Packing Bipartite Graphs with Covers of Complete Bipartite Graphs ⋆

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Abstract. For a set $\mathcal{S}$ of graphs, a perfect $\mathcal{S}$-packing ($\mathcal{S}$-factor) of a graph G is a set of mutually vertex-disjoint subgraphs of G that each are isomorphic to a member of $\mathcal{S}$ and that together contain all vertices of G. If G allows a covering (locally bijective homomorphism) to a graph H, then G is an H-cover. For some fixed H let $\mathcal{S}(H)$ consist of all connected H-covers. Let $K_{k,\ell}$ be the complete bipartite graph with partition classes of size k and ℓ, respectively. For all fixed $k, \ell \geq 1$, we determine the computational complexity of the problem that tests whether a given bipartite graph has a perfect $\mathcal{S}(K_{k,\ell})$-packing. Our technique is partially based on exploring a close relationship to pseudo-coverings. A pseudo-covering from a graph G to a graph H is a homomorphism from G to H that becomes a covering to H when restricted to a spanning subgraph of G. We settle the computational complexity of the problem that asks whether a graph allows a pseudo-covering to $K_{k,\ell}$ for all fixed $k, \ell \geq 1$.

1 Introduction

Throughout the paper we consider undirected graphs with no loops and no multiple edges. Let $G = (V, E)$ be a graph and let $\mathcal{S}$ be some fixed set of mutually vertex-disjoint graphs. A set of (not necessarily vertex-induced) mutually vertex-disjoint subgraphs of G, each isomorphic to a member of $\mathcal{S}$, is called an *$\mathcal{S}$-packing*. Packings naturally generalize matchings (the case in which $\mathcal{S}$ only contains edges). They arise in many applications, both practical ones such as exam scheduling [12], and theoretical ones such as the study of degree constraint graphs (cf. the survey of Hell [11]). If $\mathcal{S}$ consists of a single subgraph S, we write *S-packing* instead of $\mathcal{S}$-packing. The problem of finding an S-packing of a graph G that packs the maximum number of vertices of G is NP-hard for

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all fixed connected graphs S on at least three vertices, as shown by Hell and Kirkpatrick [13].

A packing of a graph is *perfect* if every vertex of the graph belongs to one of the subgraphs of the packing. Perfect packings are also called *factors* and from now on we call a perfect $\mathcal{S}$-packing an $\mathcal{S}$-*factor*. We call the corresponding decision problem the $\mathcal{S}$-FACTOR problem. For a survey on graph factors we refer to the monograph of Plummer [19].

Our Focus. We study a relaxation of $K_{k,\ell}$-factors, where $K_{k,\ell}$ denotes the *biclique* (complete connected bipartite graph) with partition classes of size k and ℓ, respectively. In order to explain this relaxation we first need to introduce some new terminology.

A *homomorphism* from a graph G to a graph H is a vertex mapping $f : V_G \to V_H$ satisfying the property that $f(u)f(v)$ belongs to E_H whenever the edge uv belongs to E_G. If for every $u \in V_G$ the restriction of f to the neighborhood of u, i.e., the mapping $f_u : N_G(u) \to N_H(f(u))$, is bijective then we say that f is a *locally bijective* homomorphism or a *covering* [2, 16]. The graph G is then called an H-*cover* and we write $G \xrightarrow{B} H$. Locally bijective homomorphisms have applications in distributed computing [1] and in constructing highly transitive regular graphs [3]. For a specified graph H, we let $\mathcal{S}(H)$ consist of all connected H-covers. In this paper we study $\mathcal{S}(K_{k,\ell})$-factors of bipartite graphs.

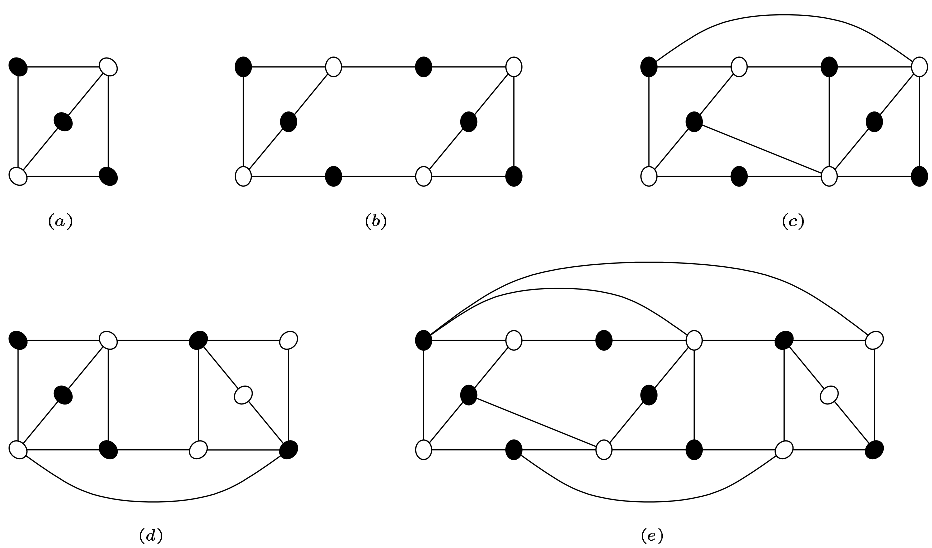

Fig. 1. Examples: (a) a $K_{2,3}$. (b) a bipartite $K_{2,3}$-cover. (c) a bipartite $K_{2,3}$-pseudo-cover that is no $K_{2,3}$-cover and that has no $K_{2,3}$-factor. (d) a bipartite graph with a $K_{2,3}$-factor that is not a $K_{2,3}$-pseudo-cover. (e) a bipartite graph with an $\mathcal{S}(K_{2,3})$-factor but with no $K_{2,3}$-factor and that is not a $K_{2,3}$-pseudo-cover.

Our Motivation. Since a $K_{1,1}$-factor is a perfect matching, $K_{1,1}$-Factor is polynomial-time solvable. The $K_{k,\ell}$-Factor problem is known to be NP-complete for all other $k, \ell \geq 1$, due to the aforementioned result of Hell and Kirkpatrick [13]. These results have some consequences for our relaxation. In order to explain this, we make the following observation, which holds because only a tree has a unique cover (namely the tree itself) and the graph $K_{k,\ell}$ is a tree if $k = 1$ or $\ell = 1$.

Observation 1 $\mathcal{S}(K_{k,\ell}) = \{K_{k,\ell}\}$ *if and only if* $\min\{k, \ell\} = 1$.

Because $\mathcal{S}(K_{1,\ell}) = \{K_{1,\ell}\}$ by Observation 1, the above results immediately imply that $\mathcal{S}(K_{1,\ell})$-Factor is only polynomial-time solvable if $\ell = 1$; it is NP-complete otherwise. What about our relaxation for $k, \ell \geq 2$? Note that, for these values of k, ℓ, the size of the set $\mathcal{S}(K_{k,\ell})$ is unbounded. The only result known so far is for $k = \ell = 2$; Hell, Kirkpatrick, Kratochvíl and Kříž [14] showed that $\mathcal{S}(K_{2,2})$-Factor is NP-complete for general graphs, as part of their computational complexity classification of finding restricted 2-factors; we explain the reason why an $\mathcal{S}(K_{2,2})$-factor is a restricted 2-factor later.

For bipartite graphs, the following is known. Firstly, Monnot and Toulouse [18] researched path factors in bipartite graphs and showed that the $K_{2,1}$-Factor problem stays NP-complete when restricted to the class of bipartite graphs. Secondly, we observed that as a matter of fact the proof of the NP-completeness result for $\mathcal{S}(K_{2,2})$-Factor in [14] is even a proof for bipartite graphs.

Our interest in bipartite graphs stems from a close relationship of $\mathcal{S}(K_{k,\ell})$-factors of bipartite graphs and so-called $K_{k,\ell}$-pseudo-covers, which originate from topological graph theory and have applications in the area of distributed computing [4, 5]. A homomorphism f from a graph G to a graph H is a *pseudo-covering* from G to H if there exists a spanning subgraph G' of G such that f is a covering from G' to H. In that case G is called an H-*pseudo-cover* and we write $G \xrightarrow{P} H$. The computational complexity classification of the H-Pseudo-Cover problem, which is to test for a fixed graph H (i.e., not being part of the input) whether $G \xrightarrow{P} H$ for some given G is still open, and our paper can also be seen as a first investigation into this question. We explain the exact relationship between factors and pseudo-coverings in detail later on; we refer to Figure 1 for some examples that illustrate the notions introduced.

Our Results and Paper Organization.
Section 2 contains additional terminology, notations and some basic observations. In Section 3 we pinpoint the relationship between factors and pseudo-coverings. In Section 4 we completely classify the computational complexity of the $\mathcal{S}(K_{k,\ell})$-Factor problem for bipartite graphs. Recall that $\mathcal{S}(K_{1,1})$-Factor is polynomial-time solvable on general graphs. We first prove that $\mathcal{S}(K_{1,\ell})$-Factor is NP-complete on bipartite graphs for all fixed $\ell \geq 2$. By applying our result of Section 3, we then show that NP-completeness of every remaining case can be shown by proving NP-completeness of the corresponding $K_{k,\ell}$-Pseudo-Cover problem. We classify the complexity of $K_{k,\ell}$-Pseudo-Cover in Section 5. We show that it is indeed NP-complete on bipartite graphs for all fixed

pairs $k, \ell \geq 2$ by adapting the hardness construction of Hell, Kirkpatrick, Kratochvíl and Kříž [14] for restricted 2-factors. In contrast to $\mathcal{S}(K_{k,\ell})$-Factor, we show that $K_{k,\ell}$-Pseudo-Cover is polynomial-time solvable for all $k, \ell \geq 1$ with $\min\{k, \ell\} = 1$. In Section 6 we further discuss the relationships between pseudo-coverings and locally constrained homomorphisms, such as the aforementioned coverings. We shall see that as a matter of fact the NP-completeness result for $K_{k,\ell}$-Pseudo-Cover for fixed $k, \ell \geq 3$ also follows from a result of Kratochvíl, Proskurowski and Telle [15] who proved that $K_{k,\ell}$-Cover is NP-complete for $k, \ell \geq 3$. This problem is to test whether $G \xrightarrow{B} K_{k,\ell}$ for a given graph G. However, the same authors [15] showed that $K_{k,\ell}$-Cover is polynomial-time solvable when $k = 2$ or $\ell = 2$. Hence, for those pairs (k, ℓ) we can only use our hardness proof in Section 5.

2 Preliminaries

From now on let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_\ell\}$ denote the partition classes of $K_{k,\ell}$. If $k = 1$ then we say that x_1 is the *center* of $K_{1,\ell}$. If $\ell = 1$ and $k \geq 2$, then y_1 is called the center. We denote the degree of a vertex u in a graph G by $\deg_G(u)$.

Recall that a homomorphism f from a graph G to a graph H is a pseudo-covering from G to H if there exists a spanning subgraph G' of G such that f is a covering from G' to H. We would like to stress that this is *not* the same as saying that f is a vertex mapping from V_G to V_H such that f restricted to some spanning subgraph G' of G becomes a covering. The reason is that in the latter setting it may well happen that f is not a homomorphism from G to H. For instance, f might map two adjacent vertices of G to the same vertex of H. However, there is an alternative definition which turns out to be very useful for us. In order to present it we need the following notations.

We let $f^{-1}(x)$ denote the set $\{u \in V_G \mid f(u) = x\}$. For a subset $S \subseteq V_G$, $G[S]$ denotes the *induced subgraph* of G by S, i.e., the graph with vertex set S and edges uv whenever $uv \in E_G$. For $xy \in E_H$ with $x \neq y$, we write $G[x, y] = G[f^{-1}(x) \cup f^{-1}(y)]$. Because f is a homomorphism, $G[x, y]$ is a bipartite graph with partition classes $f^{-1}(x)$ and $f^{-1}(y)$. We can now state the alternative definition of pseudo-coverings.

Proposition 1 ([4]). *A homomorphism f from a graph G to a graph H is a pseudo-covering if and only if $G[x, y]$ contains a perfect matching for all $x, y \in V_H$. Consequently, $|f^{-1}(x)| = |f^{-1}(y)|$ for all $x, y \in V_H$.*

Let f be a pseudo-covering from a graph G to a graph H. We then sometimes call the vertices of H *colors* of vertices of G. Due to Proposition 1, $G[x, y]$ must contain a perfect matching M_{xy}. Let $uv \in M_{xy}$ for $xy \in E_H$. Then we say that v is a *matched neighbor* of u, and we call the set of matched neighbors of u the *matched neighborhood* of u.

3 How Factors Relate to Pseudo-Covers

Our next result shows how $\mathcal{S}(K_{k,\ell})$-factors relate to $K_{k,\ell}$-pseudo-covers.

Theorem 1. *Let G be a graph on n vertices. Then G is a $K_{k,\ell}$-pseudo-cover if and only if G has an $\mathcal{S}(K_{k,\ell})$-factor and G is bipartite with partition classes A and B such that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$.*

Proof. First suppose that $G = (V, E)$ is a $K_{k,\ell}$-pseudo-cover. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then f is a homomorphism from G to $K_{k,\ell}$, which is a bipartite graph. Consequently, G must be bipartite as well. Let A and B denote the partition classes of G. Then we may assume without loss of generality that $f(A) = X$ and $f(B) = Y$. Due to Proposition 1 we then find that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$. By the same proposition we find that each $G[x_i, y_j]$ contains a perfect matching M_{ij}. We define the spanning subgraph $G' = (V, \bigcup_{ij} M_{ij})$ of G and observe that every component in G' is a $K_{k,\ell}$-cover. Hence G has an $\mathcal{S}(K_{k,\ell})$-factor.

Now suppose that G has an $\mathcal{S}(K_{k,\ell})$-factor $\{F_1, \ldots, F_p\}$. Also suppose that G is bipartite with partition classes A and B such that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$. Since $\{F_1, \ldots, F_p\}$ is an $\mathcal{S}(K_{k,\ell})$-factor, there exists a covering f_i from F_i to $K_{k,\ell}$ for $i = 1, \ldots, p$. Let f be the mapping defined on V such that $f(u) = f_i(u)$ for all $u \in V$. Let A_X be the set of vertices of A that are mapped to a vertex in X and let A_Y be the set of vertices of A that are mapped to a vertex in Y. We define subsets B_X and B_Y of B in the same way. This leads to the following equalities:

$$
\begin{aligned}
|A_X| + |A_Y| &= \tfrac{kn}{k+\ell} \\
|B_X| + |B_Y| &= \tfrac{\ell n}{k+\ell} \\
|A_Y| &= \tfrac{\ell}{k}|B_X| \\
|B_Y| &= \tfrac{\ell}{k}|A_X|.
\end{aligned}
$$

Suppose that $\ell \neq k$. Then this set of equalities has a unique solution, namely, $|A_X| = \frac{kn}{k+\ell} = |A|, |A_Y| = |B_X| = 0$, and $|B_Y| = \frac{\ell n}{k+\ell} = |B|$. Hence, we find that f maps all vertices of A to vertices of X and all vertices of B to Y. This means that f is a homomorphism from G to $K_{k,\ell}$ that becomes a covering when restricted to the spanning subgraph obtained by taken the disjoint union of the subgraphs $\{F_1, \ldots, F_p\}$. In other words, f is a pseudo-covering from G to $K_{k,\ell}$, as desired.

Suppose that $\ell = k$. In this case we have that $|V_{F_i} \cap A| = |V_{F_i} \cap B|$ for $i = 1, \ldots, p$, and since each F_i is connected by definition, either $f(V_{F_i} \cap A) = X$ and $f(V_{F_i} \cap B) = Y$, or $f(V_{F_i} \cap A) = Y$ and $f(V_{F_i} \cap B) = X$. In the second case, we can exchange the roles of X and Y and find another covering f_i from F_i such that $f(V_{F_i} \cap A) = X$ and $f(V_{F_i} \cap B) = Y$. Hence, we can assume without loss of generality that each f_i maps $V_{F_i} \cap A$ to X and $V_{F_i} \cap B$ to Y; so, $|A_X| = |A| = |B_Y| = |B|$ and $|A_Y| = |B_X| = 0$. This completes the proof of Theorem 1. $\square$

4 Classifying the $\mathcal{S}(K_{k,\ell})$-Factor Problem

Here is the main theorem of this section.

Theorem 2. *The* $\mathcal{S}(K_{k,\ell})$-Factor *problem is solvable in polynomial time for* $k = \ell = 1$*. Otherwise it is* NP*-complete, even for the class of bipartite graphs.*

Proof. We may assume without loss of generality that $k \leq \ell$. First we consider the case when $k = \ell = 1$. Due to Observation 1, the $\mathcal{S}(K_{1,1})$-Factor problem is equivalent to the problem of finding a perfect matching, which can be solved in polynomial time. We deal with the case when $k = 1$ and $\ell \geq 2$ in Proposition 2. Finally, for all $k \geq 2$ and all $\ell \geq 2$, we show in Proposition 3 that if the $K_{k,\ell}$-Pseudo-Cover problem is NP-complete, then so is the $\mathcal{S}(K_{k,\ell})$-Factor problem for the class of bipartite graphs. Then the result for this case follows from Theorem 4, in which we show that $K_{k,\ell}$-Pseudo-Cover is NP-complete for all $k \geq 2$ and all $\ell \geq 2$. □

The proof of Theorem 2 is conditional upon proving Propositions 2 and 3, and Theorem 4. We prove Theorem 4 in Section 5, and show Propositions 2 and 3 in this section.

Proposition 2 deals with the case $k = 1$ and $\ell \geq 2$. Recall that for general graphs the NP-completeness of this case immediately follows from Observation 1 and the aforementioned result of Hell and Kirkpatrick [13]. However, we consider bipartite graphs. For this purpose, a result by Monnot and Toulouse [18] is of importance for us. Here, P_k denotes a path on k vertices.

Theorem 3 ([18]). *For any fixed* $k \geq 3$*, the* P_k-Factor *problem is* NP*-complete for the class of bipartite graphs.*

We use Theorem 3 to prove Proposition 2.

Proposition 2. *For any fixed* $\ell \geq 2$*,* $\mathcal{S}(K_{1,\ell})$-Factor *and* $K_{1,\ell}$-Factor *are* NP*-complete, even for the class of bipartite graphs.*

Proof. By Observation 1, $\mathcal{S}(K_{1,\ell}) = \{K_{1,\ell}\}$ for all $\ell \geq 2$. Hence we may restrict ourselves to $K_{1,\ell}$-Factor. Clearly, $K_{1,\ell}$-Factor is in NP for all $\ell \geq 2$. Note that $P_3 = K_{1,2}$. Hence the case $\ell = 2$ follows from Theorem 3.

Let $\ell = 3$. We prove that $K_{1,3}$-Factor is NP-complete by reduction from $K_{1,2}$-Factor. Let $G = (V, E)$ be a bipartite graph with partition classes A and B. We will construct a bipartite graph G' from G such that G has an $K_{1,2}$-factor if and only if G' has a $K_{1,3}$-factor.

First we make a key observation, namely that all $K_{1,2}$-factors of G (if there are any) have the same number α of centers in A and the same number β of centers in B. This is so, because the following two equalities

$$\alpha + 2\beta = |A|$$
$$\beta + 2\alpha = |B|$$

that count the number of vertices in A and B, respectively, have a unique solution. In order to obtain G' we do as follows. Let $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_q\}$. First we consider the vertices in A. For $i = 1, \ldots p$, we introduce

- a new vertex s_i with edge $s_i a_i$
- a new vertex t_i with edge $s_i t_i$
- three new vertices u_i^1, u_i^2, u_i^3 with edges $t_i u_i^1, t_i u_i^2, t_i u_i^3$
- a new vertex w_i with edges $u_i^1 w_i, u_i^2 w_i, u_i^3 w_i$.

Finally we add $2p + \alpha$ new vertices $x_1, \ldots, x_{2p+\alpha}$ and add edges such that the subgraph induced by the w-vertices and the x-vertices is complete bipartite. We denote the set of s-vertices by S, the set of t-vertices by T, the set of u-vertices by U, the set of w-vertices by W, and the set of x-vertices by X. We repeat the above process with respect to B. For clarity we denote the new vertices with respect to B by s', t', u', w', x', and corresponding sets by S', T', U', W', X', respectively. This yields the graph G' which is bipartite with partition classes $A \cup S' \cup T \cup U' \cup W \cup X'$ and $B \cup S \cup T' \cup U \cup W' \cup X$. Also see Figure 2.

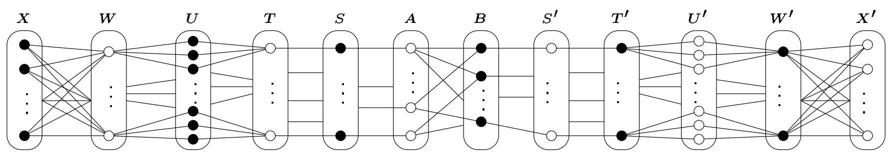

Fig. 2. The graph G'.

We are now ready to prove our claim that G has a $K_{1,2}$-factor if and only if G' has a $K_{1,3}$-factor.

Suppose that G has a $K_{1,2}$-factor. We first extend the three-vertex stars in this factor to four-vertex stars by adding the edge $a_i s_i$ for every star center a_i and the edge $b_i s_i'$ for every star center b_i. As we argued above, A contains α centers and B contains β centers. This means that we can add:

- $p - \alpha$ stars with center in T, one leaf in S and two leaves in U;
- α stars with center in T and three leaves in U;
- $p - \alpha$ stars with center in W, one leaf in U and two leaves in X;
- α stars with center in W and three leaves in X.

This is possible because $|S| = p$, $|T| = p$, $|U| = 3p$, $|W| = p$ and $|X| = 2(p - \alpha) + 3\alpha = 2p + \alpha$. With respect to B we can proceed in the same way. Hence, we obtained a $K_{1,3}$-factor of G'.

Suppose that G' has a $K_{1,3}$-factor. Let γ be the number of star centers in A that belong to stars with one leaf in S and two leafs in B. Let δ be the number of star centers in B that belong to stars with one leaf in S' and two leafs in A. We first show that $\gamma \geq \alpha$.

In order to obtain a contradiction, suppose that $\gamma < \alpha$. Because every s-vertex (resp. u-vertex) has degree two, no vertex in S (resp. U) is a star center. Let p_1 be the number of star centers in T that belong to stars with a leaf in S (and two leafs in U) and let p_2 be the number of star centers in T that belong to stars with all three leafs in U. By our construction, every star center in W belongs to a star that either has one leaf in U and two leafs in X, or else has three leafs in X. Let q_1 be the number of star centers in W of the first type, and let q_2 be the number of star centers in W of the second type. Finally, let r be the number of star centers in X (centers of stars with all leafs in W). Then by using counting arguments in combination with the equalities $|S| = |T| = |W| = p$, $|U| = 3p$ and $|X| = 2p + \alpha$, we derive the following equalities:

$$\begin{aligned}
\gamma + p_1 &= p \\
p_1 + p_2 &= p \\
2p_1 + 3p_2 + q_1 &= 3p \\
q_1 + q_2 + 3r &= p \\
2q_1 + 3q_2 + r &= 2p + \alpha
\end{aligned}$$

The last two equalities imply that $q_2 = \alpha + 5r$. Equality $\gamma + p_1 = p$ and our assumption $\gamma < \alpha$ implies that $p_1 > p - \alpha$. Equalities $p_1 + p_2 = p$ and $2p_1 + 3p_2 + q_1 = 3p$ lead to $p_1 = q_1$. Hence, we find that $q_1 > p - \alpha$. Substituting $q_1 > p - \alpha$ and $q_2 = \alpha + 5r$ into equality $q_1 + q_2 + 3r = p$ yields $8r < 0$ and this is not possible. Hence $\gamma \geq \alpha$.

By the same reasoning as above we find that $\delta \geq \beta$ holds. This has the following consequence. Let γ^* denote the number of star centers in A that belong to stars with three leaves in B and let δ^* denote the number of star centers in B that belong to stars with three leaves in A. Then we find that

$$p = \gamma + 2\delta + \gamma^* + 3\delta^* \geq \alpha + 2\beta + \gamma^* + 3\delta^*.$$

Recall that $\alpha + 2\beta = p$. If we substitute this in the above equation, we find that $p \geq p + \gamma^* + 3\delta^*$. Hence $\gamma = \alpha, \delta = \beta$ and $\gamma^* = \delta^* = 0$. This means that the restriction of the $K_{1,3}$-factor to G is a $K_{1,2}$-factor of G, which is what we had to show.

For $\ell \geq 4$ we can proceed in a similar way as for the case $\ell = 3$ (or use induction). This completes the proof of Proposition 2. □

Here is Proposition 3, which allows us to consider the $K_{k,\ell}$-Pseudo-Cover problem for all $k \geq 2$ and all $\ell \geq 2$.

Proposition 3. *Fix arbitrary integers $k, \ell \geq 2$. If the $K_{k,\ell}$-*Pseudo-Cover *problem is* NP*-complete, then so is the $\mathcal{S}(K_{k,\ell})$-*Factor *problem for the class of bipartite graphs.*

Proof. Let $k, \ell \geq 2$. Let $G = (V, E)$ be an input graph on n vertices of the $K_{k,\ell}$-Pseudo-Cover problem. By Theorem 1, we may assume without loss of generality that G is bipartite with partition classes A and B such that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$. Then, by Theorem 1, we find that $G \xrightarrow{P} K_{k,\ell}$ holds if and only if G has an $\mathcal{S}(K_{k,\ell})$-factor. This finishes the proof of Proposition 3. □

5 Classifying the $K_{k,\ell}$-Pseudo-Cover Problem

Here is the main theorem of this section.

Theorem 4. *The $K_{k,\ell}$-*Pseudo-Cover *problem can be solved in polynomial time for any fixed k, ℓ with $\min\{k, \ell\} = 1$. Otherwise it is* NP*-complete.*

Proof. When $\min\{k, \ell\} = 1$ we use Proposition 4. When $\min\{k, \ell\} \geq 2$, we use Proposition 5. □

The proof of Theorem 2 is conditional upon proving Propositions 4 and 5. The remainder of this section is devoted to these two propositions. We start with Proposition 4.

Proposition 4. *The $K_{k,\ell}$-*Pseudo-Cover *problem can be solved in polynomial time for any fixed k, ℓ with $\min\{k, \ell\} = 1$.*

Proof. Let $k = 1$, $\ell \geq 1$, and G be a graph. We show that deciding whether G is a $K_{1,\ell}$-pseudo-cover comes down to solving the problem of finding a perfect matching in a graph of size at most $\ell|V_G|$. Because the latter can be done in polynomial time, this means that we have proven the proposition.

If $\ell = 1$, then deciding whether G is a $K_{1,\ell}$-pseudo-cover is readily seen to be equivalent to finding a perfect matching in G.

Now suppose that $\ell \geq 2$. We first check in polynomial time whether G is bipartite with partition classes A and B, such that $|A| = \frac{n}{1+\ell}$ and $|B| = \frac{\ell n}{1+\ell}$. If not, then Theorem 1 tells us that G is a no-instance. Otherwise we continue as follows. Because $k = 1$ and $\ell \geq 2$, we can distinguish between A and B. We replace each vertex $a \in A$ by ℓ copies $a^1, \ldots, a^\ell$ and make each a^i adjacent to all neighbors of a. This leads to a bipartite graph G', the partition classes of which have the same size. We claim that G is a $K_{1,\ell}$-pseudo-cover if and only if G' has a perfect matching.

First suppose that G is a $K_{1,\ell}$-pseudo-cover. Then there exists a pseudo-covering f from G to $K_{1,\ell}$. Because $k = 1$ and $\ell \geq 2$, we find that $f(a) = x_1$ for all $a \in A$ and $f(B) = Y$. Consider a vertex $a \in A$. Let $b_1, \ldots, b_\ell$ be its matched neighbors. In G' we select the edges $a^i b_i$ for $i = 1, \ldots, \ell$. After having done this for all vertices in A, we obtain a perfect matching of G'.

Now suppose that G' has a perfect matching. We define a mapping f by $f(a) = x_1$ for all $a \in A$ and $f(b) = y_i$ if and only if $a^i b$ is a matching edge in G', where a^i is the ith copy of a. Then f is a pseudo-covering from G to $K_{1,\ell}$. Hence, G is a $K_{1,\ell}$-pseudo-cover. This completes the proof of Proposition 4. □

We now prove that $K_{k,\ell}$-Pseudo-Cover is NP-complete for all $k, \ell \geq 2$ (Proposition 5). Our proof is inspired by the proof of Hell, Kirkpatrick, Kratochvíl, and Kříž [14]. They consider the problem of testing if a graph has an $\mathcal{S}_L$-factor for any set $\mathcal{S}_L$ of cycles, the length of which belongs to some specified set L. This is useful for our purposes because of the following. If $L = \{4, 8, 12, \ldots, \}$, then an $\mathcal{S}_L$-factor of a bipartite graph G with partition classes A and B of size $\frac{n}{2}$ is an $\mathcal{S}(K_{2,2})$-factor of G that is also a $K_{2,2}$-pseudo-cover of G by Theorem 1. However, for $k = \ell \geq 3$, this is not longer true, and when $k \neq \ell$ the problem is not even "symmetric" anymore. Below we show how to deal with these issues. We refer to Section 6 for an alternative proof for the case $k, \ell \geq 3$. However, our construction for $k, \ell \geq 2$ does not become simpler when we restrict ourselves to $k, \ell \geq 2$ with $k = 2$ or $\ell = 2$. Therefore, we decided to present our NP-completeness result for all k, ℓ with $k, \ell \geq 2$.

Recall that we denote the partition classes of $K_{k,\ell}$ by $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_\ell\}$. We first state a number of useful lemmas. Hereby, we use the alternative definition in terms of perfect matchings, as provided by Proposition 1, when we argue on pseudo-coverings.

Let $G_1(k, \ell)$ be the graph in Figure 3. It contains a vertex a with $\ell - 1$ neighbors $b_1, \ldots, b_{\ell-1}$ and a vertex d with $k - 1$ neighbors $c_1, \ldots, c_{k-1}$. For any $i \in [1, \ell - 1]$, $j \in [1, k - 1]$, it contains an edge $b_i c_j$. Finally, it contains a vertex e which is only adjacent to d.

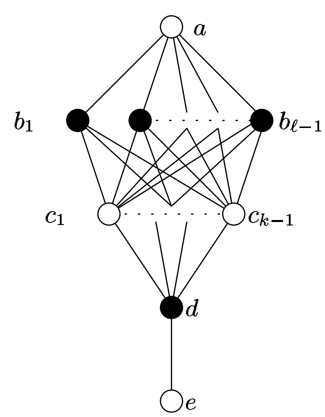

Fig. 3. The graph $G_1(k, \ell)$.

Lemma 2. *Let $G_1(k, \ell)$ be an induced subgraph of a bipartite graph G such that only a and e have neighbors outside $G_1(k, \ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then $f(a) = f(e)$. Moreover, a has only one matched neighbor outside $G_1(k, \ell)$ and this matched neighbor has color $f(d)$, where d is the only matched neighbor of e inside $G_1(k, \ell)$.*

Proof. Due to their degrees, all edges incident to the b-vertices and the c-vertices must be in a perfect matching. Since $\deg_G(d) = k$, all the edges incident to d must be in a perfect matching. Hence, we find $|f(\{a, c_1, \ldots, c_{k-1}\}| = k$ and $|f(\{d, b_1, \ldots, b_{\ell-1}\})| = \ell$. This means that $f(a)$ is the only color missing in the neighborhood of d. Consequently, $f(e) = f(a)$. Moreover, $f(d)$ is not a color of a b-vertex. Hence, $f(d)$ must be the color of the matched neighbor of a outside $G_1(k, \ell)$. □

Lemma 3. *Let G be a bipartite graph that contains $G_1(k, \ell)$ as an induced subgraph, such that only a and e have neighbors outside $G_1(k, \ell)$ and such that a and e have no common neighbor. Let G' be the graph obtained from G by removing all vertices of $G_1(k, \ell)$ and by adding a new vertex u that is adjacent to every vertex of G that is a neighbor of a or e outside $G_1(k, \ell)$. Let f be a pseudo-covering from G' to $K_{k,\ell}$, such that $f(u) \in X$ and such that u has exactly one neighbor v of a in its matched neighborhood. Then G is a $K_{k,\ell}$-pseudo-cover.*

Proof. We may assume without loss of generality that $f(u) = x_k$ and $f(v) = y_\ell$. We modify f as follows. Let $f(a) = f(e) = x_k$ and $f(d) = y_\ell$. Let $f(b_j) = y_j$ for $j = 1, \ldots, \ell - 1$ and $f(c_i) = x_i$ for all $i = 1, \ldots, k - 1$. In this way we find a pseudo-covering from G to $K_{k,\ell}$. □

Let $G_2(k, \ell)$ be the graph in Figure 4. It contains k vertices $u_1, \ldots, u_k$. It also contains $(k-1)k$ vertices $v_{h,i}$ for $h = 1, \ldots, k-1$, $i = 1, \ldots, k$, and $(k-1)(\ell-1)$ vertices $w_{i,j}$ for $i = 1, \ldots, k-1$, $j = 1, \ldots, \ell-1$. For $h = 1, \ldots, k-1$, $i = 1, \ldots, k$, $j = 1, \ldots, \ell - 1$, $G_2(k, \ell)$ contains an edge $u_i v_{h,i}$ and an edge $v_{h,i} w_{h,j}$.

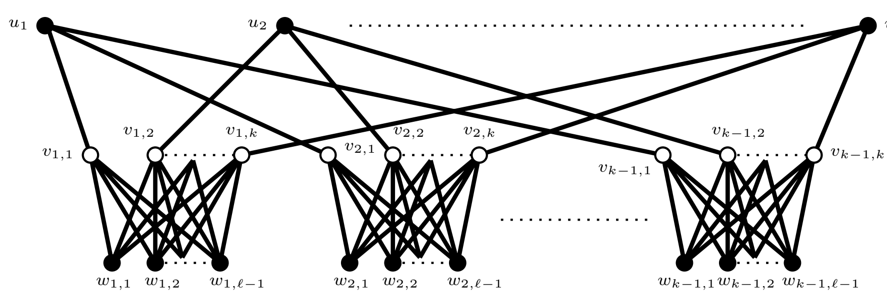

Fig. 4. The graph $G_2(k, \ell)$ from Lemma 4.

Lemma 4. *Let G be a bipartite graph that has $G_2(k, \ell)$ as an induced subgraph such that only u-vertices have neighbors outside $G_2(k, \ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then each u_i has exactly one matched neighbor t_i outside $G_2(k, \ell)$. Moreover, $|f(\{u_1, \ldots, u_k\})| = 1$ and $|f(\{t_1, \ldots, t_k\})| = k$.*

Proof. Because all v-vertices have degree ℓ and all w-vertices have degree k, all edges of $G_2(k, \ell)$ must be in perfect matchings. If $k \neq \ell$, this means that every v-vertex must get an x-color, whereas every u-vertex and every w-vertex must get a y-color. Moreover, if $k = \ell$, then we may assume this without loss of generality. As all v-vertices have degree ℓ, the vertices in any $\{u_i, w_{h,1}, \ldots, w_{h,\ell-1}\}$ have different x-colors. Moreover, the way we defined the edges between the u-vertices and the v-vertices implies that every u-vertex must have the same y-color, i.e., $|f(\{u_1, \ldots, u_k\})| = 1$. Because all edges of $G_2(k, \ell)$ are perfect matching edges and every u-vertex has degree $k-1$ in $G_2(k, \ell)$, we find that every u_i has exactly one matched neighbor t_i outside $G_2(k, \ell)$. In the (matched) neighborhood of $\{u_1, u_2, \ldots, u_k\}$ in $G_2(k, \ell)$, each color x_i appears exactly $k-1$ times. Consequently, in the matched neighborhood of $\{u_1, u_2, \ldots, u_k\}$ outside $G_2(k, \ell)$, each x_i appears once and thus $|f(\{t_1, \ldots, t_k\})| = k$.

Lemma 5. *Let G be a bipartite graph that has $G_2(k, \ell)$ as an induced subgraph, such that only u-vertices have neighbors outside $G_2(k, \ell)$ and such that no two u-vertices have a common neighbor. Let G' be the graph obtained from G by removing all vertices of $G_2(k, \ell)$ and by adding a new vertex s that is adjacent to every vertex of G that is a neighbor of some u-vertex outside $G_2(k, \ell)$. Let f be a pseudo-covering from G' to $K_{k,\ell}$, such that $f(s) \in Y$ and such that s has exactly one neighbor t_i of every u_i in its matched neighborhood. Then G is a $K_{k,\ell}$-pseudo-cover.*

Proof. We may assume without loss of generality that $f(s) = y_\ell$ and $f(t_i) = x_i$ for $i = 1, \ldots, k$. We modify f as follows. For $i = 1, \ldots, k$, we let $f(u_i) = y_1$. For $i = 1, \ldots, k-1$ and $j = 2, \ldots, \ell$ we let $f(w_{i,j}) = y_j$. For $h = 1, \ldots, k-1$ and $i = 1, \ldots, k$, we let $f(v_{h,i}) = x_{h+i}$ if $h + i \leq k$ and $f(v_{h,i}) = x_{h+i-k}$ otherwise. In this way we find a pseudo-covering from $G_2(k, \ell)$ to $K_{k,\ell}$. ☐

Let $G_3(k, \ell)$ be the graph defined in Figure 5. It contains k copies of $G_1(k, \ell)$, where we denote the a-vertex and e-vertex of the ith copy by a_i and e_i, respectively. It also contains a copy of $G_2(k, \ell)$ with edges e_iu_i and a_iu_{i+1} for $i = 1, \ldots, k$ (where $u_{k+1} = u_1$). The construction is completed by adding a vertex p adjacent to all a-vertices and by adding vertices $q, r_1, \ldots, r_{\ell-2}$ that are adjacent to all e-vertices. Here we assume that there is no r-vertex in case $\ell = 2$.

Lemma 6. *Let G be a bipartite graph that has $G_3(k, \ell)$ as an induced subgraph, such that only p and q have neighbors outside $G_3(k, \ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then either every a_i is a matched neighbor of p and no e_i is a matched neighbor of q, or else every e_i is a matched neighbor of q and no a_i is a matched neighbor of p.*

Proof. We first show the claim below.

Claim. Either every e_iu_i is in a perfect matching and no a_iu_{i+1} is in a perfect matching, or every a_iu_{i+1} is in a perfect matching and no e_iu_i is in a perfect matching.

We prove this claim as follows. Every u_i is missing exactly one color in its matched neighborhood in $G_2(k, \ell)$ by Lemma 4. This means that, for any i, either $a_{i-1}u_i$ is in a perfect matching, or else e_iu_i is in a perfect matching. We

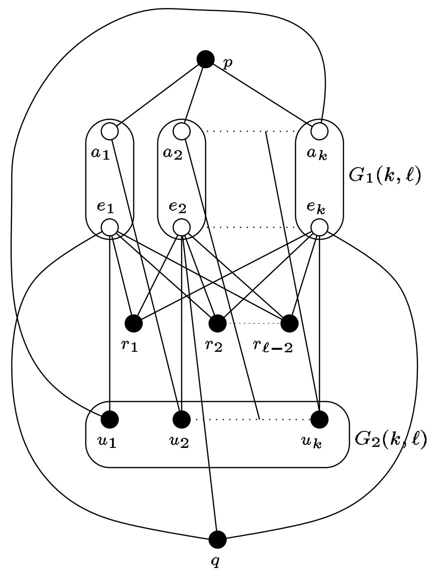

Fig. 5. The graph $G_3(k, \ell)$.

show that in the first case $e_{i-1}u_{i-1}$ is not in a perfect matching, and that in the second case $a_i u_{i+1}$ is not in a perfect matching.

Suppose that $a_{i-1}u_i$ is in a perfect matching. By Lemma 4, u_{i-1} and u_i have the same color. By Lemma 2, d_{i-1} is a matched neighbor of e_{i-1} with $f(d_{i-1}) = f(u_{i-1})$. Hence, $e_{i-1}u_{i-1}$ is not in a perfect matching. Suppose that $e_i u_i$ is in a perfect matching. Then by the same reasoning, $a_i u_{i+1}$ is not in a perfect matching.

Suppose that $e_1 u_1$ is in a perfect matching. Then $a_1 u_2$ is not in a perfect matching, and consequently $e_2 u_2$ is in a perfect matching, and so on, until we deduce that every $e_i u_i$ is in a perfect matching and no $a_i u_{i+1}$ is in a perfect matching. Suppose that $e_1 u_1$ is not in a perfect matching. Then by the same reasoning we can show the opposite. This proves the claim.

Note that every $e_i r_j$ must be in a perfect matching due to the degree of r_j. Thus, every e_i has exactly one matched neighbor in $\{q, u_i\}$. Moreover, each a_i has exactly one matched neighbor in $\{p, u_{i+1}\}$. Applying the claim then yields the desired result. $\square$

Lemma 7. *Let G be a graph that has $G_3(k, \ell)$ as an induced subgraph such that only p and q have neighbors outside $G_3(k, \ell)$ and such that p and q do not have a common neighbor. Let G' be the graph obtained from G by removing all vertices of $G_3(k, \ell)$ and by adding a new vertex r^* that is adjacent to every vertex of G that is a neighbor of p or q outside $G_3(k, \ell)$. Let f be a pseudo-covering from G' to $K_{k,\ell}$ such that $f(r^*) \in Y$ and such that either all vertices in the matched* *neighborhood of r^* in G' are all neighbors of p in G, or else are all neighbors of q in G. Then G is a $K_{k,\ell}$-pseudo-cover.*

Proof. We may assume without loss of generality that $f(r^*) = y_\ell$. We show how to modify f. Let $f(p) = f(q) = y_\ell$. Let $f(a_i) = f(e_i) = x_i$ for $1 \leq i \leq k$. Let $f(r_i) = y_{i+1}$ for $1 \leq i \leq \ell - 2$. Let $f(u_i) = y_1$ for $1 \leq i \leq k$.

First suppose that the matched neighborhood of r^* in G' is in the neighborhood of p in G. We define perfect matching edges as follows: the matched neighbor of each a_i outside the ith copy of $G_1(k, \ell)$ is u_{i+1}; the matched neighbors of each e_i outside the ith copy of $G_1(k, \ell)$ are q and the r-vertices. By Lemmas 3 and 5, we can extend f to all other vertices of $G_3(k, \ell)$. Hence, we find that G is a $K_{k,\ell}$-pseudo-cover.

Now suppose that the matched neighborhood of r^* in G' is in the neighborhood of q in G. We define perfect matching edges as follows: the matched neighbor of each a_i outside the ith copy of $G_1(k, \ell)$ is p; the matched neighbors of each e_i outside the ith copy of $G_1(k, \ell)$ are u_i and the r-vertices. By Lemmas 3 and 5, we can extend f to all other vertices of $G_3(k, \ell)$. Hence, also in this case, G is a $K_{k,\ell}$-pseudo-cover. □

Let $G_4(k, \ell)$ be the graph in Figure 6. It is constructed as follows. We take k copies of $G_3(\ell, k)$. We denote the p-vertex and the q-vertex of the ith copy by $p_{1,i}$ and $q_{1,i}$, respectively. We take ℓ copies of $G_3(k, \ell)$. We denote the p-vertex and the q-vertex of the jth copy by $p_{2,j}$ and $q_{2,j}$, respectively. We add an edge between any $p_{1,i}$ and $p_{2,j}$.

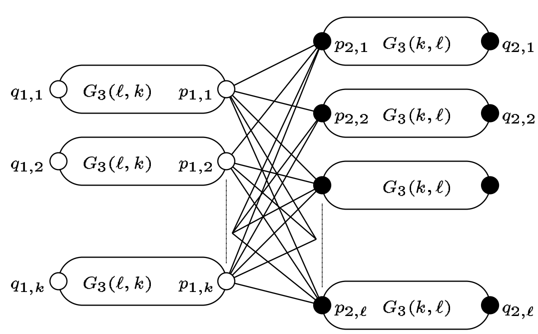

Fig. 6. The graph $G_4(k, \ell)$.

Lemma 8. *Let G be a bipartite graph that has $G_4(k, \ell)$ as an induced subgraph such that only the q-vertices have neighbors outside $G_4(k, \ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then either every $p_{1,i}p_{2,j}$ is in a perfect matching and all matched neighbors of every q-vertex are in $G_4(k, \ell)$, or else no edge $p_{1,i}p_{2,j}$*

is in a perfect matching and all matched neighbors of every q-vertex are outside $G_4(k,\ell)$.

Proof. Suppose that there is an edge $p_{1,i}p_{2,j}$ in a perfect matching. Then, $p_{1,i}$ and $p_{2,j}$ have a matched neighbor outside their corresponding copy of $G_3(\ell,k)$ and $G_3(k,\ell)$, respectively. Hence, by Lemma 6, all matched neighbors of $q_{1,i}$ and $q_{2,j}$ are inside $G_4(k,\ell)$ and all edges $p_{1,i}p_{2,j'}$ and $p_{1,i'}p_{2,j}$ are in perfect matchings. We apply Lemma 6 a number of times and are done. If no edge $p_{1,i}p_{2,j}$ is in a perfect matching, then by Lemma 6, all matched neighbors of every q-vertex are outside $G_4(k,\ell)$. □

We are now ready to show Proposition 5, where we present our NP-completeness reduction.

Proposition 5. *The $K_{k,\ell}$-Pseudo-Cover problem is* NP*-complete for any fixed k,ℓ with $k,\ell \geq 2$.*

Proof. We reduce from the problem $(k+\ell)$-Dimensional Matching, which is NP-complete as $k+\ell \geq 3$ (see [10]). In this problem, we are given $k+\ell$ mutually disjoint sets $Q_{1,1},\ldots,Q_{1,k},Q_{2,1},\ldots,Q_{2,\ell}$, all of equal size m, and a set H of hyperedges $h \in \Pi_{i=1}^{k} Q_{1,i} \times \Pi_{j=1}^{\ell} Q_{2,j}$. The question is whether H contains a $(k+\ell)$-*dimensional matching*, i.e., a subset $M \subseteq H$ of size $|M| = m$ such that for any distinct pairs $(q_{1,1},\ldots,q_{1,k},q_{2,1},\ldots,q_{2,\ell})$ and $(q'_{1,1},\ldots,q'_{1,k},q'_{2,1},\ldots,q'_{2,\ell})$ in M we have $q_{1,i} \neq q'_{1,i}$ for $i=1,\ldots,k$ and $q_{2,j} \neq q'_{2,j}$ for $j=1,\ldots,\ell$.

Given such an instance, we construct a bipartite graph G with partition classes V_1 and V_2. First we put all elements in $Q_{1,1}\cup\ldots\cup Q_{1,k}$ in V_1, and all elements in $Q_{2,1}\cup\ldots\cup Q_{2,\ell}$ in V_2. Then we introduce an extra copy of $G_4(k,\ell)$ for each hyperedge $h=(q_{1,1},\ldots,q_{1,k},q_{2,1},\ldots,q_{2,\ell})$ by adding the missing vertices and edges of this copy to G. We observe that indeed G is bipartite. We also observe that G has polynomial size.

We claim that $((Q_{1,1},\ldots,Q_{1,k},Q_{2,1},\ldots,Q_{2,\ell}),H)$ admits a $(k+\ell)$-dimensional matching M if and only if G is a $K_{k,\ell}$-pseudo-cover.

Suppose that $((Q_{1,1},\ldots,Q_{1,k},Q_{2,1},\ldots,Q_{2,\ell}),H)$ admits a $(k+\ell)$-dimensional matching M. We define a homomorphism f from G to $K_{k,\ell}$ as follows. For each hyperedge $h=(q_{1,1},\ldots,q_{1,k},q_{2,1},\ldots,q_{2,\ell})$, we let $f(p_{1,i})=f(q_{1,i})=x_i$ for $i=1,\ldots,k$ and $f(p_{2,j})=f(q_{2,j})=y_j$ for $j=1,\ldots,\ell$.

For all $h \in M$, we let every q-vertex of h has all its matched neighbors in the copy of $G_4(k,\ell)$ that corresponds to h, and we define the matched neighbors of every p-vertex of h by choosing the edges $p_{1,i}p_{2,j}$ as matching edges. Since M is a $(k+\ell)$-dimensional matching, the matched neighbors of every p-vertex and every q-vertex are now defined. We note that the restriction of f to the union S of the p-vertices of all the hyperedges is a pseudo-covering from $G[S]$ to $K_{k,\ell}$. Then, by repeatedly applying Lemma 7, we find that G is a $K_{k,\ell}$-pseudo-cover.

Conversely, suppose that f is a pseudo-covering from G to $K_{k,\ell}$. By Lemma 8, every q-vertex has all its matched neighbors in exactly one copy of $G_4(k,\ell)$ that corresponds to a hyperedge h such that the matched neighbor of *every* q-vertex in h is as a matter of fact in that copy $G_4(k,\ell)$. We now define M to be the set of all such hyperedges. Then M is a $(k+\ell)$-dimensional matching: any q-vertex appears in exactly one hyperedge of M. $\square$

6 Further Research on Pseudo-coverings

Pseudo-coverings are closely related to the so-called locally constrained homomorphisms, which are homomorphisms with some extra restrictions on the neighborhood of each vertex. In Section 1 we already defined a covering which is also called a locally bijective homomorphism. There are two other types of such homomorphisms. First, a homomorphism from a graph G to a graph H is called *locally injective* or a *partial covering* if for every $u \in V_G$ the restriction of f to the neighborhood of u, i.e., the mapping $f_u : N_G(u) \to N_H(f(u))$, is injective. Second, a homomorphism from a graph G to a graph H is called *locally surjective* or a *role assignment* if the mapping $f_u : N_G(u) \to N_H(f(u))$ is surjective for every $u \in V_G$. See [7] for a survey.

The following observation is insightful. Recall that $G[x, y]$ denotes the induced bipartite subgraph of a graph G with partition classes $f^{-1}(x)$ and $f^{-1}(y)$ for some homomorphism f from G to a graph H.

Observation 9 ([9]) *Let f be a homomorphism from a graph G to a graph H. For every edge xy of H,*

- *f is locally bijective if and only if $G[x, y]$ is 1-regular (i.e., a perfect matching) for all $xy \in E_H$;*
- *f is locally injective if and only if $G[x, y]$ has maximum degree at most one (i.e., a matching) for all $xy \in E_H$;*
- *f is locally surjective if and only if $G[x, y]$ has minimum degree at least one for all $xy \in E_H$.*

By definition, every covering is a pseudo-covering. We observe that this is in line with Proposition 1 and Observation 9. Moreover, by these results, we find that every pseudo-covering is a locally surjective homomorphism. This leads to the following result.

Proposition 6. *For any fixed graph H, if H-Cover is NP-complete, then so is H-Pseudo-Cover.*

Proof. Let H be a graph for which H-Cover is NP-complete. Let G be an instance of H-Cover. It is folklore that G and H must have the same degree refinement matrix in case $G \xrightarrow{B} H$ holds. We refer to e.g. Kristiansen and Telle [17] for the definition of a degree refinement matrix and how to compute this matrix in polynomial time. For us, it is only relevant that we may assume without loss of generality that G and H have the same degree refinement matrix. We claim that in that case $G \xrightarrow{B} H$ if and only if $G \xrightarrow{P} H$ holds.

Suppose that $G \xrightarrow{B} H$. Then by definition we have $G \xrightarrow{P} H$.

Suppose that $G \xrightarrow{P} H$. By Proposition 1 and Observation 9 we find that $G \xrightarrow{S} H$ holds. Kristiansen and Telle [17] showed that $G \xrightarrow{S} H$ implies $G \xrightarrow{B} H$ whenever G and H have the same degree refinement matrix. $\square$

Due to Proposition 6, the NP-completeness of $K_{k,\ell}$-Pseudo-Cover for $k, \ell \geq 3$ also follows from the NP-completeness of $K_{k,\ell}$-Cover for these values of k, ℓ. The latter is shown by Kratochvíl, Proskurowski and Telle [15]. However, these authors show in the same paper [15] that $K_{k,\ell}$-Cover is solvable in polynomial time for the cases k, ℓ with $\min\{k, \ell\} \leq 2$. Hence for these cases we have to rely on our proof in Section 5.

Another consequence of Proposition 6 is that H-Pseudo-Cover is NP-complete for all k-regular graphs H for any $k \geq 3$ due to a hardness result for the corresponding H-Cover [6]. However, a complete complexity classification of H-Pseudo-Cover is still open, just as dichotomy results for H-Partial Cover and H-Cover are not known, whereas for the locally surjective case a complete complexity classification has been given [8]. So far, we could obtain some partial results but a complete classification of the complexity of H-Pseudo-Cover seems already difficult for trees (we found many polynomial-time solvable and NP-complete cases).

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