

REFLECTION SUBGROUPS OF ODD-ANGLED COXETER GROUPS

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ABSTRACT. We give a criterion for a finitely generated odd-angled Coxeter group to have a proper finite index subgroup generated by reflections. The answer is given in terms of the least prime divisors of the exponents of the Coxeter relations.

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1. INTRODUCTION

Reflection subgroups of Coxeter groups arise in various contexts. As proved by Dyer [5] and Deodhar [4], subgroups of Coxeter groups generated by reflections are Coxeter groups themselves. In the arithmetic over $\mathbb{Q}$ case they provide regular subalgebras of corresponding Kac-Moody algebras (see [7, 20, 8, 10, 13]).

Classifications of reflection subgroups of Coxeter groups are known in some special cases. Namely, the reflection subgroups of finite [1, 7] and affine [5, 11, 6] Coxeter groups are completely classified. For reflection groups in the real hyperbolic space, there is a classification [13] of reflection subgroups in the case of both the group and the subgroup having simplicial fundamental domains (the simplices may have distinct dimensions).

Of special interest are reflection subgroups of finite index. In the arithmetic case, they correspond to those regular subalgebras of Kac-Moody algebras that have maximal rank. In the case of reflection

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groups acting on a space of constant curvature, a finite index reflection subgroup provides a tessellation of its fundamental polytope by copies of the fundamental polytope of the group. The same picture can be observed on the Davis complex of a general Coxeter group with finite index reflection subgroup [12].

In this paper, we solve the general problem of existence of finite index reflection subgroups in odd-angled Coxeter groups, i.e. in the groups with all orders m_{ij} of products $s_i s_j$ of generating reflections being odd (see Section 2). The answer is given in terms of a *divisibility diagram* $\text{Cox}_{\text{div}}(W)$ which is a modification of the Coxeter diagram: the edges of $\text{Cox}_{\text{div}}(W)$ are labeled by least (non-trivial) divisors of m_{ij} instead of m_{ij} themselves, and the vertices are not joined if $m_{ij} = \infty$. The connectivity of the divisibility diagram of an odd-angled Coxeter group is equivalent to the existence of one conjugacy class containing all reflections of the group.

We approach the problem by considering *special subgroups* of the given Coxeter group (they are also called *standard parabolic* in the literature) generated by a subset of the initial generating set (see Section 2 for the precise definitions). The main tool relating finite index reflection subgroups in a Coxeter group and in its special subgroups is the following *Subdiagram Lemma*:

Corollary 3.2 (Subdiagram Lemma). *Let W be a Coxeter group with set of generators S such that $\text{Cox}_{\text{div}}(W)$ is connected. Suppose that $V \subsetneq W$ is a reflection subgroup of index n , $1 < n < \infty$. Let W_1 be a special subgroup of W . Then W_1 contains a proper reflection subgroup of index at most n .*

The Subdiagram Lemma implies that the divisibility diagrams of Coxeter groups with proper finite index reflection subgroups compose a partially ordered set (with the order being inclusion), which means that it is sufficient to classify minimal (by inclusion) divisibility diagrams defining Coxeter groups without subgroups. This is done in Theorem 6.1, the result is shown in Table 6.1.

The groups with disconnected divisibility diagrams are treated based on the following lemma:

Lemma 6.3. *Let W be an odd-angled Coxeter group and $W = W_1 * W_2 * \dots * W_k$. Then W contains a proper finite index reflection subgroup if and only if at least one of $W_1, \dots, W_k$ contains one.*

This brings us to the following criterion.

Theorem 6.4. *An odd-angled Coxeter group W contains no finite index proper reflection subgroup if and only if each connected component of $\text{Cox}_{\text{div}}(W)$ contains one of the diagrams shown in Table 6.1 as a subdiagram.*

We can also reformulate the criterion to get a self-contained form of the statement, i.e., to avoid references to the table of minimal groups without subgroups.

Corollary 6.5. *An odd-angled Coxeter group W contains a finite index proper reflection subgroup if and only if $\text{Cox}_{\text{div}}(W)$ contains at least one connected component C of one of the following three types:*

1. *the order of C is 1 or 2;*
2. *C contains at most one multiple edge;*
3. *C contains a subdiagram D of order 3 with labels $(5, 5, 3)$, and every non-absent edge of C except the edges of D is simple.*

The paper is organized as follows. In Section 2, we recall necessary facts about Coxeter groups and their Davis complexes. Section 3 is devoted to the proof of the Subdiagram Lemma (Corollary 3.2). In Section 4, we construct examples of finite index reflection subgroups in two series of odd-angled Coxeter groups. Section 5 is devoted to the proof of absence of finite index reflection subgroups in most Coxeter groups with connected divisibility diagram. The combinatorial tools necessary for the proof are developed in Section 5.1. Finally, in Section 6 we combine the results of the previous two sections to obtain the list (Table 6.1) of minimal groups containing no finite index reflection subgroups (see Theorem 6.1). We also prove Lemma 6.3 (concerning disconnected divisibility diagrams) and use it to prove Theorem 6.4.

We note that some of the technical tools and partial results (especially in Section 5) still hold if we consider a larger class of groups (namely, *skew-angled* Coxeter groups, where m_{ij} may be even but not equal to 2), and some even for arbitrary Coxeter groups. However, already in rank 3 there are series of examples of finite index subgroups of skew-angled Coxeter groups (see Remark 4.2) indicating that the answer for skew-angled groups will be more complicated.

2. PRELIMINARIES

In this section, we will mainly follow [3] to reproduce definitions and essential properties of Coxeter groups and related constructions.

2.1. Coxeter system. A group W is called a *Coxeter group* if it has a representation of the form

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \forall s_i, s_j \in S \rangle$$

where S is a set, $m_{ii} = 1$ and $m_{ij} \in \mathbb{N}_{>1} \cup \{\infty\}$ for all $i \neq j$. Thereby $m_{ij} = \infty$ means that there is no relation on $s_i s_j$. Furthermore, throughout the paper we require S to be finite. A pair (W, S) of a Coxeter group W and its set of generators S is called a *Coxeter system*. The cardinality of S is called *rank* of the Coxeter system.

An element of W is said to be a *reflection* if it is conjugated in W to an element of S . A *reflection subgroup* in a Coxeter group is a proper subgroup generated by reflections, where “proper” means of index greater than one. A *special subgroup* of (W, S) is a reflection subgroup generated by elements of S' where $S' \subset S$.

A Coxeter group is called *skew-angled* if $m_{ij} \neq 2$ for every pair (i, j) and *odd-angled* if all m_{ij} are odd or infinite.

Coxeter groups are usually presented by Coxeter diagrams (see [21]). In this paper it will be convenient to use the following modification of Coxeter diagrams.

Definition 2.1 (Divisibility diagram). Let (W, S) be a Coxeter system. A *divisibility diagram* $\text{Cox}_{\text{div}}(W)$ of W is a one-dimensional simplicial complex with edges labeled by positive integers constructed in the following way:

- vertices v_i of $\text{Cox}_{\text{div}}(W)$ correspond to generating reflections $s_i \in S$;
- vertex v_i is joined with vertex v_j by an edge labeled by $k > 1$ if k is the minimal non-trivial divisor of m_{ij} (as in Coxeter diagrams, label $k = 3$ is omitted);
- v_i and v_j are not joined if the order of $(s_i s_j)$ is infinite.

We call an edge without any label *simple*, and all the other edges *multiple*. If two vertices are not joined, we say they are joined by an *absent edge*. A divisibility diagram $\text{Cox}_{\text{div}}(W)$ of a skew-angled Coxeter group can be obtained from the Coxeter diagram in the following way: substitute all labels by their least prime divisors, and delete all dashed edges (corresponding to infinite dihedral subgroups).

By a *subdiagram* of a divisibility diagram (or a Coxeter diagram) we always mean “full” subdiagram, i.e., a diagram obtained by removing some vertices and all edges emanating from them.

2.2. Length function and Exchange Condition. Since a Coxeter group W is generated by the elements of S , we can write each $w \in W$ in the form $s_1 s_2 \cdots s_k$, where $s_1, s_2, \dots, s_k$ are some (not necessarily distinct) elements of S . If k is chosen such that w cannot be written as a product of less than k elements of S , we call $s_1 s_2 \cdots s_k$ a *reduced expression* for w and say that k is the *length* of w , which we denote by $l(w)$. By convention $l(1) = 0$.

The proof of the following fundamental result on the length function for Coxeter groups can be found in [14, Theorem 5.8].

Theorem 2.2 (Strong Exchange Condition). *Let (W, S) be a Coxeter system and $w = s_1 s_2 \cdots s_n$ with $s_1, s_2, \dots, s_n$ being not necessarily distinct elements of S . If t is a reflection satisfying $l(wt) < l(w)$, then there exists $i \leq n$ for which $wt = s_1 \cdots \hat{s}_i \cdots s_n$, where $\hat{s}_i$ denotes that the element s_i is omitted.*

If the element t in the above theorem is required to be contained in S and the word $s_1 \cdots s_n$ is reduced, the resulting weaker statement is known as Exchange Condition. The following direct corollary will be useful later in this paper.

Corollary 2.3. *Let (W, S) be a Coxeter system, $w \in W$ and $t \in S$. If $l(wt) < l(w)$, then there exists a reduced expression of w that ends in t .*

To prove the corollary take a reduced expression $s_1 \cdots s_n$ for w and multiply the equation $wt = s_1 \cdots \hat{s}_i \cdots s_n$ by t from the right.

As an element of W might be represented by many different expressions, a natural question to ask is when two expressions represent the same element. The following theorem due to Tits [19], whose proof is based on the Exchange Condition (see [3, Theorem 3.4.2]), provides us with an algorithm to solve this question.

Theorem 2.4. *An expression $w = s_1 s_2 \cdots s_n$ with $s_1, \dots, s_n \in S$ is reduced if and only if it cannot be shortened by a sequence of the following two operations (called M-operations):*

- (1) *Delete $s_i s_{i+1}$ with $s_i = s_{i+1}$, $1 \leq i < n$, i.e. $w = s_1 s_2 \cdots \hat{s}_i \hat{s}_{i+1} \cdots s_n$.*
- (2) *Replace $s_i s_{i+1} \cdots s_{i+m_{ij}-1}$ with $s_{i+2m} = s_i$ for $1 \leq m \leq (m_{ij} - 1)/2$ and $s_{i+2m-1} = s_j$ for $1 \leq m \leq m_{ij}/2$ by $s_j s_i s_{i+1} \cdots s_{i+m_{ij}-2}$.*

Moreover, two reduced expressions $s_1 s_2 \cdots s_n$ with $s_1, \dots, s_n \in S$ and $t_1 t_2 \cdots t_n$ with $t_1, \dots, t_n \in S$ represent the same element in W if and only if they can be transformed into each other by a sequence of operations of the second type.

We have already observed that if $l(wt) < l(w)$ for a generator $t \in S$, then there exists a reduced expression of w that ends in t . The following property of the set of possible last letters of reduced expressions for a given element of W will be used in the sequel.

Lemma 2.5 ([3], Lemma 4.7.2). *Let $w \in W$ and denote by $In(w)$ the subset of S in which a reduced expression of w can end. Then the subgroup generated by $In(w)$ is finite.*

2.3. Davis complex. For any Coxeter system (W, S) there exists a contractible piecewise Euclidean cell complex $\Sigma(W, S)$ (called *Davis complex*) on which W acts discretely, properly and cocompactly. The construction was introduced by Davis [2]. In [15] Moussong proved that this complex yields a natural complete piecewise Euclidean metric which is $CAT(0)$. We give a brief description of this complex following [17].

For a finite group W the complex $\Sigma(W, S)$ is just one cell, which is obtained as the convex hull C of the W -orbit of a suitable point p in the standard linear representation of W as a group generated by reflections. The point p is chosen in such a way that its stabilizer in W is trivial and all the edges of C are of length 1. The faces of C are naturally identified with Davis complexes of the subgroups of W conjugated to special subgroups.

If W is infinite, the complex $\Sigma(W, S)$ is built up of the Davis complexes of maximal finite subgroups of W by gluing them together along their faces corresponding to common finite subgroups. The 1-skeleton of $\Sigma(W, S)$ considered as a combinatorial graph is isomorphic to the Cayley graph of W with respect to the generating set S .

In what follows, if W and S are fixed, we write Σ instead of $\Sigma(W, S)$.

2.4. Walls and convex polytopes. The group W admits a natural action on $\Sigma(W, S)$ by reflections. The action is an isometry with respect to $CAT(0)$ piecewise Euclidean metric. A *wall* H_w corresponding to a reflection $w \in W$ is the fixed point set of $\Sigma(W, S)$ under the action of w . In particular, two walls H_w and H_u intersect if and only if the dihedral group generated by u and w is finite. Any wall divides Σ into two connected components. We denote their closures by H_w^+ and H_w^- and call them *halfspaces*. Walls are totally geodesic, i.e. any geodesic between two points contained in the same wall lies entirely in this wall, see [16]. This implies that every intersection of walls is, in its turn, totally geodesic, and

halfspaces are convex. We note that since $\Sigma(W, S)$ is $CAT(0)$, there is a unique geodesic through every two points of $\Sigma(W, S)$.

Following [12], we call an intersection of finitely many halfspaces not contained in any wall *convex polytope*. In the sequel writing $P = \bigcap_{i \leq n} H_{w_i}^+$ we always assume that P cannot be defined by less than n walls.

A convex polytope that does not contain any other convex polytope is called a *chamber*. Chambers are fundamental domains of the action of W on Σ , and each chamber contains precisely one vertex of the 1-skeleton of Σ which corresponds to an element of the group W . Following [16], we will denote the chamber of Σ corresponding to $w \in W$ by $D(w)$. In the sequel, by abuse of notation, we will sometimes identify the chamber corresponding to $w \in W$ with w itself.

We call two elements w and v in W *neighbors* if and only if there exists $s \in S$ such that $w = vs$ (this corresponds to two neighboring chambers of Σ). Then a *gallery of length k* is a sequence of chambers $D(w_1), D(w_2), \dots, D(w_k)$, $w_1, \dots, w_k \in W$, such that for every $1 \leq i \leq k-1$ the elements w_i and w_{i+1} are neighbors. A gallery is *geodesic* if its length is equal to $l(w_1^{-1}w_k) + 1$. Note that for any two chambers there exists a geodesic gallery between them.

The following lemma is well-known (see e.g. [18, Lemma 2.5(i)] or [16, Lemma 2.2.5]).

Lemma 2.6. *A gallery is geodesic if and only if it crosses any wall at most once.*

Let $P = \bigcap_{i \leq n} H_{w_i}^+$ be a convex polytope and I be a subset of $\{1, 2, \dots, n\}$. If the intersection $\bigcap_{i \in I} H_{w_i} \cap P$ is nonempty, it is called a *face* of P . The intersection of P with one of the walls H_{w_i} is called a *facet* of P . The walls H_{w_i} itself will be called *defining walls* of P .

Following [12], we define the *dihedral angle* formed by two intersecting walls H_{w_1} and H_{w_2} to be the Euclidean dihedral angle between $H_{w_1} \cap C$ and $H_{w_2} \cap C$ in C , where C is a maximal cell of Σ containing $H_{w_1} \cap H_{w_2}$. We define analogously the *dihedral angle* formed by two intersecting facets f and g of a convex polytope P to be the dihedral angle between $f \cap \tilde{C}$ and $g \cap \tilde{C}$, where $\tilde{C}$ is a cell of Σ such that $\tilde{C} \cap f \cap g$ is nonempty. If all the dihedral angles between any two intersecting facets of a convex polytope P are less than or equal to $\pi/2$, we say that the convex polytope P is *acute-angled*. A convex polytope is a *Coxeter polytope* if all its dihedral angles are integer submultiples of π . Note that a chamber is a Coxeter polytope with angles π/m_{ij} .

Remark 2.7. For Coxeter groups W acting cocompactly on the Euclidean space $\mathbb{E}^n$ or the hyperbolic space $\mathbb{H}^n$ the cell complex structure of the Davis complex of W can be naturally identified with the cell complex constructed from fundamental polytopes of the W -action on $\mathbb{E}^n$ or $\mathbb{H}^n$. In particular, this holds for rank 3 groups with $\frac{1}{m_{12}} + \frac{1}{m_{13}} + \frac{1}{m_{23}} < 1$, $m_{ij} \neq \infty$. If some of m_{ij} are infinite in a rank 3 group W , then W acts on the hyperbolic plane $\mathbb{H}^2$ with a fundamental triangle P of finite volume, and the combinatorics (but not the topology!) of the tessellation of $\mathbb{H}^2$ by copies of P coincides with the combinatorics of the Davis complex of W .

3. SUBDIAGRAM LEMMA

In this short section we prove the Subdiagram Lemma, which states that the property of having a finite index reflection subgroup is preserved when we take a special subgroup of the group.

Let W be an arbitrary finitely generated Coxeter group, let $\Sigma(W)$ be the Davis complex of W and $D(1) \subset \Sigma(W)$ be the fundamental chamber of W corresponding to the identity element 1 of W .

Let $V \subset W$ be a reflection subgroup, and denote by R' the set of reflections in V . Consider $\Sigma(W) \setminus \bigcup_{w \in R'} H_w$. The closure of each connected component of this space in $\Sigma(W)$ is a fundamental domain for the action of V on $\Sigma(W)$. We call the closure of the connected component that contains the identity *principal fundamental domain*. Note that the principal fundamental domain is a Coxeter polytope. Conversely, every Coxeter polytope in $\Sigma(W)$ is a fundamental domain of a reflection subgroup of W .

Lemma 3.1. *Let W be a Coxeter group with set of generators S such that all elements of S are conjugate in W . Suppose that $V \subsetneq W$ is a reflection subgroup of index n , $1 < n < \infty$. Let W_1 be a special subgroup of W . Then W_1 contains a reflection subgroup of index at most n .*

Proof. Since $V \subsetneq W$ there exists $s_i \in S$ such that $s_i \notin V$. Let $S_1 = S \cap W_1$ be a set of generators for W_1 , and let $s \in S_1$. By the assumptions of the lemma, all the generating reflections are conjugate, so all the reflections are conjugate. Thus, we can find $w \in W$ such that $s = ws_iw^{-1}$. Then $V' = wVw^{-1}$ is a non-trivial finite index reflection subgroup of W . Denote its principal fundamental domain by $F_{V'}$.

Define I to be $\{w \in W \mid D(w) \in F_{V'}\}$. Set $I_1 = I \cap W_1$. Then $|I_1| \leq |I| = n$, and $F_1 = \bigcup_{w \in I_1} D(w) \subset \Sigma(W_1, S_1)$ is a Coxeter polytope (as $F_{V'}$ is a Coxeter polytope, and the reflections corresponding to the walls of F_1 form a subset of the reflections corresponding to the walls of $F_{V'}$) that contains at most n chambers. Note that $s \notin V'$ by the definition of V' . Hence $D(s) \in F_{V'}$, and therefore $D(s) \in F_1$. Thus, F_1 contains at least two chambers ($D(1)$ and $D(s)$), i.e. it is the fundamental domain for a reflection subgroup of W_1 of index at most n . $\square$

Throughout the paper we will use the following corollary of Lemma 3.1 (we will refer to it as *Subdiagram Lemma*).

Corollary 3.2. *Let W be an odd-angled Coxeter group with set of generators S such that $\text{Cox}_{\text{div}}(W)$ is connected. Suppose that $V \subsetneq W$ is a reflection subgroup of index n , $1 < n < \infty$. Let W_1 be a special subgroup of W . Then W_1 contains a proper reflection subgroup of index at most n .*

Proof. Two generating reflections $s_0, s_k \in S$ are conjugate in W if and only if there is a sequence $s_0, s_1, \dots, s_{k-1}, s_k$ of elements of S such that for every $i \in [0, k-1]$ the orders m_{ii+1} are odd (see e.g [3, Lemma 3.3.3]). Therefore, Lemma 3.1 is applicable if the following “odd version” of the Coxeter diagram of (W, S) is connected: remove from the Coxeter diagram all edges labeled by even numbers or by infinity. In the case of odd-angled groups, this diagram coincides with $\text{Cox}_{\text{div}}(W)$. $\square$

4. EXAMPLES OF SUBGROUPS IN ODD-ANGLED GROUPS

In this section we construct three series of examples of finite index reflection subgroups. Combined with the results of Section 5, this will provide the classification of all odd-angled groups with finite index reflection subgroups (see also Section 6).

Lemma 4.1. *Let k_{13} and k_{23} be positive integers not divisible by 2 or 3. Then the group $W = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1s_2)^{3k_{12}} = (s_1s_3)^{5k_{13}} = (s_2s_3)^{5k_{23}} = 1 \rangle$ has a finite index reflection subgroup.*

Proof. In order to show the existence of a finite index reflection subgroup, it suffices to construct a Coxeter polytope in $\Sigma(W, S)$ that contains finitely many, but at least 2 chambers. This polytope is then the fundamental domain of a finite index reflection subgroup. If $k_{12} = 1$, a possible choice for the fundamental chamber P of the subgroup is shown in Fig 4.1, right: it is obtained by gluing all rotation images of the shaded domain.

For the general case let (the shaded domain) P' be the union of the chambers $D(1)$, $D(s_3)$, $D(s_1)$, $D(s_1s_3)$, $D(s_1s_3s_1)$, $D(s_1s_3s_2)$, and take P as the union of (the rotations) $(s_2s_1)^k P'$ for $0 \leq k < 3k_{12}$. Then P is the union of $18k_{12}$ chambers.

For $k_{12}, k_{13}, k_{23} > 1$ the polytope P is an $(18k_{12})$ -gon. Going around the polygon counterclockwise (see Fig. 4.1), the angles are subdivided into 6-tuples and the values in every 6-tuple are $\left(\frac{\pi}{3k_{12}}, \frac{\pi}{k_{13}}, \frac{\pi}{5k_{23}}, \frac{\pi}{k_{12}}, \frac{\pi}{5k_{13}}, \frac{\pi}{k_{23}}\right)$. As these are submultiples of π , P is indeed a Coxeter polytope, so we obtain a reflection subgroup of index $18k_{12}$.

If some of k_{12}, k_{13} or k_{23} are equal to 1, then some of the above angles are π , i.e. the number of vertices of the polygon P decreases, but P is still a Coxeter polytope. More precisely, every angle of size $\frac{\pi}{k_{13}}, \frac{\pi}{k_{23}}$ or $\frac{\pi}{k_{12}}$ appears exactly $3k_{12}$ times, so with every of the numbers k_{12}, k_{13}, k_{23} equal to one the number of

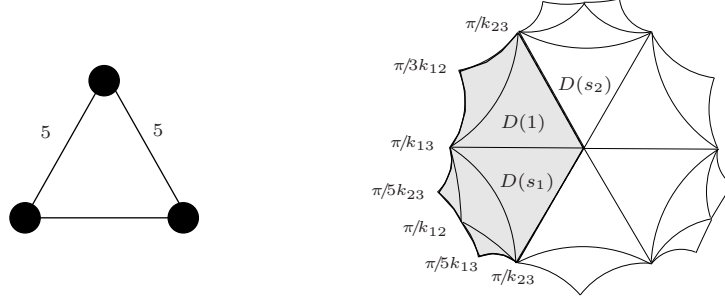


FIGURE 4.1. $\text{Cox}_{\text{div}}(W)$, and a fundamental chamber for an index 18 subgroup of W in case $k_{12} = 1$

angles decreases by $3k_{12}$. For example, if $k_{12} = k_{13} = k_{23} = 1$, then the convex polytope P is a polygon with six angles $\pi/5$ and three angles $\pi/3$. $\square$

Remark 4.2. If we allow the angles to be even, the construction from Lemma 4.1 gives rise to many other series of finite index reflection subgroups. For example, a group with presentation

$$W = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^{10k_{12}} = (s_1 s_3)^{3k_{13}} = (s_2 s_3)^{7k_{23}} = 1 \rangle$$

has a reflection subgroup of index $27k_{12}$ (see Fig. 4.2). This example suggests that, if angles are allowed to be even, the answer depends on the more subtle interplay of different divisors of the orders m_{ij} of the products of generators.

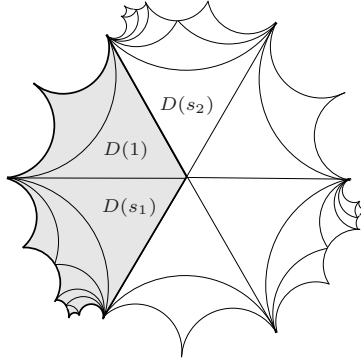


FIGURE 4.2. A fundamental chamber for an index 27 subgroup of W in case $k_{12} = 1$, see Remark 4.2

Lemma 4.3. *If at most one edge of $\text{Cox}_{\text{div}}(W)$ is neither simple nor absent then W has a finite index reflection subgroup.*

Proof. If $\text{Cox}_{\text{div}}(W)$ has only one vertex the statement is obvious. If all edges are absent the statement is also trivial: for every $s \in S$ the union $D(1) \cup D(s)$ is a Coxeter polytope.

Let v_1 and v_2 be vertices of $\text{Cox}_{\text{div}}(W)$ joined by a multiple edge, if any (if all non-absent edges are simple, take any pair of vertices joined by a simple edge). Let s_1 and s_2 be the corresponding reflections and $W_{12} = \langle s_1, s_2 \rangle$ be the special subgroup generated by these reflections. Consider the polytope

$$P_1 = \bigcup_{w \in W_{12}} D(w).$$

Let $\angle\alpha\beta$ be a dihedral angle of P_1 (formed by the facets α and β). It can be of one of the following two types:

- 1) both α and β contain facets of one copy $D(w)$ of $D(1)$; then $\angle\alpha\beta$ is a submultiple of π (since $D(w)$ is a Coxeter polytope);
- 2) α and β are not contained in defining walls of the same $D(w)$; then they are contained in walls of two adjacent copies of the fundamental chamber, and the dihedral angle is $2\pi/3k_{i1}$ or $2\pi/3k_{i2}$, where $3k_{ij} = m_{ij}$ for $j = 1$ or 2 .

Now, to construct a Coxeter polytope, we need to add some additional copies of $D(1)$. Namely, to each $D(w)$ such that $w \in W_{12}$ is of even length (we denote this subgroup of W_{12} by W_{12}^+), we glue all chambers adjacent to $D(w)$: define

$$P = P_1 \cup \left(\bigcup_{w \in W_{12}^+} P_w \right),$$

where P_w is the union of all chambers adjacent to $D(w)$. It is easy to see that P is a Coxeter polytope (each of its dihedral angles either belongs to one chamber or is dissected into exactly 3 parts of size $\pi/3k_{ij}$ for some i, j), see Fig. 4.3 for an example. It is also clear that P contains finitely many (but at least two) chambers, so it defines a finite index reflection subgroup. $\square$

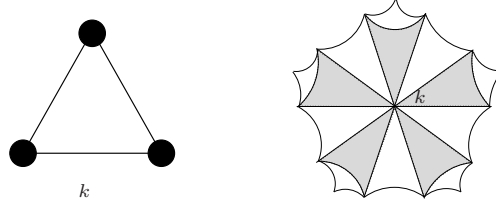


FIGURE 4.3. Example: $\text{Cox}_{\text{div}}(W)$ for a rank 3 group and fundamental chamber P constructed in the proof of Lemma 4.3 for the case $k = 5$.

Let us now generalize the example described in Lemma 4.1 to higher rank groups. Denote the diagram of the group described in Lemma 4.1 by $(5, 5, 3)$ (here we assume k_{12} to be odd).

Lemma 4.4. *Suppose that $\text{Cox}_{\text{div}}(W)$ contains $(5, 5, 3)$ as a subdiagram and all the remaining edges in $\text{Cox}_{\text{div}}(W)$ are simple (some vertices may not be joined). Then W has a finite index reflection subgroup.*

Proof. Let v_1v_2 and v_1v_3 be the edges with label 5 of $\text{Cox}_{\text{div}}(W)$. Consider the subgroup W_{123} of rank 3 generated by reflections s_1, s_2, s_3 corresponding to vertices v_1, v_2, v_3 respectively. Let $\Sigma(W_{123}) \subset \Sigma(W)$ be the subcomplex spanned by all cells corresponding to subgroups of W_{123} . According to Lemma 4.1, W_{123} has a finite index reflection subgroup V' . Denote by P' the principal fundamental chamber of V' (in $\Sigma(W_{123})$) constructed in Lemma 4.1. We assume without loss of generality that all the chambers of P' that contain a neighboring chamber outside P' correspond to group elements of odd length (see Fig. 4.1).

Denote $I = \{w \in W_{123} \mid D(w) \subset P'\}$. We can consider I as a subset of W , so we can define a polytope $P_1 \subset \Sigma(W, S)$ by

$$P_1 = \bigcup_{w \in I} D(w).$$

It is easy to see that P_1 is a convex polytope, and the angles of P_1 are either submultiples of π or of the type $2\pi/3k_{ij}$ for some integers k_{ij} . Now we use the same trick as in the proof of Lemma 4.3: attach to P_1 all the neighbors of all chambers $D(w)$ with w of even length. The procedure results in a Coxeter polytope P . $\square$

5. GROUPS WITHOUT FINITE INDEX REFLECTION SUBGROUPS

5.1. Technical tools. In this section we list technical lemmas used later to prove the absence of finite index reflection subgroups in some groups.

We will use the following notation: (W, S) is the odd-angled Coxeter system under consideration, $V \subset W$ is a finite index reflection subgroup, P is the corresponding principal fundamental domain of the V -action on $\Sigma(W)$, and I is the set of elements of W such that the corresponding chambers are contained in P (recall that P is a Coxeter polytope).

Lemma 5.1. *Suppose s_i, s_j, s_k are distinct elements of S such that m_{ij}, m_{ik}, m_{kj} are odd or infinite. Then*

- 1) s_i and $s_j s_k s_j$ generate an infinite dihedral group;
- 2) the wall H_{s_i} separating $D(1)$ from $D(s_i)$ does not intersect the wall $H_{s_i s_j s_k s_j s_i}$ separating $D(s_i s_j)$ from $D(s_i s_j s_k)$.

Proof. We will show that the walls H_{s_i} and $H_{s_j s_k s_j}$ do not intersect, which is equivalent to the first assertion. The second assertion follows since the group generated by s_i and $s_j s_k s_j$ coincides with the group generated by s_i and $s_i s_j s_k s_j s_i$.

Consider the special subgroup $W_1 \subset W$ generated by s_i, s_j and s_k . We distinguish two cases.

Case 1: $m_{ij} = m_{ik} = m_{kj} = 3$. In this case W_1 is the group generated by reflections in the sides of an equilateral triangle in the Euclidean plane, and the Davis complex $\Sigma(W_1)$ can be identified with tessellation of the plane by triangles. Then the walls $H_{s_j s_k s_j}$ and H_{s_i} are parallel lines, hence they do not intersect, and the group generated by s_i and $s_j s_k s_j$ is infinite.

Case 2: In all the other cases, W_1 can be understood as a group generated by reflections in the sides of a hyperbolic triangle.

Assume the walls $H_{s_j s_k s_j}$ and H_{s_i} have a common vertex (see Fig. 5.4). Then, together with H_{s_j} (or H_{s_k}), they form a triangle tessellated by chambers (note that the triangle contains more than one fundamental triangle). According to the results of [9], there are no triangles tessellated by hyperbolic odd-angled Coxeter triangles. The contradiction completes the proof. $\square$

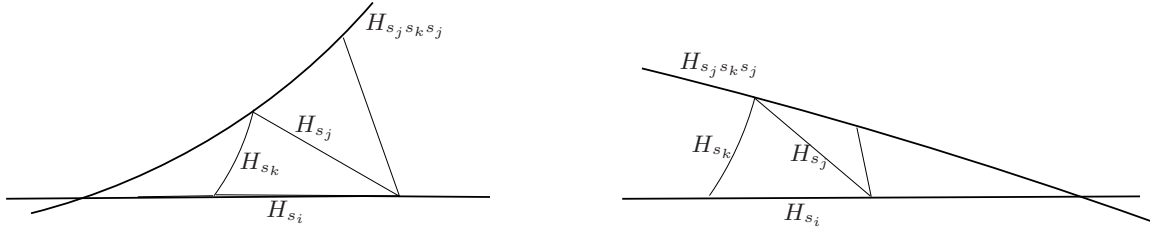


FIGURE 5.4. Towards the proof of Lemma 5.1, Case 2

Lemma 5.2. *Suppose s_i, s_j, s_k are distinct elements of S such that m_{ij}, m_{ik}, m_{kj} are odd. Then $s_j s_k s_j$ and $s_k s_i s_k s_i s_k$ generate an infinite dihedral group.*

Proof. First, suppose that $m_{ik} = 3$. Then $s_k s_i s_k s_i s_k = s_i$ and the statement follows from Lemma 5.1.

Now, suppose $m_{ik} \neq 3$. If we assume that the mirrors of the corresponding reflections intersect each other, then, as in the proof of Lemma 5.1, we can construct a hyperbolic triangle tessellated by smaller Coxeter triangles, which leads to a contradiction due to results of [9]. $\square$

Lemma 5.3. *Let (W, S) be a Coxeter system and $P = \bigcap_{i=1}^n H_{w_i}^+$ a polytope in $\Sigma(W, S)$. If v_1 and v_k are elements of W with $D(v_1), D(v_k) \subset P$, and $(D(v_1), D(v_2), \dots, D(v_k))$ is a geodesic gallery, then $D(v_i) \subset P$ for all $1 \leq i \leq k$.*

Proof. Suppose that $1 < i < k$, $D(v_i) \not\subset P$. Then there exists $1 \leq j \leq n$ such that $D(v_i) \not\subset H_{w_j}^+$. Hence $(D(v_1), \dots, D(v_i))$ and $(D(v_i), \dots, D(v_k))$ cross the wall H_{w_j} , and by Lemma 2.6 the gallery $(D(v_1), D(v_2), \dots, D(v_k))$ is not geodesic, which contradicts the assumptions. $\square$

Note also that in view of Lemma 2.6, Lemma 5.3 follows immediately from [18, Proposition 2.6].

The main technical tool will consist of the following two lemmas.

Lemma 5.4. *Let $D(w_0)$ be a chamber of $\Sigma(W, S)$. Suppose that there are two geodesic galleries $(D(w_0), D(w_1), \dots, D(w_k))$ and $(D(w_0), D(w'_1), \dots, D(w'_{k'}))$, such that $l(w_i) > l(w_j)$ and $l(w'_i) > l(w'_j)$ for $i < j$. Let H be a wall intersecting the first gallery but not the second, and H' be a wall intersecting the second gallery but not the first one. Then $H \cap H' \neq \emptyset$.*

Proof. We can extend the geodesic galleries $(D(w_0), D(w_1), \dots, D(w_k))$ and $(D(w_0), D(w'_1), \dots, D(w'_{k'}))$ to geodesic galleries $\Gamma = (D(w_0), D(w_1), \dots, D(w_k), D(w_{k+1}), \dots, D(1))$ and $\Gamma' = (D(w_0), D(w'_1), \dots, D(w'_{k'}), D(w'_{k'+1}), \dots, D(1))$. Denote by H^+ and H'^+ the halfspaces defined by H and H' that contain $D(w_0)$. As Γ crosses the wall H and Γ' crosses H' , the identity element $D(1)$ has to lie in the complementary halfspaces, i.e. in $H^- \cap H'^-$, see Lemma 2.6. This requires in particular that $H^- \cap H'^- \neq \emptyset \neq H^+ \cap H'^+$, which implies that the intersection between H and H' is nonempty. $\square$

Corollary 5.5. *Let $w \in W$, $s_i, s_j \in S$. If $l(w) = l(ws_i) + 1 = l(ws_j) + 1$, then m_{ij} is finite.*

Proof. Applying the previous lemma to the geodesic galleries $(D(w), D(ws_i))$ and $(D(w), D(ws_j))$ shows that $H_{ws_i w^{-1}} \cap H_{ws_j w^{-1}} \neq \emptyset$, and the order m_{ij} of $s_i s_j$ is finite. $\square$

Lemma 5.6. *Suppose that $\Sigma(W, S)$ has dimension two. Let $s_i, s_j \in S$, $s_i \neq s_j$, and $w \in I$ such that $l(w) = l(ws_i) + 1 = l(ws_j) + 1$. Then for every integer k the elements $w(s_i s_j)^k$ and $w(s_i s_j)^k s_i$ are in I .*

Moreover, $l(w(s_i s_j)^k) = l(w(s_j s_i)^k) = l(w) - 2k$ for $k \leq m_{ij}/2$, and $l(w(s_i s_j)^k s_i) = l(w(s_j s_i)^k s_j) = l(w) - 2k - 1$ for $k \leq (m_{ij} - 1)/2$.

Proof. We note first that $m_{ij} < \infty$ by Corollary 5.5.

Denote $l(w)$ by m . By Corollary 2.3, there exists a reduced expression E_1 for w that ends in s_i and another reduced expression E_2 that ends in s_j . Theorem 2.4 implies that E_1 can be transformed into E_2 by repeatedly applying M -operations of second type (given in Theorem 2.4). Since Σ has dimension two, every subgroup generated by at least three different elements of S is infinite. Hence, using Lemma 2.5, we see that this sequence of reduced expressions starting with E_1 , ending with E_2 and obtained by repeatedly applying M -operations of the second type contains only expressions ending in s_i or s_j . As E_1 ends in s_i and E_2 ends in s_j , there exists a reduced expression $\tilde{E} = \tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_m$ for w in the above sequence that ends in s_i and that gets transformed into an expression ending in s_j by applying M -operation of second type once. This is only possible if $\tilde{E}$ ends in $(s_j s_i)^{m_{ij}/2}$ for even m_{ij} or $(s_i s_j)^{(m_{ij}-1)/2} s_i$ if m_{ij} is odd.

Now consider $\tilde{E}$ as a geodesic gallery joining $D(1)$ with $D(w)$. In view of Lemma 5.3, we see that $w(s_i s_j)^k \in I$ and $w(s_i s_j)^k s_i \in I$ for all $k \in \mathbb{Z}$. Since the length of two neighbors differs exactly by 1, the elements $w, w(s_i s_j)^k$ have length $m - 2k$ for $k \leq m_{ij}/2$, and $w(s_i s_j)^k s_i$ has length $m - 2k - 1$ for $k \leq (m_{ij} - 1)/2$. $\square$

Remark 5.7. Similar to the last paragraph of the proof above, we make the following observation, which we will use throughout the paper. If w is a reduced expression, and $D(w) \in P$ (i.e., $w \in I$), then for any $s_i \in S$ such that $l(ws_i) = l(w) - 1$ the chamber $D(ws_i)$ is contained in P .

Remark 5.8. Lemma 5.6 still holds if we drop the assumption on $\Sigma(W, S)$ to have dimension two. This can be proved using [18, Theorem 2.9 and Lemma 2.10].

Lemma 5.9. *Suppose that $w, ws_i \in I$ and $ws_k, ws_i s_k \notin I$. Then $m_{ik} = \infty$.*

Proof. The lemma follows immediately from the assumption that W is odd-angled. Indeed, assume that $m_{ik} \neq \infty$. Then the walls $H_{ws_k w^{-1}}$ and $H_{ws_i s_k s_i w^{-1}}$ form an angle equal to $2\pi/m_{ik}$, which cannot be an angle of P as P is a Coxeter polytope (see Fig. 5.5). $\square$

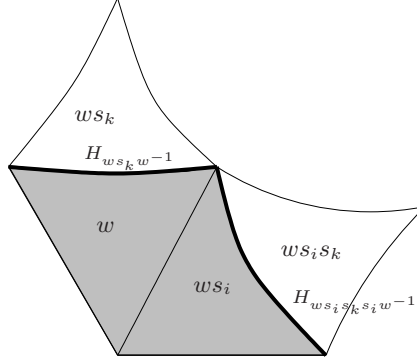


FIGURE 5.5. Towards the proof of Lemma 5.9

Lemma 5.10. *The lengths of the elements $w, ws_i, ws_i s_j, ws_i s_j s_k$ cannot be $L-1, L, L-1, L-2$ for any $L > 2$ and distinct i, j, k .*

Proof. Suppose the contrary. Then the geodesic galleries $(D(ws_i), D(w))$ and $(D(ws_i), D(ws_i s_j), D(ws_i s_j s_k))$ together with walls H_1 separating $D(ws_i)$ and $D(w)$ and H_2 separating $D(ws_i s_j)$ and $D(ws_i s_j s_k)$ satisfy the assumptions of Lemma 5.4. Therefore, these two walls must intersect. However, this contradicts Lemma 5.1, see Fig. 5.6.a. $\square$

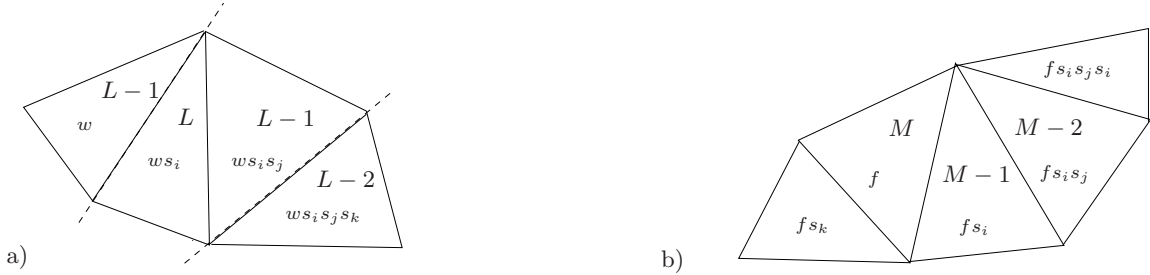


FIGURE 5.6. Towards the proofs of Lemmas 5.10 and 5.13

The following statement, which we will use throughout the section, follows immediately from the definition of a Coxeter polytope.

Lemma 5.11. *Let $w \in W$, and let $Q = H_{w_1} \cap H_{w_2} \cap P$ be a codimension 2 face of P , where $w_l = w(s_i s_j)^{k_l} s_i w^{-1}$, $l = 1, 2$ and $k_1, k_2 < m_{ij}$ are some non-negative integers. Then the number of copies of the fundamental chamber $D(1)$ in P containing Q divides m_{ij} .*

We now consider an element f of I of maximal length. We will denote its length by M .

Lemma 5.12. *Let $f \in I$ be an element of maximal length M . Assume that $fs_i \in I$ and $m_{ij} \neq \infty$. Then $fs_is_j \in I$. Moreover, if $3 \nmid m_{ij}$ then $l(fs_is_j) = l(fs_i) - 1 = M - 2$, and if $3 \nmid m_{ij}$, $5 \nmid m_{ij}$ then $l(fs_is_js_i) = M - 3$.*

Proof. Since $fs_i \in I$, we have $l(fs_i) = M - 1$. First, assume that $fs_j \in I$. Then $l(fs_j) = M - 1$ and by Lemma 5.6 $f(s_is_j)^k \in I$ and $f(s_is_j)^k s_i \in I$ for all $k \in \mathbb{Z}$, and $l(fs_is_j) = M - 2$, $l(fs_is_js_i) = M - 3$.

From now on we assume that $fs_j \notin I$. By Lemma 5.9, $fs_is_j \in I$, which proves the first part of the lemma.

Suppose that $3 \nmid m_{ij}$. Then, by Lemma 5.11, $fs_is_js_i \in I$. If $l(fs_is_j) = M$, then $l(fs_is_js_i) = M - 1$ and, by Lemma 5.6, all elements $fs_is_j(s_is_j)^k$ and $fs_is_j(s_is_j)^k s_i$ are contained in I for all $k \in \mathbb{Z}$, which contradicts the assumption that $fs_j \notin I$. This implies that $l(fs_is_j) = M - 2$.

Finally, suppose in addition that $5 \nmid m_{ij}$. We have already proved that $fs_is_js_i \in I$, so we are left to show that $l(fs_is_js_i) = M - 3$. If we assume the contrary (i.e., $l(fs_is_js_i) = M - 1$), then either $l(f(s_is_j)^2) = M - 2$ and Lemma 5.6 leads to a contradiction with $fs_j \notin I$, or $l(f(s_is_j)^2) = M$. In the latter case since $5 \nmid m_{ij}$, we see that $f(s_is_j)^2 s_i \in I$, so $l(f(s_is_j)^2 s_i) = M - 1$, and we again apply Lemma 5.6 to obtain a contradiction with $fs_j \notin I$. $\square$

Lemma 5.13. *Let $f \in I$ and $M = l(f) = l(fs_i) + 1 = l(fs_is_j) + 2$. Then*

- (a) $l(fs_k) = M + 1$, i.e. $fs_k \notin I$ for all k distinct from i and j ;
- (b) if $l(fs_is_js_i) = M - 1$, then $l(fs_j) = M + 1$ and $fs_j \notin I$.

Proof. First, $l(fs_k) = M - 1$ is impossible by Lemma 5.10 (applied to the elements fs_k, f, fs_i, fs_is_j of lengths $M - 1, M, M - 1, M - 2$, see Fig. 5.6.b), which proves the first part of the lemma. Now, suppose that $l(fs_j) = M - 1$. Then, by Lemma 5.6 (applied to the elements fs_j, f, fs_i of lengths $M - 1, M, M - 1$), we see that $l(fs_is_js_i) = M - 3$. Hence, $l(fs_is_js_i) = M - 1$ implies $l(fs_j) = M + 1$, i.e. $fs_j \notin I$. $\square$

5.2. Rank 3 groups. In Lemma 5.14 we discuss rank 3 odd-angled Coxeter groups W with 3 defining relations $(s_1 s_2)^{m_{12}} = (s_1 s_3)^{m_{13}} = (s_2 s_3)^{m_{23}} = 1$, $m_{ij} \neq \infty$. The case of two defining relations (i.e., one of m_{ij} is equal to ∞) is considered in Lemma 5.15. Rank 3 Coxeter groups with at most one $m_{ij} \in \mathbb{Z}$ have disconnected divisibility diagrams. We consider them in Section 6.

As was defined above, let $V \subset W$ be a finite index reflection subgroup, P be its fundamental domain containing $D(1)$, and let be I the set of elements w of W with $D(w) \in P$. We denote by f the element of I of maximal length M . We will also assume that $V \subset W$ is a reflection subgroup of smallest possible index, hence P does not (strictly) contain any Coxeter polytope except chambers of $\Sigma(W, S)$.

Lemma 5.14. *Let (m_{ij}, m_{ik}, m_{jk}) be distinct from $(3k_1, 3k_2, k_3)$ and $(5k_1, 5k_2, 3k_3)$. Then W has no finite index reflection subgroups.*

Proof. Suppose that $V \subset W$ is a finite index reflection subgroup. There exists s_i (say, s_1) such that $fs_1 \in I$, $l(fs_1) = M - 1$. By Lemma 5.12, there is s_j (say, s_2) such that $fs_1 s_2 \in I$, $l(fs_1 s_2) = M - 2$. By Lemma 5.13 $fs_3 \notin I$ (see Fig. 5.7). Consider the element $fs_1 s_3 \in I$. Its length is either M or $M - 2$.

Case 1: $l(fs_1 s_3) = M$.

According to Lemma 5.6, $fs_1 s_3 s_1$ does not belong to I , so Lemma 5.11 implies that m_{13} is divisible by 3. Therefore, 3 divides neither m_{12} nor m_{23} . Thus, either fs_2 or $fs_1 s_2 s_1$ belongs to I .

Case 1.1: $fs_2 \in I$ (see Fig. 5.8).

Clearly, $l(fs_2) = M - 1$. So, by Lemma 5.12, $fs_2 s_3 \in I$ and $l(fs_2 s_3) = M - 2$, which contradicts Lemma 5.10 applied to elements $fs_2 s_3, fs_2, f, fs_1$ of lengths $M - 2, M - 1, M, M - 1$ respectively.

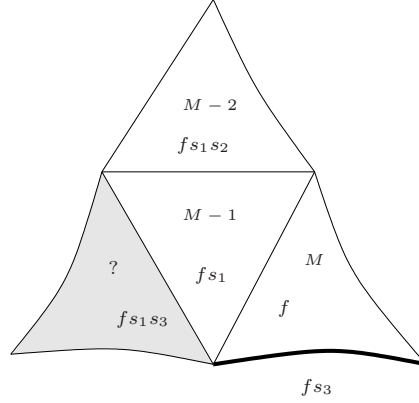


FIGURE 5.7. Towards the proof of Lemma 5.14

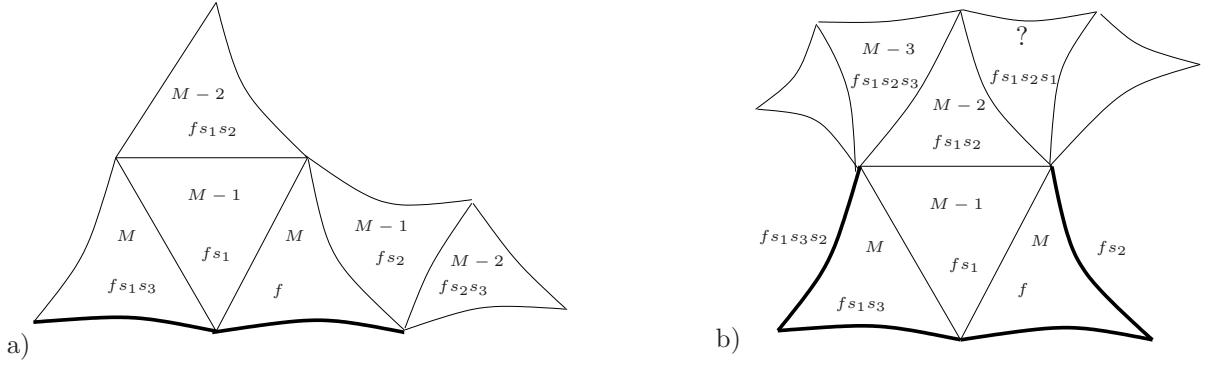


FIGURE 5.8. Proof of Lemma 5.14: (a) Case 1.1, (b) Case 1.2

Case 1.2: $fs_2 \notin I$.

By symmetry, we may also assume that $fs_1s_3s_2 \notin I$ (see Fig. 5.8b). Then all the neighbors of fs_1s_2 belong to I . One of the neighbors of fs_1s_2 (say, $fs_1s_2s_3$) has length $M-3$. Consider the second neighbor, $fs_1s_2s_1$. Its length is either $M-1$ or $M-3$.

Assume first that $l(fs_1s_2s_1) = M-1$, and consider $fs_1s_2s_1s_2$. Its length is either $M-2$ or M .

If $l(fs_1s_2s_1s_2) = M-2$ (see Fig. 5.9a), then by Lemma 5.6 $fs_2 \in I$, which contradicts the assumption of Case 1.2. Therefore, $l(fs_1s_2s_1s_2) = M$ (see Fig. 5.9b). Moreover $l(fs_1s_2s_1s_2s_1) = M+1$, and $fs_1s_2s_1s_2s_1 \notin I$, which implies that m_{12} is a multiple of 5 (again, we use Lemma 5.11). In view of the assumptions of the lemma, this means that m_{23} is neither a multiple of 3 nor a multiple of 5, so we can apply Lemma 5.12 to obtain that $l(fs_1s_2s_1s_3) = M-2$ and $l(fs_1s_2s_1s_3s_2) = M-3$. This contradicts Lemma 5.10 (applied to the elements $f_Ms_1s_2, f_Ms_1s_2s_1, f_Ms_1s_2s_1s_3, f_Ms_1s_2s_1s_3s_2$ of lengths $M-2, M-1, M-2, M-3$).

Thus, we may assume that $l(fs_1s_2s_1) = l(fs_1s_2s_3) = M-3$, see Fig. 5.10a. Consider the neighbors of the above elements, $fs_1s_2s_1s_2$ and $fs_1s_2s_3s_2$, both in I . By Lemma 5.10 (applied to the elements $fs_1s_2s_3, fs_1s_2, fs_1s_2s_1, fs_1s_2s_1s_2$) we have $l(fs_1s_2s_1s_2) = M-2$. Similarly, $l(fs_1s_2s_3s_2) = M-2$.

By the assumptions of the lemma, we may assume that either m_{12} or m_{23} (say, m_{23}) is not a multiple of 5. Therefore, we may assume that $fs_1s_2s_3s_2s_3$ and $fs_1s_2s_3s_2s_3s_2$ belong to I (see Fig. 5.10b). By

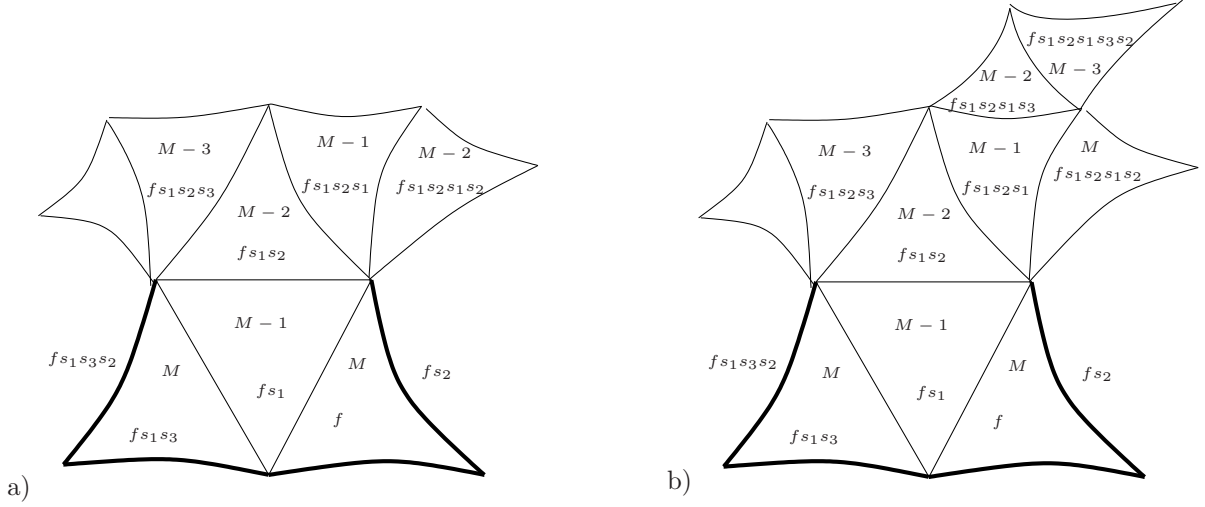


FIGURE 5.9. Proof of Lemma 5.14, Case 1.2, continuation

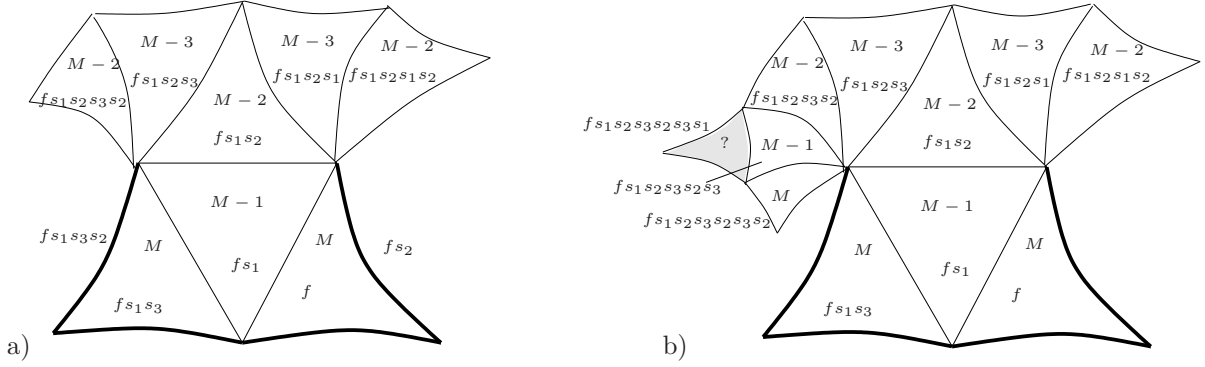


FIGURE 5.10. Case 1.2, continuation

Lemma 5.6, $l(fs_1s_2s_3s_2s_3) = M-1$, $l(fs_1s_2s_3s_2s_3s_2) = M$. Consider $fs_1s_2s_3s_2s_3s_1$, its length is either M or $M-2$.

The latter case is impossible by Lemma 5.10 (applied to the elements $fs_1s_2s_3$, $fs_1s_2s_3s_2$, $fs_1s_2s_3s_2s_3$, $fs_1s_2s_3s_2s_3s_1$ of lengths $M-3, M-2, M-1, M-2$). In the former case, $fs_1s_2s_3s_2s_3s_1 \in I$, and, reasoning as in the beginning of consideration of Case 1, we obtain that m_{12} is divisible by 3, which contradicts assumptions of the lemma.

Case 2: $l(fs_1s_3) = M-2$ (see Fig. 5.11a).

Similar to fs_3 , the element fs_2 does not belong to I either. Since at most one of the m_{ij} is divisible by 3, and the picture is symmetric with respect to the interchange of the indices 2 and 3, we may assume that m_{13} and at least one of m_{12} and m_{23} are not divisible by 3. Then $fs_1s_3s_1 \in I$, its length is either $M-1$ or $M-3$. The latter is impossible by Lemma 5.10 (applied to the elements fs_1s_2 , fs_1 , fs_1s_3 , $fs_1s_3s_1$ of lengths $M-2, M-1, M-2, M-3$). So, $l(fs_1s_3s_1) = M-1$. Consider $fs_1s_3s_1s_3 \in I$, its length is

either $M - 2$ or M . In the former case, Lemma 5.6 implies that $fs_3 \in I$, which does not hold. Hence, $l(fs_1s_3s_1s_3) = M$ (see Fig. 5.11b). In particular, $fs_1s_3s_1s_3s_1 \notin I$, and m_{13} is a multiple of 5.

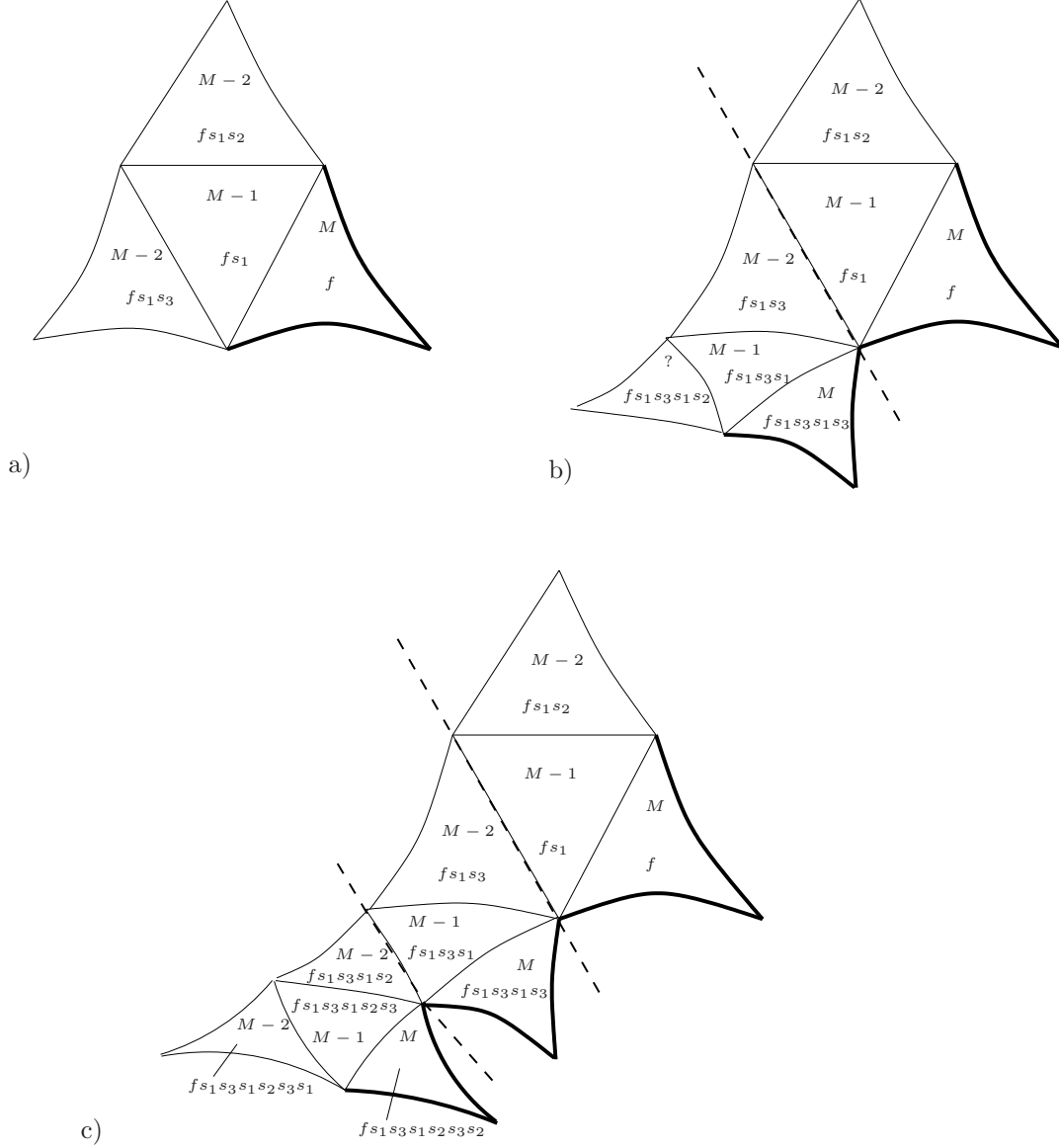


FIGURE 5.11. Proof of Lemma 5.14, Case 2.

Now consider $fs_1s_3s_1s_2 \in I$ (see Fig. 5.11b). If its length is M , then we are in Case 1, so we come to a contradiction. Therefore, $l(fs_1s_3s_1s_2) = M - 2$. Recall that, by our assumption, at least one of m_{12} and m_{23} is not divisible by 3. Consider two cases.

Case 2.1: m_{23} is not divisible by 3.

Repeating the reasonings as above (i.e., replacing the elements $f, fs_1, fs_1s_2, fs_1s_3$ by $fs_1s_3s_1s_3, fs_1s_3s_1,$

$f s_1 s_3, f s_1 s_3 s_1 s_2$), we obtain that m_{23} is divisible by 5 (see Fig. 5.11c). Therefore, by the assumptions of the lemma, m_{12} is not divisible by 3 either, and we can continue constructing elements of I in the same way (now replacing the elements $f, f s_1, f s_1 s_2, f s_1 s_3$ by $f s_1 s_3 s_1 s_2 s_3 s_2, f s_1 s_3 s_1 s_2 s_3, f s_1 s_3 s_1 s_2, f s_1 s_3 s_1 s_2 s_3 s_1$).

According to Lemma 5.1, the wall separating chambers $D(f s_1)$ and $D(f s_1 s_3)$ does not intersect the wall separating chambers $D(f s_1 s_3 s_1)$ and $D(f s_1 s_3 s_1 s_2)$. Similarly, these walls do not intersect the wall separating chambers $D(f s_1 s_3 s_1 s_2 s_3)$ and $D(f s_1 s_3 s_1 s_2 s_3 s_1)$ (see Fig. 5.11c). This implies that iterating the procedure above we can construct arbitrary many elements of I , which contradicts the finiteness of the index of V .

Case 2.2: m_{12} is not divisible by 3.

Recall that from the assumption that m_{13} is not divisible by 3 we have deduced that m_{13} is a multiple of 5. Since Figure 5.11a is symmetric with respect to the interchange of s_2 and s_3 , the assumption that m_{12} is not divisible by 3 implies that m_{12} is also divisible by 5. Then, by the assumptions of the lemma, m_{23} is not divisible by 3, and we are in the assumptions of Case 2.1. $\square$

Lemma 5.15. *Let $(W, S) = \langle s_1, s_2, s_3 \mid (s_1 s_2)^{m_{12}} = (s_2 s_3)^{m_{23}} = 1 \rangle$ be a Coxeter system such that $m_{12} m_{23}$ is not divisible by 2 and 3. Then W has no finite index reflection subgroup.*

Proof. Suppose the contrary, i.e. let V be a finite index reflection subgroup of smallest index. The complex $\Sigma(W, S)$ can be identified with the hyperbolic plane $\mathbb{H}^2$, the chamber $D(1)$ (or $D(w)$ for any $w \in W$) with a fundamental triangle of W , and P with a fundamental polygon of V . First, we show that no copy of $D(1)$ in the tessellation of P has a vertex in the interior of P .

Assume that there is a vertex X of some copy of $D(1)$ (denote this copy by F) contained in the interior of P . Let Y be the other vertex of F with non-zero angle, and let Z be the third vertex of F (it lies at the boundary of $\mathbb{H}^2$ since $m_{13} = \infty$ and the angle at Z is 0). Consider the angle $\angle XYZ$ formed by the rays YX and YZ . It is clear that $P' = P \cap \angle XYZ$ is a Coxeter polygon (see Fig. 5.12): all its angles are either angles of P , or the angle Y of F , or the zero angles at vertices at infinity (since m_{12} and m_{23} are odd). Furthermore, P' contains more than one fundamental triangle of W . So, P' is a fundamental domain of some subgroup of W with index smaller than the index of V , which contradicts the assumptions.

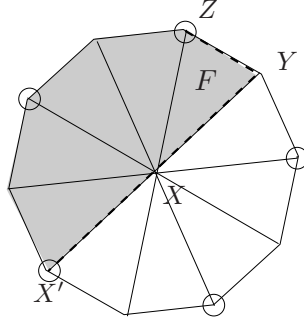


FIGURE 5.12. Proof of Lemma 5.15: no vertices in the interior of P . The circles show the vertices at infinity.

Therefore, every vertex of any fundamental chamber in the tessellation of P belongs to the boundary of P .

Now, using the notation as above, we will show that $f s_2 \notin I$. Indeed, suppose $f s_2 \in I$. Then $l(f s_2) = M - 1$. By Lemma 5.12, $f s_2 s_1 \in I$, and $l(f s_2 s_1) = M - 2$. Similarly, $f s_2 s_3 \in I$, and $l(f s_2 s_3) = M - 2$ (see Fig. 5.13.a), which contradicts Corollary 5.5.

So, one of $f s_1$ and $f s_3$ (say, $f s_3$) belongs to I . By Corollary 5.5, this implies $l(f s_1) \neq M - 1$, and hence $f s_1 \notin I$. By Lemma 5.12, $f s_3 s_2 \in I$, and $l(f s_3 s_2) = M - 2$. Taking in account the absence

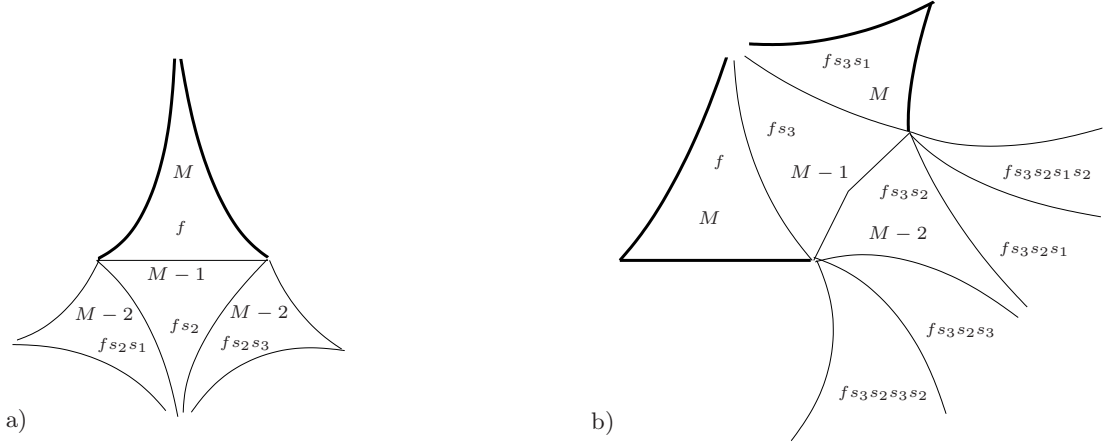


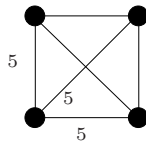
FIGURE 5.13. Towards the proof of Lemma 5.15

of interior vertices in P , this implies $l(f s_3 s_1) = M$ (by Lemma 5.6 applied to $f s_3 s_1, f s_3, f s_3 s_2$, see Fig. 5.13.b). Furthermore, since $m_{12} m_{23}$ is not divisible by 2 and 3 (and since $f s_2 \notin I$), $f s_3 s_2 s_3 \in I$, and $f s_3 s_2 s_3 s_2 \in I$. We will show that $l(f s_3 s_2 s_3) = M - 3$, and, similarly, $l(f s_3 s_2 s_1) = M - 3$, which will contradict Corollary 5.5.

Assume that $l(f s_3 s_2 s_3) = M - 1$, then $l(f s_3 s_2 s_3 s_2) = M$, otherwise by Lemma 5.6 applied to $f s_3 s_2, f s_3 s_2 s_3$ and $f s_3 s_2 s_3 s_2$ we have $f s_2 \in I$ which is false. Thus, for the word $f' = f s_3 s_2 s_3 s_2$ we have $l(f') = M$, $f', f' s_2 \in I$, which is already proven to be impossible. Hence, $l(f s_3 s_2 s_3) = M - 3$, which completes the proof. $\square$

5.3. Rank 4 groups. In this section, we describe two series of rank 4 groups without finite index reflection subgroups. We keep all the notation from the previous section.

Lemma 5.16. *Let (W, S) be an odd-angled Coxeter system of rank 4 with relations $(s_1 s_i)^{5k_i} = (s_i s_j)^{3k_{ij}} = 1$, $2 \leq i, j \leq 4$, where k_i is not divisible by 3 (see Fig. 5.14). Then W has no finite index reflection subgroups.*

FIGURE 5.14. $\text{Cox}_{\text{div}}(W)$, Lemma 5.16

Proof. As usual, suppose W has a finite index reflection subgroup.

Suppose that $f s_1 \in I$, i.e. $l(f s_1) = M - 1$. Then, by Lemma 5.12, $l(f s_1 s_i) = M - 2$ for $i = 2, 3, 4$ (since k_i has no prime divisor smaller than 5). This contradicts Lemma 2.5 and Corollary 2.3.

Therefore, $l(f s_1) = M + 1$, and $f s_1 \notin I$. Without loss of generality we assume that $f s_2 \in I$, and $l(f s_2) = M - 1$. By Lemma 5.12, this implies $f s_2 s_1 \in I$, $l(f s_2 s_1) = M - 2$. Since k_i is not divisible by 3, $f s_2 s_1 s_2, f s_2 s_1 s_2 s_1 \in I$ (see Fig. 5.15). If $l(f s_2 s_1 s_2) = M - 1$, then $l(f s_2 s_1 s_2 s_1) = M$, which is impossible by the previous paragraph applied to the element $f' = f s_2 s_1 s_2 s_1$ in place of f . Hence, $l(f s_2 s_1 s_2) = M - 3$. This implies $l(f s_2 s_3) = l(f s_2 s_4) = M$, otherwise the elements $f s_2 s_i, f s_2, f s_2 s_1, f s_2 s_1 s_2$, where $i = 3, 4$,

of lengths $M - 2, M - 1, M - 2, M - 3$ are in contradiction with Lemma 5.10. Reasoning as above (while showing $l(fs_2s_1s_2) = M - 3$), we obtain $l(fs_2s_1s_3) = l(fs_2s_1s_4) = M - 3$, which (together with $l(fs_2s_1) = M - 2$ and $l(fs_2s_1s_2) = M - 3$) is impossible by Lemma 2.5 and Corollary 2.3. $\square$

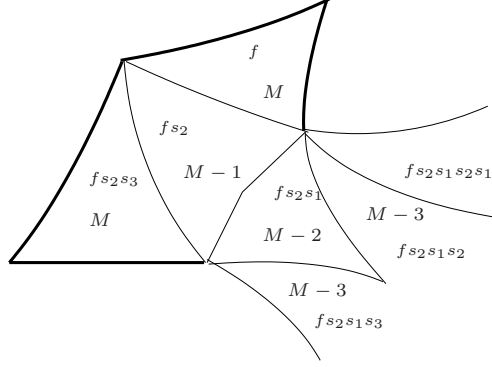


FIGURE 5.15. Towards the proof of Lemma 5.16

Lemma 5.17. *Let $W = \langle s_1, \dots, s_4 \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ be an odd-angled Coxeter system of rank 4 such that*

- $m_{12}, m_{23}, m_{34} \neq \infty$;
- m_{12} and m_{34} are not divisible by 3.

Then W has no finite index reflection subgroups.

To prove Lemma 5.17 suppose that W has a finite index reflection subgroup V .

We say that the elements $ws_i s_j, ws_i, w, ws_j, ws_j s_i$ compose an (i, j) -symmetric 5-tuple (or simply symmetric 5-tuple) if all these elements lie in I and are of length $M, M - 1, M - 2, M - 1, M$ respectively.

The plan of the proof is the following. First, we show (Claims 5.18–5.21) that there is at least one symmetric 5-tuple. Then we prove that the existence of one symmetric 5-tuple implies the existence of an infinite number of them. Since, by definition, all symmetric 5-tuples lie in I , we obtain a contradiction with the finiteness of the index of V .

Claim 5.18. *Suppose that $m_{ij}, m_{kn} \neq \infty$ and $m_{ij} m_{kn}$ is not divisible by 3, where $\{i, j, k, n\} = \{1, 2, 3, 4\}$. Let $w, ws_j \in I$, $l(w) = M - 2 = l(ws_j) - 1$. Then either $ws_j s_k \notin I$ and $ws_j s_n \notin I$, or $ws_j s_k \in I$, $ws_j s_n \in I$ and $ws_j s_k, ws_j, ws_j s_n$ are contained in some symmetric 5-tuple.*

Proof. We consider two cases: either $ws_j s_k \notin I$ and $ws_j s_n \in I$, or both $ws_j s_k$ and $ws_j s_n$ belong to I . The case $ws_j s_n \notin I$ and $ws_j s_k \in I$ is identical to the former.

Suppose that $ws_j s_k \notin I$, $ws_j s_n \in I$. Since $ws_j \in I$ and m_{kn} is not divisible by 3, we have $ws_j, ws_j s_n, ws_j s_n s_k, ws_j s_n s_k s_n, ws_j s_n s_k s_n s_k \in I$ (see Fig 5.16.a). If $l(ws_j s_n) = M$, then $l(ws_j s_n s_k) = M - 1$ and Lemma 5.6 (applied to the elements $ws_j, ws_j s_n, ws_j s_n s_k$ of the lengths $M - 1, M, M - 1$) implies that $ws_j s_k \in I$, which is false by the assumption. So, $l(ws_j s_n) = M - 2$. Similarly, $l(ws_j s_n s_k) = M - 3$. However, in this case the elements $w, ws_j, ws_j s_n, ws_j s_n s_k$ (of lengths $M - 2, M - 1, M - 2, M - 3$ respectively) are in contradiction with Lemma 5.10. Therefore, either $ws_j s_k \notin I$ and $ws_j s_n \notin I$, or $ws_j s_k \in I$ and $ws_j s_n \in I$.

Now suppose that $ws_j s_k \in I$, $ws_j s_n \in I$. As all subgroups generated by three or more different elements of S are infinite, Corollary 2.3 and Lemma 2.5 imply that at most 2 neighbors of ws_j can have length $l(ws_j) - 1 = M - 2$. Moreover, both $ws_j s_k$ and $ws_j s_n$ cannot be of length M simultaneously: indeed, in this case Lemma 5.6 implies that neither $ws_j s_n s_k$ nor $ws_j s_k s_n$ belong to I , which is impossible since m_{kn} is not divisible by 3.

Therefore, without loss of generality we may assume that $l(ws_js_k) = M$, $l(ws_j) = M - 1$, $l(ws_js_n) = M - 2$ (see Fig 5.16.b). By Lemma 5.10 (applied to the elements $w, ws_j, ws_js_n, ws_js_ns_k$) we see that $l(ws_js_ns_k) \neq M - 3$. So, $l(ws_js_ns_k) = M - 1$. Furthermore, Lemma 5.6 (applied to the elements $ws_js_n, ws_js_ns_k, ws_js_ns_ks_n$) implies that $l(ws_js_ns_ks_n) \neq M - 2$, i.e. $l(ws_js_ns_ks_n) = M$, and the elements $ws_js_k, ws_j, ws_js_n, ws_js_ns_k, ws_js_ns_ks_n$ compose a symmetric 5-tuple. $\square$

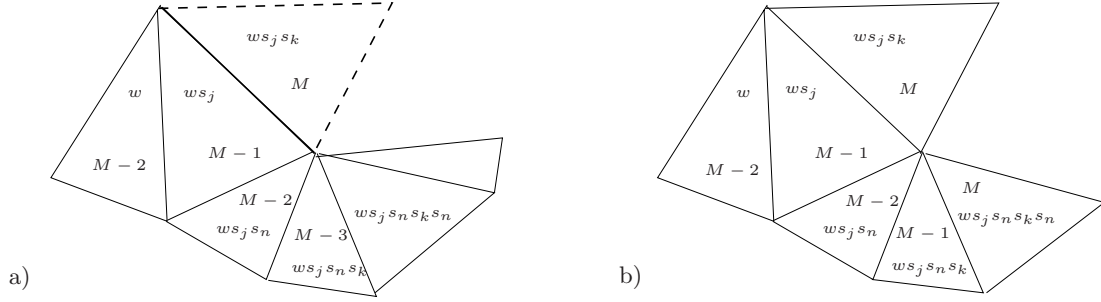


FIGURE 5.16. Towards the proof of Lemma 5.18

Claim 5.19. Suppose that $w, ws_j \in I$, $l(w) = M - 2$, $l(ws_j) = l(ws_i) = M - 1$. If $m_{ij} \neq \infty$ is not divisible by 3, then $ws_i \in I$, and $ws_is_j, ws_i, w, ws_j, ws_js_i \in I$ compose a symmetric 5-tuple in I .

Proof. First, suppose $ws_i \notin I$. Then, since m_{ij} is not divisible by 3, $ws_js_i, ws_js_is_j \in I$ (see Fig. 5.17.a). Furthermore, due to Lemma 5.6 we have $l(ws_js_i) = M$, and by the same reason $l(ws_js_is_j) = M + 1$, which is impossible since $ws_js_is_j \in I$.

Thus, $ws_i \in I$. Applying Lemma 5.6 again, we see that $l(ws_is_j) = M = l(ws_js_i)$ and $l(ws_is_js_i) = M + 1 = l(ws_js_is_j)$, which implies $ws_is_j, ws_js_i \in I$ and $ws_is_j, ws_i, w, ws_j, ws_js_i \in I$ form a symmetric 5-tuple in I . $\square$

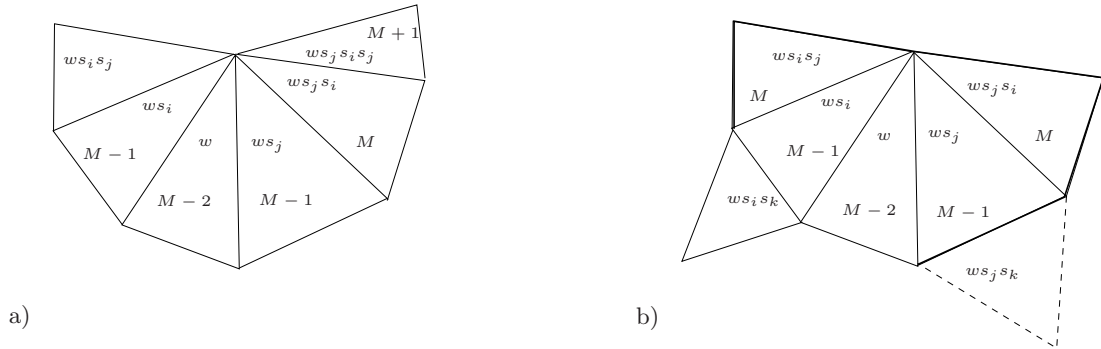


FIGURE 5.17. Towards the proof of (a) Claim 5.19 and (b) Claim 5.20

Claim 5.20. Let either $(i, j, k, n) = (1, 2, 3, 4)$ or $(i, j, k, n) = (4, 3, 2, 1)$. Suppose that an element w of length $M - 2$ is included in an (i, j) -symmetric 5-tuple. Then each of the elements ws_i and ws_j is contained in some (k, n) -symmetric 5-tuple.

In particular, all four neighbors of ws_i (and ws_j) are contained in I , and there are precisely two neighbors of ws_i (and ws_j) of length M and two neighbors of length $M - 2$.

Proof. By Lemma 5.13, each of the elements ws_js_i and ws_is_j has exactly one neighbor in I . Since $ws_i, ws_is_j \in I$ and $m_{jk} = m_{23} \neq \infty$, this implies $ws_is_k \in I$ (see Fig. 5.17.b). Applying Claim 5.18 we see that ws_i lies in a (k, n) -symmetric 5-tuple, and all four neighbors of ws_i are contained in I and have the desired length. If all neighbors of ws_j are in I , then by Claim 5.18 ws_j lies in a (k, n) -symmetric 5-tuple, and the claim is proved.

Suppose that a neighbor of ws_j does not belong to I . Then, again by Claim 5.18, $ws_js_k, ws_js_n \notin I$. Recall also that $ws_js_is_k, ws_js_is_n \notin I$, but $ws_j, ws_js_i \in I$. In view of Lemma 5.9, this implies that $m_{ik} = m_{in} = \infty$. On the other hand, we have already proved that ws_i is contained in a (k, n) -symmetric 5-tuple, so either ws_is_k or ws_is_n has length $M - 2$. Therefore, either elements w, ws_i, ws_is_k or elements w, ws_i, ws_is_n are of lengths $M - 2, M - 1, M - 2$ respectively. In view of Corollary 5.5, this contradicts $m_{ik} = m_{in} = \infty$. $\square$

Claim 5.21. *There exists a 5-tuple of elements in I composing an (i, j) -symmetric 5-tuple with $\{i, j\} = \{1, 2\}$ or $\{3, 4\}$.*

Proof. Let $fs_i \in I$ be a neighbor of $f \in I$, $l(fs_i) = M - 1 = l(f) - 1$. Choose j so that $\{i, j\} = \{1, 2\}$ or $\{i, j\} = \{3, 4\}$ (so, we need to construct either (i, j) - or (k, n) -symmetric 5-tuple), in particular m_{ij} is finite and is not divisible by 3. This implies that I contains at least 5 elements from the set $I_0 = \{f(s_is_j)^m, fs_j(s_is_j)^m, m \in \mathbb{Z}\}$. We consider two cases: either $I_0 \subset I$ or not.

First, suppose $I_0 \subset I$. Then $l(fs_i) = l(fs_j) = M - 1$. Hence, $l(fs_k) = l(fs_n) = M + 1$, and $fs_k, fs_n \notin I$. If either fs_i or fs_j has a neighbor in I other than f, fs_is_j and fs_js_i , then the statement follows from Claim 5.18. So, we assume that fs_i and fs_j has no other neighbors in I . Then Lemma 5.9 implies $m_{ik} = m_{in} = m_{jk} = m_{jn} = \infty$, which contradicts the assumption that $m_{23} \neq \infty$.

Now, suppose $I_0 \setminus I \neq \emptyset$. If I_0 contains a symmetric 5-tuple, then there is nothing to prove. So, we suppose that I_0 contains no symmetric 5-tuple. Since $I_0 \setminus I \neq \emptyset$, Lemma 5.6 implies that $l(fs_j) = M + 1$, $fs_j \notin I$. Thus, $f, fs_i, fs_is_j, fs_is_js_i, fs_is_js_is_j \in I$ (here we use the assumption that m_{ij} is not divisible by 3). Recall that $M = l(f) = l(fs_i) + 1 = l(fs_is_j) + 2$ (the last equality follows from Lemma 5.12). Furthermore, $l(fs_is_js_i) = M - 3$ (otherwise, $l(fs_is_js_i) = M - 1$, and either $l(fs_is_js_is_j) = M$ and we obtain an (i, j) -symmetric 5-tuple in I_0 , or $l(fs_is_js_is_j) = M - 2$ and by Lemma 5.6 we have $I_0 \subset I$ which contradicts our assumptions).

By Claim 5.18, either fs_i belongs to a (k, n) -symmetric 5-tuple, or fs_i has no neighbors in I except f and fs_is_j . In the former case there is nothing to prove, so suppose the latter. By Lemma 5.13, f has no other neighbors in I except fs_i . So, by Lemma 5.9, $m_{ik} = m_{in} = \infty$. Similarly, if neither $fs_is_js_k$ nor $fs_is_js_n$ belongs to I , then $m_{jk} = m_{jn} = \infty$ which is false since $m_{23} \neq \infty$ (see Fig. 5.18). Hence, at least one of $fs_is_js_k$ and $fs_is_js_n$ belongs to I . Without loss of generality we may assume $fs_is_js_k \in I$. Note, that since $l(fs_is_js_i) = M - 3$, $l(fs_is_js) = M - 2$ and $m_{ik} = \infty$, by Corollary 5.5 we have $l(fs_is_js_k) \neq M - 3$. So, $l(fs_is_js_k) = M - 1$, and, similarly, $l(fs_is_js_n) = M - 1$. Thus, by Claim 5.19 $fs_is_js_n s_k, fs_is_js_n, fs_is_js, fs_is_js_k, fs_is_js_k s_n$ compose a (k, n) -symmetric 5-tuple, and the lemma is proved. $\square$

Now, we are able to prove Lemma 5.17.

Proof of Lemma 5.17. By Claim 5.21, I contains an (i, j) -symmetric 5-tuple, say $ws_is_j, ws_i, w, ws_j, ws_js_i$, where $\{i, j\} = \{1, 2\}$ or $\{i, j\} = \{3, 4\}$. By Claim 5.20, ws_i belongs to a (k, n) -symmetric 5-tuple lying in I (where $\{k, n\} = \{1, 2, 3, 4\} \setminus \{i, j\}$). Thus, either $ws_is_k s_n$ or $ws_is_n s_k$ has length $M - 1$ and belongs again to (i, j) -symmetric 5-tuple lying in I .

By successive repetitions of this argument, we obtain that for every $p \in \mathbb{N}$ there exists $v_1, \dots, v_p \in \{s_is_j, s_js_i\}$ and $u_1, \dots, u_p \in \{s_k s_n, s_n s_k\}$ such that $wv_1 u_1 v_2 u_2 \dots v_p u_p \in I$. By Theorem 2.4, the expression $v_1 u_1 v_2 u_2 \dots v_p u_p$ is reduced, and therefore $ws_iv_1 u_1 v_2 u_2 \dots v_p u_p \neq ws_iv_1 u_1 v_2 u_2 \dots v_{p'} u_{p'}$ for $p \neq p'$. Hence, we can construct arbitrary many distinct elements contained in I , which contradicts the assumption on finiteness of I . Thus, W has no finite index reflection subgroup. $\square$

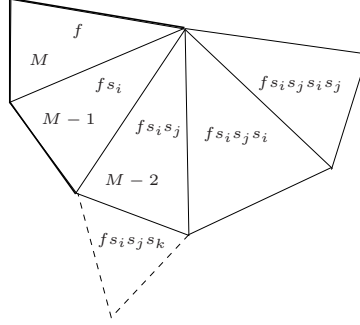
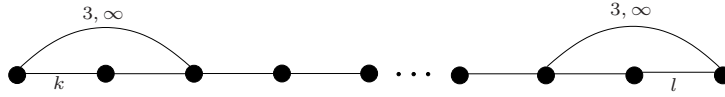


FIGURE 5.18. Towards the proof of Claim 5.21

5.4. Higher rank groups. In this section, we point out a series of groups of rank greater than four without finite index reflection subgroups.

Lemma 5.22. *Let W be the group with $\text{Cox}_{\text{div}}(W)$ shown in Fig. 5.19. Then W has no finite index reflection subgroups.*

FIGURE 5.19. $\text{Cox}_{\text{div}}(W)$ for a series of groups without finite index reflection subgroups, $k, l \geq 5$

Proof. As usual, we assume that there is a finite index reflection subgroup $V \subset W$. We keep all the notation from the previous sections.

We label the vertices of $\text{Cox}_{\text{div}}(W)$ from left to right, so that the least divisor of m_{12} is $k \geq 5$, the least divisor of $m_{n-1,n}$ is $l \geq 5$, and $m_{i,i+1} \in 3\mathbb{Z}_{\text{odd}}$ for $i = 2, \dots, n-2$, $m_{13}, m_{n-2,n} \in \{3\mathbb{Z}_{\text{odd}}, \infty\}$, $m_{ij} = \infty$ for all other pairs i, j .

To prove the lemma, we take an element f of maximal length M in I and show that all its neighbors have length $M+1$, i.e. there is no geodesic from f to 1. We consider the neighbors fs_i for $i \in [3, n-2]$ in Claim 5.23, for $i = 2, n-2$ in Claim 5.24, and for $i = 1, n$ in Claim 5.25. Note that the assertion $l(fs_i) \neq M-1$ is equivalent to $fs_i \notin I$ due to convexity of I , see Lemma 5.3.

Claim 5.23. $l(fs_i) \neq M-1$ for $3 \leq i \leq n-2$.

Proof. Suppose the contrary, i.e. there exists i , $3 \leq i \leq n-2$, such that $l(fs_i) = M-1$. Consider the elements $fs_i s_j$. Suppose that $l(fs_i s_j) = M-2$ for some $j > i$. Then, since $m_{tj} = \infty$ for all $t < i$, Corollary 5.5 implies that

$$(1) \quad l(fs_i s_t) = M \quad \text{for all } t < i,$$

see Fig. 5.20. Furthermore, by Lemma 5.12 (applied to the elements f, fs_i and $fs_i s_{i-1}$), we see that $fs_i s_{i-1} \in I$. Similarly, applying Lemma 5.12 to the elements $fs_i s_{i+1}, fs_i$ and $fs_i s_t$ we see that $fs_i s_t \in I$ for all $t < i$. In particular, $fs_i s_2, fs_i s_1 \in I$. Since m_{12} is not divisible by 3, Lemma 5.12 implies also that $l(fs_i s_1) = M-2$ which contradicts (1).

The contradiction implies that $l(fs_i s_j) = M$ for all $j > i$. Similarly, we can prove that $l(fs_i s_j) = M$ for all $j < i$. Thus, all the neighbors of fs_i have length M which is clearly impossible, so the claim follows. $\square$

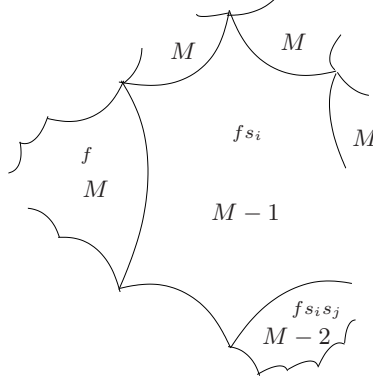


FIGURE 5.20. Towards the proof of Claim 5.23

Claim 5.24. $l(fs_2) \neq M-1$, $l(fs_{n-1}) \neq M-1$.

Proof. We will prove that $l(fs_2) \neq M-1$, the second part of the claim follows by symmetry. Suppose that $l(fs_2) = M-1$, i.e. $fs_2 \in I$.

By Lemma 5.12, $fs_2s_1 \in I$ and $l(fs_2s_1) = M-2$ (since m_{12} is not divisible by 3). By Lemma 5.5, this implies $l(fs_2s_j) = M$ for all $j > 3$ (in particular, $fs_2s_j \notin I$ for $3 < j \leq n-2$ by Claim 5.23). The element fs_2s_3 may be either of length M or of length $M-2$.

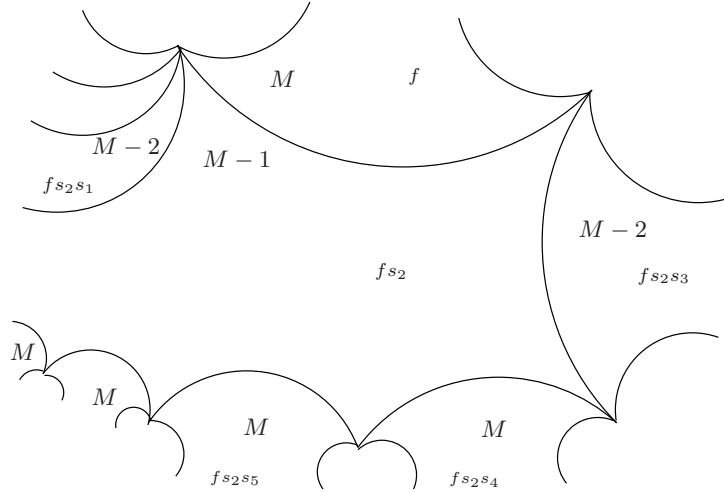


FIGURE 5.21. Towards the proof of Claim 5.24

According to Claim 5.23, $fs_3 \notin I$, which implies $fs_2s_3 \in I$. By Claim 5.23, $l((fs_2)s_3) \neq M$, so $l(fs_2s_3) = M-2$, see Fig. 5.21. In view of Corollary 5.5, this implies $m_{13} \in 3\mathbb{Z}_{\text{odd}}$. Applying Lemma 5.6 to the elements fs_2s_1 , fs_2 and fs_2s_3 of lengths $M-2$, $M-1$ and $M-2$, we obtain $fs_2(s_3s_1)^k, fs_2s_3(s_1s_3)^k \in I \forall k \in \mathbb{Z}$, and the length of all these $2m_{13}$ elements can be computed according to Lemma 5.6. In particular, $l(fs_2s_3s_1) = M-3$.

Let us figure out the lengths of the elements $fs_2s_3s_i$ for $i > 2$, see Fig. 5.22. For this, observe that $m_{1i} = \infty$ for $i > 3$, and thus we have $l(fs_2s_3s_i) = M - 1$, otherwise we get a contradiction with Corollary 5.5. For $i = 3$ we also get $M - 1$ by our assumption.

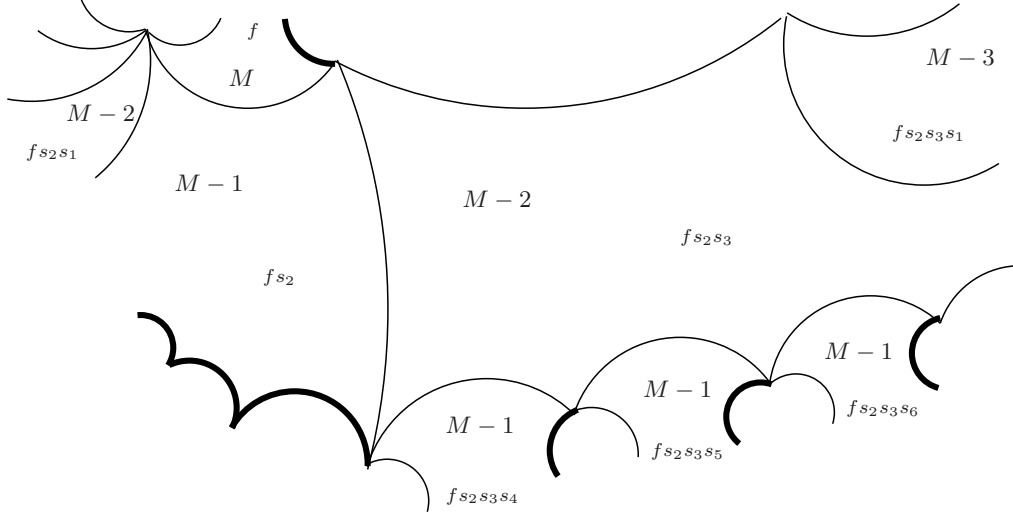


FIGURE 5.22. Towards the proof of Claim 5.24

Now we want to show that the elements $fs_2s_3s_i$ belong to I for $2 < i < n - 1$. We will prove this by induction on i . For $i = 3$ the statement is obvious. Assume the statement is true for $i - 1$. Since $m_{i-1,i}$ is a multiple of 3, either $fs_2s_3s_{i-1}s_i$ or $fs_2s_3s_i$ is in I . In the former case, $l(fs_2s_3s_{i-1}s_i) = M - 2$ by Claim 5.23, and we have $fs_2s_3s_i \in I$ by Lemma 5.6. So, in both cases $fs_2s_3s_i \in I$.

In particular, we get $fs_2s_3s_{n-2} \in I$. Let us show that $fs_2s_3s_{n-2}s_{n-1} \notin I$. Suppose the contrary, i.e. $fs_2s_3s_{n-2}s_{n-1} \in I$. Since $l(fs_2s_3s_{n-2}) = M - 1$, the length of $fs_2s_3s_{n-2}s_{n-1}$ is either M or $M - 2$. If $l(fs_2s_3s_{n-2}s_{n-1}) = M - 2$, then the four elements $fs_2s_3s_{n-2}s_{n-1}$, $fs_2s_3s_{n-2}$, fs_2s_3 , $fs_2s_3s_1$ of lengths $M - 2$, $M - 1$, $M - 2$, $M - 3$ are in contradiction with Lemma 5.10. Therefore, $l(fs_2s_3s_{n-2}s_{n-1}) = M$, and Lemma 5.12 implies that $fs_2s_3s_{n-2}s_n$ also belongs to I , and its length is $M - 2$. Thus, we obtain four elements $fs_2s_3s_{n-2}s_n$, $fs_2s_3s_{n-2}$, fs_2s_3 , $fs_2s_3s_n$ of lengths $M - 2$, $M - 1$, $M - 2$, $M - 1$ respectively, and $fs_2s_3s_{n-2} \in I$, see Fig. 5.23. This contradicts Lemma 5.6.

Therefore, $fs_2s_3s_{n-2}s_{n-1} \notin I$. This implies $fs_2s_3s_{n-1} \in I$ (see Fig. 5.23). Thus, we have two “neighboring elements” $fs_2s_3s_{n-1}$ and fs_2s_3 in I . Considering the elements obtained from fs_2s_3 by multiplication by s_{n-1} and s_n , we see that at least five such elements belong to I (since $m_{n-1,n}$ is not divisible by 3). As it was proved above, $l(fs_2s_3s_{n-1}) = l(fs_2s_3) + 1 = l(fs_2s_3s_n) = M - 1$, which means that we have an $(n, n - 1)$ -symmetric 5-tuple $(fs_2s_3s_{n-1}s_n, fs_2s_3s_{n-1}, fs_2s_3, fs_2s_3s_n, fs_2s_3s_{n-1}s_{n-1})$.

Consider now the elements obtained from fs_2s_3 by multiplication by s_{n-2} and s_n (see Fig. 5.24). Since the elements $fs_2s_3s_{n-2}$, fs_2s_3 , $fs_2s_3s_n$ all belong to I and have lengths $M - 1$, $M - 2$, $M - 1$, we see that $l(fs_2s_3s_{n-2}s_n) = M$, which implies $fs_2s_3s_{n-2}s_{n-1} \notin I$ by Claim 5.23. In view of $fs_2s_3s_n$, $fs_2s_3s_{n-1}s_{n-1} \in I$, this implies $fs_2s_3s_{n-1}s_{n-2} \in I$, and thus $l(fs_2s_3s_{n-1}s_{n-2}) = M - 1$. Since $fs_2s_3s_{n-1} \in I$ and $l(fs_2s_3s_{n-1}) = M$, this contradicts Claim 5.23, which completes the proof of the claim. $\square$

Claim 5.25. $l(fs_1) \neq M - 1$, $l(fs_n) \neq M - 1$.

Proof. We will show that $l(fs_1) \neq M - 1$, the second part is similar. Suppose that $l(fs_1) = M - 1$. By Lemma 5.12, $fs_1s_2 \in I$ and $l(fs_1s_2) = M - 2$. Furthermore, $fs_1s_2s_1, fs_1s_2s_1s_2 \in I$ (m_{12} is not divisible

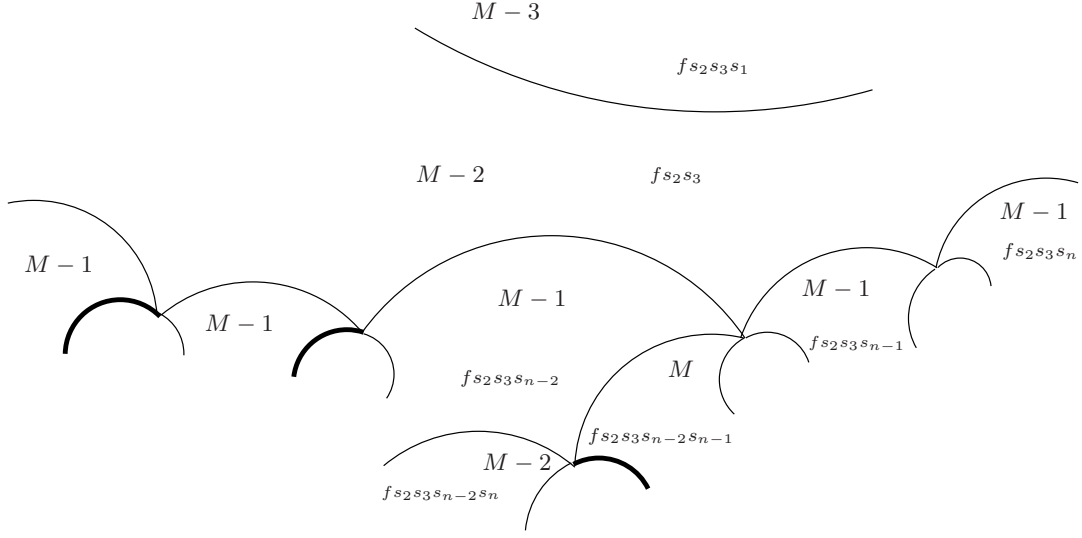


FIGURE 5.23. Towards the proof of Claim 5.24

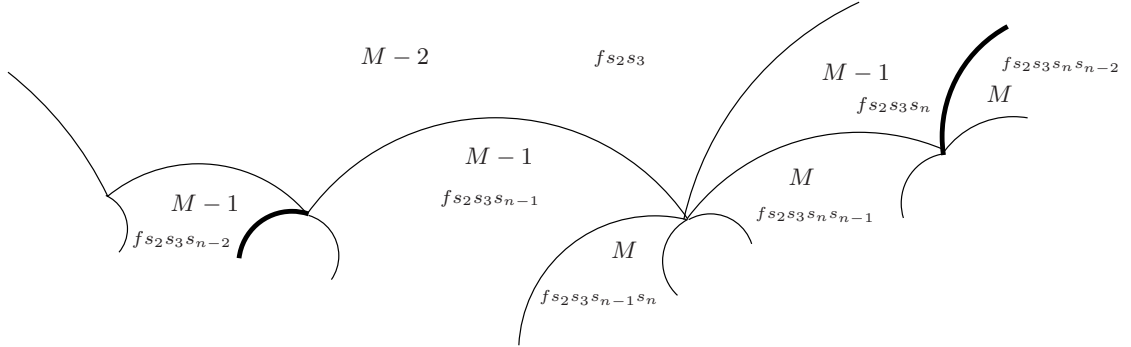


FIGURE 5.24. Towards the proof of Claim 5.24

by 3). If $l(f s_1 s_2 s_1) = M - 1$, then $l(f s_1 s_2 s_1 s_2) = M$, so that these elements are in contradiction with Claim 5.24. This implies that $l(f s_1 s_2 s_1) = M - 3$, see Fig 5.25.

Now, suppose that $m_{13} \neq \infty$. Then $f, f s_1 \in I$ together with $l(f) = l(f s_1) + 1 = M$ implies that $f s_1 s_3 \in I$. In view of Claim 5.23, we see that $l(f s_1 s_3) \neq M$, so $l(f s_1 s_3) = M - 2$. This contradicts Lemma 5.10 (applied to the elements $f s_1 s_3, f s_1, f s_1 s_2, f s_1 s_2 s_1$ of lengths $M - 2, M - 1, M - 2, M - 3$). The contradiction shows that $m_{13} = \infty$.

Now, consider the neighbors of $f s_1 s_2$. Since $l(f s_1 s_2 s_1) = M - 3$ and $m_{1j} = \infty$ for $j > 2$, we have $l(f s_1 s_2 s_j) = M - 1$.

Suppose that $f s_1 s_2 s_3 \notin I$. Then $f s_1 s_3 \in I$ (since $m_{23} \neq \infty$ and $f s_1 s_2, f s_1 \in I$). Moreover, $l(s_1 s_3) = M$ (otherwise by Lemma 5.6 we have a contradiction with the assumption $f s_1 s_2 s_3 \notin I$). However, this contradicts the statement of Claim 5.23. Hence, $f s_1 s_2 s_3 \in I$.

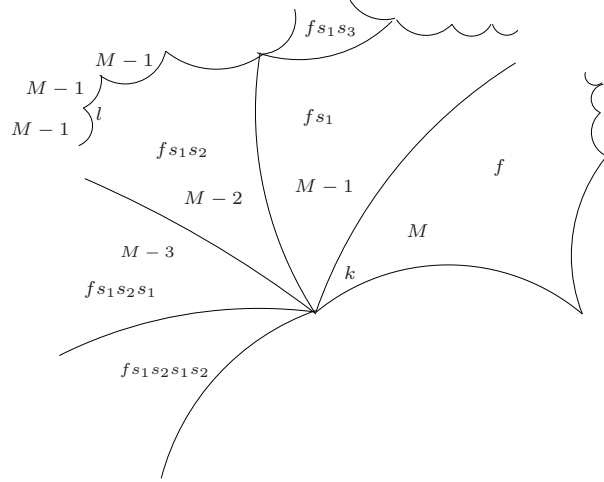


FIGURE 5.25. Towards the proof of Claim 5.25

Now we use induction on j to show that $f s_1 s_2 s_j \in I$ for all $3 \leq j \leq n-1$. Indeed, if $f s_1 s_2 s_{j-1} \in I$ but $f s_1 s_2 s_j \notin I$, then $f s_1 s_2 s_{j-1} s_j \in I$, and $l(f s_1 s_2 s_{j-1} s_j) = l(f s_1 s_2 s_{j-1}) + 1 = M$, which contradicts Claim 5.23 or Claim 5.24.

In particular, we obtain that $f s_1 s_2 s_{n-1} \in I$. Since $f s_1 s_2 \in I$ and $m_{n-1, n}$ is not divisible by 3, this implies that at least five elements obtained from $f s_1 s_2$ by multiplication by s_{n-1} and s_n belong to I . Recall that the lengths of $f s_1 s_2 s_n, f s_1 s_2, f s_1 s_2 s_{n-1}$ are $M-1, M-2, M-1$ respectively. Thus, $f s_1 s_2 s_n, f s_1 s_2 s_n s_{n-1}, f s_1 s_2 s_{n-1} s_n \in I$ with $l(f s_1 s_2 s_n s_{n-1}) = l(f s_1 s_2 s_{n-1} s_n) = M$. However, this is in contradiction with the statement of Claim 5.24, which completes the proof of the claim. $\square$

Now, combining the results of Claims 5.23–5.25 we obtain the lemma. $\square$

6. MINIMAL GROUPS WITHOUT FINITE INDEX REFLECTION SUBGROUPS

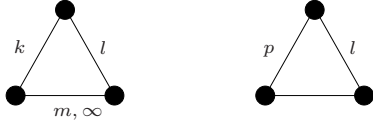
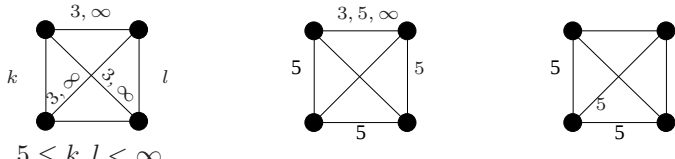
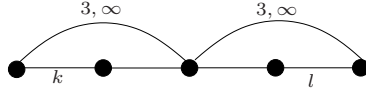
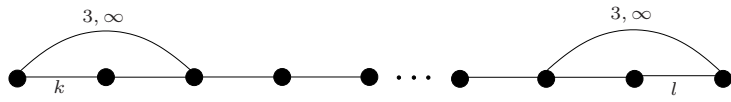
In this section we combine results of previous sections to obtain a criterion for odd-angled Coxeter groups to contain no finite index reflection subgroups. For this we list *minimal* groups containing no finite index reflections subgroups (where W is *minimal* if W has no finite index reflection subgroup while each proper special subgroup of W has one). According to Corollary 3.2, every Coxeter group with connected divisibility diagram containing such a minimal group as a special subgroup has no finite index reflection subgroup either.

The results will be formulated in terms of divisibility diagrams $\text{Cox}_{\text{div}}(W)$ of odd-angled Coxeter systems (W, S) (see Definition 2.1).

Theorem 6.1. *Let W be a minimal odd-angled Coxeter group containing no finite index reflection subgroup, and assume that $\text{Cox}_{\text{div}}(W)$ is connected. Then $\text{Cox}_{\text{div}}(W)$ is one of the diagrams shown in Table 6.1.*

Remark 6.2. It is easy to see that the diagram $\text{Cox}_{\text{div}}(W)$ is not connected (i.e., $S = S_1 \sqcup S_2$ for some nonempty S_1 and S_2 , and there are no edges between S_1 and S_2) if and only if W is the free product of the groups generated by S_1 and S_2 .

TABLE 6.1. Minimal groups without finite index reflection subgroups and with connected $\text{Cox}_{\text{div}}(W)$

$\text{rank}(W)$	$\text{Cox}_{\text{div}}(W)$
3	 $5 \leq k, l, m \leq \infty$ $7 \leq p \leq \infty$
4	 $5 \leq k, l < \infty$
5	 $5 \leq k, l < \infty$
$n \geq 5$	 $5 \leq k, l < \infty$

Proof of Theorem 6.1. We proceed by increasing the rank of W .

1. $\text{rank}(W) \leq 2$. All these groups are either finite or infinite dihedral, so any of them has a finite index reflection subgroup.

2. $\text{rank}(W) = 3$. The diagrams listed in the table can be expressed as all connected diagrams of order 3 except the ones shown in Fig. 6.26. Now the statement follows from Lemmas 5.14, 5.15 and the examples constructed in Section 4.

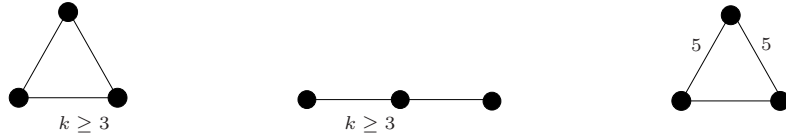


FIGURE 6.26. Rank 3 Coxeter groups having finite index subgroups

3. $\text{rank}(W) = 4$. All the diagrams of order 4 shown in Table 6.1 correspond to groups having no finite index reflection subgroups (see Lemmas 5.16 and 5.17), and no proper subdiagram of these diagrams defines a group having no finite index reflection subgroup.

To prove that the list is complete, note that by Lemma 4.3 the diagram $\text{Cox}_{\text{div}}(W)$ of minimal group W should contain at least two multiple edges. If there are two multiple edges incident to one vertex and at least one of them is labeled other than 5, then W is not minimal (see rank 3 case).

So, either there are exactly two multiple edges, and they have no common vertex (as in the left diagram in Table 6.1, second row), or all multiple edges are labeled by 5.

Suppose that all multiple edges are labeled by 5 and there exist two multiple edges incident to the same vertex. Due to the rank 3 classification, in any order 3 subdiagram containing two multiple edges the third edge should be simple. Thus, if there are three multiple edges incident to one vertex, we obtain the diagram shown in Table 6.1 (right), otherwise we obtain the diagram in the middle of row 2 in Table 6.1 (according to Lemma 4.4, in this case we need at least three multiple edges).

4. $\text{rank}(W) \geq 5$. By Lemma 5.22, the diagrams shown in Table 6.1 define Coxeter groups containing no finite index reflection subgroups. By Lemma 4.3, these diagrams are minimal connected diagrams defining groups without finite index reflection subgroups. We are left to prove that the list is complete.

The proof is by induction on the rank of W . Suppose that $\text{Cox}_{\text{div}}(W)$ is a connected diagram of a minimal group containing no finite index reflection subgroup. By Lemma 4.3, $\text{Cox}_{\text{div}}(W)$ contains at least two multiple edges. It follows from minimality and from the results for smaller ranks that $\text{Cox}_{\text{div}}(W)$ contains at most two multiple edges (if there are three multiple edges it is always possible to find a connected subdiagram containing only two of them defining a group without finite index reflection subgroups). It also follows from minimality that in case of $\text{rk}(W) = n$ the vertices of two distinct multiple edges should not be connected in $\text{Cox}_{\text{div}}(W)$ by a path containing less than $n - 3$ edges. This proves the completeness of the list. $\square$

So far we considered odd-angled Coxeter groups with connected divisibility diagrams only. Now, suppose that $\text{Cox}_{\text{div}}(W)$ has several connected components, $\text{Cox}_{\text{div}}(W_1), \dots, \text{Cox}_{\text{div}}(W_k)$, where W_i are odd-angled Coxeter groups. In other words, $W = W_1 * W_2 * \dots * W_k$.

Lemma 6.3. *Let W be an odd-angled Coxeter group and $W = W_1 * W_2 * \dots * W_k$. Then W contains a finite index reflection subgroup if and only if at least one of $W_1, \dots, W_k$ contains one.*

Proof. First, suppose that $V_1 \subset W_1$ is a finite index reflection subgroup. Let F_{V_1} be its principal fundamental domain in the Davis complex of W_1 , and define I_1 to be $\{w \in W_1 \mid D(w) \in F_{V_1}\}$. Consider now I_1 as a set of elements of W , and define the polytope $P = \{D(w) \mid w \in I_1\}$. We want to show that P is a Coxeter polytope.

The facets of P (more precisely, the reflections stabilizing the walls containing the facets) can either belong to W_1 or not, so we need to look at different types of dihedral angles of P . If both facets defining an angle of P belong to W_1 , then the value of the dihedral angle is the same as in F_{V_1} , so it is an integer part of π . If one of the facets does not belong to W_1 , then the other does not belong to W_1 either: by assumption on connected components of $\text{Cox}_{\text{div}} W$, no wall of W_i meets any wall not belonging to W_i . Since all the walls that intersect the interior of P correspond to reflections in W_1 , in this case there exists a chamber containing the intersection of the facets and whose defining walls contain the two facets. Hence the dihedral angle between the two facets is a dihedral angle of a chamber, i.e. an integer submultiple of π . Therefore, all the angles of P are integer submultiples of π , hence P is a fundamental domain of a finite index reflection subgroup of W .

Now suppose $V \subset W$ is a finite index reflection subgroup. Since V is a proper subgroup of W , the principal fundamental domain F_V of V contains a chamber $D(s)$ for some generating reflection $s \in S$. Clearly, there is a unique W_i containing s . The same reasonings as the ones used in the proof of Lemma 3.1 show that W_i has a finite index reflection subgroup. $\square$

Now we combine Theorem 6.1 with Lemma 6.3 to obtain the following criterion.

Theorem 6.4. *An odd-angled Coxeter group W contains no finite index reflection subgroup if and only if each connected component of $\text{Cox}_{\text{div}}(W)$ contains one of the diagrams shown in Table 6.1 as a subdiagram.*

We close the paper with the following reformulation of Theorem 6.4 not referring to Table 6.1.

Corollary 6.5. *An odd-angled Coxeter group W contains a finite index reflection subgroup if and only if $\text{Cox}_{\text{div}}(W)$ contains at least one connected component C of one of the following three types:*

1. *the order of C is 1 or 2;*
2. *C contains at most one multiple edge;*
3. *C contains a subdiagram D of order 3 with labels $(5, 5, 3)$, and every non-absent edge of C except the edges of D is simple.*

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