for which new phenomena emerge. We restrict to the Markovian case for simplicity of statement. The $\mathbb{R}$-valued processes that we study are, loosely speaking, two half-line processes sewn together at 0.

(B0) Let $(X_t)_{t\in\mathbb{N}}$ be an irreducible, time-homogeneous Markov chain on $\mathcal{S}$, a locally finite subset of $\mathbb{R}$ with $0 \in \mathcal{S}$, $\inf \mathcal{S} = -\infty$, and $\sup \mathcal{S} = +\infty$. Take $X_1 = 0$.

(B1) Suppose that $\mathbb{P}[X_{t+1} = y \mid X_t = x] = 0$ if x and y are separated by 0. Suppose also that $\mathbb{P}[X_{t+1} < 0 \mid X_t = 0] \in (0, 1)$ and $\mathbb{P}[X_{t+1} > 0 \mid X_t = 0] \in (0, 1)$.

Under (B1), X_t cannot jump over the origin, and from the origin jumps left or right each with positive probability. As above, write $\Delta_t := X_{t+1} - X_t$ for the increments of X_t.

(B2) Suppose that for some $p > 2$ and $\delta > 0$, $\mathbb{E}[|\Delta_t|^p \mid X_t = x] = O(|x|^{p-2-\delta})$ as $|x| \to \infty$. Suppose also that for some $c_+, c_- \in \mathbb{R}$ and $s_+^2, s_-^2 \in (0, \infty)$,

$$\mathbb{E}[\Delta_t \mid X_t = x] = |x|^{-1} \left(c_+ \mathbf{1}\{x > 0\} - c_- \mathbf{1}\{x < 0\}\right) + o(|x|^{-1} \log^{-1} |x|), \quad (3.1)$$

$$\mathbb{E}[\Delta_t^2 \mid X_t = x] = \left(s_+^2 \mathbf{1}\{x > 0\} + s_-^2 \mathbf{1}\{x < 0\}\right) + o(\log^{-1} |x|). \quad (3.2)$$

Analogously to the definition of r at (2.5), set $r_\pm := -2c_\pm/s_\pm^2$. In this section we restrict to the setting in which $r_-, r_+ \in (-1, 1]$, i.e., corresponding to null-recurrence of each of the half-line processes. Cases where one or more of r_-, r_+ is greater than 1 can be dealt with using similar methods. We assume that $-1 < r_+ < r_- \leq 1$, so that the positive half-line is 'less recurrent'. The following result demonstrates the interesting phenomenon of a separation of scales for the two sides of the process.

Theorem 3.1. *Suppose that (B0)–(B2) hold, and that $-1 < r_+ < r_- \leq 1$. Then X_t is null-recurrent, and, a.s.,*

$$\lim_{t\to\infty} \frac{\log \max_{1\leq s\leq t} X_s}{\log t} = \frac{1}{2}, \quad \textit{and}$$

$$\lim_{t\to\infty} \frac{\log |\min_{1\leq s\leq t} X_s|}{\log t} = \frac{1}{2} \cdot \frac{1 + r_+}{1 + r_-} \in (0, 1/2).$$

As a concrete example, consider a nearest-neighbour random walk on $\mathbb{Z}$ which jumps as a symmetric simple random walk when on the nonnegative integers, but from $x < 0$ jumps to $x \pm 1$ with probabilities $\frac{1}{2} \pm \frac{1}{4x}$. Then $r_+ = 0$ and $r_- = 1$; viewed separately the two half-line process are null-recurrent and have the same (diffusive) scale, but the 'combined' process has scales $t^{1/2}$ on $[0, \infty)$ and $t^{1/4}$ on $(-\infty, 0]$.

The intuition behind Theorem 3.1 is that the walk makes a comparable number of positive and negative excursions, but the positive ones have heavier-tailed durations, so occupy a dominant proportion of time. The same intuition is behind the next result, which shows that the positive sojourns dominate the path-integral asymptotics. Again we use the notation (2.14), now for X_s taking values in $\mathbb{R}$.

Theorem 3.2. *Suppose that (B0)–(B2) hold, and $-1 < r_+ < r_- \leq 1$. Then, a.s., $G_t \to +\infty$ and*

$$\lim_{t\to\infty} \frac{\log G_t}{\log t} = \frac{1}{2}.$$

Remarks 3.1. (a) We leave largely open the case $r_+ = r_-$, but see the $d = 1$ case of the model in Section 3.2. (b) Similar results to those in this section can be obtained for processes on a state space that consists of multiple copies of $[0, \infty)$, joined at a common origin, and embedded in $\mathbb{R}^d$.

3.2 Centrally biased random walks on $\mathbb{R}^d$

In this section we work in $\mathbb{R}^d$, $d \in \mathbb{N}$. For $\mathbf{x} \in \mathbb{R}^d$, write $\mathbf{x} = (x_1, \ldots, x_d)$ in Cartesian coordinates. Let $\| \cdot \|$ denote the Euclidean norm on $\mathbb{R}^d$. For a non-zero vector $\mathbf{x} \in \mathbb{R}^d$ we write $\hat{\mathbf{x}} := \mathbf{x}/\|\mathbf{x}\|$ for the corresponding unit vector. Write $\mathbf{0} := (0, \ldots, 0)$ for the origin.

(C0) Let $\Xi = (\xi_t)_{t \in \mathbb{N}}$ be an irreducible, time-homogeneous Markov process whose state space Σ is an unbounded, locally finite subset of $\mathbb{R}^d$ containing $\mathbf{0}$. Let $\xi_1 = \mathbf{0}$.

We use the notation $\theta_t := \xi_{t+1} - \xi_t$ for the increments of the walk. The assumption (C0) implies that the distribution of θ_t depends only on the position $\xi_t \in \Sigma$ and not on t. We assume that for some $p > 2$, $\delta > 0$, and $C < \infty$,

$$\mathbb{E}[\|\theta_t\|^p \mid \xi_t = \mathbf{x}] \leq C(1 + \|\mathbf{x}\|)^{p-2-\delta}. \tag{3.3}$$

Denote the one-step *mean drift* vector $\mu(\mathbf{x}) := \mathbb{E}[\theta_t \mid \xi_t = \mathbf{x}]$ for $\mathbf{x} \in \Sigma$, and denote the covariance matrix at $\mathbf{x} \in \Sigma$ by $M(\mathbf{x}) := (M_{ij}(\mathbf{x}))_{i,j} := \mathbb{E}[\theta_t^\top \theta_t \mid \xi_t = \mathbf{x}]$, for $\mathbf{x} \in \Sigma$, where θ_t is viewed as a row-vector. In vector equations such as the equation for $\mu(\mathbf{x})$ in the following assumption, an expression of the form $o(h(\|\mathbf{x}\|))$ is to be interpreted as a vector whose components are each $o(h(\|\mathbf{x}\|))$ as $\|\mathbf{x}\| \to \infty$, uniformly in $\mathbf{x}$ given $\|\mathbf{x}\|$.

(C1) Suppose that there exist $\rho \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$ for which, as $\|\mathbf{x}\| \to \infty$,

$$\begin{aligned} \mu(\mathbf{x}) &= \rho \hat{\mathbf{x}} \|\mathbf{x}\|^{-1} + o(\|\mathbf{x}\|^{-1} \log^{-1} \|\mathbf{x}\|), \\ M_{ij}(\mathbf{x}) &= \sigma^2 \mathbf{1}\{i = j\} + o(\log^{-1} \|\mathbf{x}\|). \end{aligned}$$

The assumption on M in (C1) implies that ξ_t has an asymptotically diagonal covariance structure. Processes satisfying (C0) and (C1) were studied by Lamperti [30, 31] under the name *centrally biased random walks*, due to the nature of the drift field; the name had been used earlier by Gillis [20] for a different model. Our main result on such models is the following, which will enable us to apply the results of Section 2 to generalize and sharpen Lamperti's results, among other things.

Theorem 3.3. *Suppose that (C0) and (C1) hold, and (3.3) holds for some $p > 2$. Let $X_t = \|\xi_t\|$. Then X_t satisfies the conditions (A0)–(A3), with*

$$c = \rho + (d-1)(\sigma^2/2), \quad s^2 = \sigma^2;$$

hence $r = 1 - d - (2\rho/\sigma^2)$. Moreover, (2.2) holds for the given $p > 2$.

From Theorem 3.3, we immediately deduce a series of results for Ξ from the theorems in Section 2. We state two such corollaries. Note that if we set $\eta := \min\{t \in \mathbb{N} : \xi_t = \mathbf{0}\}$, we have from (C0) that $\eta = \eta_1$ for $X_t = \|\xi_t\|$ in our previous notation, since $X_t = 0$ if and only if $\xi_t = \mathbf{0}$. Theorem 3.3 with Theorems 2.2 and 2.4 gives the following result.

Corollary 3.4. *Suppose that (C0) and (C1) hold, and (3.3) holds with $p > 2$. Then Ξ is*

(i) transient if $2\rho/\sigma^2 > 2 - d$;

(ii) null-recurrent if $-d \leq 2\rho/\sigma^2 \leq 2 - d$;

(iii) positive-recurrent if $2\rho/\sigma^2 < -d$.

Moreover, in the recurrent cases, $\mathbb{E}[\eta^q] < \infty$ if and only if $q < q_0 := 1 - (d/2) - (\rho/\sigma^2)$.

Corollary 3.4 extends results of Lamperti [30, 31], who assumed uniformly bounded increments for ξ_t and a stronger version of (B1) with the error term $\log^{-1} \|\mathbf{x}\|$ replaced by $\|\mathbf{x}\|^{-\delta}$ for $\delta > 0$: see Theorem 4.1 of [30, p. 324] and Theorem 5.1 of [31, p. 142]. Also, Lamperti's result only covers *integer* q, and is not sharp enough to determine whether $\mathbb{E}[\eta^{q_0}]$ is finite or infinite, and so cannot decide on null- or positive-recurrence at the boundary case $2\rho/\sigma^2 = -d$.

The next result follows from Theorem 3.3 with Theorem 2.7, and gives almost-sure scaling behaviour for the maximum of $\|\xi_t\|$ in the recurrent cases.

Corollary 3.5. *Suppose that (C0) and (C1) hold.*

(i) Suppose that $-d \leq 2\rho/\sigma^2 < 2 - d$ and (3.3) holds with $p > 2$. Then

$$\lim_{t\to\infty} \frac{\log \max_{1\leq s\leq t} \|\xi_s\|}{\log t} = \frac{1}{2}, \quad a.s.$$

(ii) Suppose that $2\rho/\sigma^2 < -d$ and (3.3) holds with $p > 2 - d - (2\rho/\sigma^2)$. Then

$$\lim_{t\to\infty} \frac{\log \max_{1\leq s\leq t} \|\xi_s\|}{\log t} = \frac{1}{2 - d - (2\rho/\sigma^2)}, \quad a.s.$$

Upper bounds similar to those in Corollary 3.5 can be derived from [36, Section 3]: see Theorem 2.4 of [8] for a similar application of such results, albeit under more restrictive assumptions. As far as the authors are aware, the lower bounds in Corollary 3.5 are new.

3.3 The simple harmonic urn

In this section we study a particular Markov chain (A_t, B_t) on $\mathbb{Z}^2 \setminus \{(0,0)\}$, with discrete time $t \in \mathbb{N}$. The model was introduced in [9], motivated by an urn model. The model takes as input the distribution of a $\mathbb{Z}$-valued random variable κ. We assume that, for some $\lambda > 0$, $\mathbb{E}[\mathrm{e}^{\lambda|\kappa|}] < \infty$. Let $\kappa_0, \kappa_1, \ldots$ be a sequence of independent copies of κ. The transition law of the chain is as follows. If $A_t B_t \neq 0$, i.e., the chain is not on one of the coordinate axes, it takes jumps of unit size according to the following:

$$\mathbb{P}\left[(A_{t+1}, B_{t+1}) = (a, b + \operatorname{sgn}(a)) \mid (A_t, B_t) = (a, b)\right] = \frac{|a|}{|a| + |b|}, \ (ab \neq 0);$$
$$\mathbb{P}\left[(A_{t+1}, B_{t+1}) = (a - \operatorname{sgn}(b), b) \mid (A_t, B_t) = (a, b)\right] = \frac{|b|}{|a| + |b|}, \ (ab \neq 0),$$

where $\operatorname{sgn}(x) := x/|x|$ for $x \in \mathbb{R} \setminus \{0\}$. From one of the axes, the process jumps as follows:

$$(A_{t+1}, B_{t+1}) = (\operatorname{sgn}(A_t) \max\{1, |A_t| - \kappa_t\}, \operatorname{sgn}(A_t)), \ (A_t \neq 0, B_t = 0);$$
$$(A_{t+1}, B_{t+1}) = (-\operatorname{sgn}(B_t), \operatorname{sgn}(B_t) \max\{1, |B_t| - \kappa_t\}), \ (A_t = 0, B_t \neq 0).$$

In words, the process has an approximately anti-clockwise trajectory, traversing each quadrant in sequence. When not on an axis, the process traverses the current quadrant using unit steps in two possible directions, while from an axis, the process moves one step away from the axis (in the anti-clockwise direction) and makes a special jump of size distributed as κ towards the next destination axis, truncating so as to ensure it does not actually reach the next axis in this jump.

So defined, (A_t, B_t) is an irreducible Markov chain on $\mathbb{Z}^2 \setminus \{(0,0)\}$.

The basic case has $\kappa = 0$ a.s., in which case the process is the *simple harmonic urn*; another particular case has $\kappa = 1$ a.s., which is known as the *leaky urn* [9]. The general κ model is known as the *noisy urn* [9]. In fact, the leaky urn in [9] was defined slightly differently, with an absorbing state when $|A_t| + |B_t| = 1$, but the two definitions coincide up until the time of absorption.

Let $\nu_0 := 0$ and, for $n \in \mathbb{N}$, $\nu_n := \min\{t > \nu_{n-1} : A_t B_t = 0\}$, so that $\nu_1, \nu_2, \dots$ are the successive times of visits to the axis by the process (A_t, B_t). Define the embedded process $Z_t := |A_{\nu_t}| + |B_{\nu_t}|$ for $t \in \mathbb{N}$; by construction, exactly one of $|A_{\nu_t}|$ and $|B_{\nu_t}|$ is 0. Then Z_t is an irreducible Markov chain on $\mathbb{N}$, representing the distance of the original Markov chain from the origin at those times when it visits an axis. For definiteness, we take $(A_1, B_1) = (1, 0)$, so $Z_1 = 1$.

The following result shows the connection between this model and our present setting.

Proposition 3.6. *Let $X_t = \sqrt{Z_t - 1}$. Then (A0)–(A3) hold with $\mathcal{S} = \{\sqrt{x-1} : x \in \mathbb{N}\}$, $c = \frac{1-2\mathbb{E}[\kappa]}{4}$, and $s^2 = \frac{1}{6}$; hence $r = 6\mathbb{E}[\kappa] - 3$. In addition, (2.2) holds for any $p > 0$.*

Proposition 3.6 is closely related to Lemma 7.7 in [9], but differs slightly as our embedded process Z_t is not quite the same as the one used in [9], so we sketch the proof in Section 5.3 below. For the original process, we are interested in $\tau := \min\{t \in \mathbb{N} : |A_t| + |B_t| = 1\}$. For our embedded process, define $\tau_q := \min\{t \in \mathbb{N} : Z_t = 1\}$ (where the 'q' indicates 'quadrant time'). The key relationship between the two processes is that $\tau = \nu_{\tau_q}$, since $|A_t| + |B_t| = 1$ if and only if $t = \nu_k$ for some k and $Z_k = 1$. In [9], a slightly different version of the embedded process Z_t (namely, $\tilde{Z}_k$ defined on p. 2125 of [9]) was used; for that version the analogous claim to '$\tau = \nu_{\tau_q}$' made just below equation (6) in [9] is not correct as stated, although this has no impact on the results in [9]. It is not hard to fix this small gap in the argument in [9], and the variation given in the present paper is just one way of doing so. More importantly, the results of the present paper enable us to sharpen the results in [9] and to settle a conjecture made in that paper.

Proposition 3.6 enables us to determine the tails of τ_q; some additional work is needed to account for the change of time between (A_t, B_t) and X_t and hence study the tails of $\tau = \nu_{\tau_q}$. Due to the special structure of the paths of the simple harmonic urn process, it turns out that exactly relevant to this point is an excursion sum of the type $\xi_1^{(2)}$ defined by (2.11). We prove the following result in Section 5.3. The condition $\mathbb{E}[\kappa] > \frac{1}{3}$ corresponds to $r > -1$, in which case the process is *recurrent*.

Theorem 3.7. *Suppose that $\mathbb{E}[\kappa] > \frac{1}{3}$. Let $p \geq 0$. Then $\mathbb{E}[\tau^p] < \infty$ if and only if $p < \frac{3\mathbb{E}[\kappa]-1}{2}$. In particular, the Markov chain (A_t, B_t) is null-recurrent when $\mathbb{E}[\kappa] = 1$.*

This result shows that $\mathbb{E}[\tau^p] = \infty$ for $p = \frac{3\mathbb{E}[\kappa]-1}{2}$, the boundary case not covered by Theorem 2.6 of [9]; the fact that the process is *null*-recurrent when $\mathbb{E}[\kappa] = 1$ confirms the conjecture after Corollary 2.7 in [9]. Theorem 3.7 has the following immediate corollary, which plugs the gap in leaky urn result Theorem 2.3 of [9].

Corollary 3.8. *For the leaky urn, the time to absorption is non-integrable.*

3.4 Random walk models of polymers and interfaces

The last decade or so has seen renewed interest in one-dimensional random walks with asymptotically zero drifts from a statistical physics perspective, concerning models of random polymers and interfaces, their structure, and their interactions with a medium or boundary. In the context of random polymers, the path of the process models the physical polymer chain; the asymptotically zero drift indicates the presence of *long-range interaction* with a boundary, which can be either attractive or repulsive. For a random interface, the walk models the behaviour of a liquid interface on a solid substrate (including *wetting* and *pinning* phenomena); in this context the drift may represent *affinity* for the boundary. We refer to [19, 22, 39] for recent surveys.

Much of the existing work is restricted to nearest-neighbour random walks on $\mathbb{Z}^+$, where explicit calculations are facilitated by reversibility and associated algebraic structure (such as Karlin–McGregor theory [26]); see e.g. [1, 12, 23] for models inspired directly by random polymers, and e.g. [11, 18, 40] for related work. In this section we make some brief remarks emphasizing how the present paper adds to this literature, and in particular how our results can be used to study quantities of interest in this context for a much more general class of processes; our results not only do not require the nearest-neighbour assumption, but do not need bounded jumps or even the Markov property *per se*.

A typical family of nearest-neighbour random walks X_t on $\mathbb{Z}^+$ that has been extensively studied has $\mathbb{P}[X_{t+1} = X_t \pm 1 \mid X_t = x] = \frac{1}{2} \mp \frac{\delta}{4x+2\delta}$ for $x > 0$ and a parameter δ; here (A2) holds with $c = -\delta/2$ and $s^2 = 1$, so $r = \delta$. This and closely related models were considered by Karlin and McGregor [26], and by many subsequent authors, including for instance [14–16, 18, 37, 38, 40] and, most recently, [12] and [23]. In these very special cases, Fal' [14] gives asymptotics for excursion times and the number of excursions (cf our Theorem 2.5), while several authors [16, 18, 37, 38, 40] give iterated-logarithm type upper bounds in the diffusive case (cf our Theorem 2.7(i)). Huillet [23] gives sharper versions of our Theorems 2.3, 2.4, and 2.7 in this special case: see Propositions 2, 9, 10, and 11 of [23]. The main result of [12] (see also Proposition 15 of [23]) is that, for $\delta \in (1, 2)$, $\mathbb{E}[X_t] \sim K_\delta t^{1-\frac{\delta}{2}}$, being one possible measure of the spatial extent of the polymer. Perhaps more natural (certainly more readily interpreted in terms of path properties) are the quantities $\max_{1 \le s \le t} X_s$ and $t^{-1} \sum_{s=1}^{t} X_s$ that we study in the present paper; their scaling exponents for the case $\delta \in (1, 2)$ are $\frac{1}{1+\delta}$ (our Theorem 2.7, or Proposition 10 of [23]) and $\frac{2-\delta}{1+\delta}$ (our Corollary 2.10) respectively. Note that for $\delta \in (1, 2)$, $\frac{1}{1+\delta} > 1 - \frac{\delta}{2} > \frac{2-\delta}{1+\delta}$. Alexander [1] calls such nearest-neighbour random walks with drift $O(1/x)$ at x 'Bessel-like', and gives sharp results on the asymptotics of return times, among other things. There seems to have as yet been no success in applying the methods of [1, 12, 23] beyond the nearest-neighbour setting.

4 Proofs of main results

4.1 Lyapunov functions

For $\gamma, \nu \in \mathbb{R}$ we define the function $f_{\gamma,\nu} : [0, \infty) \to [0, \infty)$ by

$$f_{\gamma,\nu}(x) := (\mathrm{e} + x)^\gamma \log^\nu(\mathrm{e} + x).$$

Our basic analytical method will be built on the fact that $f_{\gamma,\nu}(X_t)$ is a submartingale or supermartingale, for X_t outside some bounded set, for appropriate γ and ν. The following result is fundamental. The idea here is not new, although the particular form of the result is a little different from previous versions in the literature. Recall that in expressions such as (4.1), the $o(1)$ term is uniform in t and ω.

Lemma 4.1. *Suppose that (A0) and (A2) hold, $r > -1$, and (2.2) holds for some $p > \max\{2, 1+r\}$. Then for any $\nu \in \mathbb{R}$, as $X_t \to \infty$, a.s.,*

$$\mathbb{E}[f_{1+r,\nu}(X_{t+1}) - f_{1+r,\nu}(X_t) \mid \mathcal{F}_t] = \big(\nu(1+r)(s^2/2) + o(1)\big) X_t^{r-1} \log^{\nu-1} X_t. \tag{4.1}$$

In particular, for any $\nu > 0$, there exists $A < \infty$ such that, on $\{X_t \geq A\}$, a.s.,

$$\mathbb{E}[f_{1+r,\nu}(X_{t+1}) - f_{1+r,\nu}(X_t) \mid \mathcal{F}_t] \geq 0;$$
$$\mathbb{E}[f_{1+r,-\nu}(X_{t+1}) - f_{1+r,-\nu}(X_t) \mid \mathcal{F}_t] \leq 0.$$

Before proving Lemma 4.1, we state a technical result. For $\varepsilon \in (0,1)$, let $E_\varepsilon(t)$ denote the event $\{|\Delta_t| \leq (1+X_t)^{1-\varepsilon}\}$. Denote the complementary event by $E_\varepsilon^{\rm c}(t)$.

Lemma 4.2. *Suppose that (2.2) holds with $p > 2$ and $\delta > 0$. Then for some $C \in (0,\infty)$ and any $\varepsilon \in (0, \frac{\delta}{1+p})$, for any $q \in [0,p]$, $\mathbb{E}[|\Delta_t|^q \mathbf{1}_{E_\varepsilon^{\rm c}(t)} \mid \mathcal{F}_t] \leq C(1+X_t)^{q-2-\varepsilon}$, a.s.*

Proof. For $q \in [0,p]$, $|\Delta_t|^q \mathbf{1}_{E_\varepsilon^{\rm c}(t)} = |\Delta_t|^p |\Delta_t|^{q-p} \mathbf{1}_{E_\varepsilon^{\rm c}(t)} \leq |\Delta_t|^p (1+X_t)^{(1-\varepsilon)(q-p)}$, by definition of $E_\varepsilon(t)$. Taking expectations and using (2.2) we obtain

$$\mathbb{E}[|\Delta_t|^q \mathbf{1}_{E_\varepsilon^{\rm c}(t)} \mid \mathcal{F}_t] \leq C(1+X_t)^{p-2-\delta+(1-\varepsilon)(q-p)},$$

and the result follows. □

Proof of Lemma 4.1. Let $\gamma \geq 0$ and $\nu \in \mathbb{R}$. Take $\varepsilon \in (0,1)$. We estimate the expected increment of $f_{\gamma,\nu}(X_t)$ using a Taylor expansion on $E_\varepsilon(t)$, while we use Lemma 4.2 to control the expectation on $E_\varepsilon^{\rm c}(t)$. Differentiation of $f_{\gamma,\nu}$ with respect to x gives

$$f'_{\gamma,\nu}(x) = \gamma f_{\gamma-1,\nu}(x) + \nu f_{\gamma-1,\nu-1}(x);$$
$$f''_{\gamma,\nu}(x) = \gamma(\gamma-1) f_{\gamma-2,\nu}(x) + \nu(2\gamma-1) f_{\gamma-2,\nu-1}(x) + \nu(\nu-1) f_{\gamma-2,\nu-2}(x);$$

and $f'''_{\gamma,\nu}(x) = O(x^{\gamma-3}\log^\nu x)$. Thus Taylor's formula implies that

$$\begin{aligned}
&\left(f_{\gamma,\nu}(X_t+\Delta_t) - f_{\gamma,\nu}(X_t)\right) \mathbf{1}_{E_\varepsilon(t)} \\
&= \Delta_t \mathbf{1}_{E_\varepsilon(t)} f_{\gamma-1,\nu-1}(X_t)(\gamma \log(\mathrm{e}+X_t) + \nu) \\
&\quad + \frac{\Delta_t^2 \mathbf{1}_{E_\varepsilon(t)}}{2} f_{\gamma-2,\nu-2}(X_t) \left(\gamma(\gamma-1)\log^2(\mathrm{e}+X_t) + \nu(2\gamma-1)\log(\mathrm{e}+X_t) + \nu(\nu-1)\right) \\
&\quad + O(|\Delta_t|^3 \mathbf{1}_{E_\varepsilon(t)} X_t^{\gamma-3} \log^\nu X_t),
\end{aligned} \tag{4.2}$$

as $X_t \to \infty$. Since $|\Delta_t| \mathbf{1}_{E_\varepsilon(t)} = O(X_t^{1-\varepsilon})$, here we have that

$$\mathbb{E}[|\Delta_t|^3 \mathbf{1}_{E_\varepsilon(t)} X_t^{\gamma-3} \log^\nu X_t \mid \mathcal{F}_t] \leq \mathbb{E}[|\Delta_t|^2 \mid \mathcal{F}_t] O(X_t^{\gamma-2-\varepsilon}\log^\nu X_t) = O(X_t^{\gamma-2-(\varepsilon/2)}), \text{ a.s.,}$$

by (2.4). On the other hand, since (2.2) holds for some $p > 2$, we have from the $q \in \{1,2\}$ cases of Lemma 4.2 that, for $\varepsilon > 0$ small enough, a.s.,

$$\mathbb{E}[\Delta_t \mathbf{1}_{E_\varepsilon(t)} \mid \mathcal{F}_t] = \mathbb{E}[\Delta_t \mid \mathcal{F}_t] + O(X_t^{-1-\varepsilon});$$
$$\mathbb{E}[\Delta_t^2 \mathbf{1}_{E_\varepsilon(t)} \mid \mathcal{F}_t] = \mathbb{E}[\Delta_t^2 \mid \mathcal{F}_t] + O(X_t^{-\varepsilon}).$$

Taking expectations in (4.2) and using (2.3) and (2.4) we obtain, for $\varepsilon > 0$ small enough,

$$\begin{aligned} &\mathbb{E}[(f_{\gamma,\nu}(X_t+\Delta_t) - f_{\gamma,\nu}(X_t))\,\mathbf{1}_{E_\varepsilon(t)} \mid \mathcal{F}_t] \\ &= \gamma\left(c + \frac{(\gamma-1)s^2}{2}\right) X_t^{\gamma-2}\log^\nu X_t + \nu\left(c + \frac{(2\gamma-1)s^2}{2} + o(1)\right) X_t^{\gamma-2}\log^{\nu-1} X_t, \end{aligned} \tag{4.3}$$

as $X_t \to \infty$. On the other hand, for any $\varepsilon' > 0$ there exists $C < \infty$ for which

$$|f_{\gamma,\nu}(X_t+\Delta_t) - f_{\gamma,\nu}(X_t)| \le C(1+X_t)^{\gamma+\varepsilon'} + C|\Delta_t|^{\gamma+\varepsilon'}.$$

Hence $\mathbb{E}[|f_{\gamma,\nu}(X_t+\Delta_t) - f_{\gamma,\nu}(X_t)|\,\mathbf{1}_{E^c_\varepsilon(t)} \mid \mathcal{F}_t]$ is bounded above by

$$C(1+X_t)^{\gamma+\varepsilon'}\mathbb{P}[E^c_\varepsilon(t) \mid \mathcal{F}_t] + C\mathbb{E}[|\Delta_t|^{\gamma+\varepsilon'}\mathbf{1}_{E^c_\varepsilon(t)} \mid \mathcal{F}_t].$$

For $\varepsilon > 0$ small enough, both terms on the right-hand side here are $O(X_t^{\gamma+\varepsilon'-2-\varepsilon})$, by the $q = 0$ and $q = \gamma+\varepsilon'$ cases of Lemma 4.2 respectively, the latter case being applicable provided (2.2) holds for $p > \gamma$ and taking $\varepsilon' \in (0, p-\gamma)$. Taking ε' small enough ($\varepsilon' < \varepsilon/2$, say) and combining this last estimate with (4.3) we obtain, as $X_t \to \infty$,

$$\begin{aligned} &\mathbb{E}[f_{\gamma,\nu}(X_t+\Delta_t) - f_{\gamma,\nu}(X_t) \mid \mathcal{F}_t] \\ &= \gamma\left(c + \frac{(\gamma-1)s^2}{2}\right) X_t^{\gamma-2}\log^\nu X_t + \nu\left(c + \frac{(2\gamma-1)s^2}{2} + o(1)\right) X_t^{\gamma-2}\log^{\nu-1} X_t, \end{aligned} \tag{4.4}$$

provided (2.2) holds for $p > \gamma$. With the choice $\gamma = 1 + r = 1 - (2c/s^2)$, (4.4) implies (4.1) since $(c + (2\gamma-1)(s^2/2)) = (1+r)s^2/2$ for this choice of γ. Since $r > -1$ and $s^2 > 0$, the right-hand side of (4.1) has the same sign as ν, for all X_t large enough, and the conclusion of the lemma follows. □

4.2 Technical lemmas

We need some results on maxima and sums of i.i.d. random variables.

Lemma 4.3. *Let $\zeta_1, \zeta_2, \ldots$ be i.i.d. $\mathbb{R}$-valued random variables.*

(i) Suppose that, for some $\theta \in (0,\infty)$ and $\phi \in \mathbb{R}$,

$$\limsup_{x\to\infty}(x^\theta(\log x)^{-\phi}\mathbb{P}[\zeta_1 \ge x]) < \infty. \tag{4.5}$$

For any $\varepsilon > 0$, a.s., for all but finitely many n, $\max_{1\le i\le n}\zeta_i \le n^{\frac{1}{\theta}}(\log n)^{\frac{\phi+1}{\theta}+\varepsilon}$.

(ii) Suppose that, for some $\theta \in (0,\infty)$ and $\phi \in \mathbb{R}$,

$$\liminf_{x\to\infty}(x^\theta(\log x)^{-\phi}\mathbb{P}[\zeta_1 \ge x]) > 0. \tag{4.6}$$

For any $\varepsilon > 0$, a.s., for all but finitely many n, $\max_{1\le i\le n}\zeta_i \ge n^{\frac{1}{\theta}}(\log n)^{\frac{\phi-1}{\theta}-\varepsilon}$.

Proof. First we prove part (i). From (4.5), for some $C \in (0,\infty)$ and all x large enough,

$$\mathbb{P}\Big[\max_{1\le i\le n}\zeta_i \le x\Big] = \prod_{i=1}^n \mathbb{P}[\zeta_i \le x] \ge \left(1 - Cx^{-\theta}(\log x)^\phi\right)^n.$$

Set $x = n^{1/\theta}(\log n)^q$ for some $q \in \mathbb{R}$. Then, for $C' \in (0, \infty)$,

$$\begin{aligned} p(n) := \mathbb{P}\Big[\max_{1 \le i \le n} \zeta_i \ge n^{1/\theta}(\log n)^q \Big] &\le 1 - \big(1 - C' n^{-1}(\log n)^{\phi - \theta q}(1 + o(1))\big)^n \\ &= O(1 \wedge (\log n)^{\phi - \theta q}). \end{aligned}$$

Take $q > (\phi + 1)/\theta$. Then $\sum_{k \in \mathbb{N}} p(2^k) < \infty$. Hence the Borel–Cantelli lemma implies that a.s., for all but finitely many $k \in \mathbb{N}$, $\max_{1 \le i \le 2^k} \zeta_i \le (2^k)^{1/\theta}(\log 2^k)^q$. For any $n \ge 2$, $2^{k_n} \le n < 2^{k_n+1}$ for some $k_n \in \mathbb{N}$; hence, a.s., for all but finitely many $n \in \mathbb{N}$,

$$\max_{1 \le i \le n} \zeta_i \le \max_{1 \le i \le 2^{k_n+1}} \zeta_i \le (2^{k_n+1})^{1/\theta}(\log 2^{k_n+1})^q \le C n^{1/\theta}(\log n)^q,$$

where $C < \infty$ does not depend on n. Thus we obtain part (i).

Now we prove part (ii). We have from (4.6) that for some $c > 0$ and all x large enough, $\mathbb{P}[\zeta_1 \ge x] \ge c x^{-\theta}(\log x)^\phi$, so that $\mathbb{P}\left[\max_{1 \le i \le n} \zeta_i < x\right] \le \big(1 - c x^{-\theta}(\log x)^\phi\big)^n$. Taking $x = n^{1/\theta}(\log n)^q$ we obtain

$$\begin{aligned} \mathbb{P}\Big[\max_{1 \le i \le n} \zeta_i < n^{1/\theta}(\log n)^q \Big] &\le \big(1 - c n^{-1}(\log n)^{\phi - \theta q}(1 + o(1))\big)^n \\ &\le \exp\big(-c(\log n)^{\phi - \theta q}(1 + o(1))\big), \end{aligned}$$

which is summable over $n \ge 2$ if $q < (\phi - 1)/\theta$; now use the Borel–Cantelli lemma. □

The next result deals with sums of i.i.d. nonnegative random variables.

Lemma 4.4. *Let $\zeta_1, \zeta_2, \ldots$ be i.i.d. $[0, \infty)$-valued random variables.*

(i) If for some $\theta \in (0, 1)$ and $\phi \in \mathbb{R}$, (4.5) holds, then, for any $\varepsilon > 0$, a.s., for all but finitely many n, $\sum_{i=1}^n \zeta_i \le n^{\frac{1}{\theta}}(\log n)^{\frac{\phi+1}{\theta} + \varepsilon}$.

(ii) If for some $\theta \in (0, \infty)$ and $\phi \in \mathbb{R}$, (4.6) holds, then, for any $\varepsilon > 0$, a.s., for all but finitely many n, $\sum_{i=1}^n \zeta_i \ge n^{\frac{1}{\theta}}(\log n)^{\frac{\phi-1}{\theta} - \varepsilon}$.

Proof. Part (i) is part of a family of classical results related to the Marcinkiewicz–Zygmund strong laws of large numbers (see e.g. [25, p. 73]): it follows from a result of Feller [17, Theorem 2] (see also [32, p. 253] for a more general result). Part (ii) is a consequence of Lemma 4.3(ii) and the elementary bound $\sum_{i=1}^n \zeta_i \ge \max_{1 \le i \le n} \zeta_i$. □

Next we move on to some basic consequences of (A0) and (A1). Here 'i.o.' and 'f.o.' stand for 'infinitely often' and 'finitely often', respectively.

Lemma 4.5. *Suppose that (A0) and (A1) hold. Let $R, S \subset \mathcal{S}$ be finite and non-empty. Then $\{X_t \in R \text{ i.o.}\} = \{X_t \in S \text{ i.o.}\}$ up to sets of probability 0. Moreover, for any (hence every) finite, non-empty $R \subset \mathcal{S}$, the following equalities hold up to sets of probability 0:*

$$\begin{aligned} \{X_t \in R \text{ i.o.}\} &= \Big\{ \liminf_{t \to \infty} X_t = 0, \limsup_{t \to \infty} X_t = \infty \Big\}, \\ \{X_t \in R \text{ f.o.}\} &= \Big\{ \lim_{t \to \infty} X_t = \infty \Big\}. \end{aligned}$$

In particular, $\mathbb{P}\left[\{X_t \to \infty\} \cup \left\{ \liminf_{t \to \infty} X_t = 0, \limsup_{t \to \infty} X_t = \infty \right\}\right] = 1$.

Proof. Let $R, S \subset \mathcal{S}$ be finite and non-empty. Suppose that $X_t \in R$ i.o. Then, since R is finite, there exist $x \in R$, $y \in S$ (any such x, y will do) and stopping times $t_1 < t_2 < \cdots$ with $t_{i+1} > t_i + m(x,y)$ such that $X_{t_i} = x$ and $\mathbb{P}[X_{t_i+m(x,y)} = y \mid \mathcal{F}_{t_i}] \geq \varphi(x,y) > 0$ for all i, by (2.1). Then Lévy's extension of the Borel–Cantelli lemma (see e.g. [25, p. 131]) implies that $X_t = y$ i.o., a.s., giving the first statement in the lemma. Hence, a.s., either $X_t \in R$ i.o. for *all* finite non-empty $R \subset \mathcal{S}$ (including $R = \{0\}$), or for *none*. It follows that $\liminf_{t\to\infty} X_t \in \{0, \infty\}$ a.s., and the same fact also implies that $\limsup_{t\to\infty} X_t = \infty$ a.s. □

The next result says, roughly speaking, that uniformly for sites x in some interval, there is positive probability that, starting from that interval, the process hits x before leaving some larger interval. We use the notation $\mathcal{S}_x := \mathcal{S} \cap [0, x]$ for $x \geq 0$,

$$\tau_{x,t} := \min\{s \geq 0 : X_{t+s} = x\}, \text{ and } \sigma_{x,t} := \min\{s \geq 0 : X_{t+s} > x\}.$$

Lemma 4.6. *Suppose that (A0) and (A1) hold and that for some $C < \infty$, $\mathbb{E}[\Delta_t \mid \mathcal{F}_t] \leq C$, a.s., for all $t \in \mathbb{N}$. Let $A < \infty$. There exist $\varphi = \varphi(A) > 0$ and $B = B(A, C) > A$ such that for any $x \in \mathcal{S}_A$, on $\{X_t \leq A\}$, $\mathbb{P}[\tau_{x,t} < \sigma_{B,t} \mid \mathcal{F}_t] \geq \varphi$, a.s., for all $t \in \mathbb{N}$.*

Proof. From (2.1), writing $m = \max_{x,y\in\mathcal{S}_A} m(x,y)$ and $\varphi = \min_{x,y\in\mathcal{S}_A} \varphi(x,y)$ we have by (A0) and (A1) that $m < \infty$ and $\varphi > 0$ (depending on A), and, moreover, on $\{X_t \leq A\}$, for any $x \in \mathcal{S}_A$, $\mathbb{P}[\tau_{x,t} \leq m \mid \mathcal{F}_t] \geq \varphi$, a.s. In addition, by an appropriate maximal inequality [36, Lemma 3.1] and the first moment bound in the lemma, on $\{X_t \leq A\}$,

$$\mathbb{P}[\sigma_{hm,t} \leq m \mid \mathcal{F}_t] = \mathbb{P}\Big[\max_{0\leq s\leq m} X_{t+s} > hm \mid \mathcal{F}_t\Big] \leq \frac{Cm + A}{hm} \leq \frac{\varphi}{2},$$

choosing h sufficiently large (depending on A and C). Combining the two probability bounds we obtain the statement in the lemma, after a relabelling of $\varphi/2$ as φ. □

Recall that τ_n is the time of the nth return to 0 by X, and recall that N, as defined just before (A3), is the first n for which $\tau_n = \infty$.

Lemma 4.7. *Suppose that (A0), (A1), and (A3) hold, and $N = \infty$ a.s. Then for any $y \in \mathcal{S} \setminus \{0\}$, there exists $c(y) > 0$ such that, for any n,*

$$\mathbb{P}\left[(X_t)_{t\geq\tau_n} \textit{visits } y \textit{ before time } \tau_{n+1} \mid \mathcal{F}_{\tau_n}\right] = c(y), \textit{ a.s.}$$

Proof. The irreducibility assumption (2.1) implies that for any n, on $\{\tau_n < \infty\}$,

$$\mathbb{P}[X_{\tau_n+m(0,y)} = y \mid \mathcal{F}_{\tau_n}] \geq \varphi(0,y), \text{ a.s.} \tag{4.7}$$

By the regenerative assumption (A3), $\mathbb{P}\,[\text{hit } y \text{ before returning to } 0 \mid \mathcal{F}_{\tau_n}]$, on $\{\tau_n < \infty\}$, does not depend on n; call this probability $c(y)$. Then, since $N = \infty$ a.s.,

$$\mathbb{P}[\text{eventually hit } y] = \mathbb{P}\Big[\bigcup_{i=1}^{\infty}\{\text{hit } y \text{ between } \tau_i \text{ and } \tau_{i+1}\}\Big] \leq \sum_{i=1}^{\infty} c(y).$$

Thus if $c(y) = 0$, the probability of eventually hitting y is also 0, which contradicts (4.7) (cf Lemma 4.5). Hence $c(y) > 0$. □

A recurring technical component of our proofs will be controlling the process X in finite intervals such as $[0,x]$, and the exits (and overshoots) of X from such intervals. The following two lemmas give basic results in this direction.

Lemma 4.8. *Suppose that (A0) and (A1) hold. For any $x \geq 0$, there exists $\varepsilon > 0$ such that, for all t and for all s sufficiently large, $\mathbb{P}[\sigma_{x,t} > s \mid \mathcal{F}_t] \leq \mathrm{e}^{-\varepsilon s}$ a.s. In particular, there exists $K < \infty$, depending on x, for which $\mathbb{E}[\sigma_{x,t} \mid \mathcal{F}_t] \leq K$ a.s. for all t.*

Proof. Let $x \geq 0$ and $z \in \mathcal{S}$, $z > x$. By (A0) and (A1), taking $m = \max_{y \in \mathcal{S}_x} m(y,z)$ and $\delta = \min_{y \in \mathcal{S}_x} \varphi(y,z)$ we have $m \in \mathbb{N}$ and $\delta > 0$, depending on x, such that, for any $s \geq t$,

$$\mathbb{P}[\sigma_{x,t} \leq (s-t) + m \mid \mathcal{F}_s] \geq \delta \mathbf{1}\{\sigma_{x,t} > s-t\} + \mathbf{1}\{\sigma_{x,t} \leq s-t\}, \text{ a.s.}$$

Taking $s = t + \ell m$ for $\ell \in \mathbb{N}$ yields $\mathbb{P}[\sigma_{x,t} > (\ell+1)m \mid \mathcal{F}_{t+\ell m}] \leq (1-\delta)\mathbf{1}\{\sigma_{x,t} > \ell m\}$, and a telescoping conditioning argument at times $t, t+m, \ldots, t+\ell m$ gives $\mathbb{P}[\sigma_{x,t} > \ell m \mid \mathcal{F}_t] \leq (1-\delta)^\ell$. For any $s \geq 0$, there is some $\ell = \ell(s)$ for which $\ell m \leq s \leq (\ell+1)m$, so

$$\mathbb{P}[\sigma_{x,t} > s \mid \mathcal{F}_t] \leq \mathbb{P}[\sigma_{x,t} > \ell m \mid \mathcal{F}_t] \leq (1-\delta)^\ell \leq (1-\delta)^{(s/m)-1},$$

which implies the result, recalling that m and δ depend on x but not on s or t. □

Lemma 4.9. *Suppose that (A0)–(A3) hold, $r > -1$, and (2.2) holds with $p > \max\{2, 1+r\}$. Let $x \geq 0$. Then for any $\nu \in \mathbb{R}$, there exists $K < \infty$ (depending on x) such that, on $\{X_t \leq x\}$, $\mathbb{E}[f_{1+r,\nu}(X_{t+\sigma_{x,t}}) \mid \mathcal{F}_t] \leq K$, a.s.*

Proof. Under the stated conditions, Lemma 4.1 applies. In particular, (4.1) shows that for any $\varepsilon > 0$ there is $C < \infty$, not depending on x, such that for $s \geq t$,

$$\mathbb{E}[f_{1+r,\nu}(X_{(s+1)\wedge(t+\sigma_{x,t})}) - f_{1+r,\nu}(X_{s\wedge(t+\sigma_{x,t})}) \mid \mathcal{F}_s] \leq C(1+X_s)^{r-1+\varepsilon}\mathbf{1}\{s-t < \sigma_{x,t}\}.$$

Suppose that $X_t \leq x$. For $t \leq s < t + \sigma_{x,t}$, $X_s \in [0,x]$, so writing $b(x) = C\max_{y \in \mathcal{S}_x}(1+y)^{r-1+\varepsilon} < \infty$, conditioning on $\mathcal{F}_t$ and taking expectations we obtain, on $\{X_t \leq x\}$, a.s.,

$$\mathbb{E}[f_{1+r,\nu}(X_{(s+1)\wedge(t+\sigma_{x,t})}) \mid \mathcal{F}_t] - \mathbb{E}[f_{1+r,\nu}(X_{s\wedge(t+\sigma_{x,t})}) \mid \mathcal{F}_t] \leq b(x)\mathbb{P}[\sigma_{x,t} > s-t \mid \mathcal{F}_t].$$

Let $u > t$ be an integer. Summing from $s = t$ to $u-1$ we have, on $\{X_t \leq x\}$, a.s.,

$$\begin{aligned} \mathbb{E}[f_{1+r,\nu}(X_{u\wedge(t+\sigma_{x,t})}) \mid \mathcal{F}_t] &\leq \mathbb{E}[f_{1+r,\nu}(X_t) \mid \mathcal{F}_t] + b(x)\sum_{s=0}^{\infty}\mathbb{P}[\sigma_{x,t} > s \mid \mathcal{F}_t] \\ &\leq a(x) + b(x)\mathbb{E}[\sigma_{x,t} \mid \mathcal{F}_t], \end{aligned}$$

writing $a(x) = \max_{y \in \mathcal{S}_x} f_{1+r,\nu}(y) < \infty$. The final part of Lemma 4.8 then shows that there is $K < \infty$, depending on x, for which, for all $u > t$, $\mathbb{E}[f_{1+r,\nu}(X_{u\wedge(t+\sigma_{x,t})}) \mid \mathcal{F}_t] \leq K$ a.s.; letting $u \to \infty$, Fatou's lemma completes the proof. □

4.3 Proofs of main results from Section 2

Proof of Proposition 2.1. The first statement of the proposition follows from Lemma 4.5. Now from part (a) of (A3) with a repeated conditioning argument,

$$\begin{aligned} &\mathbb{P}[N > k] \\ &= \mathbb{P}[\eta_k < \infty, \eta_{k-1} < \infty, \ldots, \eta_1 < \infty] \\ &= \mathbb{P}[\eta_k < \infty \mid \tau_{k-1} < \infty]\mathbb{P}[\eta_{k-1} < \infty \mid \tau_{k-2} < \infty]\cdots\mathbb{P}[\eta_2 < \infty \mid \tau_1 < \infty]\mathbb{P}[\eta_1 < \infty] \\ &= (\mathbb{P}[\eta_1 < \infty])^k. \end{aligned}$$

If $\mathbb{P}[\eta_1 < \infty] < 1$, this implies that $N < \infty$ a.s., so that $X_t = 0$ f.o., and Lemma 4.5 shows that $X_t \to \infty$. On the other hand, if $\mathbb{P}[\eta_1 < \infty] = 1$ we have that $\mathbb{P}[N > k] = 1$ for any k, so $N = \infty$ a.s. and hence $X_t = 0$ i.o., i.e., $\liminf_{t\to\infty} X_t = 0$, a.s., as claimed. □

We now sketch the proof of Theorem 2.2.

Proof of Theorem 2.2. Under slightly different conditions, this result follows from results of [30, 31, 34]. The results in [30] apply to a more general class of processes than we consider here, with a slightly stronger version of (2.2), while [31] and [34] state their results in the Markovian setting, although their methods work (as in [30]) in the more general setting; concretely, one can use our Lemma 4.1 (and a variant thereof for $|r| = 1$, provided by calculations similar to those in [34]) together with the results from [30] or [5], for instance. These papers use a slightly different definition of recurrence to ours, but Lemma 4.5 shows that the definitions are equivalent under (A1). □

Next we give the proof of Theorem 2.3 on the tail of $M_1 = \max_{1\le s\le \eta_1} X_s$.

Proof of Theorem 2.3. Throughout the proof fix $r > -1$. First we prove the lower bound in (2.6). Fix $\nu > 0$. We ease notation by writing f for $f_{1+r,\nu}$ as defined in Section 4.1; for $r > -1$ and $\nu > 0$, f is nondecreasing on $[0, \infty)$ and $f(z) \to \infty$ as $z \to \infty$. Lemma 4.1 implies that $f(X_t)$ satisfies a local submartingale property; to achieve uniform integrability, we work with a truncated version of f, namely $h_x(z) := \min\{f(z), f(2x)\}$, for fixed $x > 0$. Recall the definition of $E_\varepsilon(t)$ from immediately above Lemma 4.2. For any $\varepsilon \in (0, 1)$, on $\{X_t \le x\}$, for all x sufficiently large, $E_\varepsilon(t)$ implies that $X_{t+1} < 2x$. Hence, for any $\varepsilon \in (0, 1)$, on $\{X_t \le x\}$,

$$h_x(X_{t+1}) - h_x(X_t) \ge (f(X_{t+1}) - f(X_t))\,\mathbf{1}_{E_\varepsilon(t)} - f(X_t)\mathbf{1}\{\Delta_t > (1 + X_t)^{1-\varepsilon}\},$$

so that

$$\mathbb{E}[h_x(X_{t+1}) - h_x(X_t) \mid \mathcal{F}_t] \ge \mathbb{E}\left[(f(X_{t+1}) - f(X_t))\,\mathbf{1}_{E_\varepsilon(t)} \mid \mathcal{F}_t\right] - f(X_t)\mathbb{P}[E^{\mathrm{c}}_\varepsilon(t) \mid \mathcal{F}_t].$$

By Lemma 4.2 with $q = 0$, for $\varepsilon > 0$ small enough, $f(X_t)\mathbb{P}[E^{\mathrm{c}}_\varepsilon(t) \mid \mathcal{F}_t] = O(X_t^{r-1-(\varepsilon/2)})$, a.s.; with the $\gamma = 1+r$ case of (4.3) this shows, as in the proof of Lemma 4.1, on $\{X_t \le x\}$,

$$\mathbb{E}[h_x(X_{t+1}) - h_x(X_t) \mid \mathcal{F}_t] \ge (\nu(1 + r)(s^2/2) + o(1))X_t^{r-1} \log^{\nu-1} X_t,$$

which is positive for all X_t sufficiently large, since $\nu > 0$, $r > -1$, and $s^2 > 0$. Thus there exists $A \in (0, \infty)$ such that, for all $x > A$,

$$\mathbb{E}[h_x(X_{t+1}) - h_x(X_t) \mid \mathcal{F}_t] \ge 0, \text{ on } \{A \le X_t \le x\}, \text{ a.s.} \tag{4.8}$$

Choose $\lambda \in (0, \infty)$ with $\lambda > \max_{z\in\mathcal{S}_A} h_x(z)$. Since $f(y) \to \infty$ as $y \to \infty$, we can (and do) choose $y \in \mathcal{S} \cap (A, \infty)$ such that $f(y) > 2\lambda$. Take $x > y$. Define the stopping times

$$\begin{aligned} \kappa_1 &:= \min\{t \in \mathbb{N} : X_t = y\}, \\ \kappa_2 &:= \min\{t > \kappa_1 : X_t \ge x\}, \\ \kappa_3 &:= \min\{t > \kappa_1 : X_t \le A\}. \end{aligned}$$

By Lemma 4.5 and the fact that for $r > -1$, X is recurrent (by Theorem 2.2), $\kappa_i < \infty$ a.s., for each $i \in \{1, 2, 3\}$. We consider $(h_x(X_{t\wedge\kappa_2\wedge\kappa_3}))_{t\ge\kappa_1}$, which is a submartingale by (4.8). Also, $(h_x(X_{t\wedge\kappa_2\wedge\kappa_3}))_{t\geq\kappa_1}$ is uniformly integrable (since it is bounded above by $f(2x)<\infty$), so $h_x(X_{t\wedge\kappa_2\wedge\kappa_3})$ converges a.s. and in L^1 to $h_x(X_{\kappa_2\wedge\kappa_3})$ as $t\to\infty$. Hence,

$$\begin{aligned}2\lambda &\leq \mathbb{E}[h_x(X_{\kappa_2\wedge\kappa_3}) \mid \mathcal{F}_{\kappa_1}]\\ &= \mathbb{E}[h_x(X_{\kappa_2})\mathbf{1}\{\kappa_2<\kappa_3\} \mid \mathcal{F}_{\kappa_1}] + \mathbb{E}[h_x(X_{\kappa_3})\mathbf{1}\{\kappa_3<\kappa_2\} \mid \mathcal{F}_{\kappa_1}]\\ &\leq f(2x)\mathbb{P}[\kappa_2<\kappa_3 \mid \mathcal{F}_{\kappa_1}] + \lambda\mathbb{P}[\kappa_3<\kappa_2 \mid \mathcal{F}_{\kappa_1}],\end{aligned}$$

since $h_x(X_{\kappa_3})\leq f(X_{\kappa_3})\leq\lambda$ and $h_x(X_{\kappa_2})\leq f(2x)$ a.s.; re-arranging we obtain

$$\mathbb{P}[\kappa_2<\kappa_3 \mid \mathcal{F}_{\kappa_1}] \geq \frac{\lambda}{f(2x)-\lambda}, \text{ a.s.} \tag{4.9}$$

Hence, by (4.9), there is a constant $C_1\in(0,\infty)$ such that $\mathbb{P}[\kappa_2<\kappa_3 \mid \mathcal{F}_{\kappa_1}]\geq 1/f(C_1x)$ for all x large enough. Finally,

$$\mathbb{P}[\kappa_2<\eta_1] \geq \mathbb{E}\left[\mathbf{1}\{\kappa_1<\eta_1\}\mathbb{P}[\kappa_2<\kappa_3 \mid \mathcal{F}_{\kappa_1}]\right] \geq \frac{\mathbb{P}[\kappa_1<\eta_1]}{f(C_1x)} \geq \frac{1}{f(C_2x)},$$

for some constant $C_2\in(0,\infty)$ and all x sufficiently large, by Lemma 4.7. On $\{\kappa_2<\eta_1\}$, we have $M_1=\max_{1\leq s\leq\eta_1}X_s\geq x$. Thus we obtain the lower bound in (2.6), since $\nu>0$ was arbitrary.

We now prove the upper bound in (2.6). Fix $\varepsilon>0$ and now write f for $f_{1+r,-\varepsilon}$. By Lemma 4.1, there is $A\in(0,\infty)$ such that $\mathbb{E}[f(X_{t+1})-f(X_t) \mid \mathcal{F}_t]\leq 0$ on $\{X_t\geq A\}$, a.s. Since $r>-1$, there exists $x_0\geq A$ such that f is increasing on $[x_0,\infty)$. Take $x>x_1>x_0$; x_1 will be fixed later. Define stopping times recursively by $\beta_0:=1$ and for $n\in\mathbb{N}$,

$$\begin{aligned}\alpha_n &:= \min\{t>\beta_{n-1} : X_t>x_1\},\\ \beta_n &:= \min\{t>\alpha_n : X_t\leq x_0\},\\ \gamma_n &:= \min\{t>\alpha_n : X_t>x\}.\end{aligned}$$

By Lemma 4.5 and the fact that X is recurrent, α_n, β_n and γ_n are a.s. finite for all n.

By Lemma 4.1, $(f(X_{t\wedge\beta_n\wedge\gamma_n}))_{t\geq\alpha_n}$ is a nonnegative supermartingale, and as $t\to\infty$ it converges a.s. to $f(X_{\beta_n\wedge\gamma_n})$. By Fatou's lemma, $\mathbb{E}[f(X_{\beta_n\wedge\gamma_n}) \mid \mathcal{F}_{\alpha_n}]\leq f(X_{\alpha_n})$. Moreover, $\mathbb{E}[f(X_{\beta_n\wedge\gamma_n}) \mid \mathcal{F}_{\alpha_n}]\geq\mathbb{P}[\gamma_n<\beta_n \mid \mathcal{F}_{\alpha_n}]f(x)$, since $x>x_0$. It follows that for all $x>x_1$,

$$\mathbb{P}[\gamma_n<\beta_n \mid \mathcal{F}_{\alpha_n}] \leq \frac{f(X_{\alpha_n})}{f(x)}. \tag{4.10}$$

Lemma 4.9 shows that $\mathbb{E}[f(X_{\alpha_n})]\leq K$ for some $K<\infty$ depending on x_1 but not on x. Thus taking expectations in (4.10) we obtain, for some $K<\infty$ and all $x>x_0$,

$$\mathbb{P}[\gamma_n<\beta_n]\leq K/f(x).$$

Moreover, it follows from Lemma 4.6 that we may choose $x_1>x_0$ large enough such that, for some $\delta>0$, $\mathbb{P}[\eta_1<\alpha_{n+1} \mid \mathcal{F}_{\beta_n}]\geq\delta$, a.s., for all n. Thus we fix such an x_1. In particular, we then have that $\mathbb{P}[\eta_1<\beta_{n+1} \mid \mathcal{F}_{\beta_n}]\geq\delta$, a.s. Let $J:=\min\{n\in\mathbb{N} : \eta_1<\beta_n\}$. Then J is stochastically dominated by a geometric random variable with parameter δ, and in particular $\mathbb{P}[J>n]\leq(1-\delta)^n\leq \mathrm{e}^{-\delta n}$. Hence

$$\mathbb{P}[M_1\geq x] \leq \mathbb{P}\Big[\bigcup_{n=1}^{J}\{\gamma_n<\beta_n\}\Big] \leq \mathbb{P}[J\geq\lfloor k\log x\rfloor] + \sum_{n=1}^{k\log x}\mathbb{P}[\gamma_n<\beta_n].$$

Now we choose k large enough so that $\mathbb{P}[J\geq\lfloor k\log x\rfloor]=O(1/f(x))$, say, for which $k>\frac{2+r}{\delta}$ suffices. It then follows that there is a constant $C\in(0,\infty)$ such that $\mathbb{P}[M_1\geq x]\leq\frac{C\log x}{f(x)}$ for all $x>x_1$, and the upper bound in (2.6) follows. □

We can now state a result on the 'maximum of the maxima' in the first n excursions.

Lemma 4.10. *Suppose that (A0)–(A3) hold. Suppose that $r > -1$ and (2.2) holds with $p > \max\{2, 1+r\}$. For any $\varepsilon > 0$, a.s., for all but finitely many n,*

$$n^{\frac{1}{1+r}} (\log n)^{-\frac{1}{1+r}-\varepsilon} \leq \max_{1 \leq i \leq n} M_i \leq n^{\frac{1}{1+r}} (\log n)^{\frac{2}{1+r}+\varepsilon}.$$

Proof. Apply the tail bounds in Theorem 2.3 together with Lemma 4.3. □

A key result for several of our remaining theorems is Lemma 4.11. It provides lower bounds on excursion functionals, and in particular gives a new approach to a lower tail bound for η_1, which has advantages over previous approaches: the results in [5] are not as sharp, while the results in [3] require uniformly bounded increments for the process.

Lemma 4.11. *Suppose that (A0) and (A1) hold, and there exists $B < \infty$ such that $\mathbb{E}[\Delta_t^2 \mid \mathcal{F}_t] \leq B$, a.s., for all $t \in \mathbb{N}$, and there exist $x_0, c \in (0, \infty)$ such that, on $\{X_t \geq x_0\}$, $\mathbb{E}[\Delta_t \mid \mathcal{F}_t] \geq -c/X_t$, a.s., for all $t \in \mathbb{N}$. Let $\Phi : \mathcal{S} \to [0, \infty)$ be a nondecreasing function. Then there exists $\varepsilon > 0$ such that for all y sufficiently large,*

$$\mathbb{P}[M_1 \geq y] \leq 2\mathbb{P}\left[\sum_{t=1}^{\eta_1} \Phi(X_t) \geq \varepsilon y^2 \Phi(y/2)\right].$$

In particular, $\mathbb{P}[\eta_1 \geq x] \geq \frac{1}{2}\mathbb{P}[M_1 \geq (x/\varepsilon)^{1/2}]$ for some $\varepsilon > 0$ and all x sufficiently large.

Proof. Let $y > 2x_0$. Define stopping times

$$\kappa_1 = \min\{t \in \mathbb{N} : X_t \geq y\}; \quad \kappa_2 = \min\{t \geq \kappa_1 : X_t \leq y/2\}.$$

Note that $\kappa_1 < \infty$ a.s., by Lemma 4.5, and $\{\kappa_1 < \eta_1\}$, the event that X reaches $[y, \infty)$ before returning to 0, is $\mathcal{F}_{\kappa_1}$ measurable. Then, for any $\varepsilon > 0$,

$$\mathbb{P}\left[\{\kappa_1 < \eta_1\} \cap \{\kappa_2 \geq \kappa_1 + \varepsilon y^2\}\right] = \mathbb{E}\left[\mathbf{1}\{\kappa_1 < \eta_1\}\mathbb{P}[\kappa_2 \geq \kappa_1 + \varepsilon y^2 \mid \mathcal{F}_{\kappa_1}]\right]. \tag{4.11}$$

We claim that we may choose $\varepsilon > 0$ for which, for all y sufficiently large,

$$\mathbb{P}[\kappa_2 \geq \kappa_1 + \varepsilon y^2 \mid \mathcal{F}_{\kappa_1}] \geq \frac{1}{2}, \text{ a.s.} \tag{4.12}$$

To verify (4.12), let $W_t = (y - X_t)^2 \mathbf{1}\{X_t < y\}$. Then on $\{X_t \geq y\}$, $W_{t+1} - W_t \leq \Delta_t^2$, so that $\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] \leq B$, a.s., on $\{X_t \geq y\}$. On the other hand, on $\{X_t < y\}$,

$$\begin{aligned} W_{t+1} - W_t &\leq (W_{t+1} - W_t)\mathbf{1}\{X_{t+1} < y\} \\ &\leq -2(y - X_t)\Delta_t \mathbf{1}\{\Delta_t < y - X_t\} + \Delta_t^2. \end{aligned}$$

Here we have that

$$\begin{aligned} -2(y - X_t)\Delta_t \mathbf{1}\{\Delta_t < y - X_t\} &= -2(y - X_t)\Delta_t + 2(y - X_t)\Delta_t \mathbf{1}\{\Delta_t > y - X_t\} \\ &\leq -2(y - X_t)\Delta_t + 2\Delta_t^2. \end{aligned}$$

Taking expectations we see that, on $\{x_0 \leq X_t < y\}$, a.s.,

$$\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] \leq -2(y - X_t)\mathbb{E}[\Delta_t \mid \mathcal{F}_t] + 3\mathbb{E}[\Delta_t^2 \mid \mathcal{F}_t] \leq \frac{2cy}{X_t} + 3B.$$

In particular, there exists $C < \infty$ such that, on $\{X_t > y/2\}$, $\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] \le C$, a.s. Hence we conclude that, for any $t \ge 0$, $\mathbb{E}[W_{(\kappa_1+t+1)\wedge\kappa_2} - W_{(\kappa_1+t)\wedge\kappa_2} \mid \mathcal{F}_{\kappa_1+t}] \le C$, a.s. Then an appropriate maximal inequality (Lemma 3.1 of [36]) implies that

$$\mathbb{P}\Big[\max_{0\le s\le t} W_{(\kappa_1+s)\wedge\kappa_2} \ge w \mid \mathcal{F}_{\kappa_1}\Big] \le Ct/w, \text{ a.s.},$$

using the fact that $W_{\kappa_1} = 0$. But $W_{\kappa_2} = (y - X_{\kappa_2})^2 \ge y^2/4$, so

$$\mathbb{P}[\kappa_2 \le \kappa_1 + t \mid \mathcal{F}_{\kappa_1}] \le \mathbb{P}\Big[\max_{0\le s\le t} W_{(\kappa_1+s)\wedge\kappa_2} \ge (y^2/4) \mid \mathcal{F}_{\kappa_1}\Big] \le \frac{4Ct}{y^2}, \text{ a.s.},$$

and choosing $t = \varepsilon y^2$ for $\varepsilon > 0$ sufficiently small (not depending on y), the claim (4.12) follows. Combining (4.11) and (4.12) we get

$$\mathbb{P}\left[\{\kappa_1 < \eta_1\} \cap \{\kappa_2 \ge \kappa_1 + \varepsilon y^2\}\right] \ge \frac{1}{2}\mathbb{P}[\kappa_1 < \eta_1] = \frac{1}{2}\mathbb{P}[M_1 \ge y].$$

On $\{\kappa_1 < \eta_1\} \cap \{\kappa_2 \ge \kappa_1 + \varepsilon y^2\}$, $X_s \ge y/2$ for all $\kappa_1 \le s < \kappa_2$, of which there are at least εy^2 values, all before time η_1; since Φ is nondecreasing we obtain the result. □

To obtain an upper tail bound on η_1, one of the technical ingredients that we need is the following consequence of Theorem 2′ of [4], which extended results in [5].

Lemma 4.12. *Suppose that $(Y_t)_{t\in\mathbb{N}}$ is an $(\mathcal{F}_t)_{t\in\mathbb{N}}$-adapted stochastic process on an unbounded subset of $[0,\infty)$. Let $T_A := \min\{t \in \mathbb{N} : Y_t \le A\}$. Suppose that there exist $p > 0$, $\nu \in \mathbb{R}$, and $\delta > 0$ such that*

$$\mathbb{E}[Y_{t+1}^{2p} \log^\nu Y_{t+1} - Y_t^{2p} \log^\nu Y_t \mid \mathcal{F}_t] \le -\delta Y_t^{2p-2} \log^{\nu-1} Y_t, \text{ on } \{T_A > t\}. \tag{4.13}$$

Then for some $C < \infty$, $\mathbb{P}[T_A \ge t \mid Y_1 = x_0] \le Ct^{-p}(\log t)^{p-\nu} x_0^{2p}(\log x_0)^\nu$.

Proof. We apply Theorem 2′ of [4] with, in the notation there (but with time denoted by t rather than n) $X_t = Y_t$, $h(x) = x^{2p}(\log x)^\nu$, $U_t = h(Y_t)$, $g(x) = x^{2p-2}(\log x)^{\nu-1}$, and $f(x) = x^p(\log x)^q$, where $p > 0$ and $\nu \in \mathbb{R}$ are as in the statement of the lemma and $q \in \mathbb{R}$ is to be chosen later. It follows from Theorem 2′ of [4] that, under the conditions of the lemma, for some $C < \infty$, for any $x_0 > A$,

$$\mathbb{E}[f(T_A) \mid Y_1 = x_0] \le Ch(x_0) = Cx_0^{2p}(\log x_0)^\nu, \tag{4.14}$$

provided that, writing f^{-1} here for the inverse function of f,

$$\liminf_{x\to\infty} \left(\frac{g(x)}{f'(f^{-1}(h(x)))} \right) > 0. \tag{4.15}$$

We verify (4.15) for the stated f, g, h. We claim there exists $c \in (0,\infty)$ such that

$$f^{-1}(x) = (c + o(1))x^{1/p}(\log x)^{-q/p}. \tag{4.16}$$

Since f is eventually increasing, to verify (4.16) it suffices to show that, for an appropriate $c \in (0,\infty)$, $f((c+\varepsilon)x^{1/p}(\log x)^{-q/p})$ is eventually greater than x if $\varepsilon > 0$ but eventually less than x if $\varepsilon < 0$. But we have, for $\alpha > 0$,

$$\begin{aligned} f(\alpha x^{1/p}(\log x)^{-q/p}) &= \alpha^p x(\log x)^{-q}\left[\log\alpha + p^{-1}\log x - (q/p)\log\log x\right]^q \\ &= (\alpha^p p^{-q} + o(1))x, \end{aligned}$$

which satisfies the desired properties with $\alpha = c$ provided $c^p p^{-q} = 1$, i.e., $c = p^{q/p}$. Thus we obtain (4.16). It follows that, for some $c' \in (0, \infty)$,

$$f^{-1}(h(x)) = (c' + o(1))x^2 (\log x)^{\frac{\nu - q}{p}}.$$

Now $f'(x) = (p + o(1))x^{p-1}(\log x)^q$, so we get, for some $c'' \in (0, \infty)$,

$$f'(f^{-1}(h(x))) = (c'' + o(1))x^{2p-2}(\log x)^{\nu + \frac{q-\nu}{p}}.$$

Then (4.15) holds provided that $\nu - 1 - (\nu + \frac{q-\nu}{p}) \geq 0$, that is, $q \leq \nu - p$. This shows that (4.14) holds, and then the result of the lemma follows by Markov's inequality. □

Now we can complete the proof of our main result on the duration of an excursion.

Proof of Theorem 2.4. First we prove the upper bound in (2.7). Our starting point will be Lemma 4.12, which deals with the hitting time of a suitably large interval $[0, A]$ starting from outside that interval. Some additional work, based on the irreducibility assumption, is needed to relate this to the return time η_1 to 0. Fix $A > 0$ and $B > A$ (to be specified later). Define stopping times α_i and β_i recursively by $\beta_0 := 1$ and for $n \in \mathbb{N}$,

$$\alpha_n := \min\{t \geq \beta_{n-1} : X_t \leq A\}, \quad \beta_n := \min\{t \geq \alpha_n : X_t \geq B\};$$

by Lemma 4.5 and the fact that X is recurrent, $\alpha_n \leq \beta_n < \infty$ for all n, a.s.

We have from the $2p = 1 + r > 0$ case of (4.4) that with $Y_t = X_t$, $f_{2p,\nu}(X_t)$ satisfies (4.13) taking $\nu < 0$, provided the A in (4.13) is large enough. Thus take A to be sufficiently large. Hence we can apply Lemma 4.12 to show that, for any $\varepsilon > 0$,

$$\mathbb{P}[\alpha_{n+1} - \beta_n \geq t \mid \mathcal{F}_{\beta_n}] \leq t^{-\frac{1+r}{2}} (\log t)^{\frac{1+r}{2} + \varepsilon} (1 + X_{\beta_n})^{1+r},$$

for all t large enough. Here the $\nu = 0$ case of Lemma 4.9 shows that $\mathbb{E}[(1 + X_{\beta_n})^{1+r}] \leq C < \infty$ for C not depending on t. So taking expectations in the last display, we obtain

$$\mathbb{P}[\alpha_{n+1} - \beta_n \geq t] \leq C t^{-\frac{1+r}{2}} (\log t)^{\frac{1+r}{2} + \varepsilon}, \tag{4.17}$$

for all t sufficiently large. On the other hand, for $B = B(A)$ as in Lemma 4.6, we have that for $\varphi > 0$, for all n, $\mathbb{P}[\eta_1 < \beta_n \mid \mathcal{F}_{\alpha_n}] \geq \varphi$, a.s. Let $K := \min\{n : \beta_n > \eta_1\}$. Then K is stochastically dominated by a geometric random variable with parameter φ, and in particular $\mathbb{P}[K > n] \leq (1 - \varphi)^n \leq \mathrm{e}^{-\varphi n}$. Moreover, $\eta_1 \leq \sum_{n=1}^{K} (\alpha_{n+1} - \alpha_n)$. So

$$\begin{aligned} \mathbb{P}[\eta_1 \geq t] &\leq \mathbb{P}[K \geq \lfloor k \log t \rfloor] + \mathbb{P}\left[\sum_{n=1}^{k \log t} (\alpha_{n+1} - \alpha_n) \geq t \right] \\ &\leq t^{-(1+r)} + k(\log t) \sup_n \mathbb{P}\left[\alpha_{n+1} - \alpha_n \geq \frac{t}{k \log t} \right], \end{aligned} \tag{4.18}$$

choosing k sufficiently large. A similar argument to Lemma 4.8 shows that $\mathbb{P}[\beta_n - \alpha_n \geq t] \leq \mathrm{e}^{-ct}$ for $c > 0$ depending on B (and hence on A). Then since

$$\mathbb{P}[\alpha_{n+1} - \alpha_n \geq t] \leq \mathbb{P}[\alpha_{n+1} - \beta_n \geq t/2] + \mathbb{P}[\beta_n - \alpha_n \geq t/2],$$

it follows that $\alpha_{n+1} - \alpha_n$ satisfies the same tail bound (4.17) as $\alpha_{n+1} - \beta_n$. The upper bound in (2.7) then follows from (4.18).

The lower bound in (2.7) follows from the final statement in Lemma 4.11 together with the lower bound in Theorem 2.3. □

Now we can give a result on the total duration of the first n excursions.

Lemma 4.13. *Suppose that (A0), (A1), (A2), and (A3) hold.*

(i) Suppose that $-1 < r \leq 1$ and (2.2) holds with $p > 2$. Then for any $\varepsilon > 0$, a.s., for all but finitely many n,

$$n^{\frac{2}{1+r}} (\log n)^{-\frac{2}{1+r}-\varepsilon} \leq \sum_{i=1}^{n} \eta_i \leq n^{\frac{2}{1+r}} (\log n)^{\frac{6+2r}{1+r}+\varepsilon}.$$

(ii) Suppose that $r > 1$ and (2.2) holds with $p > 1 + r$. Then as $n \to \infty$, a.s., $n^{-1} \sum_{i=1}^{n} \eta_i \to \mathbb{E}[\eta_1] \in (0, \infty)$.

Proof. Part (ii) follows from the strong law of large numbers since $\mathbb{E}[\eta_1] < \infty$ for $r > 1$, by Theorem 2.4, while $\mathbb{E}[\eta_1] \neq 0$ since η_1 is nondegenerate. Now suppose that $r \in (-1, 1]$. The lower bound in (i) follows from the lower bound in (2.7) with Lemma 4.4(ii), while the upper bound in (i) follows from the upper bound in (2.7) with Lemma 4.4(i). □

An inversion of the previous result enables us to complete the proof of our theorem on the number of excursions. Recall that $N_t = \max \{n \in \mathbb{N} : \sum_{i=1}^{n} \eta_i \leq t\}$.

Proof of Theorem 2.5. For part (i), fix $\varepsilon > 0$. From the lower bound in Lemma 4.13(i), we may choose $\varepsilon' > 0$ small enough for which, a.s., for all t large enough

$$\sum_{i=1}^{\lceil t^{\frac{1+r}{2}} (\log t)^{1+\varepsilon} \rceil} \eta_i \geq t (\log t)^{\frac{2\varepsilon}{1+r} - \varepsilon'} > t,$$

giving the upper bound in (2.8). The lower bound in (2.8) follows similarly from the upper bound in Lemma 4.13(i). Part (ii) follows from Lemma 4.13(ii). □

Next we turn to our results on stationary distributions.

Proof of Theorem 2.6. We verify the claimed properties of π defined at (2.10). When $r > 1$, we have from Theorem 2.4 that $\mathbb{E}[\eta_1] \in (0, \infty)$. Since, for any x, $0 \leq \ell_1(x) \leq \eta_1$ a.s., it follows that $\mathbb{E}[\ell_1(x)] < \infty$ for all $x \in \mathcal{S}$. It is clear that $\pi(x) \geq 0$ and $\sum_{x \in \mathcal{S}} \pi(x) = 1$. To show that $\pi(x) > 0$, it suffices to show that $\mathbb{E}[\ell_1(x)] > 0$. Suppose that, for some $x \in \mathcal{S}$, $\ell_1(x) = 0$ a.s. Then by (A3), $L_t(x) = 0$ a.s. for all t. But this contradicts Lemma 4.5. So $\mathbb{P}[\ell_1(x) > 0] > 0$, which implies $\mathbb{E}[\ell_1(x)] > 0$.

Next, note that for any $x \in \mathcal{S}$, a.s., $\sum_{n=1}^{N_t} \ell_n(x) \leq L_t(x) \leq \sum_{n=1}^{N_t+1} \ell_n(x)$. Here $(\ell_n(x))_{n \in \mathbb{N}}$ are i.i.d. random variables with finite means, and so it follows from the strong law of large numbers that $N_t^{-1} L_t(x) \to \mathbb{E}[\ell_1(x)]$ a.s. for $N_t \to \infty$, which with Theorem 2.5(ii) implies that $t^{-1} L_t(x) \to \frac{\mathbb{E}[\ell_1(x)]}{\mathbb{E}[\eta_1]}$ a.s., and the L^q convergence follows from the bounded convergence theorem. Finally the convergence of $\mathbb{P}[X_t = x]$ to $\pi(x)$ follows from e.g. [2, Corollary VI.1.5, p. 171] under the additional 'aperiodicity' condition. □

The proofs of our remaining theorems now involve combining our previous results.

Proof of Theorem 2.7. We have that for any $t \in \mathbb{N}$,

$$\max_{1 \leq i \leq N_t} M_i \leq \max_{1 \leq s \leq t} X_s \leq \max_{1 \leq i \leq N_t+1} M_i. \tag{4.19}$$

The result follows from Theorem 2.5 and Lemma 4.10 together with (4.19). □

Proof of Theorem 2.8. Fix $\alpha \geq 0$ and $r > -1$. First we prove the upper bound in (2.12). Clearly $\xi_1^{(\alpha)} \leq \eta_1 M_1^\alpha$. It follows that, for any $x > 1$,

$$\begin{aligned}\mathbb{P}[\xi_1^{(\alpha)} \geq x] &\leq \mathbb{P}[\{\eta_1 \geq x^{\frac{2}{\alpha+2}}(\log x)^{\frac{2\alpha}{\alpha+2}}\} \cup \{M_1 \geq x^{\frac{1}{\alpha+2}}(\log x)^{-\frac{2}{\alpha+2}}\}] \\ &\leq \mathbb{P}[\eta_1 \geq x^{\frac{2}{\alpha+2}}(\log x)^{\frac{2\alpha}{\alpha+2}}] + \mathbb{P}[M_1 \geq x^{\frac{1}{\alpha+2}}(\log x)^{-\frac{2}{\alpha+2}}].\end{aligned}$$

Now applying the upper bounds from (2.7) and (2.6) we obtain the desired upper bound.

Next we prove the lower bound in (2.12). It follows from the $\Phi(x) = x^\alpha$ case of Lemma 4.11 that there exists $C \in (0,\infty)$ such that, for all x large enough,

$$\mathbb{P}[\xi_1^{(\alpha)} \geq x] \geq \frac{1}{2}\mathbb{P}[M_1 \geq C x^{\frac{1}{\alpha+2}}]. \tag{4.20}$$

The lower bound in (2.12) now follows from (4.20) and the lower bound in (2.6). □

Recall that $S_t^{(\alpha)} = \sum_{s=1}^t X_s^\alpha$, so $S_{\tau_n}^{(\alpha)} = \sum_{i=1}^n \xi_i^{(\alpha)}$.

Lemma 4.14. *Suppose that (A0)–(A3) hold. Suppose that $r > -1$ and (2.2) holds with $p > \max\{2, 1+r\}$. Let $\alpha \geq 0$.*

(i) Suppose that $-1 < r \leq 1+\alpha$. Then for any $\varepsilon > 0$, a.s., for all but finitely many n,

$$n^{\frac{\alpha+2}{1+r}}(\log n)^{-\frac{\alpha+2}{1+r}-\varepsilon} \leq S_{\tau_n}^{(\alpha)} \leq n^{\frac{\alpha+2}{1+r}}(\log n)^{\frac{2\alpha+4}{1+r}+2+\varepsilon}.$$

(ii) Suppose that $r > 1+\alpha$. Then as $n \to \infty$, a.s.,

$$n^{-1}S_{\tau_n}^{(\alpha)} \to \mathbb{E}[\xi_1^{(\alpha)}] = \mathbb{E}[\eta_1]\sum_{x\in\mathcal{S}} x^\alpha \pi(x) \in (0,\infty), \tag{4.21}$$

where π is given by (2.10).

Proof. First we prove part (ii). For $r > 1+\alpha$, $\mathbb{E}[\xi_1^{(\alpha)}] < \infty$ by Theorem 2.8. Then, by (2.11), $\mathbb{E}[\xi_1^{(\alpha)}] = \sum_{x\in\mathcal{S}} x^\alpha \mathbb{E}[\ell_1(x)]$, so, by (2.10), the two expressions for limiting constant in (4.21) are indeed equivalent. Also, $\mathbb{E}[\xi_1^{(\alpha)}] > 0$ since, by Theorem 2.6, $\pi(x) > 0$ for all $x \in \mathcal{S}$. The convergence in (4.21) follows from the strong law of large numbers.

Now for part (i), suppose that $r \in (-1, \alpha+1]$. Then the lower bound in part (i) follows from the lower bound in (2.12) and Lemma 4.4(ii). The upper bound in part (i) follows from the upper bound in (2.12) and Lemma 4.4(i). □

Proof of Theorem 2.9. By definition of $S_t^{(\alpha)}$ and N_t, for any $t \in \mathbb{N}$,

$$S_{\tau_{N_t}}^{(\alpha)} \leq S_t^{(\alpha)} \leq S_{\tau_{N_t+1}}^{(\alpha)}. \tag{4.22}$$

For $-1 < r \leq 1+\alpha$, we have from Lemma 4.14(i) with (4.22) that for any $\varepsilon > 0$, a.s., for all but finitely many t,

$$N_t^{\frac{\alpha+2}{1+r}}(\log t)^{-\frac{\alpha+2}{1+r}-\varepsilon} \leq S_t^{(\alpha)} \leq (N_t+1)^{\frac{\alpha+2}{1+r}}(\log t)^{\frac{2\alpha+4}{1+r}+2+\varepsilon}, \tag{4.23}$$

using the fact that $N_t \leq t$ a.s. to obtain the logarithmic terms. Now from (4.23) we obtain part (i) of the theorem by applying the bounds for N_t in Theorem 2.5(i) and we obtain part (ii) of the theorem from Theorem 2.5(ii).

Finally, suppose that $r > 1 + \alpha$. We have from (4.22) that

$$(t^{-1}N_t)N_t^{-1}S^{(\alpha)}_{\tau_{N_t}} \le t^{-1}S^{(\alpha)}_t \le (t^{-1}(N_t+1))(N_t+1)^{-1}S^{(\alpha)}_{\tau_{N_t+1}}.$$

Both $t^{-1}N_t$ and $t^{-1}(N_t+1)$ converge a.s. to $\mathbb{E}[\eta_1]^{-1}$ by Theorem 2.5(ii), while Lemma 4.14(ii) and the fact that $N_t \to \infty$ a.s. as $t \to \infty$ imply that both $N_t^{-1}S^{(\alpha)}_{\tau_{N_t}}$ and $(N_t + 1)^{-1}S^{(\alpha)}_{\tau_{N_t+1}}$ converge a.s. to $\mathbb{E}[\xi_1^{(\alpha)}]$. Hence we obtain the first limit statement in (2.13); for the subsequent equality in (2.13) we use the expression for $\mathbb{E}[\xi_1^{(\alpha)}]$ given in (4.21). $\square$

5 Proofs for Section 3

5.1 Proofs for Section 3.1

In this section we prove our results on processes on the whole real line. The proofs use the same ideas as those for our results from Section 2, so we do not dwell on the details. We will use the notation $o_\omega(t)$ for a (random) sequence that satisfies $o_\omega(t) \to 0$, a.s., as $t \to \infty$ (in other words, $o_\omega(t)$ is an extension of the Landau $o(1)$ notation in which the implicit constants are allowed to depend on the sample point ω).

Proof of Theorem 3.1. We again use $\tau_0, \tau_1, \ldots$ to denote the times at which $X_t = 0$, and $\eta_n := \tau_n - \tau_{n-1}$. Conditions (B0) and (B1) ensure that $\eta_1, \eta_2, \ldots$ are i.i.d., and η_1 has the same distribution as $\theta_+\eta_+ + \theta_-\eta_- + (1 - \theta_+ - \theta_-)$, where $\theta_+ := \mathbf{1}\{X_2 > 0\}$, $\theta_- := \mathbf{1}\{X_2 < 0\}$ and (by (B2)) $\eta_\pm$ is the return time for a half-line model of Section 2 with $r = r_\pm$, independent of θ_+ and θ_-. Since $-1 < r_+ < r_- \le 1$, and $\mathbb{E}[\theta_\pm] > 0$, it follows from Theorem 2.4 that $\mathbb{P}[\eta_1 \ge x] = x^{-\frac{1+r_+}{2}+o(1)}$; hence the process is null-recurrent. By a similar argument to Theorem 2.5, since each excursion takes either sign with uniformly positive probability, there are $t^{\frac{1+r_+}{2}+o_\omega(1)}$ excursions of each sign by time t. The result then follows as in the proof of Theorem 2.7, using Lemma 4.10. $\square$

Proof of Theorem 3.2. Again, we use the fact that the numbers of positive or negative excursions up until time t are both $t^{\frac{1+r_+}{2}+o_\omega(1)}$, a.s. Then Lemma 4.14 and an argument similar to the proof of Theorem 2.9 applied separately to the positive and negative parts $\sum_{s=1}^t X_s^+$ and $\sum_{s=1}^t X_s^-$ shows that, a.s., the latter is $t^{\frac{3}{2}\frac{1+r_+}{1+r_-}+o_\omega(1)}$ while the former is $t^{\frac{3}{2}+o_\omega(1)}$, which therefore dominates the asymptotics, yielding the result. $\square$

5.2 Proofs for Section 3.2

We write $\mathbf{e}_1, \ldots, \mathbf{e}_d$ for the standard orthonormal basis of $\mathbb{R}^d$, and for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ we use $\mathbf{u} \cdot \mathbf{v}$ to denote their scalar product.

Proof of Theorem 3.3. Suppose that (C0) holds. Take $\mathcal{F}_t = \sigma(\xi_1, \xi_2, \ldots, \xi_t)$, $X_t = \|\xi_t\|$, and $\mathcal{S} = \{\|\mathbf{x}\| : \mathbf{x} \in \Sigma\} \subset [0, \infty)$. Then $0 \in \mathcal{S}$ since $\mathbf{0} \in \Sigma$, and, by local finiteness of Σ, $\{\mathbf{x} \in \Sigma : \|\mathbf{x}\| = x\}$ is finite for any $x \in \mathcal{S}$. Hence (A0) follows. Next we verify (A1). By irreducibility of Ξ, for any $\mathbf{x}, \mathbf{y} \in \Sigma$, there exist $k(\mathbf{x}, \mathbf{y}) \in \mathbb{N}$ and $\kappa(\mathbf{x}, \mathbf{y}) > 0$ such that $\mathbb{P}[\xi_{t+k(\mathbf{x},\mathbf{y})} = \mathbf{y} \mid \xi_t = \mathbf{x}] = \kappa(\mathbf{x}, \mathbf{y}) > 0$. Let $x = \|\mathbf{x}\|$ and $y = \|\mathbf{y}\|$, so $x, y \in \mathcal{S}$. Then, a.s.,

$$\begin{aligned}\mathbb{P}[X_{t+k(\xi_t,\mathbf{y})} = y \mid \mathcal{F}_t] &= \kappa(\xi_t, \mathbf{y}) \\ &\ge \min\{\kappa(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in \Sigma, \|\mathbf{x}\| = \|\xi_t\|,\ \mathbf{y} \in \Sigma, \|\mathbf{y}\| = y\};\end{aligned}$$

denote this last quantity $\varphi(\|\xi_t\|, y)$. Then $\varphi(\|\xi_t\|, y) > 0$ by the finiteness of the sets over which $\mathbf{x}$ and $\mathbf{y}$ run. We choose $\mathbf{y}$ with $\|\mathbf{y}\| = y$, and for that $\mathbf{y}$ take $m(\|\xi_t\|, y) = k(\xi_t, \mathbf{y}) < \infty$. This shows that (A1) holds. Moreover, (A3) follows from the fact that Ξ is an irreducible Markov chain. Also, by the triangle inequality, $|X_{t+1} - X_t| = |\|\xi_{t+1}\| - \|\xi_t\|| \leq \|\xi_{t+1} - \xi_t\|$, so if (3.3) holds for some $p > 0$, then so does (2.2).

It remains to show that (C1) implies (A2). Let $\gamma \in (0, 1)$, to be chosen later. We will estimate the increment $\|\xi_t + \theta_t\| - \|\xi_t\|$ by Taylor's theorem in $\mathbb{R}^d$. First observe that

$$\frac{\partial}{\partial x_i}\|\mathbf{x}\| = \frac{x_i}{\|\mathbf{x}\|}; \quad \frac{\partial^2}{\partial x_i \partial x_j}\|\mathbf{x}\| = \frac{\mathbf{1}\{i = j\}}{\|\mathbf{x}\|} - \frac{x_i x_j}{\|\mathbf{x}\|^3}; \quad \left|\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\|\mathbf{x}\|\right| = O(\|\mathbf{x}\|^{-2}).$$

Then by Taylor's formula, for any $\mathbf{x} \in \mathbb{R}^d$,

$$\begin{aligned}
(\|\mathbf{x} + \theta_t\| - \|\mathbf{x}\|)\,\mathbf{1}\{\|\theta_t\| \leq \|\mathbf{x}\|^\gamma\} &= \sum_{i=1}^d \frac{x_i}{\|\mathbf{x}\|}(\theta_t \cdot \mathbf{e}_i)\mathbf{1}\{\|\theta_t\| \leq \|\mathbf{x}\|^\gamma\} \\
&\quad + \frac{1}{2}\sum_{i=1}^d \left(\frac{1}{\|\mathbf{x}\|} - \frac{x_i^2}{\|\mathbf{x}\|^3}\right)(\theta_t \cdot \mathbf{e}_i)^2 \mathbf{1}\{\|\theta_t\| \leq \|\mathbf{x}\|^\gamma\} \\
&\quad - \sum_{i=2}^d \sum_{j=1}^{i-1} \frac{x_i x_j}{\|\mathbf{x}\|^3}(\theta_t \cdot \mathbf{e}_i)(\theta_t \cdot \mathbf{e}_j)\mathbf{1}\{\|\theta_t\| \leq \|\mathbf{x}\|^\gamma\} \\
&\quad + O\left(\|\theta_t\|^3 \|\mathbf{x}\|^{-2}\mathbf{1}\{\|\theta_t\| \leq \|\mathbf{x}\|^\gamma\}\right).
\end{aligned} \tag{5.1}$$

We will condition on $\xi_t = \mathbf{x}$ and take expectations in (5.1). To this end, note that

$$\mathbb{E}[\|\theta_t\|^3 \|\mathbf{x}\|^{-2}\mathbf{1}\{\|\theta_t\| \leq \|\mathbf{x}\|^\gamma\} \mid \xi_t = \mathbf{x}] \leq \|\mathbf{x}\|^{\gamma-2}\mathbb{E}[\|\theta_t\|^2 \mid \xi_t = \mathbf{x}] = O(\|\mathbf{x}\|^{\gamma-2}),$$

since, by (C1), $\mathbb{E}[\|\theta_t\|^2 \mid \xi_t = \mathbf{x}] = \sum_{i=1}^d M_{ii}(\mathbf{x}) < \infty$, uniformly in $\mathbf{x}$. Also, for $q \in [0, 2]$ a similar argument to Lemma 4.2 shows that, for some $\varepsilon > 0$ and γ close enough to 1,

$$\mathbb{E}[(\theta_t \cdot \mathbf{e}_i)^q \mathbf{1}\{\|\theta_t\| > \|\mathbf{x}\|^\gamma\} \mid \xi_t = \mathbf{x}] = O(\|\mathbf{x}\|^{q-2-\varepsilon}). \tag{5.2}$$

The $q = 2$ case of (5.2), with the Cauchy–Schwarz inequality, shows that

$$\mathbb{E}[(\theta_t \cdot \mathbf{e}_i)(\theta_t \cdot \mathbf{e}_j)\mathbf{1}\{\|\theta_t\| > \|\mathbf{x}\|^\gamma\} \mid \xi_t = \mathbf{x}] = O(\|\mathbf{x}\|^{-\varepsilon}).$$

Also, we have from (C1) that

$$\begin{aligned}
\mathbb{E}[\theta_t \cdot \mathbf{e}_i \mid \xi_t = \mathbf{x}] = \mathbf{e}_i \cdot \mu(\mathbf{x}) &= \frac{\rho x_i}{\|\mathbf{x}\|^2} + o(\|\mathbf{x}\|^{-1}\log^{-1}\|\mathbf{x}\|); \\
\mathbb{E}[(\theta_t \cdot \mathbf{e}_i)(\theta_t \cdot \mathbf{e}_j) \mid \xi_t = \mathbf{x}] = M_{ij}(\mathbf{x}) &= \sigma^2 \mathbf{1}\{i = j\} + o(\log^{-1}\|\mathbf{x}\|).
\end{aligned}$$

Combining these estimates, taking expectations in (5.1) yields

$$\begin{aligned}
&\mathbb{E}\left[(\|\mathbf{x} + \theta_t\| - \|\mathbf{x}\|)\,\mathbf{1}\{\|\theta_t\| \leq \|\mathbf{x}\|^\gamma\} \mid \xi_t = \mathbf{x}\right] \\
&= \sum_{i=1}^d \frac{\rho x_i^2}{\|\mathbf{x}\|^3} + \frac{1}{2}\sum_{i=1}^d \left(\frac{1}{\|\mathbf{x}\|} - \frac{x_i^2}{\|\mathbf{x}\|^3}\right)\sigma^2 + o(\|\mathbf{x}\|^{-1}\log^{-1}\|\mathbf{x}\|) \\
&= \left(\rho + \frac{\sigma^2}{2}(d-1)\right)\|\mathbf{x}\|^{-1} + o(\|\mathbf{x}\|^{-1}\log^{-1}\|\mathbf{x}\|).
\end{aligned}$$

On the other hand, by the triangle inequality,

$$\mathbb{E}\left[|\|\mathbf{x}+\theta_t\| - \|\mathbf{x}\|| \, \mathbf{1}\{\|\theta_t\| > \|\mathbf{x}\|^\gamma\} \mid \xi_t = \mathbf{x}\right] \leq \mathbb{E}\left[\|\theta_t\| \mathbf{1}\{\|\theta_t\| > \|\mathbf{x}\|^\gamma\}\right],$$

which, for γ close enough to 1, is also $o(\|\mathbf{x}\|^{-1}\log^{-1}\|\mathbf{x}\|)$ by another application of (5.2). Thus we have shown that

$$\mathbb{E}[X_{t+1} - X_t \mid \xi_t = \mathbf{x}] = \left(\rho + \frac{\sigma^2}{2}(d-1)\right)\|\mathbf{x}\|^{-1} + o(\|\mathbf{x}\|^{-1}\log^{-1}\|\mathbf{x}\|), \tag{5.3}$$

which implies that (2.3) holds with $c = \rho + (d-1)(\sigma^2/2)$.

For the second moment estimate, observe that, given $\xi_t = \mathbf{x}$,

$$\begin{aligned}(X_{t+1} - X_t)^2 &= \|\mathbf{x}+\theta_t\|^2 - \|\mathbf{x}\|^2 - 2\|\mathbf{x}\|(\|\mathbf{x}+\theta_t\| - \|\mathbf{x}\|) \\ &= \|\theta_t\|^2 + 2\mathbf{x}\cdot\theta_t - 2\|\mathbf{x}\|(X_{t+1} - X_t). \end{aligned}\tag{5.4}$$

Here we have that

$$\mathbb{E}[\|\theta_t\|^2 + 2\mathbf{x}\cdot\theta_t \mid \xi_t = \mathbf{x}] = \sum_{i=1}^d M_{ii}(\mathbf{x}) + 2\mathbf{x}\cdot\mu(\mathbf{x}) = d\sigma^2 + 2\rho + o(\log^{-1}\|\mathbf{x}\|). \tag{5.5}$$

Taking expectations in (5.4), using (5.5) and (5.3), we obtain

$$\mathbb{E}[(X_{t+1} - X_t)^2 \mid \xi_t = \mathbf{x}] = d\sigma^2 + 2\rho - 2(\rho + (d-1)(\sigma^2/2)) + o(\log^{-1}\|\mathbf{x}\|),$$

which, after simplification, shows that (2.4) holds with $s^2 = \sigma^2$. $\square$

5.3 Proofs for Section 3.3

First we prove the following analogue of Lemma 7.6 of [9]. As in [9], we relate the general version of Z_t to the special case in which $\kappa = 0$ a.s., which we denote here by Z'_t. By construction, for $x, y \in \mathbb{N}$,

$$\mathbb{P}[Z_{t+1} = y \mid Z_t = x] = \mathbb{E}\left[\mathbb{P}[Z'_{t+1} = y \mid Z'_t = x - \min\{\kappa, x-1\}]\right]. \tag{5.6}$$

Write $D_t := Z_{t+1} - Z_t$.

Lemma 5.1. *For any $\varepsilon > 0$, as $x \to \infty$,*

$$\mathbb{P}[|D_t| > x^{(1/2)+\varepsilon} \mid Z_t = x] = O(\exp\{-x^{\varepsilon/3}\}). \tag{5.7}$$

Also, for any $r \in \mathbb{N}$ there exists $C < \infty$ for which, for all $x \in \mathbb{N}$,

$$\mathbb{E}[|D_t|^r \mid Z_t = x] \leq Cx^{r/2}.$$

Moreover, as $x \to \infty$,

$$\mathbb{E}[D_t \mid Z_t = x] = \frac{2}{3} - \mathbb{E}[\kappa] + o(\log^{-1} x), \tag{5.8}$$

$$\mathbb{E}[D_t^2 \mid Z_t = x] = \frac{2}{3}x + o(x\log^{-1} x). \tag{5.9}$$

Proof. The proof is similar to that of Lemma 7.6 in [9]; we sketch the differences, which are due to the fact that we use (5.6) in place of the final statement of Lemma 7.5 in [9]. Write $D'_t := Z'_{t+1} - Z'_t$. Lemma 6.4 in [9] says that, for a given $\alpha > 0$,

$$\mathbb{E}[D'_t \mid Z'_t = x] = \frac{2}{3} + O(\mathrm{e}^{-\alpha x}). \tag{5.10}$$

We also note that, by Markov's inequality and our tail assumption on κ, there is $C < \infty$ for which, for all $r \geq 1$,

$$\mathbb{P}[|\kappa| > r] \leq C\mathrm{e}^{-\lambda r}. \tag{5.11}$$

We prove (5.7). By (5.6), for any $\varepsilon > 0$, and any $x > 1$,

$$\mathbb{P}[|D_t| > r \mid Z_t = x] \leq \mathbb{P}[|\kappa| > x^\varepsilon] + \sup_{y:|x-y|\leq x^\varepsilon} \mathbb{P}[|D'_t| > r - x^\varepsilon \mid Z'_t = x].$$

Taking $r = x^{(1/2)+\varepsilon}$, using (5.11) and the tail bound for D'_t given in Lemma 6.3 of [9], we verify (5.7). For (5.8), it follows from (5.6) that

$$\mathbb{E}[D_t \mid Z_t = x] = -\mathbb{E}[\min\{\kappa, x-1\}] + \mathbb{E}\left[\mathbb{E}[D'_t \mid Z'_t = x - \kappa]\right].$$

Here, as in the proof of Lemma 7.6 in [9], $\mathbb{E}[\min\{\kappa, x-1\}] = \mathbb{E}[\kappa] + O(\exp\{-\lambda x/2\})$. Also, using the fact that $\sup_x \mathbb{E}[D'_t \mid Z'_t = x] < C < \infty$ by (5.10), we have

$$\mathbb{E}\left[\mathbb{E}[D'_t \mid Z'_t = x - \kappa]\right] \leq C\mathbb{P}[|\kappa| > \sqrt{x}] + \sup_{y:|y-x|\leq\sqrt{x}} \mathbb{E}[D'_t \mid Z'_t = y],$$

which with (5.10) and (5.11) gives the upper bound in (5.8), a similar argument yielding the lower bound. Similar variations of the arguments in the proof of Lemma 7.6 of [9] give the remaining parts of the lemma. □

Proof of Proposition 3.6. Proposition 3.6 follows from Lemma 5.1 in exactly the same way as Lemma 7.7 in [9] follows from Lemma 7.6 there. □

Proof of Theorem 3.7. We proceed as in the proof of Theorem 2.6 of [9], but instead of Lemma 8.3 in [9], we apply our sharper Theorem 2.8. The details involve minor modifications to the arguments in [9]. Here we merely give some intuition as to why $\xi_1^{(2)}$ appears. The key fact is that, ignoring the jumps driven by κ, the original process takes $Z_t + Z_{t+1}$ steps to traverse the quadrant between times ν_t and ν_{t+1}; the correction to this due to the jump of size κ_t is small. Hence over one excursion of the embedded process $X_t = \sqrt{Z_t - 1}$, the original process accumulates time $\tau \approx \sum_{t=1}^{\tau_q} X_t^2$, which is exactly of the form of the excursion sum $\xi_1^{(2)}$. □

Acknowledgements

Most of this work was done while the third author was at the University of Strathclyde. We benefited in the early stages of this project from enjoyable discussions with Iain MacPhee, who sadly passed away on 13th January 2012; we dedicate this paper to Iain, in memory of our valued colleague and in gratitude for his generosity.

References

[1] K.S. Alexander, Excursions and local limit theorems for Bessel-like random walks, *Electron. J. Probab.* **16** (2011) 1–44.

[2] S. Asmussen, Applied Probability and Queues, 2nd edition, Springer-Verlag, New York, 2003.

[3] S. Aspandiiarov and R. Iasnogorodski, Tails of passage-times and an application to stochastic processes with boundary reflection in wedges, *Stochastic Process. Appl.* **66** (1997) 115–145.

[4] S. Aspandiiarov and R. Iasnogorodski, General criteria of integrability of functions of passage-times for non-negative stochastic processes and their applications, *Theory Probab. Appl.* **43** (1999) 343–369; translated from *Teor. Veroyatn. Primen.* **43** (1998) 509–539 (in Russian).

[5] S. Aspandiiarov, R. Iasnogorodski, and M. Menshikov, Passage-time moments for nonnegative stochastic processes and an application to reflected random walks in a quadrant, *Ann. Probab.* **24** (1996) 932–960.

[6] A.N. Borodin and I.A. Ibragimov, Limit Theorems for Functionals of Random Walks, *Proc. Steklov Inst. Math.* **195**, American Mathematical Society, Providence, Rhode Island, 1995.

[7] A.A. Borovkov, O.J. Boxma, and Z. Palmowski, On the integral of the workload process of the single server queue, *J. Appl. Probab.* **40** (2003) 200–225.

[8] F. Comets, M.V. Menshikov, S. Volkov, and A.R. Wade, Random walk with barycentric self-interaction, *J. Stat. Phys.* **143** (2011) 855–888.

[9] E. Crane, N. Georgiou, S. Volkov, A.R. Wade, and R.J. Waters, The simple harmonic urn, *Ann. Probab.* **39** (2011) 2119–2177.

[10] E. Csáki and M. Csörgő, On additive functionals of Markov chains, *J. Theoret. Probab.* **8** (1995) 905–919.

[11] E. Csáki, A. Földes, and P. Révész, Transient nearest neighbor random walk and Bessel process, *J. Theoret. Probab.* **22** (2009) 992–1009.

[12] J. De Coninck, F. Dunlop, and T. Huillet, Random walk weakly attracted to a wall, *J. Stat. Phys.* **133** (2008) 271–280.

[13] A. Dvoretzky and P. Erdős, Some problems on random walk in space, in: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 353–367. University of California Press, Berkeley and Los Angeles, 1951.

[14] A.M. Fal', Recurrence times for certain Markov random walks, *Ukrainian Math. J.* **23** (1971) 676–681; translated from *Ukrain. Mat. Zh.* **23** (1971) 824–830 (in Russian).

[15] A.M. Fal', Generalization of the results obtained by R.L. Dobrushin for additive functionals of a Markov random walk, *Theory Probab. Appl.* **22** (1978) 569–571; translated from *Teor. Veroyatn. Primen.* **22** (1977) 582–584 (in Russian).

[16] A.M. Fal', Certain limit theorems for an elementary Markov random walk, *Ukrainian Math. J.* **33** (1981) 433–435; translated from *Ukrain. Mat. Zh.* **33** (1981) 564–566 (in Russian).

[17] W. Feller, A limit theorem for random variables with infinite moments, *Amer. J. Math.* **68** (1946) 257–262.

[18] L. Gallardo, Comportement asymptotique des marches aléatoires associées aux polynômes de Gegenbauer et applications, *Adv. in Appl. Probab.* **16** (1984) 293–323 (in French).

[19] G. Giacomin, Random Polymer Models, Imperial College Press, London, 2007.

[20] J. Gillis, Centrally biased discrete random walk, *Quart. J. Math. Oxford Ser. (2)* **7** (1956) 144–152.

[21] K. Grill, On the average of a random walk, *Statist. Probab. Lett.* **6** (1988) 357–361.

[22] F. den Hollander, Random Polymers, *Lecture Notes in Mathematics* **1974**, Springer-Verlag, Berlin, 2009.

[23] T. Huillet, Random walk with long-range interaction with a barrier and its dual: Exact results, *J. Comput. Appl. Math.* **233** (2010) 2449–2467.

[24] M. Jara, T. Komorowski, and S. Olla, Limit theorems for additive functionals of a Markov chain, *Ann. Appl. Probab.* **19** (2009) 2270–2300.

[25] O. Kallenberg, Foundations of Modern Probability, 2nd edition, Springer, New York, 2002.

[26] S. Karlin and J. McGregor, Random walks, *Illinois J. Math.* **3** (1959) 66–81.

[27] C. Kipnis and S.R.S. Varadhan, Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusion, *Comm. Math. Phys.* **104** (1986) 1–19.

[28] V.S. Korolyuk, The central limit theorem for additive functionals of Markov and semi-Markov processes, *J. Soviet Math.* **38** (1987) 2299–2308; translated from pp. 229–246 Veroyatnostnye Raspredeleniya i Matematicheskaya Statistika, Papers from the third All-Union conference on probability theory and mathematical statistics held in Fergana, September 20–22, 1983, S.Kh. Sirazhdinov and V.M. Zolotarev (eds.), Fan, Tashkent, 1986 (in Russian).

[29] Y. Kovchegov and N. Michalowski, A class of Markov chains with no spectral gap, to appear in *Proc. Amer. Math. Soc.* Preprint `arXiv:1109.6050`.

[30] J. Lamperti, Criteria for the recurrence or transience of stochastic processes I, *J. Math. Anal. Appl.* **1** (1960) 314–330.

[31] J. Lamperti, Criteria for stochastic processes II: passage-time moments, *J. Math. Anal. Appl.* **7** (1963) 127–145.

[32] M. Loève, Probability Theory I, 4th edition, Springer-Verlag, New York, 1977.

[33] M. Maxwell and M. Woodroofe, Central limit theorems for additive functionals of Markov chains, *Ann. Probab.* **28** (2000) 713–724.

[34] M.V. Menshikov, I.M. Asymont, and R. Iasnogorodskii, Markov processes with asymptotically zero drifts, *Probl. Inf. Transm.* **31** (1995) 248–261; translated from *Problemy Peredachi Informatsii* **31** (1995) 60–75 (in Russian).

[35] M.V. Menshikov and S.Yu. Popov, Exact power estimates for countable Markov chains, *Markov Process. Related Fields* **1** (1995) 57–78.

[36] M.V. Menshikov, M. Vachkovskaia, and A.R. Wade, Asymptotic behaviour of randomly reflecting billiards in unbounded tubular domains, *J. Stat. Phys.* **132** (2008) 1097–1133.

[37] W.A. Rosenkrantz, A local limit theorem for a certain class of random walks, *Ann. Math. Statist.* **37** (1966) 855–859.

[38] G.J. Székely, On the asymptotic properties of diffusion processes, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **17** (1974) 69–71.

[39] Y. Velenik, Localization and delocalization of random interfaces, *Probab. Surv.* **3** (2006) 112–169.

[40] M. Voit, A law of the iterated logarithm for a class of polynomial hypergroups, *Monatsh. Math.* **109** (1990) 311–326.