GROUP SCHEMES OF PERIOD 2

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Abstract. We give an explicit construction of the antiequivalence of the category of finite flat commutative group schemes of period 2 defined over a valuation ring of a 2-adic field with algebraically closed residue field. This result extends the earlier author's approach to group schemes of period $p > 2$ from Proceedings LMS, 101, 2010, 207-259.

Introduction

0.1. **Basic notation.** Everywhere in the paper k is algebraically closed field of characteristic 2, K_{00} is the fraction field of the ring of Witt vectors $W(k)$ and $[K_0 : K_{00}] = e \in \mathbb{N}$. Let $O_0 = O_{K_0}$ be the valuation ring of K_0, π — a fixed uniformiser in K_0, $K = K_0(\pi)$, where $\pi^2 = \pi_0$, and $O = O_K$. We set $S = k[[t]]$ where t is a variable. Let $\sigma : S \longrightarrow S$ be such that $\sigma(s) = s^2$, $s \in S$. Denote by $\kappa_{SO} : S/t^{2e} \longrightarrow O/2O$ the rings isomorphism such that $\kappa_{SO}|_k = \mathrm{id}$ and $\kappa_{SO} : t \bmod t^{2e} \mapsto \pi \bmod 2$.

For a natural number u, $\underline{i}$ denotes always a vector of length u with coordinates from the set $\{0, 1\}$ and $r(\underline{i})$ denotes the sum of these coordinates.

0.2. **Categories of group schemes.** Let R be a local ring of characteristic 0 with residue field k. Denote by Gr_R the category of finite flat commutative group schemes G over R such that $2\mathrm{id}_G = 0$.

Recall that $G = \mathrm{Spec}A(G)$, where $A(G)$ is a flat R-algebra of finite rank $|G|$ and the structure of group scheme on G is given via the R-algebra morphisms $e_G : A(G) \longrightarrow O$ (counit) and $\Delta_G : A(G) \longrightarrow A(G) \otimes_R A(G)$ (coaddition) satisfying standard axioms.

Denote by Gr_R^{et} and Gr_R^{mult} the full subcategories in Gr_R of etale and, resp., multiplicative group schemes. Then any $G \in \mathrm{Gr}_R$ has the maximal etale quotient $j^{et} : G \longrightarrow G^{et}$ and the maximal multiplicative subobject $i^{mult} : G^{mult} \longrightarrow G$.

Because k is algebraically closed any etale object in Gr_R is a product of finitely many copies of the constant etale group scheme of order 2, $(\mathbb{Z}/2)_O = \mathrm{Spec}\,\mathrm{Map}(\mathbb{Z}/2, R)$. Similarly, any multiplicative group scheme in Gr_R is a product of finitely many copies of the constant multiplicative group scheme of order 2, $\mu_2 = \mathrm{Spec}R[\mathbb{Z}/2]$.

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Introduce the category Gr^*_R as follows. Its objects are the objects from Gr_R and for any $G_1, G_2 \in \mathrm{Gr}_R$,

$$\mathrm{Hom}_{\mathrm{Gr}^*_R}(G_1, G_2) = \mathrm{Hom}_{\mathrm{Gr}_R}(G_1, G_2)/\mathcal{R}(G_1, G_2)$$

where $\mathcal{R}(G_1, G_2)$ consists of the morphisms

$$G_1 \stackrel{j^{et}}{\longrightarrow} G_1^{et} \stackrel{f}{\longrightarrow} G_2^{mult} \stackrel{i^{mult}}{\longrightarrow} G_2$$

whith arbitrary $f \in \mathrm{Hom}_{\mathrm{Gr}_R}(G_1^{et}, G_2^{mult})$. Note that $\mathrm{Hom}_{\mathrm{Gr}_R}((\mathbb{Z}/2)_R, \mu_{2,R})$ has only one non-trivial element given by the embedding of R-algebras

$$R[\mathbb{Z}/2] = R\bar{0} + R\bar{1} \longrightarrow \mathrm{Map}(\mathbb{Z}/2, R) = R \oplus R$$

such that $\bar{0} \mapsto (1, 1)$ and $\bar{1} \mapsto (1, -1)$.

0.3. **Categories of filtered modules.** Let $\mathcal{MF}_S$ be the category of the triples (M^0, M^1, φ_1) such that $M^1 \subset M^0$ are S-modules and $\varphi_1 : M^1 \longrightarrow M^0$ is a σ-linear morphism of S-modules. The morphisms in $\mathcal{MF}_S$ are compatible morphisms of S-modules commuting with φ_1.

Denote by MF^e_S the full subcategory in $\mathcal{MF}_S$ consisting of the triples (M^0, M^1, φ_1) such that

- M^0 is a free S-module of finite rank;
- $M^1 \supset t^e M^0$;
- $\varphi_1(M^1)S = M^0$.

The full subcategory of etale filtered modules $\mathrm{MF}^{e,et}_S$ in MF^e_S consists of (M^0, M^1, φ_1) such that $M^1 = t^e M^0$. One can see easily that any $\mathcal{M} = (M^0, M^1, \varphi_1) \in \mathrm{MF}^e_S$ has a unique maximal etale subobject $i^{et} : \mathcal{M}^{et} = (M^{0,et}, t^e M^{0,et}, \varphi_1) \longrightarrow \mathcal{M}$. Suppose $\varphi_0 : M^0 \longrightarrow M^0$ is such that $\varphi_0(m) := \varphi_1(t^e m)$ for any $m \in M^0$. Then $\bar{M}^{0,et} := M^{0,et} \otimes_S k$ is the maximal k-submodule of $\bar{M}^0 := M^0 \otimes_S k$ such that φ_0 induces an invertible σ-linear automorphism on $\bar{M}^{0,et}$. Notice that i^{et} can be included into the following short exact sequence

$$0 \longrightarrow \mathcal{M}^{et} \stackrel{i^{et}}{\longrightarrow} \mathcal{M} \longrightarrow \mathcal{M}^{loc} \longrightarrow 0$$

where $\mathcal{M}^{loc} = (M^{0,loc}, M^{1,loc}, \varphi_1) \in \mathrm{MF}^e_S$ and $\bar{M}^{0,loc} := M^{0,loc} \otimes_S k$ can be naturally indentified with the maximal k-submodule in $\bar{M}^0$ such that φ_0 induces its nilpotent endomorphism. Any etale filtered module is a direct sum of finitely many copies of $\mathcal{S}^{et} := (Sm, St^e m, \varphi_1) \in \mathrm{MF}^{e,et}_S$, where $\varphi_1(t^e m) = m$.

The full subcategory of multiplicative filtered modules $\mathrm{MF}^{e,mult}_S$ in MF^e_S consists of (M^0, M^1, φ_1) such that $M^1 = M^0$. Any $\mathcal{M} \in \mathrm{MF}^e_S$ has a unique maximal multiplicative quotient $j^{mult} : \mathcal{M} \longrightarrow \mathcal{M}^{mult}$. Introduce the morphism $\psi_0 : \bar{M}^0 \longrightarrow \bar{M}^0$ as follows: if $m \in M^1$ and $\varphi_1(m) = n \in M^0$ then $\psi_0(n \otimes 1) = m \otimes 1$. One can verify that ψ_0 is a well-defined σ^{-1}-linear morphism of k-modules and $\bar{M}^{0,mult} := M^{0,mult} \otimes_S k$ can be identified with the maximal k-submodule in $\bar{M}^0$

such that $\psi_0|_{\bar{M}^{0,mult}}$ is invertible. Note that j^{mult} can be included into the following short exact sequence in the category MF^e_S,

$$0 \longrightarrow \mathcal{M}^u \longrightarrow \mathcal{M} \stackrel{j^{mult}}{\longrightarrow} \mathcal{M}^{mult} \longrightarrow 0$$

where $\mathcal{M}^u = (M^{0,u}, M^{1,u}, \varphi_1) \in \mathrm{MF}^e_S$ and $\bar{M}^{0,u} := M^{0,u} \otimes_S k$ is the maximal k-submodule such that $\psi_0|_{\bar{M}^{0,u}}$ is nilpotent. Any multiplicative filtered module is a direct sum of finitely many copies of $\mathcal{S}^{mult} := (Sn, Sn, \varphi_1) \in \mathrm{MF}^{e,mult}_S$, where $\varphi_1(n) = n$.

Introduce the category MF^{e*}_S as follows. Its objects are the objects of MF^e_S and for any $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{MF}^e_S$,

$$\mathrm{Hom}_{\mathrm{MF}^{e*}_S}(\mathcal{M}_1, \mathcal{M}_2) = \mathrm{Hom}_{\mathrm{MF}^e_S}(\mathcal{M}_1, \mathcal{M}_2)/R(\mathcal{M}_1, \mathcal{M}_2)$$

where $R(\mathcal{M}_1, \mathcal{M}_2)$ consists of the morphisms of MF^e_S of the form

$$\mathcal{M}_1 \stackrel{j^{mult}}{\longrightarrow} \mathcal{M}_1^{mult} \stackrel{f}{\longrightarrow} \mathcal{M}_2^{et} \stackrel{i^{et}}{\longrightarrow} \mathcal{M}_2$$

with arbitrary $f \in \mathrm{Hom}_{\mathrm{MF}^e_S}(\mathcal{M}_1^{mult}, \mathcal{M}_2^{et})$.

Note that $\mathrm{Hom}_{\mathrm{MF}^e_S}(\mathcal{S}^{mult}, \mathcal{S}^{et})$ has only one non-trivial morphism and it is given by the correspondence $n \mapsto t^{2e}m$.

0.4. **Main result.** In this paper we prove the following theorem.

Theorem 0.1. *There is an antiequivalence of categories*

$$\mathcal{F}^{O*}_{O_0} : \mathrm{Gr}^*_{O_0} \longrightarrow \mathrm{MF}^{e*}_S$$

For $p > 2$, there is an antiequivalence of categories $\mathcal{F}^O_{O_0} : \mathrm{Gr}_{O_0} \longrightarrow \mathrm{MF}^e_S$. This was proved by C.Breuil [5] and M.Kisin [7, 8] in a more general context of all p-group schemes. The proofs are obtained from the study of p-divisible groups and essentially use the crystalline Dieudonne theory which is built on a geometrical approach due to the Raynaud theorem about the existence of embedding of any p-divisible group into an abelian scheme. This approach has been generalized recently by W.Kim [6], E.Lau [9] and T.Liu [10] to the case $p = 2$. (Lau's result uses Zink's theory of displays and windows.)

On the other hand, an explicit and direct construction of the antiequivalence $\mathcal{F}^O_{O_0}$ in the case $p > 2$ was given by the author [4]. The above theorem extends that construction to the case $p = 2$. We should notice that this extension is very far from to be straightforward for the following reasons.

First, when relating group schemes over $O_0 \subset O$ and filtered S-modules we use the identification of rings S/t^{2e} and $O/2$. But when working modulo 2 we can't control quite efficiently all morphisms in the category Gr_O, e.g. both the elements of $\mathrm{Hom}_{\mathrm{Gr}_O}((\mathbb{Z}/2)_O, \mu_{2,O})$ coincide modulo $2O$. This explains why we are forced to use the quotient categories Gr^*_O, $\mathrm{Gr}^*_{O_0}$ and MF^{e*}_S. On the other hand, the above example represents essentially the only aspect we are losing in our approach and

Theorem 0.1 gives essentially complete information about the objects and morphisms of the category Gr_{O_0}.

Second, when working with odd prime numbers $p > 2$ the ideal pO is provided with nilpotent DP-structure and is considerably smaller than the maximal DP-ideal in O if $e > 1$. If $p = 2$ we have no such "safety margin" because $2O$ is already the maximal DP-ideal in O. The adjustment of methods of [4] to the case $p = 2$ required a profound revision of all constructions used in there, especially the proof of the surjectivity of the functor $\mathcal{G}^O_{O_0}$. In particular, so-called Main Lemma was restated in a more precise form and provided with an essential elaboration. As a result, all main features of our approach from [4] were preserved in the case $p = 2$.

Finally, notice that all applications developed in [4]: a criterion for the Galois module to come from the Galois module of geometric points of $G \in \mathrm{Gr}_{O_0}$, the relation between group schemes from Gr_{O_0} and Faltings's strict modules in characteristic 2 and an explicit description of the duality in Gr_{O_0} can be done along the lines of the approach from [4] in the case $p = 2$ as well.

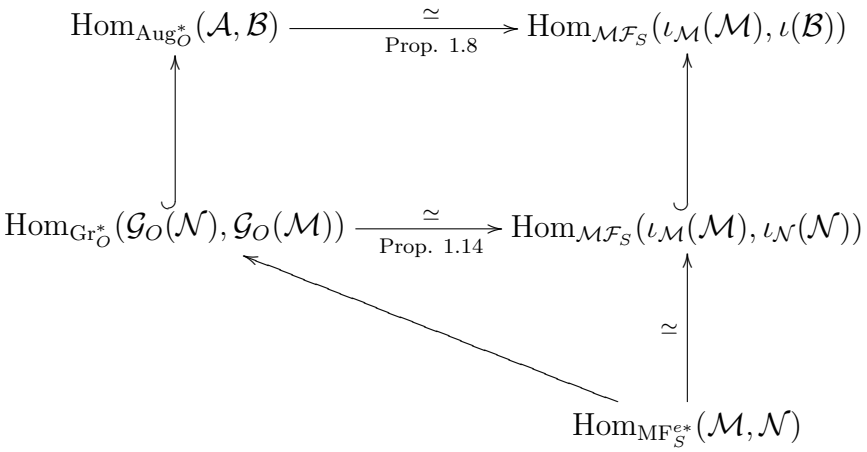
0.5. **Brief description of used methods.** Let Aug_O be the category of augmented O-algebras. Introduce the equivalence relation R such that for $f_1, f_2 \in \mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{B}_1, \mathcal{B}_2)$, $f_1 \underset{R}{\sim} f_2$ iff f_1 and f_2 coincide modulo some DP-ideal in $\mathcal{B}_2$ (cf. Subsection 1.3 for the definition of this ideal). Denote by Aug^*_O the category whose objects are the objects of Aug_O but the morphisms are the R-equivalence classes of morphisms in Aug_O. In our approach the category Aug_O (resp, Aug^*_O) relates the categories MF^e_S and Gr_O (resp., MF^{e*}_S and Gr^*_O).

As first step we associate with $\mathcal{M} = (M^0, M^1, \varphi_1) \in \mathrm{MF}^e_S$ a set of augmented O-algebras $\mathrm{Aug}_O(\mathcal{M})$ each of whose members is constructed after choosing an appropriate special basis for M^0 and a couple of other choices. One also defines for any $\mathcal{A} \in \mathrm{Aug}_O$, a canonical object $\iota(\mathcal{A}) \in \mathcal{MF}_S$ and if $\mathcal{A} \in \mathrm{Aug}_O(\mathcal{M})$ then we have a natural map $\iota_{\mathcal{M}} : \mathcal{M} \longrightarrow \iota(\mathcal{A})$ in $\mathcal{MF}_S$. Then one observes that the correspondences $\mathcal{A} \mapsto \mathrm{Aug}_O(\mathcal{M})$ and $\mathcal{B} \mapsto \iota(\mathcal{B})$ are left-adjoint. This results in the following property: if $\mathcal{N} \in \mathrm{MF}^e_S$ and $\mathcal{B} \in \mathrm{Aug}_O(\mathcal{N})$ then we have natural identifications

$$\mathrm{Hom}_{\mathrm{MF}^{e*}_S}(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathcal{MF}_S}(\iota_{\mathcal{M}}(\mathcal{M}), \iota_{\mathcal{N}}(\mathcal{N})) \subset \mathrm{Hom}_{\mathrm{Aug}^*_O}(\mathcal{A}, \mathcal{B}).$$

This allows us to show that $\mathcal{A}$, as an object of Aug^*_O, is functorial in $\mathcal{M}$ (viewed as an object of MF^{e*}), and that the assignment $\mathcal{M} \mapsto \mathcal{A}$ is functorial. It also allows us to define a family $\mathcal{L}_{\mathcal{A}}$ of Hopf algebra structures on $\mathcal{A}$ whose spectrums are group schemes over O that are isomorphic in Gr^*_O. (Use the diagonal embedding of $\mathcal{M}$ into $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}$.) In this way we obtain a faithful functor $\mathcal{G}_O : \mathrm{MF}^{e*}_S \longrightarrow \mathrm{Gr}^*_O$.

This functor is actually full. To see this, we describe $\iota_{\mathcal{M}}(\mathcal{M})$ in terms of the Hopf algebra structure on $\mathcal{A}$, and so we find that it is an object intrinsically attached to $\mathcal{G}_O(\mathcal{M})$. So we obtain the following commutative diagram:



These ideas have been applied earlier by the author in [2, 3] to describe the category Gr_O^*, where O is the valuation ring of a field extension of $\mathbb{Q}_p$ with small ramification.

Next, one shows that $\mathcal{G}_O$ factors through a functor $\mathcal{G}_{O_0}^O : \mathrm{MF}_S^{e*} \longrightarrow \mathrm{Gr}_{O_0}^*$. This amounts to showing that every group scheme $\mathcal{G}_O(\mathcal{M})$ descends to one over O_0. To accomplish this one uses tame descent and induction on rank of $\mathcal{M}$. This goes more or less along the lines of [4].

All that remains to do is to show that $\mathcal{G}_{O_0}^O$ is essentially surjective. One again uses tame descent and induction, this time on $|G_0| = p^s$. This is most difficult part of the paper, where we need essential elaboration of Main Lemma from [4] and where a special role of prime number $p = 2$ can be explained in the following way. When applying induction on s we present $G = G_0 \otimes_{O_0} O$ as an extension of a group scheme H of order p^{s-1} via a group scheme of order p. Then on the level of algebras $A(G)$ and $A(H)$ of the group schemes G and H, the Kummer theory provides us with a class of generators θ such that $A(G) = A(H)[\theta]$. Main Lemma allows us to make a very special choice of θ and then via the Lubin-Tate logarithm $l_{LT}(X) = X + X^2/p + X^{p^2}/p^2 + \ldots$, we can relate G with a group scheme of the form $\mathcal{G}_O(\mathcal{M})$, $\mathcal{M} \in \mathrm{MF}_S^e$. This special choice becomes much more delicate in the case $p = 2$ because when $p > 2$ it was enough to use only the first two terms of the above expansion of l_{LT} but in the case $p = 2$ we need to take into account one term more.

1. Construction of the functor $\mathcal{G}_O : \mathrm{MF}^e_S \longrightarrow \mathrm{Gr}_O$

We use all notation and assumptions from Introduction, in particular, the definition of the category MF^e_S and the corresponding properties of its objects. Remind also that we fixed an identification $\kappa_{SO} : S/t^{2e}S \simeq O/2O$ such that $\kappa_{SO}|_k = \mathrm{id}$ and $\kappa_{SO}(t \bmod t^{2e}) = \pi \bmod 2$.

1.1. **φ_1-lifts.** Suppose $\mathcal{M} = (M^0, M^1, \varphi_1) \in \mathrm{MF}^e_S, \mathcal{N} = (N^0, N^1, \varphi_1) \in \mathcal{MF}_S$ and $\theta \in \mathrm{Hom}_{\mathcal{MF}_S}(\mathcal{M}, \mathcal{N})$.

Definition. a) θ is a φ_1-*lift* if $\theta(M^0) = N^0$, $\theta(M^1) = N^1$ and $\mathrm{Ker}\,\theta = \mathrm{Ker}\theta|_{M^1} := T$; b) φ_1-lift θ is *nilpotent* if $\varphi_1|_T$ is topologically nilpotent, i.e. $\bigcap_n \varphi_1^n(T) = 0$; c) φ_1-lift θ is *special* if there is a submodule T' of T such that $T = T' + (T \cap t^{2e}M^{0,et})$ and φ_1 induces a topologically nilpotent endomorphism of T'.

Proposition 1.1. *Suppose for $i = 1, 2$, $\theta_i \in \mathrm{Hom}_{\mathcal{MF}_S}(\mathcal{M}_i, \mathcal{N}_i)$ are φ_1-lifts and $h \in \mathrm{Hom}_{\mathcal{MF}_S}(\mathcal{N}_1, \mathcal{N}_2)$. Then the set $\mathcal{L}(h)$ of all $f \in \mathrm{Hom}_{\mathrm{MF}^e_S}(\mathcal{M}_1, \mathcal{M}_2)$ such that $\theta_2 \circ f = h \circ \theta_1$ is not empty. If in addition:*

a) *θ_2 is nilpotent then $\mathcal{L}(h)$ consists only of one element;*

b) *θ_2 is special then all elements of $\mathcal{L}(h)$ belong to $R(\mathcal{M}_1, \mathcal{M}_2)$, i.e. coincide in MF^{e*}_S.*

Proof. We start with the following Lemma.

Lemma 1.2. *Suppose L is a finitely generated S-module and φ is a σ-linear operator on L. Then the operator $\mathrm{id}_L - \varphi$ is epimorphic. If, in addition, φ is topologically nilpotent then $\mathrm{id}_L - \mathcal{A}$ is bijective.*

Proof of Lemma. Part b) is obvious. In order to prove a) notice first that we can replace L by L/tL and, therefore, assume that L is a finite dimensional vector space over k. Then there is a decomposition $L = L_1 \oplus L_2$, where φ is invertible on L_1 and nilpotent on L_2. It remains to note that $L_1 = L_0 \otimes_{\mathbb{F}_p} k$, where L_0 is a finite dimensional $\mathbb{F}_p$-vector space such that $\varphi|_{L_0} = \mathrm{id}$. The existence of L_0 is a standard fact of σ-linear algebra: if $s = \dim_k L_1$ and $A \in M_s(k)$ is a matrix of $\varphi|_{L_1}$ in some k-basis of L_1 then $L_0 = \{(x_1, \dots, x_s) \in k^s \mid (x_1^p, \dots, x_s^p)A = (x_1, \dots, x_s)\}$; the $\mathbb{F}_p$-linear space L_0 has dimension s because the corresponding equations determine an etale algebra of rank p^s over algebraically closed field k. The Lemma is proved. □

Now suppose for $i = 1, 2$, $\mathcal{M}_i = (M_i^0, M_i^1, \varphi_1)$ and $\mathcal{N}_i = (N_i^0, N_i^1, \varphi_1)$. Let a vector $\bar{m}_1 \in (M_1^0)^s$ and a matrix $C \in M_s(S)$ be such that the coordinates of $\bar{m}_1$ and $\bar{m}_1 C$ form an S-basis of M_1^0 and, resp., M_1^1, and $\varphi_1(\bar{m}_1 C) = \bar{m}_1$.

If $\bar{n}_2 = (h \circ \theta_1)\bar{m}_1$ then $\varphi_1(\bar{n}_2 C) = \bar{n}_2$. Choose a vector $\bar{m}_2 \in (M_1^0)^s$ such that $\theta_2(\bar{m}_2) = \bar{n}_2$. Then $\bar{m}_2 - \varphi_1(\bar{m}_2 C) = \bar{t}_2 \in T_2^s$, where $T_2 =$

$\operatorname{Ker}\theta_2$. The elements of $\mathcal{L}(h)$ correspond to vectors $\bar{t} \in T_2^s$ such that $\bar{m}_2 + \bar{t} = \varphi_1((\bar{m}_2 + \bar{t})C)$ or equivalently $\bar{t} - \varphi_1(\bar{t}C) = -\bar{t}_2$.

By above Lemma such $\bar{t}$ always exists and, therefore, $\mathcal{L}(h) \neq \emptyset$. If θ_2 is nilpotent then such $\bar{t}$ is unique and part a) is proved.

Now suppose θ_2 is special. Then $T_2 = T_2' + (T_2 \cap t^{2e} M_2^{0,et})$ and $\varphi_1|_{T_2'}$ is topologically nilpotent. Clearly, we can assume that $T_2 \cap t^{2e+1} M_2^{0,et} \subset T_2'$.

Note that $\mathcal{N}_2 = (M_2^0/T_2, M_2^1/T_2, \varphi_1)$ and θ_2 appears as the composition of two natural projections in $\mathcal{MF}_S$

$$\mathcal{M}_2 \xrightarrow{\alpha} \mathcal{N}_2' := (M_2^0/T_2', M_2^1/T_2', \varphi_1) \xrightarrow{\beta} \mathcal{N}_2.$$

Here α is a nilpotent φ_1-lift and, therefore, $f \in \mathcal{L}(h)$ is uniquely determined by $\alpha \circ f \in \operatorname{Hom}_{\mathcal{MF}_S}(\mathcal{M}_1, \mathcal{N}_2')$.

Suppose for $i = 1, 2$, $f_i \in \mathcal{L}(h)$ and $f_i' = \alpha \circ f_i$. Then $f_1' - f_2' \in \operatorname{Ker}\beta_* \subset \operatorname{Hom}_{\mathcal{MF}_S}(\mathcal{M}_1, \widetilde{\mathcal{N}}_2)$, where $\widetilde{\mathcal{N}}_2 = (\widetilde{N}_2, \widetilde{N}_2, \varphi_1)$, $\widetilde{N}_2 = t^{2e} M_2^{0,et} / t^{2e+1}$ and φ_1 is σ-linear automorphism of $\widetilde{N}_2$.

Let $\widetilde{\mathcal{M}}_2 := (t^{2e} M_2^{0,et}, t^{2e} M_2^{0,et}, \varphi_1)$. Clearly it is a subobject of $\mathcal{M}_2^{et}$ in MF_S^e. Then the natural projection $\widetilde{\mathcal{M}}_2 \longrightarrow \widetilde{\mathcal{N}}_2$ is a nilpotent φ_1-lift and, therefore, $f_1 - f_2$ factors through the embedding $\widetilde{\mathcal{M}}_2 \subset \mathcal{M}_2^{et}$. Finally, $\widetilde{\mathcal{M}}_2$ is a multiplicative object in MF_S^e and this implies that $f_1 - f_2$ factors through the natural projection $\mathcal{M}_1 \longrightarrow \mathcal{M}_1^{mult}$.

The proposition is proved. □

Corollary 1.3. *a) If θ is nilpotent then $\mathcal{M}$ is defined uniquely by $\mathcal{N}$ up to a unique isomorphism in the category* MF_S^e*;*

b) If θ is special then $\mathcal{M}$ is defined uniquely by $\mathcal{N}$ up to a unique isomorphism in the category MF_S^{e*}.

1.2. **Extension of scalars.** Let $S' = S[t']$, where $t'^2 = t$. If $\mathcal{M} = (M^0, M^1, \varphi_1) \in \mathrm{MF}_S^e$ then $\mathcal{M} \otimes_S S' := (M^0 \otimes_S S', M^1 \otimes_S S', \varphi_1 \otimes \sigma) \in \mathrm{MF}_{S'}^{2e}$. (Here $\sigma(s) = s^2$ for any $s \in S'$.)

If $\mathcal{M}' = (M'^0, M'^1, \varphi_1') \in \mathrm{MF}_{S'}^{2e}$ then $\varphi_1'(M'^1) = \{\varphi_1'(m) \mid m \in M'^0\}$ has a natural structure of S-module and coincides with M^0 if $\mathcal{M}' = \mathcal{M} \otimes_S S'$. This fact implies easily the following criterion of the existence of a descent of $\mathcal{M}'$ to S.

Proposition 1.4. *If $\mathcal{M}' = (M'^0, M'^1, \varphi_1') \in \mathrm{MF}_{S'}^{2e}$ then the following two properties are equivalent:*

a) *$\mathcal{M} = (M^0, M^1, \varphi_1) \in \mathrm{MF}_S^e$ is such that $\mathcal{M}' = \mathcal{M} \otimes_S S'$;*

b) *if $M^0 = \varphi_1'(M'^1)$, $M^1 = M'^1 \cap M^0$ and $\varphi_1 = \varphi_1'|_{M^1}$ then $\mathcal{M} = (M^0, M^1, \varphi_1) \in \mathrm{MF}_S^e$.*

1.3. **The category of augmented O-algebras Aug_O.** Let Aug_O be the category of augmented O-algebras $\mathcal{B}=(B,I_B)$ such that B is a flat O-algebra of finite rank and I_B is an ideal of B such that $O\simeq B/I_B$ via the natural map $o\mapsto o\cdot 1_B$, $o\in O$.

We shall denote by $\mathcal{B}^{et}$ the subobject $(B^{et},I_{B^{et}})\in\mathrm{Aug}_O$ such that B^{et} is the maximal etale subalgebra of B and $I_{B^{et}}=B^{et}\cap I_B$.

With above notation let I_B^{loc} be the ideal of topologically nilpotent elements of I_B. Clearly, $I_B^{loc}\cap I_{B^{et}}=\pi I_{B^{et}}$. Denote by $I_B(2)$ the ideal of all $b\in I_B$ such that $b^2\in 2I_B$ and by $I_B(2)^{loc}$ — the ideal of all $b\in I_B$ such that $b^2\in 2I_B^{loc}$. Clearly, $I_B(2)=I_B(2)^{loc}+\pi^e I_{B^{et}}$.

Let $J_B=I_B(2)^2+\pi^e I_B(2)$. Then $J_B=\widetilde{J}_B+2I_{B^{et}}$, where $\widetilde{J}_B=I_B(2)^{loc}I_B(2)+\pi^e I_B(2)^{loc}$. Notice that J_B is provided with the standard DP-structure by the map $b\mapsto -b^2/2$, $b\in J_B$, and $\widetilde{J}_B$ is the maximal ideal in J_B where this DP-structure is topologically nilpotent.

We shall denote by Aug_O^* the following category. Its objects are the objects of the category Aug_O and for any $\mathcal{B}_1,\mathcal{B}_2\in\mathrm{Aug}_O$,

$$\mathrm{Hom}_{\mathrm{Aug}_O^*}(\mathcal{B}_1,\mathcal{B}_2)=\mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{B}_1,\mathcal{B}_2)/R,$$

where R is the following equivalence relation:

if $f_1,f_2\in\mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{B}_1,\mathcal{B}_2)$ *then* $f_1\underset{R}{\sim}f_2$ *iff* $f_1\equiv f_2 \bmod J_{B_2}$.

1.4. **Families of augmented O-algebras $\mathrm{Aug}_O(\mathcal{M})$, $\mathcal{M}\in\mathrm{MF}_S^e$.** Suppose $\mathcal{M}=(M^0,M^1,\varphi_1)\in\mathrm{MF}_S^e$ and the vector $\bar{m}^1=(m_1^1,\dots,m_u^1)$ is such that its coordinates form an S-basis of M^1. One can verify that $\bar{m}^1$ can be chosen in such way that the following two conditions $C1$ and $C2$ are satisfied:

$C1$: *the non-zero images of all* m_i^1, $1\le i\le u$, *in* M^0/tM^0 *are linearly independent over* k;

$C2$: $\bar{m}^1=(\bar{m}^{1,loc},t^e\bar{m}^{et})$ *where*
a) the coordinates of $\bar{m}^{et}$ *form an* S*-basis of* $M^{0,et}$ *and* $\varphi_1(t^e\bar{m}^{et})=\bar{m}^{et}$*;*
b) if $\bar{m}^{loc}=\varphi_1(\bar{m}^{1,loc})$ *then the coordinates of* $\bar{m}^{loc}\bmod t$ *form a basis of* $\bar{M}^{0,loc}=M^{0,loc}\otimes_S k$ *over* k.

Let $\bar{m}^0=(\bar{m}^{loc},\bar{m}^{et})$. Then the coordinates of $\bar{m}^0$ form an S-basis of M^0. Denote by U the $(n\times n)$-matrix with coefficients in S such that $\bar{m}^1=\bar{m}^0U$. By condition $C1$, for appropriate S-matrices U_1 and U_2, we have $\bar{m}^{1,loc}=\bar{m}^{loc}U_1+\bar{m}^{et}(tU_2)$.

The above chosen data: the vectors $\bar{m}^0,\bar{m}^1$ and the matrix $U\in M_u(S)$ — completely describe the structure of $\mathcal{M}\in\mathrm{MF}_S^e$. Choose $C\in M_u(O)$ such that $C\bmod 2=U\bmod t^{2e}$ with respect to the identification κ_{SO}. Define the O-algebra $A=O[\bar{X}]/\mathcal{I}_A$, where $\bar{X}=(X_1,\dots,X_u)$, $\mathcal{I}_A=\mathcal{I}_{A,K}\cap O[\bar{X}]$ and $\mathcal{I}_{A,K}$ is the ideal in $K[\bar{X}]$ generated by the

coordinates of the vector $(-1/2)(\bar{X}C)^{(2)} - \bar{X}$. (For any matrix $\alpha = (\alpha_{ij})$ we set $\alpha^{(2)} := (\alpha_{ij}^2)$.)

Proposition 1.5. *With above notation A is a flat algebra of rank 2^n over O.*

Proof. Indeed, it can be deduced from condition $C1$ (similarly to Lemma 2.2.2 from [4]) that $C^{(2)}$ divides the scalar matrix $2I_u$ in $M_u(O)$. (Note that C divides $\pi^e I_u$.) This implies that the ideal $\mathcal{I}_A$ is generated by the coordinates of the vector $\bar{X}^{(2)} - 2(\bar{X} + \bar{V}){C^{(2)}}^{-1}$, where $\bar{V}$ consists of O-linear combinations of X_iX_j, $1 \leqslant i < j \leqslant u$. Therefore, there is an isomorphism of O-modules

$$A \simeq \oplus_{0 \leqslant i_1, \ldots, i_u \leq 1} O X_1^{i_1} \ldots X_u^{i_u} \tag{1.1}$$

and A is flat over O. □

For the above introduced algebra A, denote by I_A the ideal in A generated by the images of $X_1, \ldots, X_u$. Then $(A, I_A) \in \mathrm{Aug}_O$.

Definition. Denote by $\mathrm{Aug}_O(\mathcal{M})$ the family of all augmented algebras $(A, I_A) \in \mathrm{Aug}_O$ obtained via the above procedure for all choices of $\bar{m}^1$ (which satisfy the conditions $C1$ and $C2$) and the corresponding matrix $C \in M_u(O)$.

1.5. **The φ_1-lift $\iota_{\mathcal{M}}$.** Define the functor $\iota : \mathrm{Aug}_O \longrightarrow \mathcal{MF}_S$ via

$$(B, I_B) \mapsto (I_B/J_B, I_B(2)/J_B, \varphi_1)$$

where $(B, I_B) \in \mathrm{Aug}_O$, J_B was introduced in Subsection 1.3 and φ_1 is induced by the correspondences $b \mapsto -b^2/2$, $b \in I_B$.

For any $\mathcal{M} \in \mathrm{MF}_S^e$ and $(A, I_A) \in \mathrm{Aug}_O(\mathcal{M})$, there is a canonical morphism $\iota_{\mathcal{M}} : \mathcal{M} \longrightarrow \iota(A(\mathcal{M}))$ in $\mathcal{MF}_S$ such that $\bar{m}^0 \mapsto \bar{X} \bmod J_A$ and $\bar{m}^1 \mapsto \bar{X}C \bmod J_A$. Clearly, the image $\iota_{\mathcal{M}}(\mathcal{M})$ is a subobject of $\iota(\mathcal{A}(\mathcal{M}))$ in the category $\mathcal{MF}_S$.

Proposition 1.6. *The map $\iota_{\mathcal{M}} : \mathcal{M} \longrightarrow \iota_{\mathcal{M}}(\mathcal{M})$ is a special φ_1-lift.*

Proof. Consider $\widetilde{\mathcal{M}} = \mathcal{M} \otimes_S S/t^{2e} \in \mathcal{MF}_S$. Clearly, the natural projection $\mathcal{M} \longrightarrow \widetilde{\mathcal{M}}$ is a special φ_1-lift and $\iota_{\mathcal{M}}$ is the composition of this projection and a unique $\tilde{\iota}_{\mathcal{M}} \in \mathrm{Hom}_{\mathcal{MF}_S}(\widetilde{\mathcal{M}}, \iota_{\mathcal{M}}(\mathcal{M}))$.

Let $\widetilde{\mathcal{M}} = (\widetilde{M}^0, \widetilde{M}^1, \varphi_1)$. Then

$$\widetilde{M}^0 = \left\{ \sum o_i \widetilde{X}_i \mid o_1, \ldots, o_n \in O, \widetilde{X}_i = X_i \bmod 2I_A \right\}$$

$$\widetilde{M}^1 = \left\{ \sum o_i \widetilde{Y}_i \mid o_1, \ldots, o_n \in O, (\widetilde{Y}_1, \ldots, \widetilde{Y}_n) = (\widetilde{X}_1, \ldots, \widetilde{X}_n)C \right\}$$

where the S-module structure is induced by the given O-module structure via the identification κ_{SO} and $\varphi_1 : \widetilde{M}^1 \longrightarrow \widetilde{M}^0$ is given via the correspondence $\sum o_i \widetilde{Y}_i \mapsto \sum o_i^2 \widetilde{X}_i$.

Suppose $\tilde{\iota}_{\mathcal{M}}=(\tilde{\iota}^0_{\mathcal{M}},\tilde{\iota}^1_{\mathcal{M}})$. Then

$$\widetilde{T}^0:=\mathrm{Ker}\,\tilde{\iota}^0_{\mathcal{M}}=\Big\{\sum o_i\widetilde{X}_i\mid \sum o_iX_i\in J_A\Big\}$$

$$\widetilde{T}^1:=\mathrm{Ker}\,\tilde{\iota}^1_{\mathcal{M}}=\Big\{\sum o_i\widetilde{Y}_i\mid \sum o_iY_i\in J_A\Big\}$$

The proposition will be proved if we show $\widetilde{T}^0=\widetilde{T}^1$, $\varphi_1(\widetilde{T}^0)\subset\widetilde{T}^0$ and $\varphi_1|_{\widetilde{T}^0}$ is nilpotent.

Suppose $\tilde{v}=\sum o_i\widetilde{X}_i\in\widetilde{T}^0$. Then $\sum o_iX_i\in J_A\subset I_A(2)$ and $\sum_i o_i^2X_i^2\in 2I_A$. Let $(G'_1,\dots,G'_u)=2(\bar{X}+\bar{V})(C^{(2)})^{-1}$, cf. the proof of Proposition 1.5. Then $\sum_i o_i^2G'_i\in 2I_A$ and due to the isomorphism (1.1) we can follow the linear terms to obtain that

$$2(o_1^2,\dots,o_u^2)(C^{(2)})^{-1}:=2(\alpha_1,\dots,\alpha_n)\in 2O^u \tag{1.2}$$

Clearly, there are $\alpha'_1,\dots,\alpha'_u\in O$ such that all $\alpha_i'^2\equiv\alpha_i \bmod 2O$ and (1.2) implies that

$$(o_1,\dots,o_u)\equiv(\alpha'_1,\dots,\alpha'_u)C \bmod \pi^e.$$

Therefore, $\sum_i o_i\widetilde{X}_i$ is congruent modulo $\pi^e\widetilde{M}^0$ to an O-linear combination of the coordinates of the vector $(\widetilde{Y}_1,\dots,\widetilde{Y}_u)=(\widetilde{X}_1,\dots,\widetilde{X}_u)C$. In other words, $\tilde{v}\in\widetilde{M}^1$, i.e. $\widetilde{T}^0=\widetilde{T}^1$.

If $\tilde{v}=\sum o'_i\widetilde{Y}_i$, then

$$\varphi_1(\widetilde{v})=\sum o_i'^2\widetilde{X}_i=-\widetilde{v}^2/2+\sum_{i,j}o'_io'_j\widetilde{Y}_i\widetilde{Y}_j\in J_A \bmod 2I_A$$

implies $\varphi_1(\widetilde{T}^0)\subset\widetilde{T}^0$. (Use that J_A is a DP-ideal and $I_A(2)^2\subset J_A$.)

Finally, let $\tilde{v}_0=\tilde{v}$ and for $n\geqslant 0$, $\tilde{v}_{n+1}=\varphi_1(\tilde{v}_n)$. We must prove that for $n\gg 0$, $\tilde{v}_n=0$.

Let $A'=O[Y_1,\dots,Y_n]$. Then A' is an O-subalgebra in A given by the equations

$$(Y_1^2,\dots,Y_n^2)+(Y_1,\dots,Y_n)(2C^{-1})=0.$$

Therefore, any element $a\in A'$ can be uniquely presented in the form

$$a=\sum_{1\leqslant i_1<\dots<i_s\leqslant u}o_{i_1\dots i_s}(a)Y_{i_1}\dots Y_{i_s},$$

where all $o_{i_1\dots i_s}(a)\in O$. Set $\mathcal{L}(a)=o_1(a)Y_1+\dots+o_u(a)Y_u$. Notice that if $I_{A'}$ is the augmentation ideal of A' generated by $Y_1,\dots,Y_u$ and $a\in I_{A'}^2$ then all $o_i(a)\equiv 0 \bmod \pi^e$ (use that $2C^{-1}\equiv 0 \bmod \pi^e$). This means that if $a_1,a_2\in I_{A'}$ and $a_1\equiv a_2 \bmod I_{A'}^2$ then $\mathcal{L}(a_1)\equiv\mathcal{L}(a_2) \bmod \pi^eI_{A'}$.

With above notation let $v_0=\sum o_iX_i$ and for all $n\geqslant 0$, $v_{n+1}=-v_n^2/2$. Clearly, all $v_n\in J_A$ and there is an $N_0\geqslant 0$ such that $v_{N_0}\in 2I_A$.

For $n\geqslant 0$, set $v_n^*=\mathcal{L}(v_n)$ and denote by $\rho: A'\longrightarrow A/2$ the composition of the natural inclusion A' into A and the reduction map $A\longrightarrow A/2$.

Lemma 1.7. $\rho(v_n^*)\equiv\tilde{v}_n \bmod(\pi^e\widetilde{M}^1)$.

Proof of Lemma. Use induction on $n \geqslant 0$.

Clearly, $v_0 = v_0^*$ and $\rho(v_0) = \tilde{v}_0$.

Suppose $\rho(v_n^*) \equiv \tilde{v}_n \bmod(\pi^e \widetilde{M^1})$.

If $v_n^* = \sum o_i^{(n)} Y_i$, where all $o_i^{(n)} \in O$, then $v_n = \sum o_i^{(n)} Y_i + \alpha$ with $\alpha \in I_{A'}^2$. Therefore,

$$v_{n+1} \equiv -v_n^2/2 \equiv \sum o_i^{(n)2} X_i \bmod I_{A'}^2$$

$$v_{n+1}^* \equiv \sum o_i^{(n)2} X_i \bmod \pi^e I_{A'}$$

and $\rho(v_{n+1}^*) \equiv \varphi_1(\rho(v_n^*)) \equiv \varphi_1(\tilde{v}_n) \bmod(\pi^e \widetilde{M^1})$. The lemma is proved. □

Finally, the above lemma implies that $\tilde{v}_{N_0} \in \pi^e \widetilde{M^1}$ and, therefore, $\tilde{v}_{N_0+1} = \varphi_1(\tilde{v}_{N_0}) \in \pi^{2e} \widetilde{M^0} = 0$. The proposition is proved. □

1.6. **The maps Θ^* and Ψ^*.** Suppose $\mathcal{B} = (B, I_B) \in \mathrm{Aug}_O$ and $\mathcal{B}^{et} = (B^{et}, I_{B^{et}})$ is the maximal etale subalgebra in $\mathcal{B}$. Then $J_{B^{et}} = 2I_{B^{et}}$ and $\iota(\mathcal{B}^{et}) = (I_{B^{et}}/2I_{B^{et}}, \pi^e I_{B^{et}}/2I_{B^{et}}, \varphi_1) \in \mathcal{MF}_S$ admits a (unique) special φ_1-lift $\mathcal{E}(\mathcal{B}^{et}) \in \mathrm{MF}_S^{e,et}$.

Introduce $\mathrm{m}(\mathcal{B}^{et}) = (2I_{B^{et}}/2\pi I_{B^{et}}, 2I_{B^{et}}/2\pi I_{B^{et}}, \varphi_1) \in \mathcal{MF}_S$, where φ_1 is induced (as usually) by the map $a \mapsto -a^2/2$, $a \in 2I_{B^{et}}$. Clearly, $\mathrm{m}(\mathcal{B}^{et})$ admits a (unique) nilpotent φ_1-lift $M(\mathcal{B}^{et}) \in \mathrm{MF}_S^{e,mult}$ and the identity morphism on $I_{B^{et}}$ induces the natural morphism

$$\omega(\mathcal{B}^{et}) : M(\mathcal{B}^{et}) \longrightarrow \mathcal{E}(\mathcal{B}^{et})$$

in the category MF_S^e.

Suppose $\mathcal{M} = (M^0, M^1, \varphi_1) \in \mathrm{MF}_S^e$ and $\mathcal{A} = (A, I_A) \in \mathrm{Aug}_O(\mathcal{M})$. Introduce the map

$$\Theta : \mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Hom}_{\mathcal{MF}_S}(\mathcal{M}, \iota(\mathcal{B}))$$

by attaching to $F \in \mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}, \mathcal{B})$ the morphism of filtered modules $\Theta(F) = \iota(F) \circ \iota_{\mathcal{M}}$.

Proposition 1.8. *With the above notation:*
a) Θ *is surjective;*
b) *if either* $\mathcal{B}^{et} = \mathcal{S}^{et}$ *or* $\mathcal{M}^{mult} = 0$ *then* θ *is bijective;*
c) *there is a natural strict action of the group* $\mathrm{Hom}_{\mathrm{MF}_S^e}(\mathcal{M}^{mult}, \mathcal{E}(\mathcal{B}^{et}))$ *on* $\mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}, \mathcal{B})$ *and the corresponding equivalence relation* R *coincides with the equivalence relation from the definition of* Aug_O^* *in Subsection* 1.3*;*
d) Θ *induces the bijection*

$$\Theta^* : \mathrm{Hom}_{\mathrm{Aug}_O^*}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Hom}_{\mathcal{MF}_S}(\iota_{\mathcal{M}}(\mathcal{M}), \iota(\mathcal{B})).$$

Proof. Suppose $\mathcal{A} = (A, I_A) \in \mathrm{Aug}_O(\mathcal{M})$ is given via a special choice of vectors $\bar{m}^0$ and $\bar{m}^1$ with the coordinates in M^0 and, resp., M^1, and the matrix $C \in M_u(O)$ from Subsection 1.4.

Lemma 1.9. *Suppose* $\bar{b}^0 \in I_B^u$ *is such that* $(-1/2)(\bar{b}^0 C)^{(2)} \equiv \bar{b}^0 \bmod J_B$. *Let* $\mathcal{L}(\bar{b}^0)$ *be the set of all* $\bar{b} \in I_B^u$ *such that* $\bar{b} \equiv \bar{b}^0 \bmod \widetilde{J}_B$ *and it holds* $(-1/2)(\bar{b}C)^{(2)} \equiv \bar{b} \bmod \widetilde{J}_B$. *Then*

a) $\mathcal{L}(\bar{b}^0) \neq \emptyset$;

b) *if* $\bar{b}_1, \bar{b}_2 \in \mathcal{L}(\bar{b}^0)$ *then* $\bar{x} = \bar{b}_1 - \bar{b}_2 \in J_B^u$ *and* $(-1/2)(\bar{x}C)^{(2)} \equiv \bar{x} \bmod \widetilde{J}_B$.

Proof of Lemma. The vector $\bar{b} = \bar{b}^0 + \bar{x} \in \mathcal{L}(\bar{b}^0)$ iff $\bar{x} \in J_B^u$ and

$$(-1/2)(\bar{x}C)^{(2)} - \bar{x} \equiv \bar{b}^0 + (1/2)(\bar{b}^0 C)^{(2)} \bmod \widetilde{J}_B.$$

Notice that $\mathcal{V} = (J_B/\widetilde{J}_B)^n$ has a natural structure of a finite dimensional vector space over k and the correspondence $\bar{x} \mapsto (-1/2)(\bar{x}C)^{(2)}$ induces a σ-linear morphism $\tilde{\varphi} : \mathcal{V} \longrightarrow \mathcal{V}$. By Lemma 1.2, $\tilde{\varphi} - \mathrm{id} : \mathcal{V} \longrightarrow \mathcal{V}$ is surjective. This proves part a). Part b) follows easily from the congruence $(\bar{b}_1 C)^{(2)} \equiv (\bar{b}_2 C)^{(2)} + (\bar{x}C)^{(2)} \bmod 2J_B I_B(2)$. □

Notice that $\mathrm{Hom}_{\mathcal{MF}_S}(\iota_{\mathcal{M}}(\mathcal{M}), \iota(\mathcal{B})) =$

$$\{\bar{b}^0 \bmod J_B \mid \bar{b}^0 \in I_B^u, (-1/2)(\bar{b}^0 C)^{(2)} \equiv \bar{b}^0 \bmod J_B\},$$

$$\mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}, \mathcal{B}) = \{\bar{b} \in I_B^u \mid (-1/2)(\bar{b}C)^{(2)} = \bar{b}\}$$

$$= \{\bar{b} \bmod \widetilde{J}_B \mid \bar{b} \in I_B^u, (-1/2)(\bar{b}C)^{(2)} \equiv \bar{b} \bmod \widetilde{J}_B\}$$

and the map Θ is given via $\bar{b} \bmod \widetilde{J}_B \mapsto \bar{b}^0 \bmod J_B$.

Therefore, part a) of Proposition 1.8 follows from part a) of above Lemma. If $\bar{x}$ is the vector from part b) of above Lemma then the correspondence $\bar{m} \mapsto \bar{x}$ identifies

$$\mathrm{Hom}_{\mathrm{MF}_S^e}(\mathcal{M}, M(\mathcal{B}^{et})) = \mathrm{Hom}_{\mathcal{MF}_S}(\mathcal{M}^{mult}, M(\mathcal{B}^{et}))$$

with $\mathrm{Hom}_{\mathcal{MF}_S}(\mathcal{M}^{mult}, \mathrm{m}(\mathcal{B}^{et}))$. This implies part b) of Proposition 1.8. With the above notation the correspondence $\bar{b} \mapsto \bar{b} + \bar{x}$ determines the action of $\mathrm{Hom}_{\mathrm{MF}_S^e}(\mathcal{M}^{mult}, \mathcal{E}(\mathcal{B}^{et}))$ on $\mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}, \mathcal{B})$. One can easily verify that this action is strict and Θ induces bijection of the corresponding quotient $\mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}, \mathcal{B})/R$ and $\mathrm{Hom}_{\mathcal{MF}_S}(\iota_{\mathcal{M}}(\mathcal{M}), \iota(\mathcal{B}))$. □

Remark. a) By condition $C2$, $\bar{m}^0 = (\bar{m}^{loc}, \bar{m}^{et})$ and therefore we have the appropriate presentation $\bar{x} = (\bar{x}^{loc}, \bar{x}^{et})$, where $\bar{x}$ is the vector from part b) of Lemma 1.9. One can easily see that $\bar{x}^{et} = \bar{0}$. In particular, the shifts by all above vectors $\bar{x}$ determine a strict action of $\mathrm{Hom}_{\mathcal{MF}_S}(\mathcal{M}^{mult}, \iota(\mathcal{B}))$ on $I_B^u \bmod \widetilde{J}_B$.

b) One can easily see that if $\mathcal{M} \in \mathrm{MF}_S^{e,mult}$ and $\mathcal{B} = \mathcal{B}^{et}$ then

$$\mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathcal{MF}_S}(\mathcal{M}, E(\mathcal{B}^{et}));$$

Corollary 1.10. *If* $\mathcal{B} \in \mathrm{Aug}_O(\mathcal{N})$ *with* $\mathcal{N} \in \mathrm{MF}_S^e$ *then the above identification* Θ^* *induces a functorial in both arguments embedding*

$$\Psi^* : \mathrm{Hom}_{\mathrm{MF}_S^{*e}}(\mathcal{M}, \mathcal{N}) \longrightarrow \mathrm{Hom}_{\mathrm{Aug}_O^*}(\mathcal{A}, \mathcal{B})$$

and the correspondences $\mathcal{M} \mapsto \mathcal{A}$ *(and* $\mathcal{N} \mapsto \mathcal{B}$*) induce a faithful functor* $\Psi^* : \mathrm{MF}_S^{e*} \longrightarrow \mathrm{Aug}_O^*$.

1.7. **Group schemes** Spec$A$**,** $(A, I_A) \in \mathrm{Aug}_O(\mathcal{M})$**.** Suppose $\mathcal{M} \in \mathrm{MF}_S^e$ and $\mathcal{A} = (A, I_A) \in \mathrm{Aug}_O(\mathcal{M})$ is given via a special choice of vectors $\bar{m}^0$, $\bar{m}^1$ and matrices $U \in M_u(S)$, $C \in M_u(O)$ under assumptions $C1$ and $C2$ from Subsection 1.4.

We can describe the structures of $\mathcal{M} \oplus \mathcal{M}$ and $A \otimes_O A$ via the S-basis $\bar{m}^0 \oplus \{0\}$, $\{0\} \oplus \bar{m}^0$ for $M^0 \oplus M^0$, the S-basis $\bar{m}^1 \oplus \{0\}, \{0\} \oplus \bar{m}^1$ for $M^1 \oplus M^1$ and the corresponding matrices $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \in M_{2u}(S)$ and $\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \in M_{2u}(O)$. (One can easily see that these data satisfy assumptions $C1$ and $C2$ from Subsection 1.4.) Note that $A = O[\bar{X}]/\mathcal{I}_A$, where the ideal $\mathcal{I}_A$ is generated by the coordinates of the vector $((\bar{X}C)^{(2)} + 2\bar{X})C^{(2)^{-1}}$,

$$\mathcal{A} \otimes \mathcal{A} = (A \otimes_O A, I_A \otimes_O A + A \otimes_O I_A) \in \mathrm{Aug}_O(\mathcal{M})$$

and

$$A \otimes_O A = O[\bar{X} \otimes 1, 1 \otimes \bar{X}]/(\mathcal{I}_A \otimes 1, 1 \otimes \mathcal{I}_A).$$

Let $e : A \longrightarrow A/I_A = O$ be the natural projection and let $\Delta_{\mathcal{A}}^* = \Psi^*(\bigtriangledown) \in \mathrm{Hom}_{\mathrm{Aug}_O^*}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$, where $\bigtriangledown : \mathcal{M} \longrightarrow \mathcal{M} \oplus \mathcal{M}$ is the class of the diagonal morphism in the category MF_S^{e*}.

Let $\mathcal{L}_{\mathcal{A}}$ be the set of all $\Delta \in \mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ such that:

- $\Delta \bmod R = \Delta_{\mathcal{A}}^*$;
- $G = \mathrm{Spec}A$ becomes an object of the category Gr_O when provided with the counit e and the coaddition Δ.

Proposition 1.11. a) $\mathcal{L}_{\mathcal{A}} \neq \emptyset$*;*

b) *If* $\Delta_1, \Delta_2 \in \mathcal{L}_{\mathcal{A}}$ *then the corresponding coalgebra structures on* $\mathcal{A}$ *are transformed one into another via an automorphism* $f \in \mathrm{Aug}_O(\mathcal{A})$ *such that* $f \underset{R}{\sim} \mathrm{id}_{\mathcal{A}}$ *(i.e.* f *and* $\mathrm{id}_{\mathcal{A}}$ *coincide in* Aug_O^**).*

Proof. Let $\bar{X} = (\bar{X}^{loc}, \bar{X}^{et})$ with respect to the presentation $\bar{m} = (\bar{m}^{loc}, \bar{m}^{et})$ from condition $C2$ in Subsection 1.4. For $\Delta \in \mathcal{L}_{\mathcal{A}}$, set

$$\Delta(\bar{X}) = (\Delta(\bar{X}^{loc}), \Delta(\bar{X}^{et})) = \bar{X} \otimes 1 + 1 \otimes \bar{X} + \bar{\jmath},$$

where $\bar{\jmath} = (\bar{\jmath}^{loc}, \bar{\jmath}^{et})$.

Note that $\bar{\jmath}^{et}$ does not depend on a choice of $\Delta \in \mathcal{L}_{\mathcal{A}}$. This implies that $G^{et} := \mathrm{Spec}A^{et} \in \mathrm{Gr}_O^{et}$ when provided with the coaddition $\Delta^{et} = \Delta|_{A^{et}}$ and the counit $e^{et} = e|_{A^{et}}$. More explicitly, $A^{et} = O[\bar{X}^{et}]$ with the equations $\eta \bar{X}^{et(2)} \equiv \bar{X}^{et} \bmod 2I_{A^{et}}$, where $\eta = -\pi^{2e}/2 \in O^*$, and $\Delta(\bar{X}^{et}) \equiv \bar{X}^{et} \otimes 1 + 1 \otimes \bar{X}^{et} \bmod 2I_{A^{et} \otimes A^{et}}$.

Let $G^{et}_k := G^{et} \otimes_O k = \mathrm{Spec}A^{et}_k$. Remind that the k-module of symmetric Hochschild 2-cocycles $Z^2_{sym}(G^{et}_k)$ consists of symmetric $\gamma \in I_{A^{et}_k \otimes A^{et}_k}$ such that

$$(\Delta \otimes \mathrm{id})\gamma + \gamma \otimes 1 = 1 \otimes \gamma + (\mathrm{id} \otimes \Delta)\gamma.$$

The corresponding k-module of 2-coboundaries equals

$$B^2(G^{et}_k) = \{\Delta^{et}(\gamma) - \gamma \otimes 1 - 1 \otimes \gamma \mid \gamma \in I_{A^{et}_k}\} \subset Z^2_{sym}(G^{et}_k)$$

We have the following two facts:

• Suppose $\gamma \in Z^2_{sym}(G^{et}_k)$ and $\mathrm{mult} : A^{et}_k \otimes A^{et}_k \longrightarrow A^{et}_k$ is the morphism of multiplication. Then $\gamma \in B^2(G^{et}_k) \Leftrightarrow \mathrm{mult}(\gamma) = 0$.

• If $\bar{X}^{et} = (X^{et}_1, \dots, X^{et}_{u_{et}})$ then the elements $\delta^+(X^{et}_{i_1} \dots X^{et}_{i_s}) \bmod \pi$, where $s \geqslant 2$, $1 \leqslant i_1 < \dots < i_s \leqslant u_{et}$ and $\delta^+ = \Delta - \mathrm{id} \otimes 1 - 1 \otimes \mathrm{id}$, form a k-basis of $B^2(G^{et}_k)$ and the correspondences $\delta^+(X^{et}_{i_1} \dots X^{et}_{i_s}) \mapsto X^{et}_{i_1} \dots X^{et}_{i_s}$ determine the k-linear embedding $\omega : B^2(G^{et}_k) \longrightarrow I^{et}_A \bmod \pi$. Note that for any $\alpha \in B^2(G^{et}_k)$, $\alpha^2 \in B^2(G^{et}_k)$ and $\omega(\alpha^2) = \omega(\alpha)^2$.

Now notice that Δ depends only on the residue

$$\bar{\jmath} \bmod \widetilde{J}_{A\otimes A} \in J^u_{A\otimes A} \bmod \widetilde{J}_{A\otimes A} = 2I^u_{A^{et}\otimes A^{et}} \bmod (2\pi I_{A^{et}\otimes A^{et}})$$

Let $\bar{\jmath} = 2\bar{\alpha} \bmod 2\pi I_{A^{et}\otimes A^{et}}$, where $\bar{\alpha}(\Delta) = (\bar{\alpha}^{loc}(\Delta), \bar{\alpha}^{et}) \in I^u_{A^{et}\otimes A^{et}}$.

We have the following properties:

a) Δ defines a morphism of augmented algebras iff $(\bar{\alpha}C)^{(2)} + \bar{\alpha}$ has all its coordinates in $\pi I_{A^{et}\otimes A^{et}}$.

b) Δ determines a structure of commutative group scheme on $G = \mathrm{Spec}A$ iff $\bar{\alpha} \bmod \pi I_{A^{et}\otimes A^{et}}$ has all its coordinates in $Z^2_{sym}(G^{et}_k)$.

c) $2\mathrm{id}_G = 0$ iff $\bar{\alpha}^{loc} \bmod \pi I_{A^{et}\otimes A^{et}}$ has all its coordinates in $B^2(G^{et}_k)$.

The proof of property a) uses the equations $(-1/2)(\bar{X}C)^{(2)} = \bar{X}$ for A, property b) is equivalent to the axioms of coassociativity and cocommutativity for G. As for property c), note that $\bar{0} = 2\mathrm{id}_G(\bar{X}) = \mathrm{mult}(\Delta(\bar{X})) = 2\bar{X} + \mathrm{mult}(\bar{\jmath})$ and, therefore, $\mathrm{mult}\bar{\jmath}^{loc}$ has all its coordinates in $\widetilde{J}_{A\otimes A}$ or, equivalently, $\mathrm{mult}(\bar{\alpha}^{loc}) \equiv 0 \bmod \pi I_{A^{et}}$.

Now we can finish the proof of Proposition 1.11.

Let $\Delta_0 \in \Delta^*_{\mathcal{A}}$ be such that

$$\Delta_0(\bar{X}^{loc}) \equiv \bar{X}^{loc} \otimes 1 + 1 \otimes \bar{X}^{loc} \bmod \widetilde{J}_{A\otimes A}.$$

This means that $\bar{\alpha}(\Delta_0) = (\bar{0}, \bar{\alpha}^{et})$ and by above properties a)-c), we have $\Delta_0 \in \mathcal{L}_{\mathcal{A}}$.

Suppose $\Delta \in \mathcal{L}_{\mathcal{A}}$ and $\bar{\alpha}(\Delta) = (\bar{\alpha}^{loc}(\Delta), \bar{\alpha}^{et})$. Let $\bar{\gamma} = (\gamma^{loc}, \bar{0}) \in I^n_{A^{et}}$ be such that $\delta^+(\bar{\gamma}^{loc} \bmod \pi I_{A^{et}}) = \bar{\alpha}^{loc} \bmod \pi I_{A^{et}\otimes A^{et}}$. We can assume that $\bar{\gamma}^{loc} = \omega(\delta^+\bar{\alpha}^{loc})$ and, therefore, $(\bar{\gamma}C)^{(2)} + \bar{\gamma}$ has all its coordinates in $\pi I_{A^{et}}$. Therefore, there is a unique $F \in \mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ such that $F(\bar{X}) \equiv \bar{X} + 2\bar{\gamma} \bmod \widetilde{J}_{A\otimes A}$. Clearly, $F \underset{R}{\sim} \mathrm{id}_{\mathcal{A}}$ in Aug^*_O.

In addition, $\Delta_0(\bar{F}(\bar{X})) \equiv \Delta_0(\bar{X} + 2\bar{\gamma}) \equiv$

$$(\bar{X}^{loc} \otimes 1 + 1 \otimes \bar{X}^{loc} + 2\Delta^{et}(\bar{\gamma}^{loc}), \Delta_0(\bar{X}^{et}))$$

$$\equiv (\bar{X}^{loc} \otimes 1 + 1 \otimes \bar{X}^{loc} + 2\bar{j}^{loc} + 2(\bar{\gamma}^{loc} \otimes 1 + 1 \otimes \bar{\gamma}^{loc}), \Delta(\bar{X}^{et}))$$

$$\equiv (F \otimes F)(\Delta(\bar{X})) \bmod \widetilde{J}_{A\otimes A}.$$

Therefore, $F \circ \Delta_0 = \Delta \circ (F \otimes F)$. The Proposition is proved. □

1.8. **Functor $\mathcal{G}_O$.**

Proposition 1.12. *There is a functor $\mathcal{G}_O : \mathrm{MF}_S^{e*} \longrightarrow \mathrm{Gr}_O^*$ such that its compositoon with the forgetful functor $\mathrm{Gr}_O^* \longrightarrow \mathrm{Aug}_O^*$ coincides with the functor Ψ^* from Corollary 1.10.*

Proof. For $i = 1, 2$, let

• $\mathcal{M}_i \in \mathrm{MF}_S^e$ with specially chosen vectors $\bar{m}_i^0$, $\bar{m}_i^1$ satisfying the conditions $C1$, $C2$ fom Subsection 1.4;

• $\mathcal{A}_i = (A_i, I_{A_i})$ be the corresponding augmented O-algebras with the coalgebra structures uniquely given by the coadditions $\Delta_i : A_i \longrightarrow A_i \otimes A_i$ such that $\Delta_i \in \Psi^*(\bigtriangledown_i)$ (where $\bigtriangledown_i$ are the diagonal maps from $\mathcal{M}_i$ to $\mathcal{M}_i \oplus \mathcal{M}_i$) and $\Delta_i(\bar{X}_i^{loc}) \equiv \bar{X}_i^{loc} \otimes 1 + 1 \otimes \bar{X}_i^{loc} \bmod \widetilde{J}_{A_i\otimes A_i}$.

Denote the corresponding group schemes $\mathrm{Spec}A_i = G_i \in \mathrm{Gr}_O$.

Suppose $f \in \mathrm{Hom}_{\mathcal{MF}_S}(\iota(\mathcal{M}_1), \iota(\mathcal{M}_2)) = \mathrm{Hom}_{\mathrm{MF}_S^{e*}}(\mathcal{M}_1, \mathcal{M}_2)$ and $F \in \mathrm{Hom}_{\mathrm{Aug}_O}(\mathcal{A}_1, \mathcal{A}_2)$ is such that $F \in \Psi^*(f)$. Then

$$(F \otimes F)(\Delta_1(\bar{X}_1)) - \Delta_2(F(\bar{X}_1)) \in J_{A_2\otimes A_2}.$$

Let $\bar{\alpha} = (\bar{\alpha}^{loc}, \bar{\alpha}^{et})$ be the vector with the coordinates in $I_{A_2^{et}\otimes A_2^{et}}$ such that

$$(F \otimes F)(\Delta_1(\bar{X}_1)) - \Delta_2(F(\bar{X}_1)) \equiv 2\bar{\alpha} \bmod \widetilde{J}_{A_2\otimes A_2}.$$

Note that if $F^{et} := F|_{A_1^{et}}$ then the congruence $(F^{et}\otimes F^{et})(\Delta_1(\bar{X}_1^{et})) \equiv \Delta_2(F^{et}(\bar{X}_1^{et})) \bmod 2A_2^{et}$ implies that $\bar{\alpha}^{et} = \bar{0}$ and F^{et} induces a morphism of etale group schemes $G_2^{et} \longrightarrow G_1^{et}$.

Using that F is a morphism of augmented O-algebras we obtain that $(\bar{\alpha}C)^{(2)} + \bar{\alpha}$ has all coordinates in $\pi I_{A^{et}\otimes A^{et}}$.

One can verify easily that $\bar{\alpha} \bmod \pi I_{A_2^{et}\otimes A_2^{et}}$ has all coordinates in $Z^2_{sym}(G_2^{et} \otimes k)$ and using that for $i = 1, 2$, $2\,\mathrm{id}_{G_i} = \Delta_{G_i} \circ \mathrm{mult}$ we obtain that $\bar{\alpha} \bmod \pi I_{A_2^{et}\otimes A_2^{et}}$ has all coordinates in $B^2(G_2^{et} \otimes k)$.

Let $\bar{\gamma} = (\gamma^{loc}, \bar{0})$ be the vector with coordinates in $I_{A_2^{et}}$ such that $\bar{\gamma} \bmod \pi I_{A_2^{et}} = \omega(\bar{\alpha} \bmod \pi I_{A_2^{et}\otimes A_2^{et}})$. Then

$$\Delta_2^{et}(\bar{\gamma}) \equiv \bar{\gamma} \otimes 1 + 1 \otimes \bar{\gamma} + \bar{\alpha} \bmod \pi I_{A_2^{et}\otimes A_2^{et}}$$

and $(\bar{\gamma}C)^{(2)} + \bar{\gamma}$ has all coordinates in $\pi I_{A_2^{et}}$. This implies that there is an $F' \in \mathrm{Aug}_O(\mathcal{A}_1, \mathcal{A}_2)$ such that $F'(\bar{X}_1^{et}) = F(\bar{X}_1^{et})$ and $F'(\bar{X}^{loc}) \equiv F(\bar{X}_1^{loc}) + 2\bar{\gamma} \bmod \widetilde{J}_{A_2\otimes A_2}$. Therefore,

$$(F' \otimes F')(\Delta_1(\bar{X}_1)) \equiv \Delta_2(F'(\bar{X}_1)) \bmod \widetilde{J}_{A_2\otimes A_2},$$

and $(F' \otimes F') \circ \Delta_1 = \Delta_2 \circ F'$. This proves the existence of $F' \in \Psi^*(f)$ such that $\mathrm{Spec}F' \in \mathrm{Hom}_{\mathrm{Gr}_O}(G_2, G_1)$.

Similarly, one can verify that if $F'' \in \Psi^*(f)$ is such that $\mathrm{Spec}(F'') \in \mathrm{Hom}_{\mathrm{Gr}_O}(G_2, G_1)$ then F' and F'' are equivalent in the category Gr_O^* with the obvious inverse statement. The proposition is proved. $\square$

1.9. **Full faithfulness of $\mathcal{G}_O$.** Suppose $\mathcal{N} = (N^0, N^1, \varphi_1) \in \mathrm{MF}_S^e$ and $\mathcal{G}_O(\mathcal{N}) = \mathrm{Spec}B$.

1.9.1. *Special construction of B.* Use the following special case of the construction of the O-algebra B from Subsection 1.4.

Let $n_1^1, \dots, n_u^1$ be an S-basis of N^1 such that there are $\tilde{s}_1, \dots, \tilde{s}_u \in S$ and an S-basis $n_1, \dots, n_u$ of N^0 such that for $1 \leqslant i \leqslant u$, $n_i^1 = \tilde{s}_i n_i$. One can easily see that this choice of $\bar{n}^1 = (n_1^1, \dots, n_u^1)$ can be made in such a way that conditions $C1$ and $C2$ from Subsection 1.4. are satisfied. Also notice that all $\tilde{s}_i$ divide t^e.

Set $\bar{n}^0 = \varphi_1(\bar{n}^1)$ and let the matrices $U \in M_u(S)$ and $U_0 \in \mathrm{GL}_u(S)$ be such that $\bar{n}^1 = \bar{n}^0 U$ and $\bar{n}^0 U_0 = \bar{n} := (n_1, \dots, n_u)$. Then $U = U_0 U_1$, where $U_1 = (\tilde{s}_i \delta_{ij})_{1 \leqslant i,j \leqslant u}$ is diagonal. Choose $\widetilde{\Omega} = (\tilde{\eta}_i' \delta_{ij})_{1 \leqslant i,j \leqslant u} \in M_u(O)$ and $D \in \mathrm{GL}_u(O)$ such that $\kappa_{SO}(U_1 \bmod t^{2e}) = \widetilde{\Omega} \bmod 2$ and $\kappa_{SO}(U_0 \bmod t^{2e}) = D \bmod 2$. Then for $\bar{Y} = (Y_1, \dots, Y_u)$, the O-algebra B is the quotient of $O[\bar{Y}] := O[Y_1, \dots, Y_u]$ by the ideal generated by the coordinates of the vector

$$\left((\bar{Y} D \widetilde{\Omega})^{(2)} + 2\bar{Y} \right) (D\widetilde{\Omega})^{(2)^{-1}}$$

Then in the new coordinates $\bar{X} = (X_1, \dots, X_u) := \bar{Y} D$, B is the quotient of $O[\bar{X}] := O[X_1, \dots, X_u]$ by the ideal generated by the elements

$$X_i^2 - \eta_i \sum_j X_j c_{ji}, \quad 1 \leqslant i \leqslant u,$$

Here $C = (c_{ij}) = D^{-1}$, and for all i, $\eta_i = -2/\tilde{\eta}'^2$. With this notation, the counit $e : B \longrightarrow O$ and the coaddition $\Delta : B \longrightarrow B \otimes_O B$ are uniquely recovered (in the category Gr_O^*) from the conditions $e(X_i) = 0$ and $\Delta(X_i) \equiv X_i \otimes 1 + 1 \otimes X_i \bmod J_{B \otimes B}$.

Remind that $\bar{X} = (\bar{X}^{loc}, \bar{X}^{et})$, where for $u_0 = \dim_k \bar{N}^{0,loc}$, $\bar{X}^{loc} = (X_1, \dots, X_{u_0})$ and $\bar{X}^{et} = (X_{u_0+1}, \dots, X_u)$. Then condition $C2$ implies:

- for $1 \leqslant i \leqslant u_0$, $X_i, X_i^2/\eta_i \in I_B^{loc}$;
- for $u_0 < i \leqslant u$, $\eta_i \in O^*$ and $X_i \in I_{B^{et}}$.

Therefore, the matrix $C = (c_{ij})$ has the following block structure $C = \begin{pmatrix} C_0 & 0 \\ C' & C^{et} \end{pmatrix}$, where $C_0 \in \mathrm{GL}_{u_0}(O)$, $C^{et} \in \mathrm{GL}_{u^{et}}(O)$ with $u^{et} := u - u_0$, $C' \equiv 0 \bmod \pi$. In particular,

(**C**) *if $C = (c_{ij})$ and $D = C^{-1} = (d_{ij})$ then $c_{ij} \equiv d_{ij} \equiv 0 \bmod \pi$ if either $i \leqslant u_0 < j$ or $j \leqslant u_0 < i$.*

Let $\delta^+ X_i = j_i \in I_{B\otimes B}$, $1 \leqslant i \leqslant u$. The coordinates of the vector $\bar{\jmath} = (\bar{\jmath}^{loc}, \bar{\jmath}^{et})$, where $\bar{\jmath}^{loc} = (j_1, \dots, j_{u_0})$ and $\bar{\jmath}^{et} = (j_{u_0+1}, \dots, j_u)$, appear as the solutions in $I_{B\otimes B}$ of the system of equations

$$\sum_s j_s c_{si} = -\tilde{\eta}_i X_i \otimes X_i - (\tilde{\eta}_i X_1 \otimes 1 + 1 \otimes \tilde{\eta}_i X_i) j_i - \tilde{\eta}_i j_i^2/2, \tag{1.3}$$

where $1 \leqslant i \leqslant u$ and $\tilde{\eta}_i = -2/\eta_i = \tilde{\eta}'^2$.

The coordinates of $\bar{\jmath}^{et}$ are determined by these equations uniquely and belong to the ideal J_B. The coordinates of $\bar{\jmath}^{loc}$ are unique under the assumption that $j_1, \dots, j_{u_0} \in \widetilde{J}_{B\otimes B} \subset J_{B\otimes B}$.

Remark. a) One can easily verify that the above system of equations when considered modulo any DP-ideal $\widetilde{I}$ of B such that $\widetilde{I} \subset \widetilde{J}_{B\otimes B}$ has a unique solution $\bar{\jmath} \bmod \widetilde{I}$ under the assumption that for $1 \leqslant i \leqslant u_0$, all $j_i \in \widetilde{J}_{B\otimes B}$.

b) The above description of the coalgebra B is related to a very special choice of S-bases in N^0 and N^1. This choice is sufficient for the formal construction of the algebras $\mathcal{A}(\mathcal{M})$ in Subsection 1.3. But when proving the full faithfulness of $\mathcal{G}_O$ below in Section 3 we need a choice of appropriate bases which is compatible with extensions in MF^e_S. Such choice is possible under more general assumptions from Subsection 1.3.

1.9.2. *Recovering $\mathcal{N}$.* In the above construction of the O-algebra B, any $a \in I_B$ can be uniquely written as

$$a = \sum_{\underline{i}} o_{\underline{i}} X^{\underline{i}} = \sum_{\underline{i}} o_{\underline{i}} X_1^{i_1} \dots X_u^{i_u} \tag{1.4}$$

where (by our general agreement from Introduction) $\underline{i} = (i_1, \dots, i_u)$ is a non-zero vector with the coordinates $i_1, \dots, i_u \in \{0, 1\}$, and all coefficients $o_{\underline{i}} = o_{\underline{i}}(a)$ belong to O. Similarly, any $a \in I_{B\otimes B}$ can be uniquely written as an O-linear combination of the elements $X^{\underline{i}_1} \otimes 1$, $1 \otimes X^{\underline{i}_2}$ and $X^{\underline{i}_1} \otimes X^{\underline{i}_2}$

Consider the following property of ideals $I \subset I_B$ in B (or with the obvious changes in $B \otimes B$, $\bar{B}$, $\bar{B} \otimes \bar{B}$, etc.).

$$\sum_{\underline{i}} o_{\underline{i}} X^{\underline{i}} \in I \;\Leftrightarrow\; \forall \underline{i}, \;\; o_{\underline{i}} X^{\underline{i}} \in I \tag{1.5}$$

Remark. Suppose I_1 and I_2 satisfy property (1.5). Then

a) $I_1 + I_2$ satisfies property (1.5);

b) for any monomial $o_{\underline{i}} X^{\underline{i}}$, $o_{\underline{i}} X^{\underline{i}} \in I_1 + I_2$ iff either $o_{\underline{i}} X^{\underline{i}} \in I_1$ or $o_{\underline{i}} X^{\underline{i}} \in I_2$.

Proposition 1.13. *The ideals J_B and $\widetilde{J}_B$ satisfy property* (1.5).

Proof. Any element of J_B is a sum of "elementary" elements of the form $o_{\underline{i}_1}X^{\underline{i}_1}o_{\underline{i}_2}X^{\underline{i}_2}$ and $\pi^e o_{\underline{i}_1}X^{\underline{i}_1}$, where $o_{\underline{i}_1}X^{\underline{i}_1}, o_{\underline{i}_2}X^{\underline{i}_2} \in I_B(2)$. It will be sufficient to verify property (1.5) for such elementary elements.

If $a = \pi^e o_{\underline{i}_1}X^{\underline{i}_1}$ there is nothing to prove.

Suppose $a = o_{\underline{i}_1}X^{\underline{i}_1}o_{\underline{i}_2}X^{\underline{i}_2}$ and $\underline{i}_1 = (i_{11}, \dots, i_{u1})$, $\underline{i}_2 = (i_{12}, \dots, i_{u2})$. Use induction on the number $\nu = \nu(a)$ of $1 \leqslant j \leqslant u$ such that $i_{j1} = i_{j2} = 1$.

If $\nu(a) = 0$ then $\underline{i}_1 + \underline{i}_2 = \underline{i}$ and there is nothing to prove. If $\nu(a) \geqslant 1$ and, say, $i_{j1} = i_{j2} = 1$ use the identity $X_j^2 = \eta'_j \cdot \eta'_j \sum_s X_s c_{sj}$ to rewrite a as a sum of elements with smaller ν's and, perhaps, elements of the form $\pi^e I_B(2)$.

The case of the ideal $\widetilde{J}_B$ can be considered similarly. □

Proposition 1.14. *If* $\iota_{\mathcal{N}}(\mathcal{N}) = (\widetilde{N}^0, \widetilde{N}^1, \varphi_1)$ *then*

$$\widetilde{N}^0 = \{a \bmod J_B \mid a \in I_B, \delta^+(a) \in J_{B\otimes B}\}$$

Proof. Suppose $a = \sum_{\underline{i}} o_{\underline{i}}X^{\underline{i}} \in I_B$ and $\delta^+ a \in J_{B\otimes B}$. Note that

$$\delta^+(a) \equiv \sum_{\underline{i}_1+\underline{i}_2=\underline{i}} o_{\underline{i}}X^{\underline{i}_1} \otimes X^{\underline{i}_2} \bmod J_{B\otimes B}$$

Then Proposition 1.13 implies that all $o_{\underline{i}}X^{\underline{i}_1} \otimes X^{\underline{i}_2} \in J_{B\otimes B}$ and, therefore, all $o_{\underline{i}}X^{\underline{i}_1} \cdot X^{\underline{i}_2} \in J_B$. This means that all non-linear terms amongst $o_{\underline{i}}X^{\underline{i}}$ (i.e. the terms with $r(\underline{i}) = i_1 + \dots + i_u \geqslant 2$) belong to J_B. □

Using that the ideals J_B and $J_{B\otimes B}$ depend functorially on the group scheme $\mathcal{G}_O(\mathcal{N})$ (i.e. do not depend on a choice of the special construction in Subsection 1.9) we obtain the following property.

Corollary 1.15. *The functor* $\mathcal{G}_O$ *is fully faithful.*

Proof. Suppose $G_1 = \mathcal{G}_O(\mathcal{M}_1)$, $G_2 = \mathcal{G}_O(\mathcal{M}_2)$, $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{MF}_S^e$, $g \in \mathrm{Hom}_{\mathrm{Gr}_O}(G_1, G_2)$ and $A(g) : A_2 \longrightarrow A_1$ is the corresponding morphism of O-algebras. Then $\iota(A(g)) \in \mathrm{Hom}_{\mathcal{MF}_S}(\iota(A_2), \iota(A_1))$ maps $\iota_{\mathcal{M}_2}(\mathcal{M}_2)$ to $\iota_{\mathcal{M}_1}(\mathcal{M}_1)$ (use Proposition 1.14) and by Proposition 1.1 can be lifted uniquely $F \in \mathrm{Hom}_{\mathrm{MF}_S^e}(\mathcal{M}_2, \mathcal{M}_1)$. Clearly, $\mathcal{G}_O(F) = g$. □

In Subsection 3 we need the following version of Proposition 1.14.

Let $\bar{B} = B \otimes_O \bar{O}$. Denote by $\bar{\mathcal{J}}$ the ideal in $\bar{B}$ generated by $\tilde{\eta}_i X_i \otimes X_i$, $1 \leqslant i \leqslant u_0$, and all elements of $2I^{loc}_{\bar{B}\otimes\bar{B}}$. One can easily prove (use relation (1.3)) that all $j_1, \dots, j_{u_0}$ belong to $\bar{\mathcal{J}}$ and $\pi j_{u_0+1}, \dots, \pi j_u \in 2\pi I_{B\otimes B} \subset \bar{\mathcal{J}}$.

Proposition 1.16. *The ideal* $\bar{\mathcal{J}}$ *consists of all* $\bar{O}$*-linear combinations of monomials in* $\bar{B} \otimes \bar{B}$ *which either belong to* $2I^{loc}_{\bar{B}\otimes\bar{B}}$ *or are divisible by one of* $\tilde{\eta}_i X_i \otimes X_i$*, where* $1 \leqslant i \leqslant u_0$.

Proof. Suppose an element $a \in \bar{\mathcal{J}}$ is an $\bar{O}$-linear combination of the form

$$a = \sum_{\underline{i}} o'_{\underline{i}}(X^{\underline{i}} \otimes 1) + \sum_{\underline{i}} o''_{\underline{i}}(1 \otimes X^{\underline{i}}) + \sum_{\underline{i}_1,\underline{i}_2} o_{\underline{i}_1\underline{i}_2} X^{\underline{i}_1} \otimes X^{\underline{i}_2} \tag{1.6}$$

Let $M(i,b) = \tilde{\eta}_i(X_i \otimes X_i)b$, where $1 \leqslant i \leqslant u_0$ and b is a monomial from $\bar{B} \otimes \bar{B}$.

Clearly, any $M(i,b)$ can't contribute to the coefficients of the first two sums in (1.6). Therefore, their summands belong to $2I^{loc}_{\bar{B}\otimes\bar{B}}$.

Suppose $o_{\underline{i}_1\underline{i}_2} X^{\underline{i}_1} \otimes X^{\underline{i}_2} \notin 2I^{loc}_{\bar{B}\otimes\bar{B}}$ and satisfies the following condition (The proposition is proved if there are no such monomials.):

if $X^{\underline{i}_1} \otimes X^{\underline{i}_2}$ is divisible by $X_i \otimes X_i$, $1 \leqslant i \leqslant u_0$, then $o_{\underline{i}_1\underline{i}_2}$ is not divisible by $\tilde{\eta}_i$.

Then there is $M(i_0,b)$ with $1 \leqslant i_0 \leqslant u_0$, which contributes to the coefficient for $X^{\underline{i}_1} \otimes X^{\underline{i}_2}$ and this contribution divides $o_{\underline{i}_1\underline{i}_2}$.

If b is not divisible by either $X_{i_0} \otimes 1$ or $1 \otimes X_{i_0}$ then $M(i_0,b) = oX^{\underline{i}_1} \otimes X^{\underline{i}_2}$ and $o \in \bar{O}$ is divisible by $\tilde{\eta}_{i_0}$. The contradiction. But otherwise, $M(i_0,b) \in 2I^{loc}_B$ because $\tilde{\eta}_{i_0} X^2_{i_0} \in 2I^{loc}_B$. The proposition is proved. □

Corollary 1.17. *The ideal $\bar{\mathcal{J}}$ satisfies condition* (1.5).

2. Functor $\mathcal{G}^O_{O_0}$

In this section we prove that any $G \in \mathrm{Gr}_O$ from the image of $\mathcal{G}_O$ has a canonical descent to O_0, $G_0 \in \mathrm{Gr}_{O_0}$. Therefore, the fully faithful functor $\mathcal{G}_O$ appears as the composition of the fully faithful functor $\mathcal{G}^O_{O_0} : \mathrm{MF}^{e*}_S \longrightarrow \mathrm{Gr}^*_{O_0}$ and the extension of scalars $\mathrm{Gr}^*_{O_0} \longrightarrow \mathrm{Gr}^*_O$.

2.1. Uniqueness of descent to O_0.

Proposition 2.1. a) *Suppose $G = \mathrm{Spec}A \in \mathrm{Im}\,\mathcal{G}_O$, $e : A \longrightarrow O$ is the counit and $I_A = \mathrm{Ker}\, e$. If there is an $(A_0, I_{A_0}) \in \mathrm{Aug}_{O_0}$ such that $(A_0, I_{A_0}) \otimes_{O_0} O = (A, I_A)$ then $G_0 = \mathrm{Spec}A_0$ has a natural structure of object of the category Gr_{O_0} such that $G_0 \otimes_{O_0} O = G$.*

b) *Suppose $G_0, H_0 \in \mathrm{Gr}_{O_0}$ and $G = G_0 \otimes_{O_0} O$ and $H = H_0 \otimes_{O_0} O$ are in the image of $\mathcal{G}_O$. Then the natural map $f \mapsto f \otimes_{O_0} O$ induces identification $\mathrm{Hom}_{\mathrm{Gr}_{O_0}}(G_0, H_0) = \mathrm{Hom}_{\mathrm{Gr}_O}(G, H)$.*

Proof. It will be sufficient to prove that $\Delta(I_{A_0}) \subset I_{A_0 \otimes A_0}$, where Δ is the coaddition on A.

Let $\mathrm{Gal}(K/K_0) = \{\mathrm{id}, \tau\}$ and $\Delta^{(\tau)} = (\tau \otimes \tau) \circ \Delta \circ \tau$ is the conjugate to Δ. In other words, if $b_1, \ldots, b_u$ is an O_0-basis of I_{A_0} and for $1 \leq i \leq u$, $\Delta(b_i) = b_i \otimes 1 + 1 \otimes b_i + \sum_{k,l} \alpha_{kl} b_k \otimes b_l$ with all $\alpha_{kl} \in O$, then $\Delta^{(\tau)}(b_i) = b_i \otimes 1 + 1 \otimes b_i + \sum_{k,l} \tau(\alpha_{kl}) b_k \otimes b_l$. Using that all $\tau(\alpha_{kl}) \equiv \alpha_{kl} \bmod 2\pi O$, we conclude that for any $a \in I_A$, $\Delta^{(\tau)}(a) \equiv \Delta(a) \bmod \widetilde{J}_{A \otimes A}$. Therefore,

by results of Subsection 1.7, $\Delta = \Delta^{(\tau)}$ and all $\alpha_{kl} \in O_0$. Part a) is proved.

Part b) follows by similar arguments. $\square$

2.2. **Existence of descent to O_0.**

Proposition 2.2. *Suppose $G = \mathrm{Spec}A \in \mathrm{Im}\ \mathcal{G}_O$ and $I_A = \mathrm{Ker}\, e$, where $e : A \longrightarrow O$ is the counit. Then there is an $(A_0, I_{A_0}) \in \mathrm{Aug}_{O_0}$ such that $I_A = I_{A_0} \otimes_{O_0} O$.*

Proof. Use induction on the order $|G|$ of G.

2.2.1. *The case $|G| = 2$.* Here $A = O[X]$, where $X^2 = \eta cX$ with $\eta \in O_0$, $\eta | 2$ and $c \in O^*$. Clearly, we can take $A_0 = O_0[c^{-1}X]$.

2.2.2. *Tame descent.* Suppose K_0' is a tamely ramified extension of K_0 of degree e_0. Let π_0' be a uniformising element of K_0' such that $\pi_0'^{e_0} = \pi_0$. Let $K' = K_0'(\pi')$, where $\pi'^2 = \pi_0'$. We can assume that $\pi'^{e_0} = \pi$. The field extensions K_0'/K_0 and K'/K are Galois, their Galois groups are cyclic of order e_0 and can be naturally identified. Denote by O_0' and O' the valuation rings of K_0' and, resp., K'.

Lemma 2.3. *Let $G \in \mathrm{Gr}_O$ and $G' = G \otimes_O O' \in \mathrm{Gr}_{O'}$. Then:*
a) *G is in the image of $\mathcal{G}_O$ if and only if G' is in the image of $\mathcal{G}_{O'}$;*
b) *G admits a descent to O_0 if and only if G' admits a descent to O_0'.*

Proof. The proof is based on an application of the criterion of tamely ramified descent. In the case of O'-algebras this criterion can be stated as follows:

• *Suppose A' is a flat O'-algebra and τ is a generator of $\mathrm{Gal}(K'/K)$. Then the existence of a flat O-algebra A such that $A' = A \otimes_O O'$ is equivalent to the existence of a τ-linear automorphism f of A' such that $f^{e_0} = \mathrm{id}_{A'}$ and $f \otimes_O k = \mathrm{id}_{A' \otimes k}$.*

Then one can state a similar criterion for objects of $\mathrm{MF}_{S'}^{e_0 e}$ where S' is a tamely ramified extension of S of degree e_0 and deduce part a) from the fact that tame descent data for $G' = \mathcal{G}_{O'}(\mathcal{M}')$ induce tame descent data for $\mathcal{M}'$. Similarly, part b) can be proved from the fact that tamely ramified descent data for G' induce tamely ramified descent data for G_0' (use the uniqueness of G_0' given by Proposition 2.1). Cf. for more detailed explanation in [4], Proposition 4.3. $\square$

2.2.3. *Lubin-Tate formal group law.* Consider the formal Lubin-Tate group law with the logarithm

$$l_{LT}(X) = X + X^2/2 + \cdots + X^{2^n}/2^n + \cdots \in \mathbb{Q}_2[[X]]$$

This means that for any 2-adic ring R its topological nilradical $\mathrm{m}(R)$ is provided with the structure of abelian group via the operation $[f]+[g] = [P(f,g)]$, where $f, g \in \mathrm{m}(R)$ and for indeterminants X, Y,

$$P(X,Y) = l_{LT}^{-1}(l_{LT}(X) + l_{LT}(Y)) \in \mathbb{Z}_2[[X,Y]].$$

We have the following simple properties:

• $[f]+[g] = [P_0(f,g)]+\cdots+[P_n(f,g)]+\ldots$, where all $P_n \in \mathbb{Z}_2[X,Y]$ and $\deg P_n = 2^n$. In particular, $P_0 = X+Y$, $P_1 = -XY$, $P_2 = -XY(X+Y)^2$;

• $-[f] = [-f] + [-f^2] + \cdots + [-f^{2^n}] + \ldots$;

• $[2](f) = [f]+[f] \equiv [2f]+[f^2]+[-2f^2]+\cdots+[-2f^{2^n}]+\ldots \bmod 4\mathrm{m}(R)$.

2.2.4. *The case* $|G| > 2$. By replacing O by its suitable tamely ramified extension we can assume that $G = \mathcal{G}_O(\mathcal{M})$, where $\mathcal{M} = (M^0, M^1, \varphi_1) \in \mathrm{MF}_S^e$ is such that for $u \geqslant 1$:

• there is an S-basis $m^1, n_1^1, \ldots, n_u^1$ of M^1 and an S-basis $m, n_1, \ldots, n_u$ of M^0 such that $\varphi_1(m^1) = m$ and for $1 \leqslant i \leqslant u$, there are $\tilde{s}_i \in S$, $\tilde{s}_i | t^e$ such that $n_i^1 = \tilde{s}_i n_i$, $\varphi_1(n_1^1) = \sum_j n_j u_{ji}$, where $(u_{ij}) \in \mathrm{GL}_u(S)$;

• there is an $\tilde{s} \in S$, $\tilde{s} | t^e$ such that $m^1 = \tilde{s}m + \sum_i \alpha_i n_i$, where for $1 \leqslant i \leqslant u$, the coefficients $\alpha_i \in S$ and $t^e \tilde{s}^{-1}\alpha_i \equiv 0 \bmod \tilde{s}_i$.

The above conditions simply mean that $\mathcal{N} = (N^0, N^1, \varphi_1) \in \mathrm{MF}_S^e$, where $N^0 = \sum_i Sn_i$, $N^1 = \sum_i Sn_i^1$ and there is a short exact sequence in MF_S^e

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}_{\tilde{s}} \longrightarrow 0$$

where $\mathcal{M}_{\tilde{s}} = (S\tilde{m}, S\tilde{m}^1, \varphi_1)$ with $\tilde{m}^1 = \tilde{s}\tilde{m}$ and $\varphi_1(\tilde{m}^1) = \tilde{m}$. We shall always assume that the above data for the structure of $\mathcal{N} \in \mathrm{MF}_S^e$ satisfy assumptions $C1$ and $C2$ from Subsection 1.4.

Note that in the above description of $\mathcal{M} \in \mathrm{Ext}_{\mathrm{MF}_S^e}(\mathcal{M}_{\tilde{s}}, \mathcal{N})$ we can replace m by $m' = m + v$ and m^1 by $m^{1\prime} = m^1 + v^1$, where $v \in N^0$, $v^1 \in N^1$ and $\varphi_1(v^1) = v$. Then $m^{1\prime} = \tilde{s}m' + \sum_i \alpha_i' n_i$, where

$$\sum_i \alpha_i' n_i = \sum_i \alpha_i n_i + v^1 - \tilde{s}\varphi_1(v^1).$$

In particular, if $\tilde{s} \in S^*$ we can always assume that

$$\sum_i \alpha_i n_i \in N^1 + tN^0 \tag{2.1}$$

In terms of the corresponding O-algebras we have:

• $A = A(G)$ contains the O-algebra $B = O[X_1, \ldots, X_u]$, where $X_i^2 = \eta_i \sum_j X_j c_{ji}$ with $\eta_i = -2/\tilde{\eta}_i'^2 \in O_0$ and $\tilde{\eta}_i' \bmod 2 = \kappa_{SO}(\tilde{s}_i \bmod t^{2e})$, $1 \leqslant i \leqslant u$, and $C = (c_{ij}) \in \mathrm{GL}_u(O)$;

• $A = B[Y]$, where $(\tilde{\eta}'Y + \sum_i r_i X_i)^2 + 2Y = 0$ or, equivalently,

$$\left(Y + \tilde{\eta}'^{-1} \sum_i r_i X_i\right)^2 - \eta Y = 0 \tag{2.2}$$

with $\tilde{\eta}' \bmod 2 = \kappa_{SO}(\tilde{s} \bmod t^{2e})$, $r_i \bmod 2 = \kappa_{SO}(\alpha_i \bmod t^{2e})$ and

$$\eta r_i^2 \equiv 0 \bmod \tilde{\eta}_i \text{ (or, equivalently, } \eta_i r_i^2 \equiv 0 \bmod \tilde{\eta}) \tag{2.3}$$

for all $1 \leqslant i \leqslant u$ (this follows from the above congruences $t^e \tilde{s}^{-1} \alpha_i \equiv 0 \bmod \tilde{s}_i$). Recall that as usually $\eta\tilde{\eta} = \eta_i\tilde{\eta}_i = -2$ for all i.

Let $h = \sum_i r_i X_i \in I_B$. Then (2.3) implies that $h^2 \in \tilde{\eta} I_B$. In particular, if $\tilde{\eta} \notin O_0^*$ then $h \in I_B^{loc}$. If $\tilde{\eta} \in O_0^*$ then (2.1) implies that again $h \in I_B^{loc}$. Therefore, $\tilde{\eta}'Y \in I_A^{loc}$.

By inductive assumption, there is an augmented flat O_0-algebra (B_0, I_{B_0}) such that $I_B = I_{B_0} \otimes_{O_0} O$. Therefore, $h = b_0 + \pi b_1$, where $b_0 \in I_{B_0}^{loc}$ and $b_1 \in I_{B_0}$.

From now on we use the Lubin-Tate group law. Clearly, there is $Y' \in I_A$ such that $[\tilde{\eta}Y'] = [\tilde{\eta}Y + \tilde{\eta}'h] - [\tilde{\eta}'b_0] - [\tilde{\eta}'\pi b_1]$. If $\tilde{\eta} \notin O_0^*$ then $Y' \equiv Y \bmod \pi I_A$ and if $\tilde{\eta} \in O_0^*$ then $Y' \equiv Y \bmod (Y I_A^{loc} + \pi I_B^{loc})$. Therefore, $A = B[Y']$.

The equation for Y' can be found as follows. From (2.2) we obtain that $(\tilde{\eta}Y + \tilde{\eta}'h)^2 = -2\tilde{\eta}Y$. Then using the properties of the Lubin-Tate group law from Subsection 2.2.3 we obtain:

$$[2](\tilde{\eta}Y') = [2(\tilde{\eta}Y + \tilde{\eta}'h)] + [-2\tilde{\eta}Y] - ([2](\tilde{\eta}'b_0) + [2](\tilde{\eta}'\pi b_1)) \equiv$$

$$-[\tilde{\eta}b_0^2] - [\tilde{\eta}\pi_0 b_1^2] - \sum_{n \geqslant 0} \left[-2(\tilde{\eta}b_0^2)^{2^n}\right] - \sum_{n \geqslant 0} \left[-2(\tilde{\eta}\pi_0 b_1^2)^{2^n}\right] \bmod 4(\tilde{\eta}I_A)^{loc}$$

Here $(\tilde{\eta}I_A)^{loc}$ coincides with $\tilde{\eta}I_A$ if $\tilde{\eta} \notin O_0^*$ and with I_A^{loc}, otherwise.

Notice that $h^2 \in \tilde{\eta}I_B$ and, therefore, the right hand side of the above congruence equals $\tilde{\eta}^2 b^* \in (\tilde{\eta}^2 I_{B_0})^{loc}$. Using that $[2](2(\tilde{\eta}I_A)^{loc}) = 4(\tilde{\eta}I_A)^{loc}$ we can replace Y' by $Y_1 = Y' - 2a$ with a suitable $a \in I_A$ such that $[2](\tilde{\eta}Y_1) = \tilde{\eta}^2 b^*$.

Finally, if $1 + \tilde{\eta}Y_2 = \exp(l_{LT}(\tilde{\eta}Y_1))$ then we still have $A = B[Y_2]$ and $(1 + \tilde{\eta}Y_2)^2 = \exp(l_{LT}(\tilde{\eta}^2 b^*))$ implies that $Y_2^2 - \eta Y_2 = b_1^*$ with $b_1^* \in B_0$. So, for $A_0 = B_0[Y_2]$, $A_0 \otimes_{O_0} O = A$. □

3. Surjectivity of $\mathcal{G}_{O_0}^O$

In order to establish that $\mathcal{G}_{O_0}^O$ is antiequivalence of the categories MF_S^{e*} and $\mathrm{Gr}_{O_0}^*$ it remains only to prove the following result.

Theorem 3.1. *If $G_0 \in \mathrm{Gr}_{O_0}$ then there is $\mathcal{M} \in \mathrm{MF}_S^e$ such that $\mathcal{G}_O(\mathcal{M}) \simeq G_0 \otimes_{O_0} O$.*

The proof will be given in Subsections 3.1-3.11. It uses induction on the order $|G_0|$ of G_0.

3.1. **The case $|G_0| = 2$.** If $|G_0| = 2$ then there is $\eta \in O_0$, $\eta|2$, such that $G_0 = \mu_\eta$, where $\mu_\eta = \mathrm{Spec}\, O_0[X]$, $X^2 = \eta X$, $e(X) = 0$ and $\Delta(X) = X \otimes 1 + 1 \otimes X + \tilde{\eta} X \otimes X$ with $\eta\tilde{\eta} = -2$. We can assume that $\tilde{\eta} = \pi_0^r$, $0 \leqslant r \leqslant e$ (because $\mu_\eta \simeq \mu_{\eta'}$ iff $\eta^{-1}\eta' \in O_0^*$). Then for $\mathcal{M} = (Sm, St^r m, \varphi_1) \in \mathrm{MF}_S^e$ with $\varphi_1(t^r m) = m$, one has $\mathcal{G}_O(\mathcal{M}) = \mu_\eta \otimes_{O_0} O$.

3.2. **Basic strategy I.** When studying the case $|G_0| > 2$ we can replace O_0 by its tamely ramified extension, cf. Subsection 2.2.2. In particular, we can assume that in Gr_{O_0} there is a short exact sequence

$$0 \longrightarrow \mu_\eta \longrightarrow G_0 \longrightarrow H_0 \longrightarrow 0,$$

where $H_0 = \mathrm{Spec} B_0$, and $H = H_0 \otimes_{O_0} O = \mathcal{G}_O(\mathcal{N})$ with $\mathcal{N} \in \mathrm{MF}_S^e$.

Use the description of $B = B_0 \otimes_{O_0} O$ from Subsection 1.9.1.

Namely, $B = O[X_1, \dots, X_u]$ with the relations $X_i^2 = \eta_i \sum_j X_j c_{ji}$, where all $\eta_i \in O_0$, $\eta_i|2$, $C = (c_{ij}) \in \mathrm{GL}_u(O)$, $e(X_i) = 0$ and $j_i = \delta^+(X_i) = \Delta(X_i) - X_i \otimes 1 - 1 \otimes X_i \in J_{B\otimes B}$. In addition, if $\bar{X}^{loc} = (X_1, \dots, X_{u_0})$ then $j_1, \dots, j_{u_0} \in \widetilde{J}_{B\otimes B}$. For $1 \leqslant i \leqslant u$, the elements η_i are defined up to units in O_0 and we can assume that $\tilde{\eta}_i = \tilde{\eta}_i'^2 = -2/\eta_i$ with $\tilde{\eta}_i' \in O$. Our strategy is to use the explicit construction of $H = H_0 \otimes_{O_0} O$ from Subsection 1.9.1 to describe $G = G_0 \otimes_{O_0} O$ as an element of the group $\mathrm{Ext}_{\mathrm{Gr}_O}(H, \mu_\eta \otimes_{O_0} O)$. If $O' = O[\pi']$ with $\pi'^2 = \pi$, we shall prove then that $G' = G \otimes_O O'$ appears in the form $\mathcal{G}_{O'}(\mathcal{M}')$, where $\mathcal{M}' \in \mathrm{MF}_{S'}^{2e}$ with $S' = S[t']$, $t'^2 = t$.

Finally, we shall prove that the fact that G' admits a descent G_0 to O_0 implies that $\mathcal{M}'$ admits a descent to S, i.e. $\mathcal{M}' = \mathcal{M} \otimes_S S'$ with $\mathcal{M} \in \mathrm{MF}_S^e$ and, therefore, $G = \mathcal{G}_O(\mathcal{M})$.

3.3. **The group $\mathrm{Ext}_{\mathrm{Gr}_{O_0}}(H_0, \mu_\eta)$.** Let $B' = B \otimes_O O'$, $I_{B'} = I_B \otimes_O O'$. Let $\bar{O}$ be the valution ring of an algebraic closure of $K' = K(\pi')$, $\bar{B} = B' \otimes_{O'} \bar{O}$ and $I_{\bar{B}} = I_{B'} \otimes_{O'} \bar{O}$.

Introduce the (multiplicative) group $\mathcal{H} = \mathcal{H}(H_0, \mu_\eta)$ of all elements $f \in (1 + \tilde{\eta} I_{\bar{B}})^\times$ such that $f^2 \in 1 + \tilde{\eta}^2 I_{B_0}$ and $\delta^\times(f) := \Delta(f)(f \otimes f)^{-1} \in 1 + \tilde{\eta} I_{B_0 \otimes B_0}$.

Then there is a group epimorphism $\Theta : \mathcal{H} \longrightarrow \mathrm{Ext}_{\mathrm{Gr}_{O_0}}(H_0, \mu_\eta)$ attaching to $f \in \mathcal{H}$, the group scheme $\Theta(f) = \mathrm{Spec} A_0 \in \mathrm{Gr}_{O_0}$ such that:

- $A_0 = B_0[X]$ where $(1 + \tilde{\eta} X)^2 = f^2$;
- $e(X) = 0$ and $\Delta(1 + \tilde{\eta} X) = (1 + \tilde{\eta} X) \otimes (1 + \tilde{\eta} X) \cdot \delta^\times f$;
- $\mathrm{Ker}\Theta = (1 + \tilde{\eta} I_{B_0})^\times$.

This result was proved in a very detailed way in Section 5 of [4] in a more general context of p-group schemes, where p is any prime number. Note that the case $p = 2$ is slightly easier to obtain.

If $\tilde{\eta} \notin O_0^*$ (i.e. if μ_η is not multiplicative) then obviously $f \in 1 + I_{\bar{B}}^{loc}$. If $\tilde{\eta} \in O_0^*$ then we can always multiply f by a suitable element from $(1 + I_{B^{et}})^\times \subset (1 + I_B)^\times$ to assume again that $f \in 1 + I_{\bar{B}}^{loc}$. This allows us to replace in the above description of $\mathrm{Ext}_{\mathrm{Gr}_{O_0}}(H_0, \mu_\eta)$, the multiplicative group $(1 + I_{\bar{B}})^\times$ by $\mathrm{m}(I_{\bar{B}}) = I_{\bar{B}}^{loc}$ with the Lubin-Tate group law from Subsection 2.2.3.

More precisely, let $\delta_{LT} : I_{\bar{B}}^{loc} \longrightarrow I_{\bar{B}\otimes\bar{B}}^{loc}$ and $[2] : I_{\bar{B}}^{loc} \longrightarrow I_{\bar{B}}^{loc}$ be such that for any $f \in I_{\bar{B}}^{loc}$,

— $\delta_{LT}(f) = [\Delta(f)] - [f \otimes 1] - [1 \otimes f]$;

— $[2](f) = [f] + [f]$.

Let $\mathcal{H}_{LT} = \mathcal{H}_{LT}(H_0, \mu_\eta)$ be the subgroup of $I_{\bar{B}}^{loc}$ (with respect to the Lubin-Tate group law) consisting of $f \in I_{\bar{B}}^{loc}$ such that $[2](f) \in \tilde{\eta}^2 I_{B_0}$ and $\delta_{LT}(f) \in \tilde{\eta} I_{B_0 \otimes B_0}$. Let $\Theta_{LT} : \mathcal{H}_{LT} \longrightarrow \mathrm{Ext}_{\mathrm{Gr}_{O_0}}(H_0, \mu_\eta)$ be such that for all $f \in \mathcal{H}_{LT}$, $\Theta_{LT}(f) = \Theta(E(f))$, where $E(X) = \exp(l_{LT}(X))$ is the Artin-Hasse exponential. Our description of $\mathrm{Ext}_{\mathrm{Gr}_{O_0}}(H_0, \mu_\eta)$ will be used below in the following form.

Proposition 3.2. *Θ_{LT} is a group epimorphism and its kernel equals $\mathcal{H}_{LT} \cap (\tilde{\eta} I_{B_0}) = (\tilde{\eta} I_{B_0})^{loc}$.*

3.4. **Main Lemma.** In next subsections we work systematically with the Lubin-Tate group law from Subsection 2.2.3. We always bear in mind the following agreement: if, say, $a \in I_B$ appears in the form $[a]$ then a is assumed automatically to be an element of I_B^{loc}, that is the corresponding result of the Lubin-Tate addition is automatically well-defined.

Recall that we use the multi-indices $\underline{i} = (i_1, \dots, i_u)$, where all coordinates of the vector $\underline{i}$ belong to $\{0, 1\}$. We shall use such indices for the abbreviation $X^{\underline{i}} := X_1^{i_1} \dots X_u^{i_u}$, especially, when $r(\underline{i}) := i_1 + \dots + i_u \geqslant 2$. If $r(\underline{i}) = 1$ then the multi-index $\underline{i}$ appears just as index j, $1 \leqslant j \leqslant u$, such that $\underline{i} = (\delta_{1j}, \dots, \delta_{uj})$ (where δ is the Kronecker symbol).

The following statement is very similar to the statement appeared in Subsection 6.1 of [4] as Main Lemma.

Lemma 3.3 (Main Lemma). *If $f \in \mathcal{H}_{LT} = \mathcal{H}_{LT}(H_0, \mu_\eta)$ then there are:*

— *$f_0 \in \tilde{\eta} I_B$;*

— *for $1 \leqslant i \leqslant u$, $o'_{i0} \in O$ and $o'_{i\underline{1}} \in \pi' O$;*

— *for all multi-indices $\underline{i}$, $D_{\underline{i}} \in O$,*

such that $o'^2_{i0}, o'^2_{i1} \in \tilde{\eta}O$, $D_iX_i \in I_{\bar{B}}(2)^{loc}$, $D_{\underline{i}}X^{\underline{i}} \in I_{\bar{B}}^{loc}$ *and*

$$f = [f_0] + \sum_{1 \leqslant i \leqslant u} ([o'_{i0}X_i] + [o'_{i1}X_i] + [D_iX_i]) + \left[2 \sum_{r(\underline{i}) \geqslant 2} D_{\underline{i}}X^{\underline{i}}\right]$$

Remark. If $\tilde{\eta} \in O_0^*$ we can always assume that all $o'_{i0} = 0$, because all $o'_{i0}X_i$ can disappear by contributing to f_0. In other words, we assume that $o'^2_{i0}, o'^2_{i1} \in (\tilde{\eta}O)^{loc}$, i.e. these elements belong to $\tilde{\eta}O$ if $1 \leqslant i \leqslant u_0$ and belong to the maximal ideal m of O, otherwise.

The proof of this Lemma uses the auxiliary statements from Subsection 3.6 and will be given in Subsection 3.7 below. This is a simplified version of the proof of Main Lemma in [4], where we studied the group schemes of period $p > 2$. As a matter of fact, this simplified version works equally well also in the case $p > 2$.

3.5. **Basic strategy II.** Via Main Lemma we shall prove below the existence of $\mathcal{M}' \in \mathrm{MF}^{2e}_{S'}$ such that $G' \simeq \mathcal{G}_{O'}(\mathcal{M}')$. The description of G_0 as an element of $\mathrm{Ext}_{\mathrm{Gr}_{O_0}}(H_0, \mu_\eta)$ from Subsection 3.3 is given in terms of Kummer's theory and, therefore, is of multiplicative nature. On the other hand, the construction of the algebra of $\mathcal{G}_{O'}(\mathcal{M}')$ as extension of $B' = B \otimes_O O'$ should be given (by the definition of $\mathcal{G}_{O'}$) in additive terms. Therefore, the description of $A(G_0)$ as an extension of $B_0 = A(H_0)$ in terms of the Lubin-Tate group is a natural step towards presentation of $G \otimes_O O'$ in the form $\mathcal{G}_{O'}(\mathcal{M}')$.

If G_0 is given via $f \in \mathcal{H}_{LT}(H_0, \mu_\eta)$ then $A_0 = B_0[Y]$ with equation for Y coming from the relation $2l_{LT}(\tilde{\eta}Y) = l_{LT}(f)$ in $A_0 \otimes_{O_0} K_0$. If μ_η is multiplicative, e.g. $\tilde{\eta} = 1$, then it is much easier to obtain an "integral" version of the above relation. The left-hand side $2l_{LT}(Y) = 2Y + Y^2 + 2(Y^2/2)^2 + \dots$ looks nicely related to operations in filtered modules of the form $\iota_{\mathcal{M}}(\mathcal{M})$. As for the right-hand side, we need it to belong to $\iota_{\mathcal{N}}(\mathcal{N})$, i.e. to be congruent to a linear combination of all X_i. By Main Lemma after replacing f by $\tilde{f} = [f] - [f_0]$, $l_{LT}(\tilde{f})$ is a linear combination of all $l_{LT}(o'_{i0}X_i)$, $l_{LT}(o'_{i1}X_i)$ and $l_{LT}(D_iX_i)$ modulo $2I_{\bar{B}}$. First two logarithms can give non-trivial denominators but in $A' = A \otimes_O O'$ we can consider the element $[g] = [\tilde{f}] - \sum_i([o'_{i0}X_i] + [o'_{i1}X_i])$ and because all $D_iX_i \in I_{\bar{B}}(2)$, $l_{LT}(g)$ is an O'-linear combination of all $X_1, \dots, X_u$ modulo $J_{B'}$, i.e. gives already an element of $\iota_{\mathcal{N}}(\mathcal{N}) \otimes_O O'$.

If μ_η is not multiplicative the calculations should be more precise because of the extra factor $\tilde{\eta}$. (Here we can see a crucial difference with the case $p > 2$, where ideals of the form $(p/\tilde{\eta})I_B$ are still DP-ideals.) In particular, we can't ignore the quadratic forms in X_i coming from the third term $(D_iX_i)^4/4$ of the expansion of $l_{LT}(D_iX_i)$ and the second term g^2 in $l_{LT}(2g)$, where $g = \sum_{\underline{i}} D_{\underline{i}}X^{\underline{i}}$. The elaboration of Main Lemma from Subsection 3.8 relates these quadratic forms and allows

us to prove in Subsection 3.9 that they, as a matter of fact, kill one-another. This provides us in Subsection 3.11 with explicit construction of $\mathcal{M}' \in \mathrm{MF}^{2e}_{S'}$ and the existence of $\mathcal{M} \in \mathrm{MF}^{e}_{S}$ such that $\mathcal{M}' = \mathcal{M} \otimes_S S'$. A formal verification that $G \simeq \mathcal{G}_O(\mathcal{M})$ is done in Subsection 3.12.

3.6. **Auxiliary statements.** Follow Subsection 3.3 of [4] to introduce the ideals $I_B(\alpha)$ and $I_B(\alpha)^{loc}$ in B, where $\alpha \in O$. Recall that any $a \in I_B$ can be uniquely written as $a = \sum_{\underline{i}} o_{\underline{i}}(a) X^{\underline{i}}$ with the coefficients $o_{\underline{i}} = o_{\underline{i}}(a) \in O$.

Definition. For any $\alpha \in O$, set

a) $I_B(\alpha) := \{a \in I_B \mid \text{all } (o_{\underline{i}}(a)X^{\underline{i}})^2 \in \alpha I_B\}$

b) $I_B(\alpha)^{loc} := \{a \in I_B \mid \text{all } (o_{\underline{i}}(a)X^{\underline{i}})^2 \in \alpha I_B^{loc}\}$.

Note (use property (**C**) from Subsection 1.9.1), that for any $1 \leqslant i \leqslant u_0$, $X_i \in I_B(\eta_i)^{loc}$ but for $u_0 < i \leqslant u$, $X_i \notin I_B(\eta_i)^{loc} = I_B^{loc}$. In addition, for arbitrary $\alpha \in O$, the O-modules $I_B(\alpha)$ and $I_B(\alpha)^{loc}$ depend generally on the above chosen special construction of B. Nevertheless, one can verify that:

- for any $\alpha \in O$, $I_B(\alpha)$ and $I_B(\alpha)^{loc}$ are ideals in B;

- for $\alpha|2$, we have $I_B(\alpha) = \{a \in I_B \mid a^2 \in \alpha I_B\}$ and, similarly, $I_B(\alpha)^{loc} = \{a \in I_B \mid a^2 \in \alpha I_B^{loc}\}$.

For obvious reasons, the above ideals $I = I_B(\alpha)$ and $I = I_B(\alpha)^{loc}$ satisfy property (1.5) from Subsection 1.9.2.

Remark. a) For any $\alpha \in \bar{O}$ we shall denote below by $I_{\bar{B}}(\alpha)$ and $I_{\bar{B}}(\alpha)^{loc}$ the similar ideals of the $\bar{O}$-algebra $\bar{B} = B \otimes_O \bar{O}$. Clearly, they also satisfy property (1.5).

b) Using the special basis $\{X^{\underline{i}_1} \otimes 1, 1 \otimes X^{\underline{i}_2}, X^{\underline{i}_1} \otimes X^{\underline{i}_2} \mid \underline{i}_1, \underline{i}_2\}$ of $I_{B \otimes B}$ and $I_{\bar{B} \otimes \bar{B}}$ we can introduce similarly the ideals $I_{B \otimes B}(\alpha)$ and $I_{\bar{B} \otimes \bar{B}}(\alpha)$.

The following lemmas admit straightforward proofs and are quite analogous to the lemmas from [4] Subsection 6.2, Lemmas 6.2-6.6.

Lemma 3.4. *Suppose* $C_1, \dots, C_u \in \bar{O}$, $g \in I_{\bar{B}}^{loc}$, $\beta_0 \in \bar{\mathfrak{m}}$ *and* $g \equiv \sum_i [C_i X_i] \bmod (I_{\bar{B}}(\beta_0)^{loc} + \bar{\mathcal{I}})$, *where* $\bar{\mathcal{I}} \subset I_{\bar{B}}^{loc}$ *is an ideal satisfying condition* (1.5) *from Subsection 1.9.2. Then*

$$g \equiv \sum_{1 \leqslant i \leqslant u} [C'_i X_i] + \left[\sum_{r(\underline{i}) \geqslant 2} C'_{\underline{i}} X^{\underline{i}} \right] \bmod \bar{\mathcal{I}}$$

with all $C'_i, C'_{\underline{i}} \in \bar{O}$, $C_i X_i \equiv C'_i X_i \bmod I_{\bar{B}}(\beta_0)^{loc}$ *and* $C_{\underline{i}} X^{\underline{i}} \in I_{\bar{B}}(\beta_0)^{loc}$.

Lemma 3.5. *Suppose l equals either 0 or 1.*
a) *If $o'_1, o'_2 \in \pi'^l O$ are such that $o'^2_1, o'^2_2 \in \tilde{\eta}O$, then for any $a \in I_B$, $[o'_1 a] + [o'_2 a] - [(o'_1 + o'_2)a] \in \tilde{\eta} I_B$;*
b) *If $o' \in \pi'^l O$, $o'^2 \in \tilde{\eta}O$ and $a_1, a_2 \in I_B$ then $[o'a_1] + [o'a_2] - [o'(a_1 + a_2)] \in \tilde{\eta} I_B$.*

Remark. a) When proving Lemma 3.4 use first remark from Subsection 1.9.2; b) note the following special cases of above Lemma 3.5:
— $[o'a] + [-o'a] \in \tilde{\eta} I_B$;
— $[\delta_{LT}(o'X_i)] - [o'j_i] \in \tilde{\eta} I_{B\otimes B}$.

Lemma 3.6. *If $C \in \bar{O}$, $\alpha_1 \in \bar{\mathfrak{m}}$ and $CX_i \in I_{\bar{B}}(\alpha_1)^{loc}$ then*

$$\delta_{LT}(CX_i) \equiv C^2 X_i \otimes X_i \bmod I_{\bar{B}}(\alpha_1^4)^{loc} + \bar{\mathcal{J}},$$

where $\bar{\mathcal{J}}$ is the ideal defined in the end of Subsection 1.9.2.

Remark. Note that if $CX_i \in I_{\bar{B}}(2)^{loc}$ then $C^2 X_i \otimes X_i \in \bar{\mathcal{J}}$.

Proof. From the definition of the Lubin-Tate group law, cf. Subsection 2.2.3 it follows that

$$\begin{aligned}(3.1)\quad \delta_{LT}(CX_i) &= [Cj_i] - [-C^2(X_i \otimes 1 + 1 \otimes X_i)j_i] - [-C^2 X_i \otimes X_i] \\ &- \sum_{n\geqslant 2} [C^{2^n} P_n(X_i \otimes 1 + 1 \otimes X_i, j_i)] - \sum_{n\geqslant 2} [C^{2^n} P_n(X_i \otimes 1, 1 \otimes X_i)]\end{aligned}$$

We know that for $1 \leqslant i \leqslant u_0$, $j_i \in \bar{\mathcal{J}}$ and if $u_0 < i \leqslant u$ then $C \in \bar{\mathfrak{m}}$ and again $Cj_i \in \bar{\mathcal{J}}$. Therefore, the first, second and forth terms of the right-hand side of (3.1) belong to $\bar{\mathcal{J}}$. As for the last term of that formula it remains to note that $\left(I_{\bar{B}}(\alpha_1)^{loc}\right)^4 \subset I_{\bar{B}}(\alpha_1^4)^{loc}$. □

3.7. **Proof of Main Lemma.** Prove that for any $\alpha \in \bar{\mathfrak{m}}$, one has

$$(3.2)\quad f \equiv [f_\alpha] + \sum_{1\leqslant i\leqslant u} ([o'_{i0}X_i] + [o'_{i1}X_i] + [D_i X_i]) \bmod (2I^{loc}_{\bar{B}} + I_{\bar{B}}(\alpha^2)^{loc})$$

where
• $f_\alpha \in \tilde{\eta} I_B$;

• all $o'_{i0} = o'_{i0}(\alpha) \in O$, $o'_{i1} = o'_{i1}(\alpha) \in \pi' O$ are such that $o'^2_{i0}, o'^2_{i1} \in \tilde{\eta}O$; (If $\tilde{\eta} \in O_0^*$ then we can assume that all $o'_{i0} = 0$.)

• all $D_i \in \bar{O}$ are such that $D_i X_i \in I_{\bar{B}}(\alpha, 2)^{loc} := I_{\bar{B}}(\alpha)^{loc} + I_{\bar{B}}(2)^{loc}$.

First, there is an $\alpha_0 \in \mathfrak{m}$ such that (3.2) holds for trivial reasons with $\alpha = \alpha_0$. Indeed, if $\tilde{\eta} \notin O_0^*$ then use that $f^2 \equiv [2](f) \equiv 0 \bmod \tilde{\eta} I_{\bar{B}}$; otherwise, use that $[f] - [f_0] \in \bar{\mathfrak{m}} I_{\bar{B}}$, where $f_0 \in I_B$ and $f \equiv f_0 \bmod \bar{\mathfrak{m}} I_{\bar{B}}$.

By induction on α it will be sufficient to prove that if (3.2) holds with $\alpha = \alpha_1 \in \bar{\mathfrak{m}}$ then it also holds with $\alpha = \alpha_1^2$.

Apply Lemma 3.4 with $\bar{\mathcal{I}} = 2I_{\bar{B}}^{loc}$ to obtain

(3.3)

$$f = [f_{\alpha_1}] + \sum_{1 \leqslant i \leqslant u} ([o'_{i0}X_i] + [o'_{i1}X_i] + [D'_iX_i]) + \left[\sum_{r(\underline{i}) \geqslant 2} D'_{\underline{i}} X^{\underline{i}} \right] \bmod 2I_{\bar{B}}^{loc},$$

where all $D'_i, D'_{\underline{i}} \in \bar{O}$, $D'_iX_i \in I_{\bar{B}}(\alpha_1, 2)^{loc}$ and $D'_{\underline{i}}X^{\underline{i}} \in I_{\bar{B}}(\alpha_1^2)^{loc}$.

By Lemma 3.5 all $\delta_{LT}(o'_{i0}X_i), \delta_{LT}(o'_{i1}X_i) \in \tilde{\eta}I_{B\otimes B} + \bar{\mathcal{J}}$. Therefore, the condition $\delta_{LT}(f) \in \tilde{\eta}I_{B_0\otimes B_0} \subset \tilde{\eta}I_{B\otimes B}$ implies (use Lemma 3.6) that

$$\sum_{1 \leqslant i \leqslant u} D_i'^2 X_i \otimes X_i + \sum_{\underline{i}_1 + \underline{i}_2 = \underline{i}} D'_{\underline{i}} X^{\underline{i}_1} \otimes X^{\underline{i}_2} \in \tilde{\eta}I_{B\otimes B} \bmod (I_{\bar{B}\otimes\bar{B}}(\alpha_1^4)^{loc} + \bar{\mathcal{J}})$$

(Recall that all multi-indices $\underline{i}$, $\underline{i}_1$, $\underline{i}_2$ are non-zero vectors with coordinates 0 or 1.)

This implies the following properties (cf. first Remark in Subsection 1.9.2):

a) for $1 \leqslant i \leqslant u$, $D_i'^2 X_i \otimes X_i \equiv o_iX_i \otimes X_i \bmod (I_{\bar{B}\otimes\bar{B}}(\alpha_2^2)^{loc} + \bar{\mathcal{J}})$, where $o_i \in \tilde{\eta}O$;

b) if $r(\underline{i}) \geqslant 2$ then $D'_{\underline{i}} X^{\underline{i}_1} \otimes X^{\underline{i}_2} \equiv o_{\underline{i}} X^{\underline{i}_1} \otimes X^{\underline{i}_2} \bmod (I_{\bar{B}\otimes\bar{B}}(\alpha_2^2)^{loc} + \bar{\mathcal{J}})$, where $o_{\underline{i}} \in \tilde{\eta}O$.

Consider the morphism of multiplication $m : \bar{B} \otimes \bar{B} \longrightarrow \bar{B}$. Then $m(I_{\bar{B}\otimes\bar{B}}(\alpha_2^2)^{loc}) = I_{\bar{B}}(\alpha_2^2)^{loc}$, $m(\bar{\mathcal{J}}) = 2I_{\bar{B}}^{loc}$. Therefore, b) implies that $(D'_{\underline{i}} - o_{\underline{i}})X^{\underline{i}} \in I_{\bar{B}\otimes\bar{B}}(\alpha_2^2)^{loc} + 2I_{\bar{B}}^{loc}$ and the last summand in (3.3) disappears modulo $I_{\bar{B}}(\alpha_2^2)^{loc}$ by contributing to the corresponding $f_{\alpha_2} \in \tilde{\eta}I_B$. If $2|\alpha_1$ formula (3.2) has been already proved for $\alpha = \alpha_2$ because we don't need to change the terms D'_iX_i.

If $\alpha_1|2$ we should continue with property a). If $D'_iX_i \otimes X_i \in \bar{\mathcal{J}}$ keep D'_i the same. Otherwise, by first Remark from Subsection 1.9.2 we can assume that $D'_iX_i \otimes X_i \in I_{\bar{B}\otimes\bar{B}}(\alpha_2^2)^{loc}$. Consider the elements $o''_{i0} \in O$ and $o''_{i1} \in \pi'O$ such that $o_i \equiv o_{i0}''^2 + o_{i1}''^2 \bmod 2o_i\pi$ and $o_{i0}''^2, o_{i1}''^2 \equiv 0 \bmod o_i$.

Lemma 3.7. $[D'_iX_i] + [-o''_{i0}X_i] + [-o''_{i1}X_i] \equiv \sum_j [D'_{ij}X_j] \bmod I_{\bar{B}}(\alpha_2^2)^{loc}$, *where all* $D'_{ij} \in \bar{O}$ *and* $D'_{ij}X_j \in I_{\bar{B}}(\alpha_2^2)^{loc}$.

Proof. Suppose $1 \leqslant i \leqslant u_0$. In this case $X_i^2 \in I_B(\eta_i)^{loc}$ and property a) means that $(D_i'^2 - o_i)\eta_i \in \alpha_2\bar{O}$. Therefore (use that $\alpha_2|4$), $(D'_i - o''_{i0} - o''_{i1})^2\eta_i \in \alpha_2\bar{O}$, i.e. $(D'_i - o''_{i0} - o''_{i1})X_i \in I_{\bar{B}}(\alpha_2)^{loc}$. Also, $0 \equiv D_i'^2\eta_i \equiv o_i\eta_i \equiv o_{i0}''^2\eta_i \equiv o_{i1}''^2\eta_i \bmod \alpha_1\bar{O}$, i.e. $o''_{i0}X_i, o''_{i1}X_i \in I_{\bar{B}}(\alpha_1)^{loc}$.

Suppose $u_0 < i \leqslant u$. In this case $\eta_i \in O^*$, $D_i \in \bar{\mathfrak{m}}$ and $X_i, X_i^2 \in I_B^{et}$. The relation $(D_i'^2 - o_i)X_i \otimes X_i \in I_{\bar{B}}(\alpha_2^2)^{loc}$ means that $(D_i'^2 - o_i)\eta_i \in \alpha_2\bar{\mathfrak{m}}$, $D_i'^2\eta_i \equiv (o_{i0}''^2 + o_{i1}''^2)\eta_i \bmod \alpha_2\bar{\mathfrak{m}}$, $(D'_i - o''_{i0} - o''_{i1})^2\eta_i \in \alpha_2\bar{\mathfrak{m}}$ and again $(D_i - o''_{i0} - o''_{i1})X_i \in I_{\bar{B}}(\alpha_2^2)^{loc}$. Similarly, $D'_iX_i \in I_{\bar{B}}(\alpha_1)^{loc}$ means that $0 \equiv D_i'^2\eta_i \equiv o_i\eta_i \equiv o_{i0}''^2\eta_i \equiv o_{i1}''^2\eta_i \bmod \alpha_1\bar{\mathfrak{m}}$, i.e. $o''_{i0}X_i, o''_{i1}X_i \in I_{\bar{B}}(\alpha_1)^{loc}$.

Now for any $1 \leqslant i \leqslant u$,

$$[D'_iX_i] + [-o''_{i0}X_i] + [-o''_{i1}X_i] \equiv [(D'_i - o''_{i0} - o''_{i1})X_i]$$
$$-[D'_io''_{i0}X_i^2] - [D'_io''_{i1}X_i^2] - [-o''_{i0}o''_{i1}X_i^2] \bmod I_{\bar{B}}(\alpha_2^2)^{loc}$$

where all terms in the right-hand side belong to $\left(I_{\bar{B}}(\alpha_1)^{loc}\right)^2 \subset I_{\bar{B}}(\alpha_2)^{loc}$. Thus the right-hand side is congruent modulo $I_{\bar{B}}(\alpha_2^2)^{loc}$ to $\sum_j [D'_{ij}X_j]$ with all $D'_{ij} \in \bar{O}$ and $D'_{ij}X_j \in I_{\bar{B}}(\alpha_2)^{loc}$. Lemma 3.7 is proved. □

Finally, we finish the proof of formula (3.2) with $\alpha = \alpha_2$ by noting that (use Lemma 3.5)

$$\sum_i ([o'_{i0}X_i] + [o'_{i1}X_i]) - \sum_i ([-o''_{i0}X_i] + [-o''_{i1}X_i]) =$$
$$[\tilde{a}] + \sum_i ([(o'_{i0} + o''_{i0})X_i] + [(o'_{i1} + o''_{i1})X_i]),$$

where $\tilde{a} \in \tilde{\eta}I_B$.

Clearly, there is $\alpha \in \bar{\mathfrak{m}}$ such that $I_{\bar{B}}(\alpha^2)^{loc} \subset 2I_{\bar{B}}^{loc}$. It remains to apply Lemma 3.4 with $\bar{\mathcal{I}} = 0$ to finish the proof of Main Lemma.

3.8. **Elaboration of Main Lemma.** Let $D = C^{-1} = (d_{ij})$ and for $1 \leqslant i \leqslant u$,

$$R_i = \sum_{\substack{t \\ s_1<s_2}} (X_{s_1} \otimes X_{s_2} + X_{s_2} \otimes X_{s_1})c_{s_1t}c_{s_2t}d_{ti}^2 \in I_{B\otimes B}.$$

Then property (**C**) of the matrix C from Subsection 1.9.1 implies that all $\tilde{\eta}_iR_i \in I_{B\otimes B}^{loc}$. Indeed, if $\tilde{\eta}_iX_{s_1}\otimes X_{s_2} \notin I_{B\otimes B}^{loc}$ then $1 \leqslant i \leqslant u_0$ (because $\tilde{\eta}_i$ must belong to O_0^*) and $u_0 < s_1, s_2 \leqslant u$ (because $X_{s_1}, X_{s_2} \notin I_B^{loc}$), but then $c_{s_1t}d_{ti}, c_{s_2t}d_{ti} \equiv 0 \bmod \pi$. Therefore, the system of congruences

$$\sum_s B_sc_{si} \equiv \tilde{\eta}_i(R_i + B_i^2) \bmod I_{B\otimes B}(4)^{loc}, \quad 1 \leqslant i \leqslant u, \tag{3.4}$$

has a unique solution $(B_1, \dots, B_u) \bmod I_{B\otimes B}(4)^{loc}$ with all $B_i \in I_{B\otimes B}^{loc}$.

We shall use below the following agreement: if $1 \leqslant i \leqslant u$ and $\alpha \in O$ then $I_B(\alpha)^{(i)}$ will be equal to $I_B(\alpha)^{loc}$ if $i \leqslant u_0$ and to $I_B(\alpha)$ if $i > u_0$. Same agreement will be used for similar ideals $I_{B\otimes B}(\alpha)$, $I_{\bar{B}}(\alpha)$, etc.

Lemma 3.8. *With above notation one has*

$$j_i \equiv \sum_t \tilde{\eta}_tX_t \otimes X_td_{ti} + 2B_i + \sum_{t,u} \tilde{\eta}_u(X_u \otimes X_u)d_{ut}(\tilde{\eta}_tX_t \otimes 1 + 1 \otimes \tilde{\eta}_tX_t)d_{ti}$$
$$+2\sum_t (\tilde{\eta}_tX_t \otimes 1 + 1 \otimes \tilde{\eta}_tX_t)B_td_{ti} \bmod I_{B\otimes B}(16)^{(i)}$$

Proof. It will be sufficient to verify by direct calculations that the right-hand sides of above congruences give solutions of equalities (1.3) modulo $I_{B\otimes B}(16)^{(i)}$. (When calculating use that the first two summands belong to $I_{B\otimes B}(4)^{(t)}$ and the last two ones – to $I_{B\otimes B}(8)^{(t)}$.) □

One can easily see that the above elements $B_i \in I_{B\otimes B}^{loc}$ appear in the form

$$B_i \equiv \sum_{s_1<s_2} \gamma_{s_1 s_2 i}(X_{s_1}\otimes X_{s_2} + X_{s_2}\otimes X_{s_1}) \bmod I_{B\otimes B}(4)^{loc}$$

where all $\gamma_{s_1 s_2 i} \in O$. Introduce the elements

$$\widetilde{R}_i = \sum_{\substack{t\\ s_1<s_2}} X_{s_1}X_{s_2}c_{s_1 t}c_{s_2 t}d_{ti}^2, \quad \widetilde{B}_i = \sum_{s_1<s_2} \gamma_{s_1 s_2 i}X_{s_1}X_{s_2}$$

Then

• $(\widetilde{B}_1,\dots,\widetilde{B}_u) \bmod I_B(4)^{loc}$ is a unique solution in $I_B^{loc} \bmod I_B(4)^{loc}$ of the congruences $\sum_s \widetilde{B}_s c_{si} \equiv \tilde{\eta}_i(\widetilde{R}_i + \widetilde{B}_i^2) \bmod I_B(4)^{loc}$;

• for all i, $\delta^+(\widetilde{B}_i) \equiv B_i \bmod I_{B\otimes B}(4)^{loc}$.

Now we can state the following elaboration of Main Lemma.

Proposition 3.9. *In Lemma* 3.3 *the elements $D_i \in \bar{O}$, $1 \leqslant i \leqslant u$, and $g := \sum_{r(\underline{i})\geqslant 2} D_{\underline{i}}X^{\underline{i}} \in I_{\bar{B}\otimes\bar{B}}$ can be taken in such a way that*

a) $(D_i^2 + \sum_s (o'_{s0} + o'_{s1} + D_s)d_{is}\tilde{\eta}_i - \tilde{o}_i)X_i^2 \in 4I_{\bar{B}}$, *where $1 \leqslant i \leqslant u$ and all $\tilde{o}_i \in \tilde{\eta}O$;*

b) $g \equiv \sum_i (o'_{i0} + o'_{i1} + D_i)\widetilde{B}_i \bmod I_{\bar{B}}(4)^{loc}$.

Remark. Part a) implies that all $\tilde{o}_i \equiv 0 \bmod \tilde{\eta}_i$ and if $u_0 < t \leqslant u$ then $\tilde{o}_i \equiv 0 \bmod \pi$. In other words, all $\tilde{o}_i \in (\tilde{\eta}_i O)^{loc}$.

Proof. Let $\tilde{f} := [f] - [f_0]$. Then

$$\delta_{LT}(\tilde{f}) \equiv \sum_i (o'_{i0} + o'_{i1} + D_i)j_i + \sum_i D_i^2 X_i \otimes X_i$$
$$+ \sum_i D_i^2(X_i\otimes 1 + 1\otimes X_i)j_i + 2\delta^+(g) \bmod I_{\bar{B}\otimes\bar{B}}(16)^{loc}.$$

For $1 \leqslant t \leqslant u$, let $s(t) = \sum_i (o'_{i0} + o'_{i1} + D_i)d_{ti}\tilde{\eta}_t$. Then using the explicit formulas for $j_i \bmod I_{B\otimes B}(16)^{(i)}$, $1 \leqslant i \leqslant u$, from Lemma 3.8 we obtain that $\delta_{LT}(\tilde{f})$ is congruent modulo $I_{\bar{B}\otimes\bar{B}}(16)^{loc}$ to

$$\sum_t (D_t^2 + s(t))X_t\otimes X_t + 2\sum_t (o'_{t0} + o'_{t1} + D_t)B_t +$$

$$\sum_t (D_t^2 + s(t))(X_t\otimes 1 + 1\otimes X_t)\left(2B_t + \sum_u d_{ut}\tilde{\eta}_u X_u\otimes X_u\right) + 2\delta^+(g).$$

Follow the coefficient for $X_i \otimes X_i$, $1 \leqslant i \leqslant u$.

Verify that only the first sum contributes to this coefficient.

Indeed, the second sum does not contribute because $B_t \bmod I_{B\otimes B}(4)^{loc}$ is a linear combination of the terms $X_{s_1}\otimes X_{s_2}$ with $s_1 \neq s_2$. The remaining big sum also does not contribute modulo $4I_{\bar{B}\otimes\bar{B}}^{loc}$ because:

— $2B_t(X_t\otimes 1)$ and $2B_t(1\otimes X_t)$ contribute in the same way;

— for similar reason it will be sufficient to verify that $X_i \otimes X_i$ can appear in $(\tilde{\eta}_u X_u \otimes X_u)(X_t \otimes 1)$ with coefficient from $2O$; but if this coefficient is not 0 then $u = t$ and we can use that $\eta_u \tilde{\eta}_u = -2$.

So, for $1 \leqslant i \leqslant u$, $(D_i^2 + s(i))X_i \otimes X_i \in \tilde{\eta} I_{B\otimes B} \bmod I_{\bar{B}\otimes\bar{B}}(16)^{loc}$. Therefore, there is $\tilde{o}_i \in \tilde{\eta}O$ such that $(D_i^2 + s(i) - \tilde{o}_i)\eta_i$ belongs to $4\bar{O}$. This proves part a).

The remaining terms in the relation $\delta_{LT}(\tilde{f}) \in \tilde{\eta} I_{B\otimes B} \bmod I_{\bar{B}\otimes\bar{B}}(16)^{loc}$ give

$$\sum_i (o'_{i0} + o'_{i1} + D_i)B_i + \delta^+(g) \in I_{B\otimes B} \bmod I_{\bar{B}\otimes\bar{B}}(4)^{loc}$$

(use property a) to eliminate the last big sum).

We know that $h = \sum_i (o'_{i0} + o'_{i1} + D_i)\widetilde{B}_i + g$ is of the form $\sum_{r(\underline{i})\geqslant 2} C_{\underline{i}} X^{\underline{i}}$. Therefore, the relation $\delta^+ h \in I_{B\otimes B} \bmod I_{\bar{B}\otimes\bar{B}}(4)^{loc}$ implies that $h \equiv h_0 \bmod I_{\bar{B}}(4)^{loc}$ with $h_0 \in I_B$. So, replacing f_0 by $f_0 - 2h_0$ we obtain property b). □

3.9. **Explicit calculation of $l_{LT}(f)$.** As earlier, $f \in \mathcal{H}_{LT}(H_0, \mu_\eta)$ is given via Main Lemma where we can now assume that $D_1, \ldots, D_u \in \bar{O}$ and $g \in I_{\bar{B}}^{loc}$ satisfy Proposition 3.9. We also set $\tilde{f} = [f] - [f_0]$, $L_{\bar{B}} = \sum_i \bar{O}X_i$, $L_{\bar{B}}^{loc} = \sum_{1\leqslant i\leqslant u_0} \bar{O}X_i + \sum_{u_0<i\leqslant u} \bar{\mathfrak{m}}X_i$ and define

$$(\tilde{\eta}L_{\bar{B}})^{loc} = \begin{cases} \tilde{\eta}L_{\bar{B}} & \text{if } \tilde{\eta} \notin O^* \\ L_{\bar{B}}^{loc} & \text{if } \tilde{\eta} \in O^* \end{cases} \qquad (\tilde{\eta} I_{\bar{B}}(4))^{loc} = \begin{cases} \tilde{\eta} I_{\bar{B}}(4) & \text{if } \tilde{\eta} \notin O^* \\ I_{\bar{B}}(4)^{loc} & \text{if } \tilde{\eta} \in O^* \end{cases}$$

For $1 \leqslant i \leqslant u$ and $\tilde{o}_i \in O$ from Proposition 3.9, let $o'_{i2} \in O$ and $o'_{i3} \in \pi' O$ be such that $\tilde{o}_i \equiv o'^2_{i2} + o'^2_{i3} \bmod 2\tilde{\eta}\pi O$ and $o'^2_{i2}, o'^2_{i3} \in \tilde{o}_i O$.

Proposition 3.10. *With above notation*

$$l_{LT}(\tilde{f}) \equiv -\sum_{i,l} o'_{il} X_i + \sum_{i,l} l_{LT}(o'_{il} X_i) \bmod ((\tilde{\eta}L_{\bar{B}})^{loc} + (\tilde{\eta} I_{\bar{B}}(4))^{loc}).$$

(In both sums $1 \leqslant i \leqslant u$ and $0 \leqslant l \leqslant 3$.)

Proof. Note that (use that all $D_i X_i \in I_{\bar{B}}(2)$)

$$l_{LT}(D_i X_i) \equiv D_i X_i + \frac{D_i^2 X_i^2}{2} + \left(\frac{D_i^2 X_i^2}{2}\right)^2 + 2\left(\frac{D_i^2 X_i^2}{2}\right)^4 \bmod 4 I_{\bar{B}}$$

Rewrite property a) of Proposition 3.9 in the following form

$$\frac{D_i^2 X_i^2}{2} \equiv \frac{\tilde{o}_i X_i^2}{2} - \sum_s (o'_{s0} + o'_{s1} + D_s) d_{is} \tilde{\eta}_i X_i^2 \bmod 2L_{\bar{B}}$$

Then

$$\sum_i \frac{D_i^2 X_i^2}{2} \equiv \sum_i \frac{\tilde{o}_i X_i^2}{2} - \sum_s (o'_{s0} + o'_{s1} + D_s) X_s \bmod 2L_{\bar{B}}$$

Property a) also implies that

$$\left(\frac{D_i^2X_i^2}{2}\right)^2 \equiv \left(\frac{\tilde{o}_i^2}{\tilde{\eta}_i} - \sum_s (o'_{s0} + o'_{s1} + D_s)d_{is}\right)^2 \left(X_i^2/\eta_i\right)^2 \bmod 4I_{\bar{B}}$$

The first factor in the right-hand side of the last congruence can be written in the form

$$\left(\frac{o'^2_{i2}}{\tilde{\eta}_i}\right)^2 + \left(\frac{o'^2_{i3}}{\tilde{\eta}_i}\right)^2 + \sum_s D_s^2 d_{is}^2 + \tilde{\eta}A_i$$

where $A_i \in \bar{O}$ and belongs to the maximal ideal $\bar{\mathfrak{m}}$ of $\bar{O}$ if $u_0 < i \leqslant u$. Then

$$\left(\frac{D_i^2X_i^2}{2}\right)^2 \equiv \left(\frac{o'^2_{i2}X_i^2}{2}\right)^2 + \left(\frac{o'^2_{i3}X_i^2}{2}\right)^2 + 2\sum_{\substack{s\\ u_1<u_2}} D_s^2 d_{is}^2 X_{u_1}X_{u_2}c_{u_1i}c_{u_2i}$$

modulo $(\tilde{\eta}L_{\bar{B}})^{loc} + 2(\tilde{\eta}I_{\bar{B}})^{loc}$, because

$$\sum_{i,s,u} D_s^2 d_{is}^2 X_u^2 c_{ui}^2 \equiv \sum_s D_s^2 X_s^2 \equiv 0 \bmod 2L_{\bar{B}}.$$

Note also that

$$2\left(\frac{D_i^2X_i^2}{2}\right)^4 = 2\left(\frac{D_i^2\eta_i}{2}\right)^4 \left(\sum_s X_s^2c_{si}^2 + 2\sum_{s_1<s_2} X_{s_1}X_{s_2}c_{s_1i}c_{s_2i}\right)^2$$

belongs to $2L_{\bar{B}} + 4I_{\bar{B}}$.

Taking above calculations together we obtain

$$\sum_i l_{LT}(D_iX_i) \equiv -\sum_i (o'_{i0} + o'_{i1})X_i + \sum_{i,l} \frac{o'^2_{il}X_i^2}{2} + 2\sum_s D_s^2\widetilde{R}_s \equiv$$

$$-\sum_{i,l} o'_{il}X_i + \sum_i l_{LT}([o'_{i2}X_i] + [o'_{i3}X_i]) + 2\sum_s D_s^2\widetilde{R}_s$$

modulo $(\tilde{\eta}L_{\bar{B}})^{loc} + 2(\tilde{\eta}I_{\bar{B}})^{loc}$ because all $2(o'^2_{i2}X_i^2/2)^4$ and $2(o'^2_{i3}X_i^2/2)^4$ belong to $2L_{\bar{B}} + 4I_{\bar{B}}$.

On the other hand we have:

$g^2 \equiv \sum_t D_t^2\widetilde{B}_t^2 \bmod (\tilde{\eta}I_{\bar{B}})^{loc}$, use Proposition 3.9b);

$\sum_t D_t^2\widetilde{B}_t^2 \equiv \sum_{t,s}(D_t^2/\tilde{\eta}_t)\widetilde{B}_s c_{st} - \sum_t D_t^2\widetilde{R}_t \bmod I_{\bar{B}}(4)^{loc}$, cf. Subsection 3.9;

$D_t^2/\tilde{\eta}_t \equiv -\sum_s (o'_{s0} + o'_{s1} + D_s)d_{is} \bmod \tilde{\eta}\bar{O}$, use Proposition 3.9a).

Therefore,

$$l_{LT}(2g) \equiv 2(g^2 + g) \equiv -2\sum_t D_t^2\widetilde{R}_t \bmod (\tilde{\eta}I_{\bar{B}}(4))^{loc}$$

because $2(\tilde{\eta}I_{\bar{B}})^{loc}$, $2I_{\bar{B}}(4)^{loc}$, $2\tilde{\eta}I^{loc}_{B\otimes B}$ are contained in $(\tilde{\eta}I_{\bar{B}}(4))^{loc}$.

The Proposition is proved. □

3.10. **Special element $h \in I_{\bar{B}}$ and its properties.** With above notation set

$$h = [\tilde{f}] - \sum_{i,l} [o'_{il} X_i].$$

Note that $[2](h) = [2](f) - [2](f_0) - \sum_{i,l} [2](o'_{il} X_i) \in I_{B'} \subset I_{\bar{B}}$ and, therefore, $l_{LT}(h) = (1/2) l_{LT}([2]h) \in I_{B'} \otimes_O K$. Then by Proposition 3.10

$$l_{LT}(h) \equiv \tilde{\eta}' l_0 \bmod (\tilde{\eta} I_{B'}(4))^{loc},$$

where $l_0 = \sum_i a_i X_i$ with $a_i \in O'$ and one has

$$\tilde{\eta}' a_i \equiv -\sum_l o'_{il} \bmod \tilde{\eta} O'. \tag{3.5}$$

In addition, if $\tilde{\eta} \in O_0^*$ then $l_0 \in I_{B'}^{loc}$.

Proposition 3.11. a) $[2](h) \equiv 2\tilde{\eta}' l_0 \bmod 2(\tilde{\eta} I_{B'})^{loc}$;

b) $\delta_{LT}(h) \equiv \tilde{\eta}' \delta^+(l_0) \bmod (\tilde{\eta}' I_{B' \otimes B'}(4))^{loc}$.

Proof. a) By Main Lemma, $h \in I_{B'}(2)^{loc}$ and, therefore, $[2](h) \in 2I_{B'}(2)^{loc} + \left(I_{B'}(2)^{loc}\right)^2 \subset I_{B'}(4)^{loc}$, cf. Subsection 2.2.3. The standard DP-structure on this ideal is topologically nilpotent and $2(\tilde{\eta}' I_{B'})^{loc}$ is its DP-subideal. Therefore, the congruence $l_{LT}([2]h) \equiv 2\tilde{\eta}' l_0 \bmod 2(\tilde{\eta} I_{B'}(4))^{loc}$ implies that $[2](h) \in 2(\tilde{\eta}' I_{B'})^{loc}$ (use that l_{LT} is bijective on any ideal with nilpotent divided powers). If $[2](h) = 2A$ with $A \in (\tilde{\eta}' I_{B'})^{loc}$ then $l_{LT}([2](h)) \equiv 2A \bmod 2(\tilde{\eta} I_{B'})^{loc}$ and, therefore, $A \equiv \tilde{\eta}' l_0 \bmod (\tilde{\eta} I_{B'})^{loc}$.

b) Similarly, Main Lemma implies that

$$\delta_{LT}(h) \in I_{B'}(2)^{loc} \otimes I_{B'}(2)^{loc} \subset I_{B' \otimes B'}(4)^{loc}.$$

Again this ideal has topologically nilpotent DP-structure and in addition we have

$$l_{LT}(\delta_{LT}(h)) = \delta^+ l_{LT}(h) \equiv \tilde{\eta}' \delta^+ l_0 \bmod (\tilde{\eta} I_{B' \otimes B'}(4))^{loc}$$

Proceeding as in a) we obtain $\delta_{LT}(h) \in (\tilde{\eta}' I_{B' \otimes B'}(4))^{loc}$. It remains to note that then $\delta_{LT}(h)^2/2 \in (\tilde{\eta} I_{B' \otimes B'}(4))^{loc}$ and, therefore, $l_{LT}(\delta_{LT}(h)) \equiv \delta_{LT}(h) \bmod (\tilde{\eta} I_{B' \otimes B'}(4))^{loc}$. □

Proposition 3.12. *For* $1 \leqslant i \leqslant u$, $(\sum_l o'^2_{il})\, \eta_i \in \tilde{\eta}^2 O_0 \bmod 2\tilde{\eta} O$.

Proof. We know that $[2](f) \in \tilde{\eta}^2 I_{B_0}$. On the other hand,

$$[2](f) = [2](f_0) + [2](h) + \sum_{i,l} [2](o'_{il} X_i),$$

where $f_0 \in \tilde{\eta} I_B$. Note that:

$[2](f_0) \equiv f_0^2 \bmod 2\tilde{\eta} I_B$ implies that $[2](f_0) \in \tilde{\eta}^2 I_{B_0} \bmod 2(\tilde{\eta} I_B)^{loc}$;

$[2](o'_{il} X_i) \equiv [o'^2_{il} X_i^2] + [2o'_{il} X_i] \bmod 2(\tilde{\eta} I_B)^{loc}$ (use that all $o'^2_{il} \in \tilde{\eta} O$);

$[2](h) \equiv -\sum_{i,l} [2o'_{il} X_i] \bmod 2(\tilde{\eta} I_{B'})^{loc}$ by the above proposition.

So,

$$\sum_{i,l}[o'^2_{il}X_i^2] \in \tilde{\eta}^2 I_{B_0} \bmod 2(\tilde{\eta} I_B)^{loc}. \tag{3.6}$$

Note that for all i (use just that $\pi^2 = \pi_0$),

$o'^2_{i0}, o'^2_{i2} \in \tilde{\eta} O_0 + 2\tilde{\eta} O$;

$o'^2_{i1}, o'^2_{i3} \in \pi\tilde{\eta} O_0 + 2\tilde{\eta} O$;

$X_i^2 \in \eta_i I_{B_0} + 2I_B$.

This implies that:

$$\sum_i \left([o'^2_{i0}X_i^2] + [o'^2_{i2}X_i^2]\right) \equiv \sum_i \left(o'^2_{i0} + o'^2_{i2}\right) X_i^2 \bmod(\tilde{\eta}^2 I_{B_0} + 2\tilde{\eta} I_B)$$

$$\sum_i \left([o'^2_{i1}X_i^2] + [o'^2_{i3}X_i^2]\right) \equiv \sum_i \left(o'^2_{i1} + o'^2_{i3}\right) X_i^2 \bmod(\tilde{\eta}^2 I_{B_0} + 2\tilde{\eta} I_B)$$

Clearly, $\sum_i (o'^2_{i0} + o'^2_{i2}) X_i^2 \in \tilde{\eta} I_{B_0} + 2\tilde{\eta} I_B$ and $\sum_i (o'^2_{i0} + o'^2_{i2}) X_i^2 \in \pi\tilde{\eta} I_{B_0} + 2\tilde{\eta} I_B$. Therefore, (3.6) implies that for any $1 \leqslant i \leqslant u$, $(o'^2_{i0} + o'^2_{i2})\eta_i \in \tilde{\eta} O_0 + 2\tilde{\eta} O$ and $(o'^2_{i1} + o'^2_{i3})\eta_i \in 2\tilde{\eta} O$. This proves the proposition. □

3.11. **Introducing $\mathcal{M} = \mathcal{M}(G) \in \mathrm{MF}^e_S$.** Choose a rings identification $\kappa_{S'O'} : S' \bmod t^e \longrightarrow O' \bmod 2$ such that $\kappa_{S'O'}|_{S \bmod t^e} = \kappa_{SO}$. Consider $l_0 = \sum_i a_i X_i \in I_{B'}$ from Subsection 3.10 and set

$$l_1 = \sum_i b_i X_i \equiv (-l_0 - \tilde{\eta}' l_0^2 - \dots - \tilde{\eta}'^{2^n-1} l_0^{2^n} - \dots) \bmod 2(\tilde{\eta}' I_{B'})^{loc}$$

Let $\tilde{s}', \alpha_1, \dots, \alpha_u \in S'$ be such that $\kappa_{S'O'}(\tilde{s}' \bmod t^e) = \tilde{\eta}' \bmod 2$ and $\kappa_{S'O'}(\alpha_i \bmod t^e) = b_i \bmod 2$ for $1 \leqslant i \leqslant u$.

Introduce $\mathcal{M}' = \mathcal{M}'(G) = (M'^0, M'^1, \varphi'_1) \supset \mathcal{N}' := \mathcal{N} \otimes_S S'$ such that $M'^0 = m'S' \oplus (N^0 \otimes_S S')$, $M'^1 = m'^1 S' + (N^1 \otimes_S S') \subset M'^0$ with $m'^1 = \tilde{s}' m' + \sum_i \alpha_i n_i$, and $\varphi'_1(m'^1) = m'$.

Proposition 3.13. a) $\mathcal{M}' \in \mathrm{MF}^{2e}_{S'}$;

b) *there is* $\mathcal{M} = \mathcal{M}(G) \in \mathrm{MF}^e_S$ *such that* $\mathcal{M}' = \mathcal{M} \otimes_S S'$.

Proof. a) $\mathcal{M}' \in \mathrm{MF}^{2e}_{S'}$ means that $t^e m' \in M'^1$, i.e. for $1 \leqslant i \leqslant u$, one has $t^e \tilde{s}^{-1} \alpha_i \equiv 0 \bmod \tilde{s}_i$ or, equivalently,

$$(t^e \tilde{s}_i^{-1})\alpha_i \equiv 0 \bmod \tilde{s}.$$

Using (3.5) and the identification $\kappa_{S'O'}$ we can rewrite these congruences in the form $(\sum_l o'_{il})\, \eta'_i \equiv 0 \bmod \tilde{\eta}$ or equivalently, $(\sum_l o'^2_{il})\, \eta_i \equiv 0 \bmod \tilde{\eta}^2$. Clearly, these conditions follow from Proposition 3.12.

b) The criterion of the existence of a descent $\mathcal{M}$ of $\mathcal{M}'$ to S from Proposition 1.4 can be specified in our case in the following form: for all $1 \leqslant i \leqslant u$, $\alpha_i \in S \bmod \tilde{s}_i S'$.

Using the identification $\kappa_{S'O'}$ we rewrite these conditions in the form $\tilde{b}_i \bmod \tilde{\eta}'_i \in O \bmod \tilde{\eta}'$. By (3.5) we can replace them by $\sum_l o'_{il} \in O \bmod \tilde{\eta}'_i\tilde{\eta}'$, then by $\sum_l o'^2_{il} \in O_0 \bmod \tilde{\eta}_i\tilde{\eta}$ and finally by $(\sum_l o'^2_{il})\eta_i \in O_0 \bmod 2\tilde{\eta}O$. But this is given again by Proposition 3.12. □

3.12. **Construction of isomorphism $\mathcal{G}_O(\mathcal{M}) \simeq G$.** We know that $G = \mathrm{Spec}A$, where $A = B[X]$, $[2](\tilde{\eta}X) = [2](f) \in (\tilde{\eta}^2 I^{loc}_{B_0} \subset (\tilde{\eta}^2 I_B)^{loc}$ and $\delta_{LT}(\tilde{\eta}X) = \delta_{LT}(f) \in (\tilde{\eta} I_{B_0\otimes B_0})^{loc} \subset (\tilde{\eta} I_{B\otimes B})^{loc}$.

Let $A' = A \otimes_O O'$ and let $Z \in I_{A'}$ be such that $\tilde{\eta}'Z = [\tilde{\eta}X] - \sum_{i,l}[o'_{il}X_i]$. Note that

$$Z + \tilde{\eta}'^{-1}\sum_{i,l} o'_{il}X_i \equiv Z - l_0 \equiv Z + l_1 \equiv 0 \bmod \tilde{\eta}' I_{A'} \tag{3.7}$$

Proposition 3.14. a) $\tilde{\eta}'Z \in (\tilde{\eta}' I_{A'}(2))^{loc}$;
b) $\delta^+(Z - l_0) \equiv \tilde{\eta}' I_{A'\otimes A'}(4)$;
c)$\tilde{\eta}'(Z^2/2) \equiv Z + l_1 \bmod (\tilde{\eta}' I_{A'}(4))^{loc}$.

Proof. a) By Proposition 3.11a), $[2](\tilde{\eta}'Z) = [2](h) \equiv 2\tilde{\eta}' l_0 \bmod 2(\tilde{\eta} I_{B'})^{loc}$. This implies $(\tilde{\eta}'Z)^2 \in 2(\tilde{\eta}' I_{A'})^{loc}$, i.e. $\tilde{\eta}'^{1/2}Z \in I_{A'}(2)^{loc}$. If $\tilde{\eta} \in O_0^*$ then a) is proved. Continue with the following congruence

$$l_{LT}(\tilde{\eta}'Z) = l_{LT}(h) \equiv \tilde{\eta}' l_0 \bmod (\tilde{\eta} I_{B'}(4))^{loc}$$

(cf. Proposition 3.10). It implies that

$$Z - l_0 + \tilde{\eta}'(Z^2/2)(1 + \tilde{\eta}Z^2/2) \in (\tilde{\eta}' I_{A'}(4))^{loc}. \tag{3.8}$$

By (3.7) $(Z - l_0)/\tilde{\eta}' \in I_{A'}$ and $1 + \tilde{\eta}(Z^2/2) \equiv 1 \bmod \tilde{\eta}' I_{A'}$ is invertible (use that $\tilde{\eta} \notin O_0^*$). Therefore, $Z^2/2 \in I_{A'}$ and a) is completely proved.

b) By proposition 3.11b), $\delta_{LT}(\tilde{\eta}'Z) \equiv \tilde{\eta}'\delta^+ l_0 \bmod \tilde{\eta} I_{B'\otimes B'}(4)$. It remains to note that $Z \in I_{A'}(2)$ implies that

$$\delta_{LT}(\tilde{\eta}'Z) \equiv \delta^+(\tilde{\eta}'Z) \bmod \tilde{\eta} I_{A'\otimes A'}(4).$$

c) Iterating (3.8) we obtain

$$\tilde{\eta}'(Z^2/2) \equiv Z - l_0 - \tilde{\eta}'(\tilde{\eta}'Z^2/2)^2 \equiv Z - l_0 - \tilde{\eta}' l_0^2 - \tilde{\eta}'^3(\tilde{\eta}'Z^2/2)^4$$

$$\equiv Z - l_0 - \tilde{\eta}' l_0^2 - \cdots - \tilde{\eta}'^{2^n-1} l_0^{2^n} - \cdots \equiv Z + l_1 \bmod (\tilde{\eta}' I_{A'}(4))^{loc}.$$

The proposition is proved. □

Consider the ideals $J_{A'}$ and $J_{A'\otimes A'}$, cf. Subsection 1.3.

Corollary 3.15. *There is $\widetilde{Z} \in I_{A'}$ such that*
a) $\widetilde{Z} \equiv (\tilde{\eta}' I_{A'}(4))^{loc}$;
b) $\tilde{\eta}'(\widetilde{Z}^2/2) \equiv \widetilde{Z} + l_1 \bmod \tilde{\eta}' J_{A'}$;
c) $\delta^+(\widetilde{Z}), \delta^+(\widetilde{Z}^2/2) \in J_{A'\otimes A'}$.

Proof. Let $\widetilde{Z} = \tilde{\eta}'(Z^2/2) - l_1$.

Part a) follows directly from Proposition 3.14.

Because $(\tilde{\eta}' I_{A'}(4))^{loc} \subset I_{A'}(2)$ this implies that

$$\widetilde{Z}^2/2 \equiv Z^2/2 \bmod J_{A'} \tag{3.9}$$

and we obtained part b).

By Proposition 3.14b), $\delta^+ Z \in I_{A'\otimes A'}(4)$. This implies that $\delta^+(Z^2/2) \in J_{A'\otimes A'}$, use Proposition 3.14a). So, $\delta^+\widetilde{Z} \in J_{A'\otimes A'}$ and by (3.9) we obtain $\delta^+(\widetilde{Z}^2/2) \in J_{A'\otimes A'}$. □

By above Corollary the correspondences $m' \mapsto \widetilde{Z}^2/2 \bmod J_{A'}$ and $m'^1 \mapsto \widetilde{Z} \bmod J_{A'}$ give a map of filtered modules $\mathcal{M}' \longrightarrow \iota(A')$. This gives a morphism of O'-algebras $\Pi' : A(\mathcal{G}_{O'}(\mathcal{M}')) \longrightarrow A'$. By Proposition 1.12 we can assume that Π' is also a morphism of coalgebras. Both these coalgebras contain $B' = B \otimes_O O'$ and $\Pi'|_{B'}$ is isomorphism. Similarly, both the coalgebras have as their quotient the coalgebra $A(\mu_\eta) \otimes O'$ and Π' induces on it a coalgebra isomorphism as well. This implies that Π' is isomorphism of coalgebras and Theorem 3.1 is completely proved.

References

[1] V. Abrashkin *Group schemes of period p*, Math. USSR Izvestiya **20** (1983) no. 3, 411-433

[2] V. Abrashkin *Honda systems of group schemes of period p*, Math. USSR Izvestiya **30** (1988) no. 3, 419-453

[3] V. Abrashkin *Group schemes over a discrete valuation ring with small ramification*, Leningrad Math. J. **1** (1990) no. 1, 1990, 57-97

[4] V. Abrashkin *Group schemes of period $p > 2$*, Proc. London Math. Soc. **101** (2010) no. 3, 207-259

[5] C. Breuil, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. **152** 2000 no. 2, 489-549

[6] W. Kim *The classification of p-divisible groups over 2-adic discrete valuation rings*, arXiv:1007.1904. Preprint.

[7] M. Kisin *Modularity of 2-adic Barsotti-Tate representations*, Inv. Math. **178** (2009), 587-634

[8] M. Kisin *Moduli of finite flat group schemes, and modularity*, Ann. of Math. **170** (2009), 1085-1180

[9] E. Lau, *A relation between crystalline Dieudonné theory and Dieudonné displays*, arXiv:1006.2720. Preprint.

[10] T. Liu, *The correspondence between Barsotti-Tate groups and Kisin modules when $p = 2$*, available at math.purdue.edu/ tongliu/pub/2BT.pdf. Preprint.

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