SKEW-SYMMETRIC CLUSTER ALGEBRAS OF FINITE MUTATION TYPE

ANNA FELIKSON, MICHAEL SHAPIRO, AND PAVEL TUMARKIN

Abstract. In the famous paper [FZ2] Fomin and Zelevinsky obtained Cartan-Killing type classification of all cluster algebras of finite type, i.e. cluster algebras having only finitely many distinct cluster variables. A wider class of cluster algebras is formed by cluster algebras of finite mutation type which have finitely many exchange matrices (but are allowed to have infinitely many cluster variables). In this paper we classify all cluster algebras of finite mutation type with skew-symmetric exchange matrices. Besides cluster algebras of rank 2 and cluster algebras associated with triangulations of surfaces there are exactly 11 exceptional skew-symmetric cluster algebras of finite mutation type. More precisely, 9 of them are associated with root systems E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$; two remaining were found by Derksen and Owen in [DO]. We also describe a criterion which determines if a skew-symmetric cluster algebra is of finite mutation type, and discuss growth rate of cluster algebras.

Anna Felikson, Independent University of Moscow, B. Vlassievskii 11, 119002 Moscow, Russia, and Max-Planck Institute for Mathematics, Vivatsgasse 7, D-53111, Germany;
felikson@mccme.ru.

Michael Shapiro, Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA; (corresponding author)
mshapiro@math.msu.edu.

Pavel Tumarkin, School of Engineering and Science, Jacobs University Bremen, Campus Ring 1, D-28759, Germany;
p.tumarkin@jacobs-university.de.

Research supported by grants INTAS YSF-06-10000014-5916 (A. F.), RFBR 07-01-00390-a (A. F. and P. T.), PHY 0555346 and DNS 0800671 (M. S.), and INTAS YSF-06-10000014-5766 (P. T.).

Contents

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in the sequel of papers [FZ1], [FZ2], [BFZ], [FZ3].

We think of cluster algebra as a subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ determined by generators ("cluster coordinates"). These generators are collected into n-element groups called *clusters* connected by local transition rules which are determined by an $n \times n$ skew-symmetrizable exchange matrix associated with each cluster. For precise definitions see Section 2.

In [FZ2], Fomin and Zelevinsky discovered a deep connection between cluster algebras of *finite type* (i.e., cluster algebras containing finitely many clusters) and Cartan-Killing classification of simple Lie algebras. More precisely, they proved that there is a bijection between Cartan matrices of finite type and cluster algebras of finite type. The corresponding Cartan matrices can be obtained by some symmetrization procedure of exchange matrices.

Exchange matrices undergo *mutations* which are explicitly described locally. Collection of all exchange matrices of a cluster algebra form a *mutation class* of exchange matrices. In particular, mutation class of a cluster algebra of finite type is finite. In this paper, we are interested in a larger class of cluster algebras, namely, cluster algebras whose exchange matrices form finite mutation class. We will assume that exchange matrices are skew-symmetric.

Besides cluster algebras of finite type, there exist other series of algebras belonging to the class in consideration. One series of examples is provided by cluster algebras corresponding to Cartan matrices of affine Kac-Moody algebras with simply-laced Dynkin diagrams. It was shown in [BR] that these examples exhaust all cases of acyclic skew-symmetric cluster algebras of finite mutation type. Furthermore, Seven in [S2] has shown that acyclic skew-symmetrizable cluster algebras of finite mutation type correspond to affine Kac-Moody algebras.

One more large class of infinite type cluster algebras of finite mutation type was studied in the paper [FST], where, in particular, was shown that signed adjacency matrices of arcs of a triangulation of a bordered two-dimensional surface have finite mutation class.

In the same paper, Fomin, Shapiro and Thurston discussed the conjecture [FST, Problem 12.10] that besides adjacency matrices of triangulations of bordered two-dimensional surfaces and matrices mutation-equivalent to one of the following nine types: E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ (see [FST, Section 12]), there exist finitely many skew-symmetric matrices of size at least 3×3 with finite mutation class. Notice that the first three types in the list correspond to cluster algebras of finite type.

In the preprint [DO] Derksen and Owen found two more skew-symmetric matrices (denoted by X_6 and X_7) with finite mutation class that are not included in the previous conjecture. The authors also ask if their list of 11 mutation classes contains all the finite mutation classes of skew-symmetric matrices of size at least 3×3 not corresponding to triangulations.

The main goal of this paper is to prove the conjecture by Fomin, Shapiro and Thurston by showing the completeness of the Derksen-Owen list, i.e. to prove the following theorem:

Main Theorem (Theorem 6.1). *Any skew-symmetric $n \times n$ matrix, $n \geq 3$, with finite mutation class is either an adjacency matrix of triangulation of a bordered two-dimensional surface or a matrix mutation-equivalent to a matrix of one of the following eleven types: E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$, X_6, X_7.*

Remark 1.1. The same approach that we used for skew-symmetric matrices is applicable (after small changes) for the more general case of skew-symmetrizable matrices. The complete list of skew-symmetrizable matrices with finite mutation class will be published elsewhere.

We also show a way to classify all minimal skew-symmetric $n \times n$ matrices with infinite mutation class. In particular, we prove that $n \leq 10$. This gives rise to the following criterion for a large skew-symmetric matrix to have finite mutation class:

Theorem 7.4. *A skew-symmetric* $n \times n$ *matrix* B*,* $n \geq 10$*, has finite mutation class if and only if a mutation class of every principal* 10×10 *submatrix of* B *is finite.*

As an application of the classification of skew-symmetric matrices of finite mutation type we characterize skew-symmetric cluster algebras of polynomial growth, i.e. cluster algebras for which the number of distinct clusters obtained from the initial one by n mutations grows polynomially in n.

The paper is organized as follows. In Section 2, we provide necessary background in cluster algebras, and reformulate the classification problem of skew-symmetric matrices in terms of *quivers* by assigning to every exchange matrix an oriented weighted graph.

In Section 3, we present the sketch of the proof of the Main Theorem. We list all the key steps, and discuss the main combinatorial and computational ideas we use. Sections 4–6 contain the detailed proofs.

Section 4 is devoted to the technique of block-decomposable quivers. We recall the basic facts from [FST] and prove several properties we will heavily use in the sequel. Section 5 contains the proof of the key theorem classifying minimal non-decomposable quivers. Section 6 completes the proof of the Main Theorem.

In Section 7 we provide a criterion for a skew-symmetric matrix to have finite mutation class. Section 8 is devoted to growth rates of cluster algebras.

Finally, in Section 9 we use the results of the previous section to complete the description of mutation classes of quivers of order 3.

We would like to thank M. Barot, V. Fock, S. Fomin, C. Geiss, A. Goncharov, B. Keller, A. Seven, and A. Zelevinsky for their interest in the problem and many fruitful discussions, and H.Thomas for explaining us that linear growth of affine cluster algebras is a corollary of categorification theory. The first and the third authors are grateful to the University of Fribourg for a great atmosphere during their visit, and for a partial support by SNF projects 200020-113199 and 200020-121506/1. The second author thanks EPFL, Stockholm University, and Royal Institute of Technology, the third author thanks Michigan State University for support during the work on this paper.

2. Cluster algebras, mutations, and quivers

We briefly remind the definition of coefficient-free cluster algebra.

An integer $n \times n$ matrix B is called *skew-symmetrizable* if there exists an integer diagonal $n \times n$ matrix $D = diag(d_1, \ldots, d_n)$, such that the product DB is a skew-symmetric matrix, i.e., $d_i b_{i,j} = -b_{j,i} d_j$.

A seed is a pair (f, B), where $f = \{f_1, \ldots, f_n\}$ form a collection of algebraically independent rational functions of n variables $x_1, \ldots, x_n$, and B is a skew-symmetrizable matrix.

The part f of seed (f, B) is called *cluster*, elements f_i are called *cluster variables*, and B is called *exchange matrix*.

Definition 2.1. For any k, $1 \le k \le n$ we define *the mutation* of seed (f, B) in direction k as a new seed (f', B') in the following way:

$$B'_{i,j} = \begin{cases} -B_{ij}, & \text{if } i = k \text{ or } j = k; \\ B_{ij} + \frac{|B_{ik}|B_{kj} + B_{ik}|B_{kj}|}{2}, & \text{otherwise.} \end{cases} \tag{2.1}$$

$$f'_i = \begin{cases} f_i, & \text{if } i \ne k; \\ \frac{\prod_{B_{ij}>0} f_j^{B_{ij}} + \prod_{B_{ij}<0} f_j^{-B_{ij}}}{f_i}, & \text{otherwise.} \end{cases} \tag{2.2}$$

We write $(f', B') = \mu_k((f, B))$. Notice that $\mu_k(\mu_k((f, B))) = (f, B)$. We say that two seeds are *mutation-equivalent* if one is obtained from the other by a sequence of seed mutations. Similarly we say that two clusters or two exchange matrices are *mutation-equivalent*.

Notice that exchange matrix mutation 2.1 depends only on the exchange matrix itself. The collection of all matrices mutation-equivalent to a given matrix B is called the *mutation class* of B.

For any skew-symmetrizable matrix B we define *initial seed* (x,B) as $(\{x_1, \ldots, x_n\},B)$, B is the *initial exchange matrix*, $x = \{x_1, \ldots, x_n\}$ is the *initial cluster*.

Cluster algebra $\mathfrak{A}(B)$ associated with the skew-symmetrizable $n \times n$ matrix B is a subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all cluster variables of the clusters mutation-equivalent to the initial cluster (x, B).

Cluster algebra $\mathfrak{A}(B)$ is called *of finite type* if it contains only finitely many cluster variables. In other words, all clusters mutation-equivalent to initial cluster contain totally only finitely many distinct cluster variables.

In [FZ2], Fomin and Zelevinsky proved a remarkable theorem that cluster algebras of finite type can be completely classified. More excitingly, this classification is parallel to the famous Cartan-Killing classification of simple Lie algebras.

Let B be an integer $n \times n$ matrix. Its *Cartan companion* $C(B)$ is the integer $n \times n$ matrix defined as follows:

$$C(B)_{ij} = \begin{cases} 2, & \text{if } i = j; \\ -|B_{ij}|, & \text{otherwise.} \end{cases}$$

Theorem 2.2 ([FZ2]). *There is a canonical bijection between the Cartan matrices of finite type and cluster algebras of finite type. Under this bijection, a Cartan matrix A of finite type corresponds to the cluster algebra $\mathfrak{A}(B)$, where B is an arbitrary skew-symmetrizable matrix with $C(B) = A$.*

The results by Fomin and Zelevinsky were further developed in [S1] and [BGZ], where the effective criteria for cluster algebras of finite type were given.

A cluster algebra of finite type has only finitely many distinct seeds. Therefore, any cluster algebra that has only finitely many cluster variables contains only finitely many distinct exchange matrices. Quite the contrary, the cluster algebra with finitely many exchange matrices is not necessarily of finite type.

Definition 2.3. A cluster algebra with only finitely many exchange matrices is called *of finite mutation type*.

Example 2.4. One example of infinite cluster algebra of finite mutation type is the Markov cluster algebra whose exchange matrix is

$$\begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

It was described in details in [FZ1]. Markov cluster algebra is not of finite type, moreover, it is even not finitely generated. Notice, however, that mutation in any direction leads simply to sign change of exchange matrix. Therefore, the Markov cluster algebra is clearly of finite mutation type.

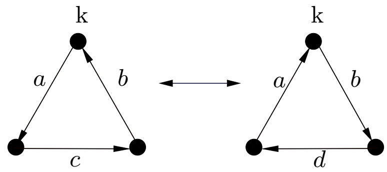
Remark 2.5. Since the orbit of an exchange matrix depends on the exchange matrix only, we may speak about skew-symmetrizable matrices of finite mutation type.

Therefore, the Main Theorem describes *all skew-symmetric integer matrices whose mutation class is finite.*

For our purposes it is convenient to encode an $n \times n$ skew-symmetric integer matrix B by a finite oriented multigraph without loops and $2-$cycles called *quiver*. More precisely, a *quiver* S is a finite 1-dimensional simplicial complex with oriented weighted edges, where weights are positive integers.

Vertices of S are labeled by $[1, \dots, n]$. If $B_{i,j} > 0$, we join vertices i and j by an edge directed from i to j and assign to this edge weight $B_{i,j}$. Vice versa, any quiver with integer positive weights corresponds to a skew-symmetric integer matrix. While drawing quivers, usually we draw edges of weight $B_{i,j}$ as edges of multiplicity $B_{i,j}$, but sometimes, when it is more convenient, we put the weight on simple edge.

Mutations of exchange matrices induce *mutations of quivers*. If S is the quiver corresponding to matrix B, and B' is a mutation of B in direction k, then we call the quiver S' associated to B' a *mutation of S in direction k*. It is easy to see that mutation in direction k changes weights of quiver in the way described in the following picture (see e.g. [K2]):



$$c + d = ab$$

Figure 2.1. Mutations of quivers

Clearly, for given quiver the notion of *mutation class* is well-defined. We call a quiver *mutation-finite* if its mutation class is finite. Thus, we are able to reformulate the problem of classification of exchange matrices of finite type in terms of quivers: *find all mutation-finite quivers.*

The following criterion for a quiver to be mutation-finite is well-known (see e.g. [DO, Corollary 8])

Theorem 2.6. *A quiver S of order at least 3 is mutation-finite if and only if any quiver in the mutation class of S contains no edges of weight greater than 2.*

One can use linear algebra tools to describe quiver mutations.

Let $e_1, \dots, e_n$ be a basis of vector space V over field k equipped with a skew-symmetric form Ω. Denote by B the matrix of the form Ω with respect to the basis e_i, i.e. $B_{ij} = \Omega(e_i, e_j)$.

For each $i \in [1, n]$ we define new basis $e'_1, \dots, e'_n$ in the following way:

$$\begin{aligned} e'_i &= -e_i \\ e'_j &= e_j, \text{ if } \Omega(e_i, e_j) \geq 0 \\ e'_j &= e_j - \Omega(e_i, e_j)e_i, \text{ if } \Omega(e_i, e_j) < 0. \end{aligned}$$

Note that matrix B' of the form Ω in basis e'_k is the mutation of matrix B in direction i.

From now on, we use language of quivers only. Let us fix some notations we will use throughout the paper.

Let S be a quiver. A *subquiver* $S_1 \subset S$ is a subcomplex of S. The *order* $|S|$ is the number of vertices of quiver S. If S_1 and S_2 are subquivers of quiver S, we denote by $\langle S_1, S_2 \rangle$ the subquiver of S spanned by all the vertices of S_1 and S_2.

Let S_1 and S_2 be subquivers of S having no common vertices. We say that S_1 and S_2 are *orthogonal* ($S_1 \perp S_2$) if no edge joins vertices of S_1 and S_2.

We denote by $\mathrm{Val}_S(v)$ the unsigned valence of v in S (a double edge adds two to the valence). A *leaf* of S is a vertex joined with exactly one vertex in S.

3. Ideas of the proof

In this section we present all key steps of the proof.

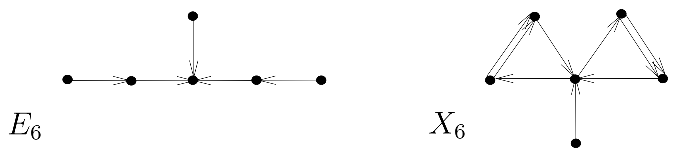
We need to prove that all mutation-finite quivers except some finite number of mutation classes satisfy some special properties, namely they are block-decomposable (see Definition 4.1). In Section 5.11 we define a *minimal non-decomposable quiver* as a non-decomposable quiver minimal with respect to inclusion (see Definition 5.1). By definition, any non-decomposable quiver contains a minimal non-decomposable quiver as a subquiver. First, we prove the following theorem:

Theorem 5.2. *Any minimal non-decomposable quiver contains at most* 7 *vertices.*

The proof of Theorem 5.2 contains the bulk of all the technical details in the paper. We assume that there exists a minimal non-decomposable quiver of order at least 8, and investigate the structure of block decompositions of proper subquivers of S. By exhaustive case-by-case consideration we prove that S is also block-decomposable. The main tools are Propositions 4.6 and 4.8 which under some assumptions produce a block decomposition of S from block decompositions of proper subquivers of S.

The next step is to prove the following key theorem:

Theorem 5.11. *Any minimal non-decomposable mutation-finite quiver is mutation-equivalent to one of the two quivers* X_6 *and* E_6 *shown below.*



The proof is based on the fact that the number of mutation-finite quivers of order at most 7 is finite, and all such quivers can be easily classified. For that, we use an inductive procedure: we take one representative from each finite mutation class of quivers of order n and attach a vertex by edges of multiplicity at most 2 in all possible ways (here we use Theorem 2.6). For each obtained quiver we check if its mutation class is finite (by using Keller's applet for quivers mutations [K1]). In this way we get all the finite mutation classes of quivers of order $n+1$. After collecting all finite mutation classes of order at most 7, we analyze whether they are block-decomposable. It occurs that all the classes except ones containing X_6 and E_6 are block-decomposable. The two quivers X_6 and E_6 are non-decomposable by [DO, Propositions 4 and 6]).

Therefore, we proved that each mutation-finite non-decomposable quiver contains a subquiver mutation-equivalent to X_6 or E_6 (Corollary 5.13). This allows us to use the same inductive procedure to get all the finite classes of non-decomposable quivers. We attach a vertex to X_6 and E_6 by edges of multiplicity at most 2 in all possible ways. In this way we get all the finite mutation classes of non-decomposable quivers of order 7. More precisely, there are 3 of them, namely those containing X_7, E_7 and $\widetilde{E}_6$. Any mutation-finite non-decomposable quiver of order 8 should contain a subquiver mutation-equivalent to one of these 3 quivers due to the following lemma:

Lemma 6.4. *Let S_1 be a proper subquiver of S, let S_0 be a quiver mutation-equivalent to S_1. Then there exists a quiver S' which is mutation-equivalent to S and contains S_0.*

Using the same procedure, we list one-by-one all the mutation-finite non-decomposable quivers of order 8, 9 and 10. The results are the entries from the list by Derksen-Owen (see Fig. 6.1). Applying the inductive procedure to a unique mutation-finite non-decomposable quiver $E_8^{(1,1)}$ of order 10, we obtain no mutation-finite quivers. Now we use the following statement:

Corollary 6.3. Suppose that for some $d \geq 7$ there are no non-decomposable mutation-finite quivers of order d. Then order of any non-decomposable mutation-finite quiver does not exceed $d-1$.

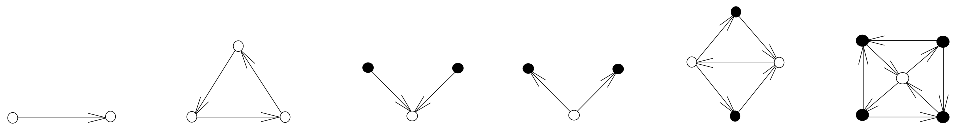
Corollary 6.3 implies that there is no non-decomposable mutation-finite quiver of order at least 11, which completes the proof of the Main Theorem.

4. Block decompositions of quivers

Let us start with definition of block-decomposable quivers (we rephrase Definition 13.1 from [FST]).

Definition 4.1. A *block* is a quiver isomorphic to one of the quivers with black/white colored vertices shown on Fig. 4.1, or to a single vertex. Vertices marked in white are called *outlets*. A connected quiver S is called *block-decomposable* if it can be obtained from a collection of blocks by identifying outlets of different blocks along some partial matching (matching of outlets of the same block is not allowed), where two edges with same endpoints and opposite directions cancel out, and two edges with same endpoints and same directions form an edge of weight 2. A non-connected quiver S is called block-decomposable either if S satisfies the definition above, or if S is a disjoint union of several mutually orthogonal quivers satisfying the definition above. If S is not block-decomposable then we call S *non-decomposable*. Depending on a block, we call it *a block of type* I, II, III, IV, V, or simply *a block of n-th type*.

We denote by $\mathcal{B}_{\mathrm{I}}$, $\mathcal{B}_{\mathrm{II}}$ etc. the isomorphism classes of blocks of types I, II, etc. respectively. For a block B we write $B \in \mathcal{B}_{\mathrm{I}}$ if B is of type I.



$\mathcal{B}_{\mathrm{I}}$ $\mathcal{B}_{\mathrm{II}}$ $\mathcal{B}_{\mathrm{IIIa}}$ $\mathcal{B}_{\mathrm{IIIb}}$ $\mathcal{B}_{\mathrm{IV}}$ $\mathcal{B}_{\mathrm{V}}$

FIGURE 4.1. Blocks. Outlets are colored white, dead ends are black.

Remark 4.2. It is shown in [FST] that block-decomposable quivers have a nice geometrical interpretation: they are in one-to-one correspondence with adjacency matrices of arcs of ideal (tagged) triangulations of bordered two-dimensional surfaces with marked points (see [FST, Section 13] for the detailed explanations). Mutations of block-decomposable quivers correspond to flips of triangulations.

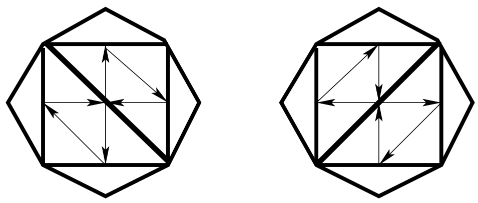


FIGURE 4.2. Fat sides denote arcs of triangulations. Arrows form corresponding quiver.

In particular, this description implies that mutation class of any block-decomposable quiver is finite (indeed, the absolute value of an entry of adjacency matrix can not exceed 2). Another immediate corollary is that any subquiver of a block-decomposable is block-decomposable too.

Remark 4.3. As the following example shows, a block decomposition of quiver (if exists) may not be unique.

Example 4.4. There are two ways to decompose an oriented triangle into blocks, see Fig. 4.3.

We say that a vertex of a block is a *dead end* if it is not an outlet.

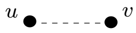
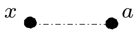
Remark 4.5. Notice that if S is decomposed into blocks, and $u \in S$ is a dead end of some block B, then any edge of B incident to u can not cancel with any other edge of another block and therefore must appear in S with weight 1.

FIGURE 4.3. Two different decompositions of an oriented triangle into blocks

We call a vertex $u \in S$ an *outlet of* S if S is block-decomposable and there exists a block decomposition of S such that u is contained in exactly one block B, and u is an outlet of B.

We use the following notations. For two vertices u_i, u_j of quiver S we denote by (u_i, u_j) a directed arc connecting u_i and u_j which may or may not belong to S. It may be directed either way. By (u_i, u_j, u_k) we denote oriented triangle with vertices u_i, u_j, u_k which is oriented either way and whose edges also may or may not belong to S. We use standard notation $\langle u_i, u_j \rangle$ for an edge of S.

While drawing quivers, we keep the following notation:

- a non-oriented edge is used when orientation does not play any role in the proof;
- an edge  is an edge of a block containing u and v, where u and v are not joined in the quiver. The figure assumes a fixed block decomposition;
- an edge  means that x is joined with a by some edge.

Proposition 4.6. *Let S be a connected quiver with n vertices, and let b be a vertex of S satisfying the following properties:*

(0) *$S \setminus b$ is not connected;*
(1) *for any $u \in S$ the quiver $S \setminus u$ is block-decomposable;*
(2) *at least one connected component of $S \setminus b$ has at least 3 vertices;*
(3) *each connected component of $S \setminus b$ has at most $n - 3$ vertices.*

Then S is block-decomposable.

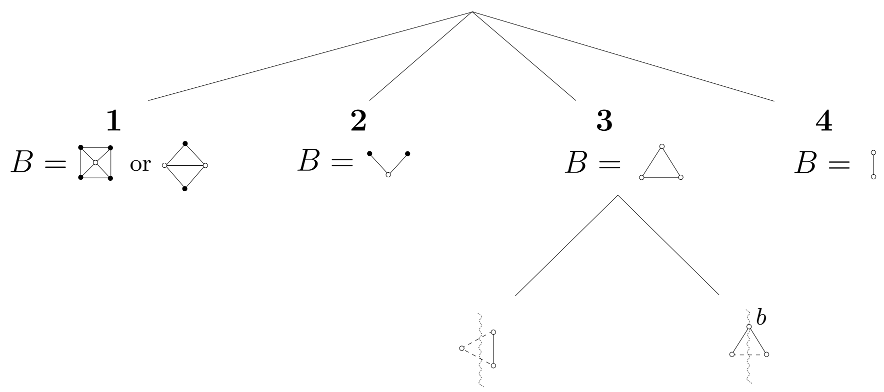
Proof. We divide $S \setminus b$ into two parts S_1 and S_2 in the following way: S_1 is any connected component of $S \setminus b$ with at least 3 vertices (it exists by assumption (2)), and $S_2 = S \setminus \langle b, S_1 \rangle$. Notice that assumption (3) implies that $|S_2| \geq 2$.

Now choose vertices $a_1 \in S_1$ and $a_2 \in S_2$ satisfying the following conditions: $S \setminus a_i$ is connected, and $S \setminus a_i$ does not contain leaves attached to b and belonging to S_1. We always can take as a_2 a vertex of S_2 at the maximal distance from b. To choose $a_1 \in S_1$, we look at the vertices on maximal distance from b in $\langle S_1, b\rangle$. If the maximal distance from b in $S_1 \cup b$ is greater than 2, then we may take as a_1 any vertex of S_1 at the maximal distance from b: in this case $\langle S_1, b\rangle \setminus a_1$ does not contain leaves attached to b. If S_1 contains a leaf of S, then we can take as a_1 this leaf. This does not produce leaves of $\langle S_1, b\rangle \setminus a_1$ since S_1 is connected and $|S_1| \geq 3$. Finally, if the maximal distance from b in $S_1 \cup b$ is 2, and each vertex on distance 2 is not a leaf, we take as a_1 any neighbour of b with minimal number of neighbours in S_1. Again, this does not produce leaves of $\langle S_1, b\rangle \setminus a_1$.

We will prove now that each $\langle S_i, b\rangle$ is block-decomposable with outlet b. Since b is the only common vertex of $\langle S_1, b\rangle$ and $\langle S_2, b\rangle$, this will imply that S is block-decomposable.

Consider the quiver $S \setminus a_2$. It is block-decomposable by assumption (1). Choose any its decomposition into blocks. Let us prove that for any block B either $B \cap S_1 = \emptyset$ or $B \cap (S_2 \setminus a_2) = \emptyset$. In particular, this will imply that S_1 is block-decomposable, and b is an outlet (since $|S_2| \geq 2$ and $S \setminus a_2$ is connected). Suppose that for some B both intersections $B \cap S_1$ and $B \cap (S_2 \setminus a_2)$ are not empty. We consider below all possible types of block B. Table 4.1 illustrates the plan of the proof.

Table 4.1. To the proof of Proposition 4.6



Case 1: $B \in \mathcal{B}_{\mathrm{IV}}$, or $B \in \mathcal{B}_{\mathrm{V}}$. At least one edge of B having a dead end runs from S_1 to $S_2 \setminus a_2$. By Remark 4.5 this edge appears in S contradicting assumption $S_1 \perp S_2$.

Case 2: $B \in \mathcal{B}_{\text{III}}$. A unique outlet of B cannot coincide with b due to the way of choosing a_1 and a_2. Therefore, some edge of B joins vertices of S_1 and $S_2 \setminus a_2$, which is impossible since both edges of B have dead ends.

Case 3: $B \in \mathcal{B}_{\text{II}}$. Suppose first that b is not a vertex of B. Then one vertex of B (say w) belongs to one part of $S \setminus a_2$ (i.e. in either $S_2 \setminus a_2$ or S_1), and the remaining two (w_1 and w_2) to the other. Since S does not contain edges (w, w_1) and (w, w_2) these edges must cancel out with edges from other blocks of block decomposition. Since all these additional blocks contain w, the only way is to attach a block B_1 of second type along all the three outlets of B. This results in three vertices w, w_1 and w_2 unjoined with all other vertices of $S \setminus a_2$. In particular, $\langle S_1, b \rangle$ is not connected (since $|S_1| \geq 2$), which implies that S is not connected either.

Now suppose that b is a vertex of B. Then we may assume that other vertices w_1 and w_2 of B belong to S_1 and $S_2 \setminus a_2$ respectively. S does not contain edge (w_1, w_2). The only way to avoid it in S is to glue an edge (w_1, w_2) (which is a block B_1 of the first type) to B (since blocks of all other types are already prohibited by Cases 1,2 and the past of this one). Then w_1 is a leaf of $\langle S_1, b \rangle$ attached to b in contradiction with the way of choosing of a_2.

Case 4: $B \in \mathcal{B}_{\text{I}}$. Let $B = (w_1, w_2)$, $w_1 \in S_1$ and $w_2 \in S_2 \setminus a_2$. The only way to avoid this edge in S is to glue another edge (w_1, w_2) (which is a block B_1 of the same type) to B (all other blocks are already prohibited by previous cases). Then $\langle S_1, b \rangle$ is not connected, which implies that S is not connected.

Since all the four cases are done, we obtain that S_1 is block-decomposable with outlet b. Considering $S \setminus a_1$ instead of $S \setminus a_2$, in a similar way we conclude that S_2 is also block-decomposable with outlet b. Gluing these decompositions together along b we obtain a block decomposition of S.

□

Remark 4.7. In all the situations where we will apply Proposition 4.6 connectedness of S and assumption (1) will be stated in advance. Usually it is sufficient only to point out the vertex b (in this case we say that S is block-decomposable by Proposition 4.6 applied to b) and all the assumptions are evidently satisfied. Only in the proofs of Lemma 5.7 and Theorem 5.2 assumption (3) requires additional explanations.

Proposition 4.8. *Let S be a connected quiver $S = \langle S_1, b_1, b_2, S_2\rangle$, where $S_1 \perp S_2$, and S has at least 8 vertices. Suppose that*

(0) b_1 and b_2 are not joined in S;

(1) for any $u \in S$ the quiver $S \setminus u$ is block-decomposable;

(2) there exist $a_1 \in S_1, a_2 \in S_2$ such that

(2a) $S \setminus a_i$ is connected;

(2b) either $\langle S_i, b_1, b_2\rangle \setminus a_i$ or $\langle S_j, b_1, b_2\rangle$ (for $i, j = 1, 2,\ j \neq i$) contains no leaves attached to b_1;

similarly, either $\langle S_i, b_1, b_2\rangle \setminus a_i$ or $\langle S_j, b_1, b_2\rangle$ (for $j \neq i$) contains no leaves attached to b_2;

(2c) if a_i is joined with b_j (for $i, j = 1, 2$), then there is another vertex $w_i \in S_i$ attached to b_j.

Then S is block-decomposable.

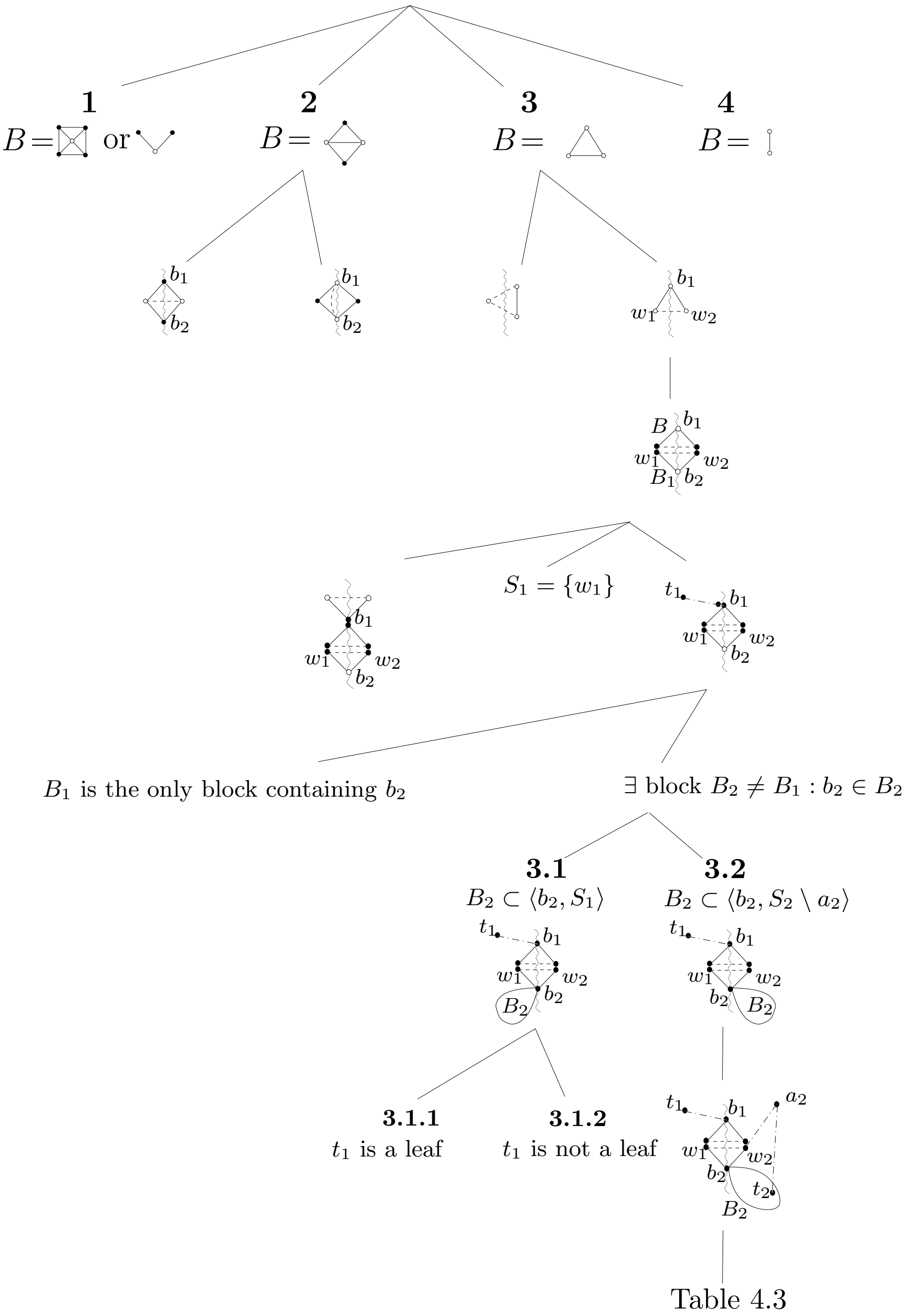
Proof. The plan is similar to the proof of Proposition 4.6. The idea is to prove that each S_i together with those of b_1, b_2 which are attached to S_i is block-decomposable with outlets b_1 and b_2 (or just one of them if the second is not joined with S_i). Then we combine together these block decompositions to obtain a decomposition of S. First we show that for any block decomposition of $S \setminus a_2$ any block B is contained entirely either in $\langle S_1, b_1, b_2\rangle$ or in $\langle S_2, b_1, b_2\rangle \setminus a_2$. For this, we consider any block decomposition of $S \setminus a_2$, assuming that for a block B both intersections $B \cap S_1$ and $B \cap (S_2 \setminus a_2)$ are not empty. We consider all possible types of block B (see Table 4.2) and obtain contradiction for each type.

Notice that the assumption (2c) implies that $|S_i| \geq 2$. We will refer this fact as assumption (3).

Case 1: $B \in \mathcal{B}_{\mathrm{V}}$, or $B \in \mathcal{B}_{\mathrm{III}}$. The proof is the same as in Proposition 4.6.

Case 2: $B \in \mathcal{B}_{\mathrm{IV}}$. Evidently, both $b_1, b_2 \in B$. We may assume that both b_1 and b_2 are either dead ends of B, or outlets of B (otherwise, by Remark 4.5 S contains simple edge (b_1, b_2) contradicting assumption (0)). First suppose that b_1 and b_2 are dead ends of B. Then we may assume that the remaining vertices w_1, w_2 of B lie in S_1 and $S_2 \setminus a_2$ respectively. The edge (w_1, w_2) of B must be canceled out by an edge of another block B_1, otherwise (w_1, w_2) appears in S contradicting $S_1 \perp S_2$. Block B_1 cannot be of type IV (otherwise none of its vertices

TABLE 4.2. To the proof of Proposition 4.8



is b_1, so the block directly connects S_1 and S_2). Therefore it is either of type II or I. If B_1 is of type II (i.e. a triangle with vertices w_1, w_2 and w, $w \neq b_i$), then we note that edges (w_1, w) and (w, w_2) are not canceled out by other blocks and appear in S. Therefore either $(w_1 w)$ or $(w w_2)$ connects S_1 to S_2, and, hence, contradicts the assumption $S_1 \perp S_2$. If we glue an edge (w_1, w_2) as a block of the first type, then w_1 is not joined with any other vertex of S_1. Since b_1 and b_2 are dead ends of B, they are not joined with any other vertex of S_1 either. Thus, due to assumption (3) the subquiver $\langle S_1, b_1, b_2 \rangle$ is not connected, so S is not connected either.

Now suppose that b_1 and b_2 are outlets of B. Denote by $w_1 \in S_1$ and $w_2 \in S_2 \setminus a_2$ the other vertices of B. To avoid the edge (b_1, b_2) in S, some block should be glued along this edge. If we glue a block of first or forth type, then we obtain a quiver with respectively 4 and 6 vertices without outlets. Since S has at least 8 vertices the complement of this quiver in $S \setminus a_2$ is nonempty and $S \setminus a_2$ is not connected contradicting the choice of a_2. If we glue a block of the second type (a triangle $b_1 b_2 w$), then we obtain a quiver with 5 vertices, the only outlet is w. Therefore, all the remaining vertices of $S \setminus a_2$ are not joined with vertices of B. In particular, $w \in S_1$ (otherwise S_1 consists of w_1 only), so S_2 consists of w_2 and a_2. Hence, S is block-decomposable by Proposition 4.6 applied to w.

Case 3: $B \in \mathcal{B}_{\mathrm{II}}$. We may assume that vertices of B are b_1, $w_1 \in S_1$ and $w_2 \in S_2 \setminus a_2$. Since edge (w_1, w_2) is not in S it must be canceled by a block B_1. It is either of type I or II (since all other types are already excluded above). B_1 is not of the first type, otherwise w_1 and w_2 are leaves in S_1 and S_2, resp., attached to b_1, contradicting (2b). Therefore, B_1 is of type II, and the remaining vertex of B_1 is either b_1 or b_2. If it is b_1 then S is not connected. We conclude that $b_2 \in B_1$.

Any other block B_1' with vertex b_1 is contained in either $\langle S_1, b_1 \rangle$ or $\langle b_1, S_2 \setminus a_2 \rangle$ (otherwise, if B_1' containing b_1 has nonempty intersection with both S_1 and $S_2 \setminus a_2$, then B_1' is again of type II. As above there exists B_2' containing b_2 completing B_1' in such a way that $\langle B, B_1, B_1', B_2' \rangle$ form a six vertex subquiver without outlets. Since $|S| \geq 8$, this implies that $S \setminus a_2$ is not connected.) Similarly, any block with vertex b_2 (other than B_1) is contained either in $\langle b_1, b_2, S_1 \rangle$ or in $\langle b_1, b_2, S_2 \setminus a_2 \rangle$. Further, no vertex of $S \setminus a_2$ except b_1, b_2 is joined with w_1 and w_2.

Therefore, $S \setminus a_2$ consists of $\langle b_1, b_2, w_1, w_2 \rangle$, blocks attached to b_1 and b_2, and vertices not joined with $\langle b_1, b_2, w_1, w_2 \rangle$. Combining that with assumption (3), we see that there is at least one vertex $t_1 \in S_1$ distinct from w_1 which is joined with at least one of b_1 and b_2 by a simple edge. We may assume that t_1 is attached to b_1. Since b_1 is contained in two blocks of $S \setminus a_2$, no vertex of S_2 except possibly a_2 is joined with b_1.

Suppose that no block except B_1 contains b_2. Then the subquiver $S \setminus \langle b_1, b_2, w_1, w_2, a_2 \rangle$ belongs to S_1 and is joined with b_1 only. Applying Proposition 4.6 to b_1 we conclude that S is block-decomposable.

Let us consider the case when some block B_2 is attached to b_2. As we have already shown, B_2 is entirely contained either in $\langle S_1, b_1, b_2 \rangle$ or in $\langle b_1, b_2, S_2 \setminus a_2 \rangle$.

Case 3.1: B_2 is contained in $\langle S_1, b_2 \rangle$. In this case S_2 consists of w_2 and a_2 only, so $|S_2| = 2$. Recall that w_1 is joined with b_1 and b_2 only, and define new decomposition of $S = \langle S_1', b_1, b_2, S_2' \rangle$, where $S_1' = S_1 \setminus w_1$, and $S_2' = \langle S_2, w_1 \rangle$. We will show that this decomposition satisfies all the assumptions of Proposition 4.8. Since $|S_2| = 2$ and $|S| \geq 8$, this decomposition satisfies $|S_1'|, |S_2'| \geq 3$. Therefore we may avoid Case 3.1.

Clearly, assumptions (0) and (1) hold. We need to choose vertices a_1' and a_2' satisfying conditions $(2a)$–$(2c)$. We keep $a_2' = a_2$. Recall that $t_1 \in S_1'$ is attached to b_1, and some vertex of B_2 (say t_2) is attached to b_2. So, if a_2 is not a leaf of S attached to exactly one of b_1 and b_2 we may choose as a_1' any vertex of S_1' different from both t_1 and t_2 and satisfying condition $(2a)$, see Fig. 4.4.

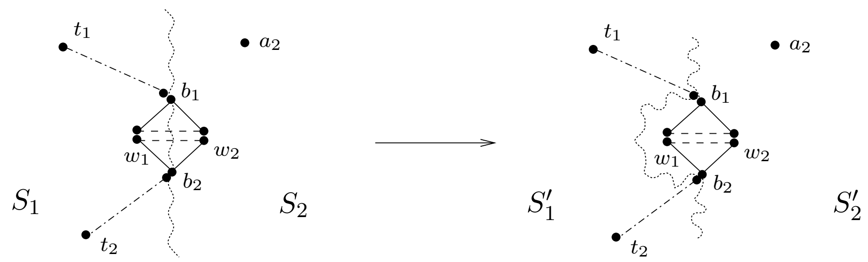


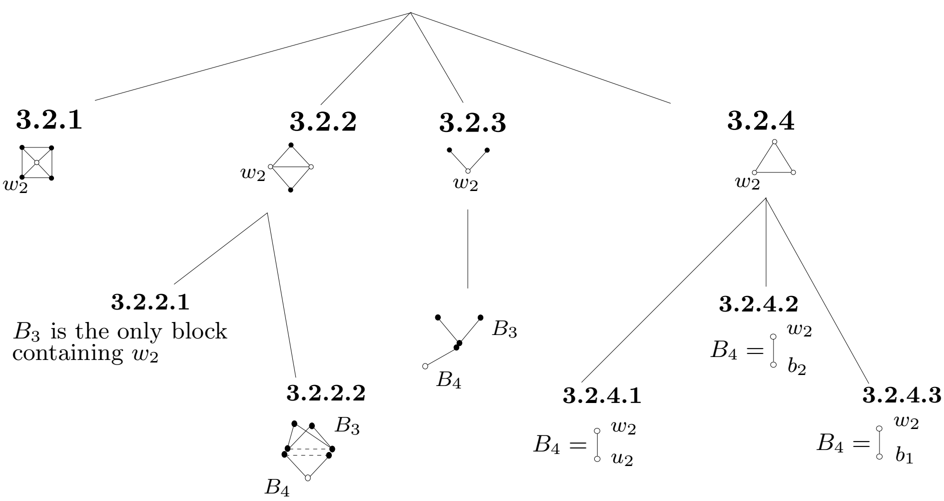
FIGURE 4.4. To the proof of Proposition 4.8, Case 3.1

Suppose that a_2 is a leaf of S attached to exactly one of b_1 and b_2, say to b_1. Notice that there exists at most one leaf of S in S_1' attached to b_1, otherwise assumption $(2b)$ does not hold for the initial decomposition of S. We may assume that if there is a leaf of S in S'_1 attached to b_1, then it is t_1. Consider the following two cases.

Case 3.1.1: t_1 is a leaf of S. If there exists another vertex of S'_1 attached to b_1, then $a'_1 = t_1$ satisfies all the assumptions. So, t_1 is the only vertex of S'_1 attached to b_1. Further, as we have seen above, $S \setminus \langle t_1, a_2 \rangle$ is block-decomposable with outlet b_1, vertices t_1 and a_2 are joined with b_1 only. Therefore, depending on the orientation of edges $\langle b_1, t_1 \rangle$ and $\langle b_1, a_2 \rangle$, we may glue either a block of the third type composed by b_1, t_1 and a_2, or a composition of a second type block with an extra edge (t_1, a_2) in appropriate direction, which implies that S is block-decomposable.

Case 3.1.2: t_1 is not a leaf of S. Denote by r_1 any vertex of S'_1 attached to t_1, and consider the following quiver $S' = \langle r_1, t_1, b_1, w_1, w_2, a_2 \rangle$. As a proper subquiver of S, S' should be block-decomposable. However, using the Java applet [K1] by Keller one can easily check that the mutation class of S' is infinite, so S' is non decomposable.

TABLE 4.3. To the proof of Proposition 4.8, Case 3.2



Case 3.2: B_2 is contained in $\langle b_2, S_2 \setminus a_2 \rangle$. If t_1 is not a leaf of S then applying Proposition 4.6 to b_1 we see that S is block-decomposable. Thus, we may assume that t_1 is a leaf of S. In this case S_1 consists of w_1 and t_1 only, so $|S_1| = 2$. If a_2 is joined with neither b_1 nor w_2, then S is block-decomposable by Proposition 4.6 applied to b_2. By the

same reason, a_2 is joined with some vertex $t_2 \in S \setminus \langle t_1, b_1, w_1, w_2, b_2 \rangle$. If a_2 is not joined with w_2, then switching S_1 and S_2 leads us back to Case 3.1. Thus, we may assume that a_2 is attached to w_2 by a simple edge.

Now take any block decomposition of $S \setminus t_1$ and consider all possible types of blocks containing vertex w_2 (see Table 4.3). Recall that the valence $\mathrm{Val}_{S\setminus t_1}(w_2) = 3$.

Case 3.2.1: w_2 lies in block B_3 of type V. In this case w_2 is a dead end of B_3 (due to its valence), one of b_1 and b_2 is a dead end of B_3, and another one is an outlet (since the orientations of edges $\langle w_2, b_1 \rangle$ and $\langle w_2, b_2 \rangle$ differ, see Fig. 4.5). Then the edge (b_1, b_2) in B_3 is not canceled out by any other edge. Therefore, b_1 and b_2 are joined in S contradicting assumption (0).

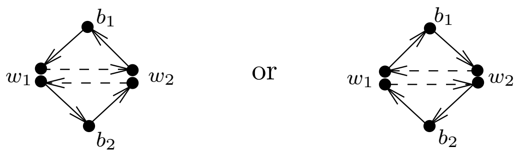


FIGURE 4.5. To the proof of Proposition 4.8, Case 3.2.1. Orientations of the edges $\langle w_2, b_1 \rangle$ and $\langle w_2, b_2 \rangle$ are different.

Case 3.2.2: w_2 is contained in block B_3 of type IV. In this case w_2 is an outlet of B_3 (since $\mathrm{Val}_{S\setminus t_1}(w_2) = 3$). Consider two cases.

Case 3.2.2.1: w_2 is contained in block B_3 only. Then one of b_1 and b_2 is a dead end of B_3, and another one is an outlet (since the orientations of edges $\langle w_2, b_1 \rangle$ and $\langle w_2, b_2 \rangle$ are different). Therefore, b_1 is joined with b_2, which contradicts assumption (0).

Case 3.2.2.2: w_2 is contained simultaneously in two blocks B_3 and B_4, $B_4 \neq B_3$. Since $\mathrm{Val}_{S\setminus t_1}(w_2) = 3$, B_4 is of second type. Again, the orientations of edges $\langle w_2, b_1 \rangle$ and $\langle w_2, b_2 \rangle$ are different, so a_2 is a dead end of B_3. This implies that valence of a_2 is 2, so only t_2 can be outlet of B_3. The second dead end of B_3 should be joined with both t_2 and w_2. Since b_1 is not joined with t_2, b_2 is a dead end of B_3. But this contradicts existence of the edge joining b_2 and w_1.

Case 3.2.3: w_2 is contained in block B_3 of type III. Since $\mathrm{Val}_{S\setminus t_1}(w_2) =$ 3, vertex w_2 is the outlet of B_3. By the same reason at least one of b_1

and b_2 is a dead end of B_3, hence a leaf of $S \setminus t_1$. But neither b_1 nor b_2 is a leaf and such B_3 does not exist.

Case 3.2.4: w_2 is contained in block B_3 of type II. Recall that $\mathrm{Val}_{S\setminus t_1}(w_2) = 3$, so in this case w_2 is also contained in block B_4 of first type. There are exactly three ways to place vertices w_2, b_1, b_2, a_2 into two blocks.

Case 3.2.4.1: $B_4 = (w_2, a_2)$. Then B_3 has an edge (b_1, b_2) which must cancel out in $S \setminus t_1$. Since b_1 is joined with w_1, the only way to cancel out edge (b_1, b_2) in $S \setminus t_1$ is to attach along this edge a second type block $B_5 = (b_1, b_2, w_1)$. Then $b_2 \in B_3 \cap B_5$, so it must be disjoint from B_2. Hence $B_2 = \emptyset$.

Case 3.2.4.2: $B_4 = (w_2, b_2)$. Then $B_3 = (w_2, b_1, a_2)$. Since w_1 is not joined with a_2 and $\mathrm{Val}_{S\setminus t_1}(b_1) \le 3$, vertices b_1 and w_1 compose a block B_5 of first type. Since $\mathrm{Val}_{S\setminus t_1}(w_1) = 2$, vertices w_1 and b_2 also compose a block B_6 of first type. Again, this implies that $b_2 \in B_3 \cap B_6$, so $B_2 = \emptyset$.

Case 3.2.4.3: $B_4 = (w_2, b_1)$. Then $B_3 = (w_2, b_2, a_2)$. The proof is similar to the previous case. Since w_1 is not joined with a_2, and $\mathrm{Val}_{S\setminus t_1}(b_1) \le 3$, vertices b_1 and w_1 compose a block B_5 of first type. Since $\mathrm{Val}_{S\setminus t_1}(w_1) = 2$, vertices w_1 and b_2 also compose a block B_6 of first type. This implies that $b_2 \in B_3 \cap B_6$, so again $B_2 = \emptyset$.

Case 4: $B \in \mathcal{B}_{\mathrm{I}}$. The proof is the same as in Proposition 4.6.

We call the connected component of $\langle S_i, b_1, b_2 \rangle$ containing S_i *the closure* of S_i and denote it by $\widetilde{S}_i$. We proved above that any block in the decomposition of $S \setminus a_2$ is entirely contained in exactly one of $\widetilde{S}_1$ and $\widetilde{S}_2$. Consider the union of all the blocks with vertices from $\widetilde{S}_1$ only. They form a block decomposition either of $\widetilde{S}_1$, or of $\widetilde{S}_1 \cup (b_1, b_2)$, i.e. $\widetilde{S}_1$ with edge (b_1, b_2). Due to assumption $(2c)$, in both cases vertices b_1 and b_2 are outlets. Similarly, considering a block decomposition of $S \setminus a_1$, we obtain a block decomposition either of $\widetilde{S}_2$, or of $\widetilde{S}_2 \cup (b_1, b_2)$ where both b_1 and b_2 are outlets.

Suppose that in the way described above we got block decompositions of $\widetilde{S}_1$ and $\widetilde{S}_2$. Then we can glue these decompositions to obtain a block decomposition of S. Now we will prove that in all other cases S is also block-decomposable.

Now suppose that for one of $\widetilde{S}_1$ and $\widetilde{S}_2$ (say $\widetilde{S}_1$) we got a block decomposition of $\widetilde{S}_1$ with an edge (b_1, b_2). Consider the corresponding decomposition of $S \setminus a_2$. Clearly, there is a block B_1 in the decomposition of $\widetilde{S}_1$ with an edge (b_1, b_2) containing both b_1 and b_2 as outlets. Since b_1 and b_2 are not joined in S, there exists a block B_2 with vertices from $\widetilde{S}_2$ containing the edge (b_1, b_2), again both b_1, b_2 are outlets. Notice that B_1 and B_2 are blocks of second or fourth type (block of third or fifth type has one outlet only; if B_i is a block of first type then no vertex of $S_i \setminus a_2$ can be attached to b_j, so $|S_i| \leq 1$).

First, we prove that if S is not block-decomposable then $|S_1| = 2$. Indeed, no vertex of S_1 except vertices of B_1 can be attached to b_1 or b_2. If B_1 is of fourth type, then both its vertices belonging to S_1 are dead ends, so $|S_1| = 2$. If B_1 is of second type with third vertex v_1, then S is block-decomposable by Proposition 4.6 applied to v_1 unless $|S_1| = 2$.

Finally, we look at the type of B_2. Again, no vertex of $S_2 \setminus a_2$ except vertices of B_2 can be attached to b_1 or b_2. If B_2 is of fourth type, we see that $|S_2| \leq 3$, so $|S| < 8$, which contradicts assumptions of the proposition. Therefore, we may assume that B_2 is of second type with third vertex v_2. In particular, v_2 is the only vertex of $S_2 \setminus a_2$ joined with b_1 and b_2. We will prove that S is block-decomposable.

If a_2 is joined with neither b_1 nor b_2, then S is block-decomposable by Proposition 4.6 applied to v_2. So, we may assume that a_2 is joined with one of b_1 and b_2, say b_1. If a_2 is joined with no vertex of $S_2 \setminus v_2$, then again S is block-decomposable by Proposition 4.6 applied to v_2. Hence, there is $t_2 \in S_2 \setminus v_2$ attached to a_2. Further, since $|S_1| = 2$ and $|S| \geq 8$, there exists a vertex $u_2 \in S_2 \setminus \langle v_2, a_2 \rangle$ joined with v_2 (see Fig. 4.6), otherwise S is block-decomposable by Proposition 4.6 applied to a_2.

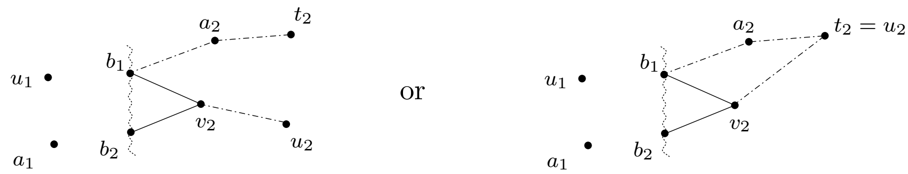


FIGURE 4.6. To the proof of Proposition 4.8.

Take $a_1 \in S_1$, and consider a block decomposition of $S \setminus a_1$. Denote by u_1 the remaining vertex of S_1, and consider all blocks containing b_1. Since $\mathrm{Val}_S(b_1) \leq 4$ and no block contains vertices from S_1 and S_2 simultaneously, b_1 does not belong to a block of fifth type. Since

a_2 and v_2 are not leaves, b_1 does not belong to a block of third type. Moreover, b_1 does not belong to a block of fourth type: in this case a_2 and v_2 are dead ends contradicting existence of u_2.

Block (u_1, b_1, b_2) can not exist, otherwise b_1 enters three blocks simultaneously. Therefore, (b_1, u_1) is a block of first type while b_1, a_2, v_2 compose a block of second type (since $S \setminus a_1$ is connected). Notice that v_2 is the only vertex of $S_2 \setminus a_2$ joined with b_1 and b_2, which implies that either (b_2, v_2) is a block of first type, or (b_2, v_2, a_2) is a block of second type (see Fig. 4.7). In both cases v_2 is contained in two blocks, so it cannot be attached to u_2. This contradiction completes the proof of the Proposition.

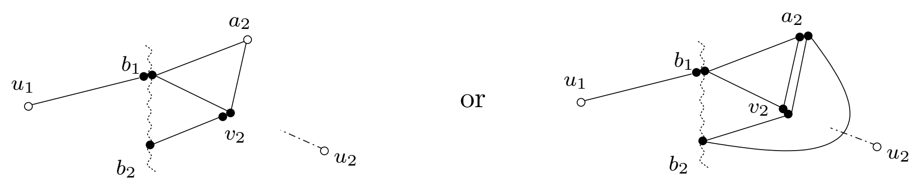


FIGURE 4.7. To the proof of Proposition 4.8, v_2 belongs to two blocks and cannot be joined with u_2.

□

Corollary 4.9. Suppose that $S = \langle S_1, b_1, b_2, S_2 \rangle$ satisfies all the assumptions of Proposition 4.8 except (2). Suppose also that $|S_1| \geq 2, |S_2| \geq 3$, and there exists $c_1 \in S_1$ such that the following holds:

(a) $S_1 \setminus c_1$ is connected;

(b) S_1 contains no leaves of S attached to b_1 or b_2, and $S_1 \setminus c_1$ contains no leaves of $S \setminus c_1$ attached to b_1 or b_2;

(c) $S_1 \setminus c_1$ is attached to both b_1 and b_2.

Then S is block-decomposable.

Proof. We will show how to choose a_1 and a_2 to fit into assumption (2) of Proposition 4.8.

As a_1 we can always take c_1. Clearly, assumptions (a)–(c) imply corresponding assumptions (2a)–(2c) of the Proposition 4.8 for a_1.

To choose a_2, we look how S_2 is attached to b_1 and b_2. If either S_2 is not attached to one of them (say b_2) or there is a vertex $v_2 \in S_2$ joined with both b_1 and b_2, then we take as a_2 any vertex of S_2 being at the maximal distance from b_1; otherwise, we fix two vertices

$v_1, v_2 \in S_2$ joined with b_1 and b_2 respectively, and take as a_2 any vertex of $S_2 \setminus \langle v_1, v_2 \rangle$ being at the maximal distance from b_1.

□

5. Minimal non-decomposable quivers

Our aim is to prove that any non-decomposable quiver contains a subquiver of relatively small order which is non-decomposable either.

Definition 5.1. A *minimal non-decomposable quiver* S is a quiver that

- is non-decomposable;
- for any $u \in S$ the quiver $S \setminus u$ is block-decomposable.

Notice that a minimal non-decomposable quiver is connected. Indeed, if S is non-connected and non-decomposable, then at least one connected component of S is non-decomposable either.

Theorem 5.2. *Any minimal non-decomposable quiver contains at most* 7 *vertices.*

The plan of the proof is the following. We assume that there exists a quiver S of order at least 8 satisfying the assumptions of Theorem 5.2, and show for each type of block that if a block-decomposable subquiver $S \setminus u$ contains block of this type then S is also block-decomposable.

Throughout this section we assume that S satisfies the assumptions of Theorem 5.2. Here we emphasize that we do not assume the mutation class of S to be finite.

A *link* $L_S(v)$ of vertex v in S is a subquiver of S spanned by all neighbors of v. If S is block-decomposable, we introduce for a given block decomposition a quiver $\Theta_S(v)$ obtained by gluing all blocks either containing v or having at least two points in common with $L_S(v)$. Notice that $\Theta_S(v)$ may not be a subquiver of S. Clearly, $L_S(v)$ is a subquiver of $\Theta_S(v)$ for any block decomposition of S.

Lemma 5.3. *For any $x \in S$ any block decomposition of $S \setminus x$ does not contain blocks of type* V*.*

To prove the lemma we use the following proposition.

Proposition 5.4. *Suppose that $S \setminus x$ contains a subquiver S_1 consisting of a block B of type* V *(with dead ends $v_1, \dots, v_4$ and outlet v) and a vertex t joined with v (and probably with some of v_i). Then* *for any $u \in S \setminus S_1$ and any block decomposition of $S \setminus u$ a subquiver $\langle v, v_1, \dots, v_4 \rangle$ is contained in one block of type* V. *In particular, t does not attach to any of v_i, $i = 1, \dots, 4$.*

Proof. Take any $u \in S \setminus S_1$ and consider any block decomposition of $S_2 = S \setminus u$. Since valence of v in S_2 is at least 5, v is contained in exactly two blocks B_1 and B_2, at least one of which is of the type V or IV. Suppose that none of B_1 and B_2 is of the type V, and let B_1 be of the type IV. Then for any choice of B_2 the number of vertices of S_2 which are neighbors of v and have valence at least three in S_2 does not exceed 3. However, there are at least four such vertices $v_1, \dots, v_4$, so the contradiction implies that we may assume B_1 to be of the type V with outlet v.

Since block B of $S \setminus x$ is of type V, the subquiver $\langle v_1, v_2, v_3, v_4 \rangle \subset S$ is a cycle. At the same time, the link $L_{S_2}(v)$ is a disjoint union of a cycle of order 4 (composed by dead ends of B_1) and another quiver with at most 4 vertices (composed by vertices of $B_2 \setminus v$). If we assume that v_1, v_2, v_3, v_4 are not contained in one block B_1 (or B_2) in S_2, then v_1, v_2, v_3, v_4 do not compose a cycle, and we come to a contradiction. To complete the proof it is enough to notice that only block of type V contains chordless cycle of length 4.

□

Proof of Lemma 5.3. Suppose that a block decomposition of $S \setminus x$ contains a block B of type V with dead ends $v_1, \dots, v_4$ and outlet v. Consider two cases: either $\mathrm{Val}_S(v) \geq 5$ or $\mathrm{Val}_S(v) = 4$.

Case 1: $\mathrm{Val}_S(v) \geq 5$. Then there exists $u \in S$, $u \notin B$ that is joined with v. Denote $S_1 = \langle v, v_1, v_2, v_3, v_4, u \rangle$, and consider any block decomposition of $S_2 = S \setminus w_2$ for any $w_2 \notin S_1$. By Proposition 5.4, $\langle v, v_1, v_2, v_3, v_4 \rangle$ form a block B_1 of type V with the outlet v. Therefore, no vertex of $\langle u, S_2 \setminus S_1 \rangle$ is joined with v_1, v_2, v_3, v_4. Since $|S| \geq 8$, we have $S \setminus \langle w_2, S_1 \rangle \neq \emptyset$. Consider a block decomposition of $S_3 = S \setminus w_3$ for some $w_3 \in S \setminus \langle S_1, w_2 \rangle$. No vertex of $S_3 \setminus S_1$ is joined with v_1, v_2, v_3, v_4. In particular, we obtain that none of w_2, w_3, u is joined with v_1, v_2, v_3, v_4. Moreover, since w_2 and w_3 are arbitrary vertices of $S \setminus S_1$, this implies that no vertex of $\langle u, S \setminus S_1 \rangle$ is joined with v_1, v_2, v_3, v_4. Thus, S is block-decomposable by Proposition 4.6 applied to v.

Case 2: $\mathrm{Val}_S(v) = 4$. Fix a block decomposition of $S \setminus x$ containing B. Since $\mathrm{Val}_S(v) = 4$ and $v_1, \dots, v_4$ are dead ends, no vertex of

$S\backslash\{x\cup B\}$ is joined with vertices of B. Again, S is block-decomposable by Proposition 4.6 applied to x.

□

Lemma 5.5. *For any $x \in S$ no block decomposition of $S \setminus x$ contains blocks of type* IV*.*

Proof. Consider any block decomposition of $S \setminus x$. Any vertex is contained in at most two blocks. By Lemma 5.3, any block decomposition of $S \setminus x$ does not contain blocks of type V. This implies that the valence $\mathrm{Val}_{S\setminus x}(v)$ does not exceed 6 for any $v \in S\setminus x$. Thus, $\mathrm{Val}_S(v) \le 8$ (recall that any proper subquiver of S, and therefore S itself, does not contain edges of multiplicity greater than 2 due to Theorem 2.6).

Now let B be a block of type IV in some block decomposition of $S\setminus x$, denote by v_1 and v_2 the outlets of B, and assume that $\mathrm{Val}_{S\setminus x}(v_2) \le \mathrm{Val}_{S\setminus x}(v_1) \le 6$. If $\mathrm{Val}_{S\setminus x}(v_2) = \mathrm{Val}_{S\setminus x}(v_1)$, we assume that $\mathrm{Val}_S(v_2) \le \mathrm{Val}_S(v_1)$. We analyze the situation case by case with respect to the valence $\mathrm{Val}_{S\setminus x}(v_1)$ decreasing. Each case splits in two: either v_1 is joined with x or not (see Table 5.1).

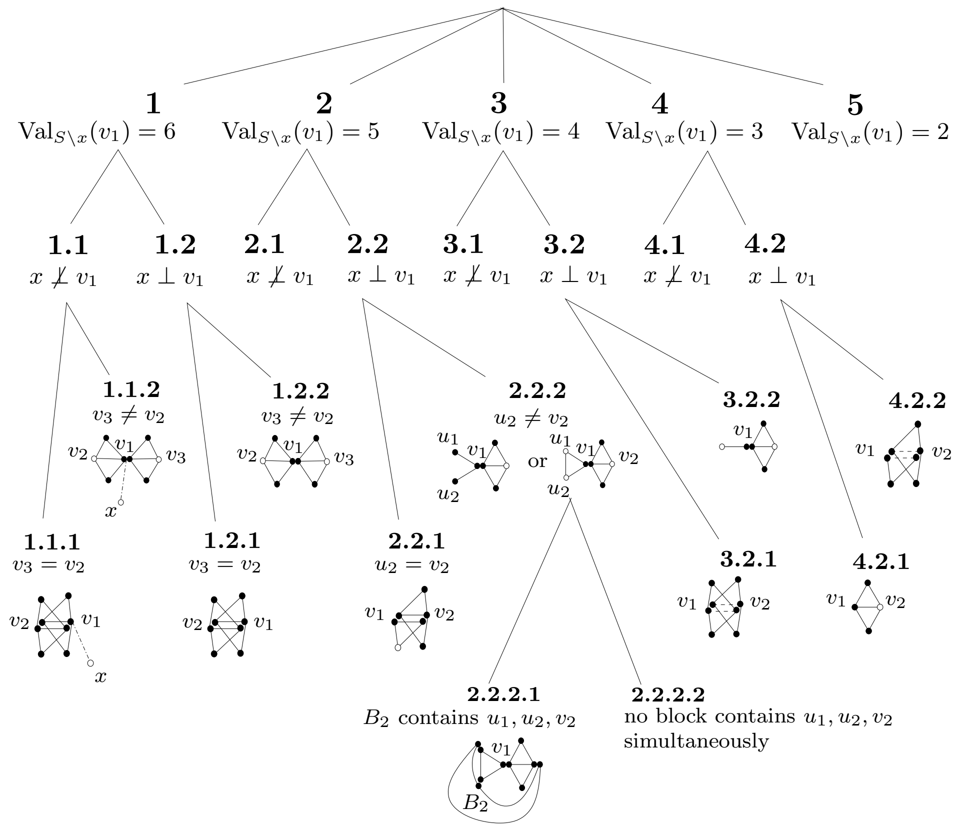
Notice that x may be joined with v_1 by a simple edge only. Indeed, suppose that x is joined with v_1 by a double edge. Denote by w_1 and w_2 dead ends of B. Since the subquiver $\langle x, v_1, w_1\rangle$ is decomposable and its mutation class is finite, x is joined with w_1 and the triangle composed by x, v_1 and w_1 is oriented (this follows from the fact that the mutation class of a chain of a double edge and a simple edge is infinite independently of the orientations of edges, and the mutation class of a non-oriented triangle containing a double edge is also infinite). By the same reason, v_2 is joined with both x and v_1, and the triangle composed by x, v_1 and v_2 is oriented. Thus, directions of edges $\langle v_1, w_1\rangle$ and $\langle v_1, v_2\rangle$ induce opposite orientations of edge $\langle x, w_1\rangle$ (see Fig. 5.1). Obtained contradiction shows that $\mathrm{Val}_S(v_1) \le \mathrm{Val}_{S\setminus x}(v_1) + 1$.

Case 1: $\mathrm{Val}_{S\setminus x}(v_1) = 6$.

Case 1.1: $x \not\perp v$, hence $\mathrm{Val}_S(v_1) = 7$. Then v_1 is contained in block B_1 of type IV, and v_1 is joined with x. Denote the dead ends of B_1 by w_3 and w_4, and the remaining outlet by v_3.

Case 1.1.1: v_3 coincides with v_2. Since $\mathrm{Val}_S(v_1) = 7$, v_1 and v_2 are joined by a double edge. Consider $S_1 = S\setminus w_1$ with some block decomposition. Clearly, $\mathrm{Val}_{S_1}(v_1) = 6$, and subquiver $\langle v_1, v_2, x, w_2, w_3, w_4\rangle \subset S_1$ is obtained by gluing two blocks of fourth type along the edge (v_1, v_2). In particular, x is a dead end of one of these blocks, so x is

TABLE 5.1. To the proof of Lemma 5.5



joined with v_2 and is not joined with w_2, w_3, w_4. Similarly, considering $S_2 = S\setminus w_2$, we see that x is not joined with w_1 either.

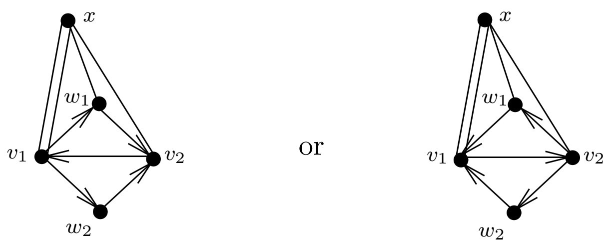


FIGURE 5.1. To the proof of Lemma 5.5. One of the triangles xv_1v_2 and xv_1w_1 is non-oriented.

Now consider subquiver $S' = \langle v_1, x, w_1, w_2, w_3, w_4 \rangle$. The only edges in S' are those joining v_1 with other vertices. Using Keller's applet [K1] we can check that mutation class of S' is infinite. Recall that S' is a subquiver of block-decomposable quiver and, hence, is block-decomposable itself. Therefore, its mutation class must be finite, and obtained contradiction eliminates the considered case.

Case 1.1.2: v_3 does not coincide with v_2. Consider $S_1 = S \setminus v_3$. Since $\mathrm{Val}_{S_1}(v_1) = 6$, subquiver $\langle v_1, v_2, x, w_1, w_2, w_3, w_4 \rangle \subset S_1$ is obtained by gluing of two blocks of fourth type at v_1. Vertices w_3 and w_4 are not joined, so x is an outlet of block $\langle v_1, x, w_3, w_4 \rangle$. In particular, x is joined with w_3 and w_4 and is not joined with w_1 and w_2. Now, considering $S_2 = S \setminus v_2$, we obtain in a similar way that x is joined with w_1 and w_2 and is not joined with w_3 and w_4, so we come to a contradiction.

Case 1.2: $x \perp v_1$, hence $\mathrm{Val}_S(v_1) = 6$. As in the previous case, v_1 is contained in block B_1 of type IV. Denote the dead ends of B_1 by w_3 and w_4, and the remaining outlet by v_3.

Case 1.2.1: v_3 coincides with v_2. Since $\mathrm{Val}_S(v_1) = 6$, v_1 and v_2 are joined by a double edge. The only outlets of B and B_1 are v_1 and v_2, and they are already contained in two blocks each. Therefore, the quiver spanned by B and B_1 has no outlets, so any other vertex of S except x is not joined with $S' = \langle v_1, v_2, w_1, w_2, w_3, w_4 \rangle$. Now take any vertex u distinct from x which is not contained in B or B_1, and consider any block decomposition of $S_1 = S \setminus u$. Since $\mathrm{Val}_{S_1}(v_1) = \mathrm{Val}_{S_1}(v_2) = 6$, the subquiver $S' \subset S_1$ is again obtained by gluing of two blocks of fourth type along the edge (v_1, v_2). By the reasons described above, no vertex of S except u is joined with vertices of S'. This implies that neither x nor u is attached to S', so no vertex of $S \setminus S'$ is attached to S' and S is not connected.

Case 1.2.2: v_3 does not coincide with v_2. Recall that none of vertices w_1, w_2, w_3, w_4 is joined with vertices of $S \setminus \{x \cup (B_1 \cap B_2)\}$. Let us prove that they are not joined with x either.

There are two options for link $L_{S \setminus x}(v_1)$: either v_2 is joined with v_3 or not. Notice that if two vertices u and v are dead ends of a block of type IV with outlet v_1 in some block decomposition of any quiver S', then u and v are leaves of the link $L_{S'}(v)$, and the distance between them equals two.

Consider the quiver $S_1 = S \setminus w_1$ with some block decomposition. Since $\mathrm{Val}_{S_1}(v_1) = 5$, subquiver $\langle v_1, v_2, v_3, w_2, w_3, w_4 \rangle \subset S_1$ is obtained by gluing a block of fourth type and a block of second or third type at v_1. Looking at the link $L_{S_1}(v_1)$, we see that there is only one pair of leaves at distance two, namely w_3 and w_4. Therefore, vertices v_1, v_3, w_3, w_4 are contained in one block of fourth type. In particular, x is not joined with w_3 and w_4. Similarly, considering $S_2 = S \setminus w_3$, we obtain that x is not joined with w_1 and w_2.

Let us take another one look at block decomposition of S_1. Since w_2 is joined with v_2, vertices v_1, v_2, w_2 are contained in a block of second type, i.e. triangle. Since none of w_1 and w_2 is joined with any other vertex of S than v_1 and v_2 we can replace triangle $v_1v_2w_2$ by block (v_1, v_2, w_1, w_2) of type IV to obtain a block decomposition of S.

Case 2: $\mathrm{Val}_{S\setminus x}(v_1) = 5$.

Case 2.1: $x \not\perp v_1$, $\mathrm{Val}_S(v_1) = 6$. Since $|S| \geq 8$, there exists $y \in S$ which is not joined with v_1. Since $\mathrm{Val}_{S\setminus y}(v_1) = 6$, in any block decomposition of $S \setminus y$ vertex v_1 is an outlet of a block of type IV, so we may refer to Case 1.2.

Case 2.2: $x \perp v_1$, $\mathrm{Val}_S(v_1) = 5$. Vertex v_1 is contained in block B and in block B_1 of type II or III. Vertices w_1 and w_2 are dead ends of B, so they are joined in $S \setminus x$ with v_1 and v_2 only.

Denote by u_1 and u_2 the remaining vertices of B_1, and consider the quiver $\Theta_{S\setminus x}(v_1)$. Since the union of B and B_1 has at most 3 outlets, $\Theta_{S\setminus x}(v_1)$ consists of blocks B, B_1 and probably some block B_2 containing at least two of vertices v_2, u_1 and u_2. Consider the following three cases.

Case 2.2.1: Vertex v_2 coincides with u_2. In this case v_1 and v_2 are joined by a double edge, B_1 is a block of second type (which implies that u_1 is joined with v_2, so $\mathrm{Val}_S(v_2) \geq 5$), and the union of B and B_1 has a unique outlet u_1. Thus, $\langle L_S(v_1), v_1\rangle \setminus u_1$ may be joined with x only. Since $\mathrm{Val}_S(v_2) \leq \mathrm{Val}_S(v_1) = 5$, x is not joined with v_1 and v_2. If x is not joined with w_1 and w_2 either, then we can apply Proposition 4.6 to u_1. Therefore, we may assume that x is joined with at least one of w_1 and w_2, say w_1.

Now take any $y \notin \langle B, u_1, x\rangle$ and consider $S_1 = S \setminus y$ with some block decomposition. Recall that y cannot be joined with any vertex of B. Since $\mathrm{Val}_{S_1}(v_1) = 5$, v_1 is contained in some fourth type block of this decomposition together with v_2 and two of w_1, w_2, u_1. But w_1 is joined with x, so it cannot be a dead end of the block. Hence, v_1, v_2, w_2, u_1 compose a block of type IV with outlets v_1, v_2 and dead ends w_2, u_1. In particular, neither w_2 nor u_1 is attached to x. If y is not joined with u_1, then we can apply Proposition 4.6 to w_1. Therefore, we may assume that y is joined with u_1.

By assumption, $|S| \geq 8$. Thus, we can take a vertex z which does not coincide with any of preceding ones, and consider $S_2 = S \setminus z$ (see Fig. 5.2). As explained above, any block decomposition of subquiver $\langle B, u_1 \rangle$ is a union of two blocks with one outlet only. However, w_1 is joined with x, and u_1 is joined with y, so we come to a contradiction.

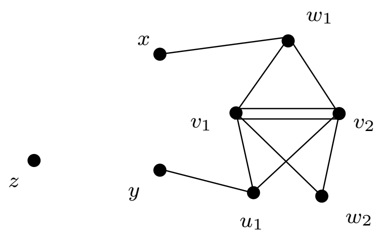


FIGURE 5.2. To the proof of Lemma 5.5, Case 2.2.1

Case 2.2.2: Neither u_1 nor u_2 coincides with v_2. The case also splits into the following two.

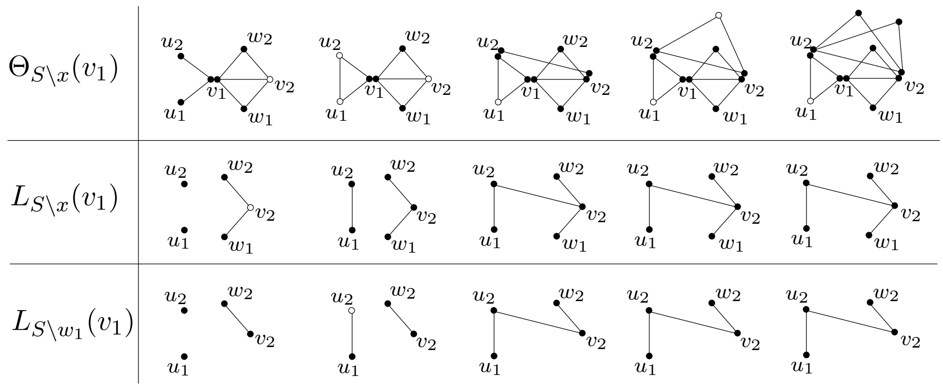
Case 2.2.2.1: $\Theta_{S\setminus x}(v_1)$ consists of three blocks B, B_1, B_2, where B_2 is a triangle with vertices v_2, u_1 and u_2. In this case the quiver $\Theta_{S\setminus x}(v_1)$ has no outlets, so no vertex of $S \setminus \Theta_{S\setminus x}(v_1)$ except x is joined with $\Theta_{S\setminus x}(v_1)$.

Take any vertex $y \in S \setminus \Theta_{S\setminus x}(v_1)$ distinct from x, and consider quiver $S_1 = S \setminus y$. The quiver $\Theta_{S\setminus y}(v_1)$ is spanned by $\Theta_{S\setminus x}(v_1)$ and probably x. Since $\mathrm{Val}_{S_1}(v_1) = 5$, three of vertices w_1, w_2, v_2, u_1, u_2 should compose a block B' of type IV together with v_1. Since none of w_1, w_2 is joined with any of u_1, u_2, that block contains v_2. Vertex v_1 is also contained in some block B'' of type II or III which contains two remaining vertices of $\Theta_{S\setminus x}(v_1)$. Similarly, the same two vertices lie in some block B''' of type II or III containing v_2. In particular, all vertices of $L_S(v_1)$ are either dead ends of block B', or are already contained in two blocks, so $x \notin \Theta_{S\setminus y}(v_1)$, and, moreover, x is not joined with $\Theta_{S\setminus y}(v_1)$ implying that S is not connected.

Case 2.2.2.2: no block contains vertices v_2, u_1 and u_2 simultaneously. In this case $L_S(v_1)$ contains exactly two leaves on distance two from each other, namely w_1 and w_2, see Table 5.2. This means that for any $y \notin L_S(v_1)$ distinct from v_1 and x and any block decomposition of

$S_1 = S \setminus y$ vertices v_1, v_2, w_1, w_2 compose a block of forth type, which implies that x is not joined with w_1 and w_2.

TABLE 5.2. To the proof of Lemma 5.5, Case 2.2.2.2



Now consider $S_2 = S \setminus w_1$ with a block decomposition. Clearly, $\mathrm{Val}_{S_2}(v_1) = 4$. Looking at possible quivers $\Theta_{S\setminus x}(v_1)$ (see Table. 5.2), one can notice that $L_{S_2}(v_1)$ does not contain a pair of leaves on distance two, so v_1 is contained in two blocks of second or third type. Since w_2 is joined with v_1 and v_2 only, it is easy to see that w_2, v_1, v_2 compose a block of type II. Now recall that neither w_1 nor w_2 are joined with any vertex of S except v_1 and v_2, so we can replace triangle v_1, v_2, w_2 by block v_1, v_2, w_1, w_2 of type IV to obtain a block decomposition of S.

Notice that we do not show in Table. 5.2 quivers $\Theta_{S\setminus x}(v_1)$ in which vertices u_1 and u_2 are contained in two blocks simultaneously. The case of a triangle with vertices u_1, u_2 and v_2 is treated in Case 2.2.2.1, and it is easy to see that all the others do not produce new leaves.

Case 3: $\mathrm{Val}_{S\setminus x}(v_1) = 4$.

Case 3.1: $x \not\perp v_1$, $\mathrm{Val}_S(v_1) = 5$. The proof is the same as in Case 2.1. Namely, since $|S| \geq 8$, there exists $y \in S$ which is not joined with v_1. Since $\mathrm{Val}_{S\setminus y}(v_1) = 5$, in any block decomposition of $S \setminus y$ vertex v_1 is an outlet of a block of type IV, so we may refer to Case 2.2.

Case 3.2: $x \perp v_1$, $\mathrm{Val}_S(v_1) = 4$. In this case vertex v_1 is contained in block B and in block B_1 of type I or IV. Consider these two cases separately.

Case 3.2.1: Block B_1 is of type IV. The case is similar to Case 1.2.1, the only difference is in the orientation of B_1: instead of getting double edge, the edge (v_1, v_2) cancels out. Vertex v_2 is also contained in B_1, denote by w_3 and w_4 dead ends of B_1. The only vertex joined with B and B_1 is x. We want to show that x is not joined with $L_S(v_1)$, which will imply that S is not connected.

Take any vertex $y \notin \langle L_S(v_1), v_1, x\rangle$ and consider $S_1 = S \setminus y$ with some block decomposition. If v_1 is contained in a block of fourth type, then the second block containing v_1 is also of forth type (otherwise the remaining block is an edge, and the link $L_{S_1}(v_1) = L_S(v_1)$ contains at most 3 edges, contradicting the fact that $L_S(v_1)$ contains 4 edges), and we see that x is not joined with $L_S(v_1)$. Therefore, v_1 is contained in two blocks of type II or III. More precisely, since valence of all neighbors of v_1 is at least two, v_1 is contained in two blocks B' and B'_1 of type II.

Block B' contains one of w_1, w_2 and one of w_3, w_4 (due to orientation, see Fig. 5.3 a)), assume that it contains w_1 and w_3. To avoid the edge (w_1, w_3) which does not appear in S, another block B'_1 should contain w_1 and w_3. Since w_1 and w_3 are joined with v_2, B'_1 is either of second or fourth type. In the latter case v_2 is a dead end of B'_1, but v_2 is joined with w_2 and w_4 also. Hence, B'_1 is a triangle containing v_2, w_1, w_3, see Fig. 5.3 b). Similarly, w_2, w_4 and v_2 are contained in block B''_1 of second type. In particular, all the vertices of $L_S(v_1)$ are already contained in two blocks, so x is not joined with $L_S(v_1)$.

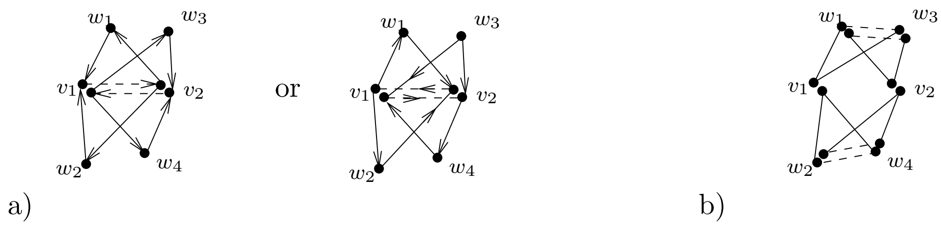


FIGURE 5.3. To the proof of Lemma 5.5, Case 3.2.1

Case 3.2.2: Block B_1 is of type I. Denote by u the second vertex of B_1. If u coincides with v_2, then $\langle L_S(v_1), v_1\rangle$ has no outlets, so we can apply Proposition 4.6 to x. Now we may assume that $u \neq v_2$.

If u is joined with v_2, then the edge $\langle u, v_2\rangle$ must form a block of first type, otherwise $\mathrm{Val}_S(v_2) > \mathrm{Val}_S(v_1)$. Therefore, no vertex except x is attached to $\langle L_S(v_1), v_1\rangle$. Since $|S| \geq 8$, in this case we can apply

Proposition 4.6 to x. Thus, we may assume that u is not joined with v_2.

Take any vertex $y \notin \langle L_S(v_1), v_1, x\rangle$ and consider $S_1 = S \setminus y$ with some block decomposition. It is easy to see that v_1 should be contained in a block of type IV. Furthermore, looking at $L_S(v_1)$ one can notice that there is exactly one pair of leaves on distance two, namely w_1 and w_2, which implies that vertices v_1, v_2, w_1, w_2 belong to one block, and w_1 and w_2 are dead ends. Therefore, x is not joined with w_1 and w_2, so only v_1 and v_2 are attached to w_1 and w_2.

Now we proceed as in Case 2.2.2.2. We consider quiver $S_1 = S \setminus w_1$ with some block decomposition and show that w_2, v_1, v_2 compose a block of type II. Since neither w_1 nor w_2 are not joined with any vertex of S except v_1 and v_2, we can replace triangle v_1, v_2, w_2 by block v_1, v_2, w_1, w_2 of type IV to obtain a block decomposition of S.

Case 4: $\mathrm{Val}_{S\setminus x}(v_1) = 3$.

Case 4.1: $x \not\perp v_1$, $\mathrm{Val}_S(v_1) = 4$. Since $\mathrm{Val}_{S\setminus x}(v_2) \le \mathrm{Val}_{S\setminus x}(v_1)$, we have $\mathrm{Val}_{S\setminus x}(v_2) = 3$. This means that vertices of B may be joined with x only. Thus, we may apply Proposition 4.6 to x.

Case 4.2: $x \perp v_1$, $\mathrm{Val}_S(v_1) = 3$. In this case vertex v_1 is contained either in block B only, or in block B and in second type block B_1. Consider the two cases.

Case 4.2.1: Vertex v_1 is contained in one block. This implies that v_1 and v_2 are joined in S, so no vertex except x is attached to B. We apply Proposition 4.6 to x.

Case 4.2.2: Vertex v_1 is contained in two blocks. In this case the second block B_1 is of type II, it contains v_1, v_2 and some vertex u. The case is similar to Case 2.2.1, the difference is in orientation of B_1.

The union of B and B_1 has a unique outlet u. Thus, $\langle L_S(v_1), v_1\rangle \setminus u$ may be joined with x only. Further, x is not joined with v_1 and v_2 (since $\mathrm{Val}_S(v_2) \le \mathrm{Val}_S(v_1)$. If x is not joined with w_1 and w_2 either, then we can apply Proposition 4.6 to u. Therefore, we may assume that x is joined with one of w_1 and w_2, say w_1. Moreover, we may assume that some vertex $y \notin \langle B, u, x\rangle$ is joined with u, otherwise we can apply Proposition 4.6 to x.

Now take any $z \notin \langle B, u, x, y\rangle$ and consider $S_1 = S \setminus z$ with some block decomposition. Vertex v_1 is contained either in a blocks of fourth and second type, or in blocks of second and first type. In the first case, due to orientations of edges (see Fig. 5.4) block of type IV should contain all vertices of B, which is impossible since dead end w_1 is joined with x. Hence, v_1 is contained in a triangle B' and an edge B'_1. Again, because of orientations of edges, w_1 and w_2 cannot belong to B' simultaneously, so B' contains u and w_i. To avoid the edge (w_i, u) these two vertices should be contained in some block B'_2. Since u is joined with v_2 and y, B'_2 contains v_2 and y also. Therefore, B'_2 is a block of fourth type with outlets u, w_i and dead ends v_2, y. But this implies that y attaches to w_i which is impossible.

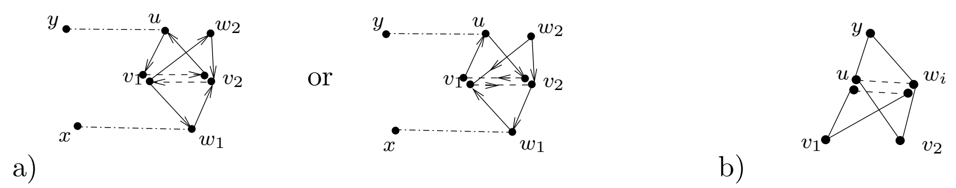


FIGURE 5.4. To the proof of Lemma 5.5, Case 4.2.2

Case 5: $\mathrm{Val}_{S\setminus x}(v_1) = 2$. In this case vertex v_1 is contained in block B and block B_1 of type I, where B_1 is an edge joining v_1 and v_2 with suitable orientation. The union of B and B_1 has no outlets, so we apply Proposition 4.6 to x.

Clearly, valence of v_1 in $S \setminus x$ is at least two, so all cases are studied and the lemma is proved.

□

Corollary 5.6. Valence of any vertex v of a minimal non-decomposable quiver S does not exceed 4.

Proof. The proof is evident. Indeed, take any x which is not joined with v, and consider any block decomposition of $S \setminus x$. Vertex v is contained in at most two blocks, valence of any vertex of blocks does not exceed two.

□

Lemma 5.7. *Let $v \in S$ be a vertex of valence 4. Then for any non-neighbor x of v and any block decomposition of $S \setminus x$ vertex v is not contained in a block of third type.*

Proof. Denote by v_1 and w_1 dead ends of block B of type III with outlet v, and denote by v_2 and w_2 vertices of block B_1 with outlet v. Clearly, v_1 and w_1 are not joined with v_2 and w_2. Denote $S' = \langle v_1, v_2, v, w_1, w_2 \rangle$.

If no vertex of $S \setminus \langle S', x\rangle$ is joined with v_2 and w_2, then we apply Proposition 4.6 to x. Thus, we may assume that some vertex u_2 attaches to one of v_2 and w_2, say v_2, see Fig. 5.5 a). In particular, this implies that B_1 is a block of type II.

Suppose that x is not joined with v_1 and w_1. Since $\mathrm{Val}_S(v) = 4$, the quiver $S \setminus v$ has at most 4 connected components. Two of them are v_1 and w_1. The remaining two (or one) contain at least 5 vertices (due to $|S| \geq 8$), so at least one connected component has at least 3 vertices. Hence, we can apply Proposition 4.6 to v.

Therefore, we assume that x attaches to at least one of v_1 and w_1, say w_1. We want to prove that S is block-decomposable by applying Proposition 4.8 to $S = \langle S_1 = \langle v_1, w_1\rangle, b_1 = v, b_2 = x, S_2 = S \setminus \langle S_1, v, x\rangle\rangle$, see Fig. 5.5b. For this take $a_1 = v_1$, and try to choose a_2. The choice of a_2 will depend on $\Theta_{S\setminus x}(v)$.

If v_2 and w_2 are joined in S, then we choose from non-attached to x vertices of S_2 (if they do exist) those which is at maximal distance from v in S. Clearly, such a vertex can be taken as a_2. If each vertex of S_2 is joined with x, we take as a_2 any vertex of $S_2 \setminus \langle v_2, w_2\rangle$.

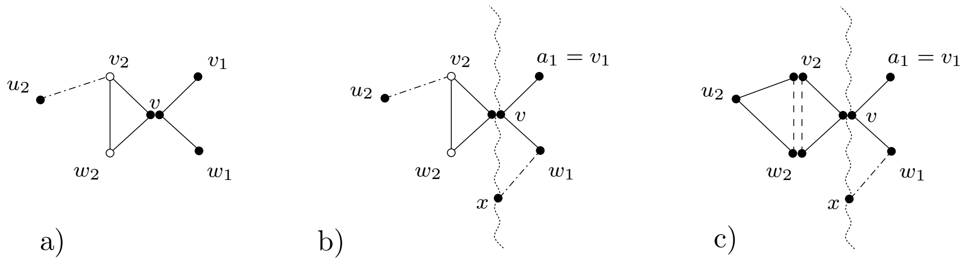


FIGURE 5.5. To the proof of Lemma 5.7

Now suppose that v_2 and w_2 are not joined in S (in particular, $\Theta_{S\setminus x}(v)$ contains also some block B_2 composed by v_2, w_2 and probably some other vertex of S_2). B_2 cannot be of first type since v_2 attaches to u_2. Thus, w_2 is joined with u_2, see Fig. 5.5 c). Moreover, no vertex of $S \setminus x$ attaches to any of v_2 and w_2 since both of them belong already to two blocks of considered block decomposition of $S \setminus x$. So, $S \setminus v_2$ is connected and we may take $a_2 = v_2$.

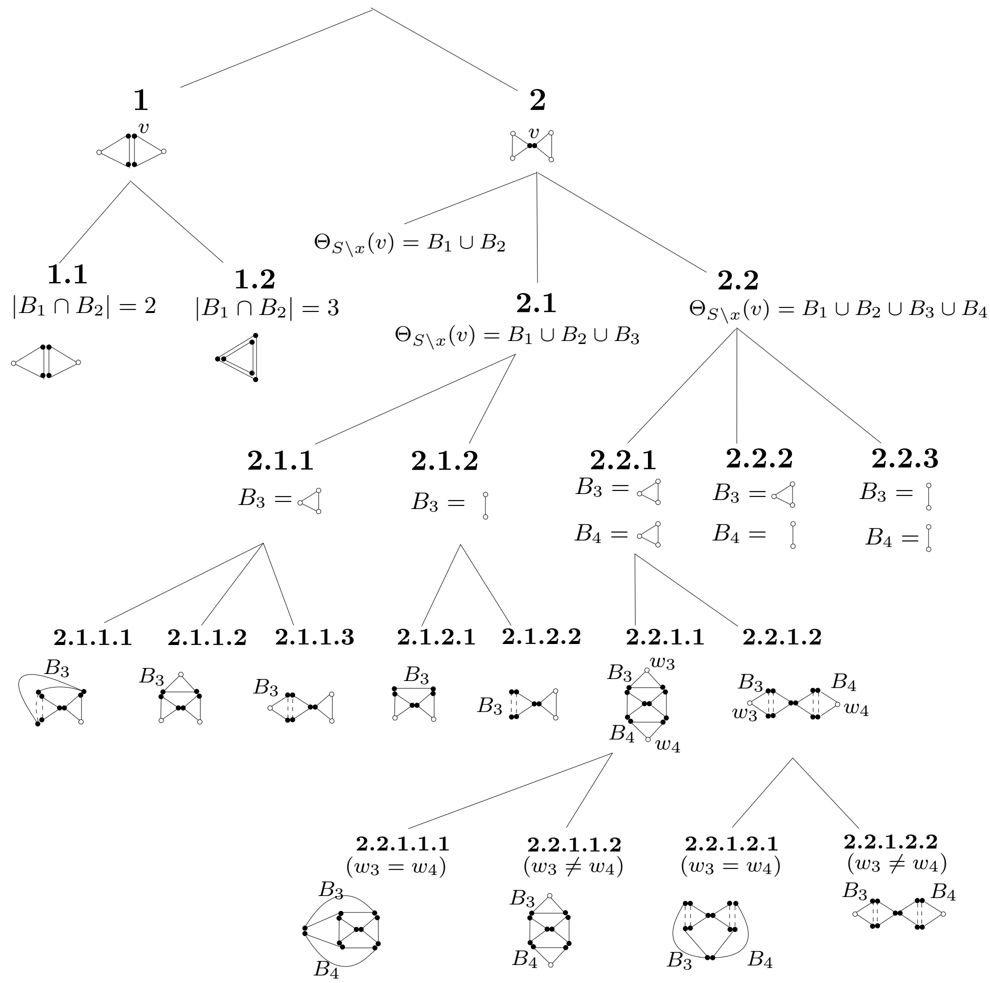
□

Lemma 5.8. *Let $v \in S$ be a vertex of valence 4. Then for any non-neighbor x of v and any block decomposition of $S \setminus x$ the diagram*

$\Theta_{S\setminus x}(v)$ consists of exactly two blocks of type II *having the only vertex v in common.*

Proof. Lemmas 5.7, 5.5, and 5.3 rule out blocks of types III, IV, and V from decomposition of $S \setminus x$. The only possibility left is that v_1 is contained in two blocks B_1 and B_2 of second type. Clearly, they have at most 4 outlets, so $\Theta_{S\setminus x}(v)$ may contain at most 2 additional blocks. Denote by v_1, u_1 and v_2, u_2 remaining vertices of B_1 and B_2 respectively, and consider the following cases (see Table 5.3).

TABLE 5.3. To the proof of Lemma 5.8



Case 1: B_1 and B_2 have at least two vertices in common.

Case 1.1: B_1 and B_2 have exactly two vertices in common. We may assume that $v_1 = v_2$. Then $\mathrm{Val}_S(v) = \mathrm{Val}_S(v_1) = 4$ is the maximal possible valence in S, so x is not attached to v and v_2. Consider all options for $\Theta_{S\setminus x}(v)$ (see Fig. 5.6).

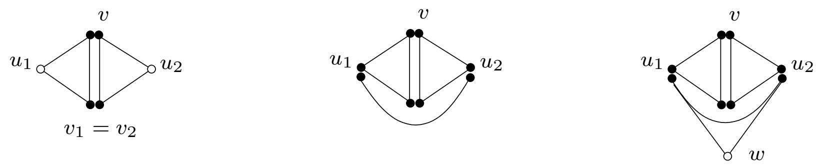


FIGURE 5.6. To the proof of Lemma 5.8, Case 1.1

If u_1 and u_2 are not joined (i.e. $\Theta_{S\setminus x}(v)$ consists of B_1 and B_2 only, see the left picture of Fig.5.6), then S is block-decomposable by Corollary 4.9 applied to $S=\langle S_1=\langle v, v_2\rangle, b_1=u_1, b_2=u_2, S_2=S\setminus\langle S_1, u_1, u_2\rangle\rangle$ with $c_1 = v$.

If the edge $\langle u_1, u_2\rangle$ forms a block of first type (see Fig.5.6, the middle picture), then they can be connected only with vertex $x \in S$. Hence, all the vertices of $\Theta_{S\setminus x}(v)$ are dead ends, so S is block-decomposable by Proposition 4.6 applied to x.

If u_1 and u_2 are contained in a block of second type with additional vertex w (see Fig.5.6, right), then u_1 and u_2 have valence 4, so they are not joined with x. Therefore, S is block-decomposable by Proposition 4.6 applied to w.

Case 1.2: B_1 and B_2 have three vertices in common. In this case union of B_1 and B_2 has no outlets, so S is block-decomposable by Proposition 4.6 applied to x.

Case 2: B_1 and B_2 intersect at v only. The quiver $\Theta_{S\setminus x}(v)$ consists of two, three, or four blocks. We are going to prove that there are no other blocks in $\Theta_{S\setminus x}(v)$ except B_1 and B_2, this will imply our lemma. For that we consider the two remaining cases and find a contradiction.

Case 2.1: $\Theta_{S\setminus x}(v)$ consists of three blocks B_1, B_2 and B_3. We go through different types of B_3 and the way it attaches to B_1 and B_2.

Case 2.1.1: $B_3 \in \mathcal{B}_{\mathrm{II}}$

Case 2.1.1.1: B_3 has 3 points in common with the union of B_1 and B_2. Let v_1, u_1, v_2 be vertices of B_3. Either v_1 and u_1 are joined by a double edge or they are no joined at all. If they are joined by a double edge, then valence of u_1 equals 4, so we are in assumptions of Case 1, which implies that S is block-decomposable. Hence, we may assume that v_1 and u_1 are not joined in S. Thus, $\Theta_{S\setminus x}(v)$ is the quiver shown on Fig. 5.7. The only outlet is u_2. We may assume that some $y \notin \langle \Theta_{S\setminus x}(v), x\rangle$ is joined with u_2, otherwise S is block-decomposable by Proposition 4.6 applied to x.

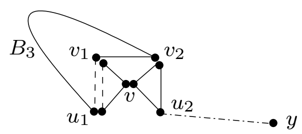


FIGURE 5.7. To the proof of Lemma 5.8, Case 2.1.1.1

Consider any $z \notin \langle \Theta_{S\setminus x}(v), x, y\rangle$ and some block decomposition of $S \setminus z$. By Lemmas 5.7, 5.5, and 5.3, v is contained in two blocks of second type. One of these blocks contains v, v_2 and one of u_1, u_2, v_1. Another one contains v and two remaining vertices from u_1, u_2, v_1. Similarly, v_2 is also contained in block of second type with vertices v_2 and two of u_1, u_2, v_1. This imply that only two of u_1, u_2, v_1 are dead ends of $\langle v, v_1, v_2, u_1, u_2\rangle$, and only one of them is outlet.

Note that no other vertex than u_1, u_2, v_1, v is joined with v_2 otherwise $\mathrm{Val}_S(v_2) > 4$. If u_2 is an outlet, then x may be joined with u_2 only in $\Theta_{S\setminus x}(v)$, so S is block-decomposable by Proposition 4.6 applied to u_2.

If u_1 or v_1 is an outlet, then u_2 is dead end, but y is attached to u_2, so we get a contradiction.

Case 2.1.1.2: B_3 has exactly 1 point in common with each of B_1 and B_2. We may assume that vertices of B_3 are v_1, v_2 and w. Since $\mathrm{Val}_S(v_1) = \mathrm{Val}_S(v_2) = 4$, no vertex of $S \setminus \langle \Theta_{S\setminus x}(v), w\rangle$ is joined with v_1 or v_2.

Consider the quiver $S_1 = S \setminus v_1$ with a block decomposition. Since $\mathrm{Val}_{S_1}(v) = 3$ and v_2 is joined with u_2 in S, v is contained in one block of second type and in one of first type. But u_1 is joined neither with v_2 nor with u_2, so v, v_2, u_2 are vertices of one block, and v, u_1 compose another one block. Looking at vertex v_2 we see that v_2 and w compose a block of first type, too. Replace the block v_2w by B_3, and block vu_1 by B_1, and obtain a block decomposition of S.

Case 2.1.1.3: B_3 does not intersect one of B_1 and B_2. In this case we may assume that vertices of B_3 are v_1, u_1 and w. Similarly to Case 2.1.1.1, we conclude that either this situation is already considered in Case 1, or v_1 and u_1 are not joined in S. Take any $y \notin \langle \Theta_{S\setminus x}(v), x\rangle$ and consider $S_1 = S \setminus y$. Since v_1 and u_1 are not joined with v_2 and u_2, vertices v, v_2 and u_2 compose one block (otherwise in order to cancel two edges joining $\langle v_1, u_1\rangle$ with $\langle v_2, u_2\rangle$ we have to glue in two blocks such that neither of them contains simultaneously u_2 and v_2. Then both u_2 and v_2 are dead ends with no edge between them contradicting the fact u_2 and v_2 are joined in S). Therefore, v, v_1 and u_1 compose a block, so v_1, u_1 and w also form a block. In particular, v_1 and u_1 are dead ends of $\langle u_1, w, v_1, v\rangle$, and x is not attached to v_1 and u_1. Recall also that no vertex except x, v, w could be attached to v_1 and u_1 since v_1 and u_1 are dead ends of $\Theta_{S\setminus x}(v)$. Hence, both u_1 and v_1 are joined with v and w only.

Consider $S_2 = S \setminus v_1$ with some block decomposition. Similarly to Case 2.1.1.2, it easy to see that v, v_2, u_2 compose one block of type II, and u_1, v form another block of type I. Since u_1 is not joined with any vertex except v and w, $\langle u_1, w\rangle$ is a block. Notice that blocks $\langle u_1, w, v_1\rangle$ and $\langle u_1, v_1, v\rangle$ are oriented so that sides $\overrightarrow{v_1u_1}$ of one triangle cancels out with the side $\overrightarrow{u_1v_1}$ of the other. This yields that one of the edges $\langle u_1, w\rangle$ and $\langle u_1, v\rangle$ is directed towards u_1 while the other is directed from u_1. Replacing (u_1, w) by B_3, and (u_1, v) by B_1, we obtain a block decomposition of S.

Case 2.1.2: $B_3 \in \mathcal{B}_{\mathrm{I}}$. There are two possibilities to attach B_3 to B_1 and B_2.

Case 2.1.2.1: B_3 has exactly one point in common with each of B_1 and B_2. We may assume that vertices of B_3 are v_1 and v_2 (see Fig. 5.8a). If x is not joined with v_1 and v_2, then S is block-decomposable by Corollary 4.9 applied to $S = \langle S_1 = \langle v, v_1, v_2\rangle, b_1 = u_1, b_2 = u_2, S_2 = S \setminus \langle S_1, u_1, u_2\rangle\rangle$ with $c_1 = v_1$. So, we may assume that x is joined with at least one of v_1 and v_2, say v_1.

Take any $y \notin \langle \Theta_{S\setminus x}(v), x\rangle$ and consider $S_1 = S \setminus y$. Valence of v_1 is equal to four, which means that v_1 is contained in two blocks of second type. Since v_1 is joined with v, one of these blocks (call it B') contains both v_1 and v. The third vertex of B' is either v_2 or u_1 (since $\mathrm{Val}_S(v) = \mathrm{Val}_S(v_1) = 4$ is the maximal possible and only u_1 and v_2 are joined with both v_1 and v).

If B' contains v_2, then vertices u_1, v_1, x compose one block. The second block containing v is composed by v, u_1, u_2. In particular, u_1 is joined with u_2, which contradicts the assumption, see Fig. 5.8 b) . Therefore, B' is composed by u_1, v_1 and v. The second block containing v is composed by v, v_2, u_2, and the second block containing v_1 is composed by v_1, v_2, x, see Fig. 5.8 c) We obtain a quiver considered above in Case 2.1.1.2.

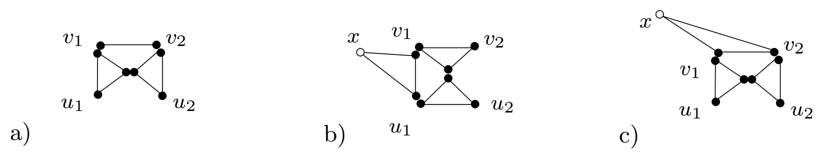


FIGURE 5.8. To the proof of Lemma 5.8, Case 2.1.2.1

Case 2.1.2.2: B_3 does not intersect one of B_1 and B_2. The proof repeats the proof of Lemma 5.7.

Case 2.2: $\Theta_{S\setminus x}(v)$ consists of four blocks B_1, B_2, B_3, and B_4. We go through all the different types of B_3 and B_4 and all the ways to assemble $\Theta_{S\setminus x}(v)$ from them.

Case 2.2.1: Both $B_3 \in \mathcal{B}_{\rm II}$ and $B_4 \in \mathcal{B}_{\rm II}$. There are two possibilities to attach B_3 and B_4 to B_1 and B_2.

Case 2.2.1.1: Each of B_3 and B_4 has exactly one vertex in common with each of B_1 and B_2. Denote by w_3 and w_4 the remaining vertices of B_3 and B_4 respectively, and consider two possibilities.

Case 2.2.1.1.1: Vertices w_3 and w_4 coincide. In this case valences of all the 6 vertices of $\Theta_{S\setminus x}(v)$ are equal to 4 yielding that $\langle\Theta_{S\setminus x}(v)\rangle \perp (S \setminus \langle\Theta_{S\setminus x}(v)\rangle)$ and S is not connected.

Case 2.2.1.1.2: Vertices w_3 and w_4 do not coincide. We may assume that B_3 contains vertices w_3, u_1 and u_2. Consider $S_1 = S \setminus u_1$ with some block decomposition. Since valences of v, v_1, u_1, v_2, u_2 are equal to 4, no vertex from $S \setminus \Theta_{S\setminus x}(v)$ attaches to any of these 5 vertices. We want to prove that edges $\langle w_3, u_2\rangle$ and $\langle v_1, v\rangle$ are blocks of first type of S_1, see Fig. 5.9. Then replacing (w_3, u_2) and (v_1, v) by B_3 and B_1 respectively we get a block decomposition of S.

Since $\mathrm{Val}_{S_1}(u_2) = 3$, u_2 is contained in one block of second type and one of first type. The edge $\langle v_2, u_2\rangle$ belongs to block of type II because of $\mathrm{Val}_{S_1}(v_2) = 4$. By the same reason, w_3 and v_2 are not contained in

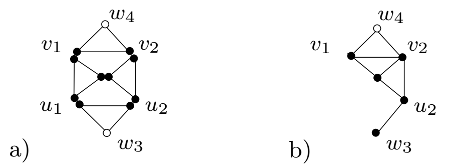


FIGURE 5.9. To the proof of Lemma 5.8, Case 2.2.1.1.2

one block. Therefore, v_2, v, u_2 compose one block, and w_3, u_2 compose another one block. Looking at vertex v_2 we see that the second block containing v_2 is spanned by v_2, v_1 and w_4. Recall that vertices v and v_1 are not joined with any vertex of S_1 except w_3, v_2 and u_2. Thus, v and v_1 compose a block, which completes the case.

Case 2.2.1.2: One of B_3 and B_4 (say B_3) does not intersect B_2 and the other (namely, B_4) does not intersect B_1. Denote by w_3 and w_4 the remaining vertices of B_3 and B_4 respectively. Taking into account Case 1, we may assume that u_2 is not joined with v_2 in S, and u_1 does not attach to v_1. We consider two possibilities.

Case 2.2.1.2.1: Vertices w_3 and w_4 coincide. Notice that each vertex of $\Theta_{S\setminus x}(v)$ is already contained in two blocks, so no vertex of $S\setminus\langle\Theta_{S\setminus x}(v)\rangle$ except x is attached to $\langle\Theta_{S\setminus x}(v)\rangle$. To show that x is not joined with $\langle\Theta_{S\setminus x}(v)\rangle$ either, take any $y\notin\langle\Theta_{S\setminus x}(v), x\rangle$ (such y does exist since $|S|\geq 8$) and consider $S\setminus y$ with some block decomposition. Valences of v and w_3 are equal to 4, so they belong to two blocks of second type each. Further, suppose that some pair of u_1, u_2, v_1, v_2 compose a block with w_3. To avoid an edge between this pair of vertices, they should compose also a block with v. Therefore, each vertex of $\Theta_{S\setminus x}(v)$ is contained in two blocks, so no vertex of $S\setminus\langle\Theta_{S\setminus x}(v)\rangle$ except y attaches to $\langle\Theta_{S\setminus x}(v)\rangle$. In particular, $x\perp\langle\Theta_{S\setminus x}(v)\rangle$.

Thus, $S\setminus\langle\Theta_{S\setminus x}(v)\rangle\perp\langle\Theta_{S\setminus x}(v)\rangle$, and S is not connected.

Case 2.2.1.2.2: Vertices w_3 and w_4 do not coincide. Clearly, no vertex of $(S\setminus x)\setminus\Theta_{S\setminus x}(v)$ is joined with v or any of its neighbors, see Fig. 5.10 a). Suppose that x is joined with any of neighbors of v, say with u_1. Consider $S_1 = S\setminus w_4$ with some block decomposition. Since u_1 does not attach to any of neighbors of v and $\mathrm{Val}_{S_1}(u_1) = 3$, the edge $\langle u_1, v\rangle$ is not contained in block of second type. However, $\mathrm{Val}_{S_1}(v) = 4$ so the edge $\langle u_1, v\rangle$ should be contained in some block of second type. The contradiction shows that $x\perp u_1$. Similarly, x is not joined with any other neighbor of v (and with v itself, of course).

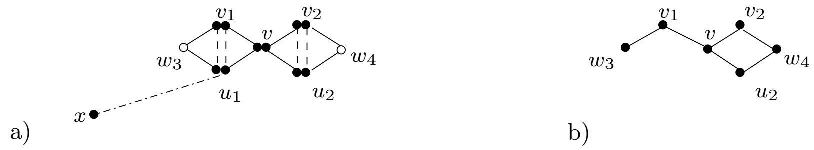


FIGURE 5.10. To the proof of Lemma 5.8, Case 2.2.1.2.2

Now consider $S_2 = S \setminus u_1$, see Fig. 5.10 b). Since $v_1 \perp v_2$, $v_1 \perp u_2$ and v is the only common neighbor of v_1 and v_2 (or v_1 and u_2), a block containing the edge $\langle v, v_1 \rangle$ contains neither v_2 nor u_2. Therefore, v_1 and v compose a block of first type. As we have proved above, $\mathrm{Val}_S(v_1) = 2$ so v_1 and w_3 also compose a block of first type. Now replacing the edge (v_1, v) by B_1, and (v_1, w_3) by B_3, we obtain a block decomposition of S.

Case 2.2.2: $B_3 \in \mathcal{B}_{\mathrm{II}}$, and $B_4 \in \mathcal{B}_{\mathrm{I}}$. Denote by w_3 the remaining vertex of B_3. If B_4 does not intersect one of B_1 or B_2 (say B_1), then it must coincide with the edge $\langle u_2, v_2 \rangle$ of B_2, and we get a situation described in Lemma 5.7 (notice that we did not use orientations of edges while proving Lemma 5.7). Hence, we may assume that B_4 intersects both B_1 and B_2. This implies that B_3 does the same. Let u_1 and u_2 be vertices of B_4. Then $\Theta_{S \setminus x}(v)$ is a quiver shown on Fig. 5.11. Consider $\Theta_{S \setminus x}(v_2)$. Notice, that $\mathrm{Val}_{S \setminus x}(v_2) = 4$ and $\Theta_{S \setminus x}(v_2)$ was treated in Case 2.1.1.2.

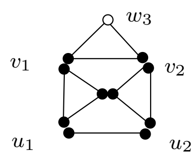


FIGURE 5.11. To the proof of Lemma 5.8, Case 2.2.2

Case 2.2.3: Both $B_3, B_4 \in \mathcal{B}_{\mathrm{I}}$. In this case all the vertices of $\Theta_{S \setminus x}(v)$ are dead ends, so S is block-decomposable by Proposition 4.6 applied to x.

Since all possibilities are exhausted, the lemma is proved.

□

Lemma 5.9. *Let $v \in S$ be a vertex of valence* 3*. Then for any non-neighbor x of v and any block decomposition of $S \setminus x$ the diagram $\Theta_{S\setminus x}(v)$ consists of exactly two blocks, one of type* II *and the other of type* I *having only vertex v in common.*

Proof. Since valence of v equals 3, v is contained in block B_1 of second or third type with two other vertices v_1 and u_1, and in block B_2 of first type with second vertex v_2. We consider both types of B_1 and all possible quivers $\Theta_{S\setminus x}(v)$ below (see Table 5.4).

Case 1: $B_1 \in \mathcal{B}_{\mathrm{III}}$. In this case v_1 and u_1 are dead ends of the union of B_1 and B_2, so $\{v_1, u\} \perp (S \setminus \{x, v\})$, and $\Theta_{S\setminus x}(v)$ consists of B_1 and B_2 only. If x is not joined with v_1 and u_1, then S is block-decomposable by Proposition 4.6 applied to v_2. Therefore, we may assume that x is joined with at least one of v_1 and u_1, say u_1. If $v_2 \perp S\setminus\langle\Theta_{S\setminus x}(v), x\rangle$, then again S is block-decomposable by Proposition 4.6 applied to x. Thus, we can assume that v_2 is joined with some vertex distinct from v and x, see Fig. 5.12. Now we can apply Corollary 4.9 to $S = \langle S_1 = \langle u_1, v_1, v\rangle, b_1 = v_2, b_2 = x, S_2 = S \setminus \langle S_1, x, v_2\rangle\rangle$ with $c_1 = v_1$ to show that S is block-decomposable.

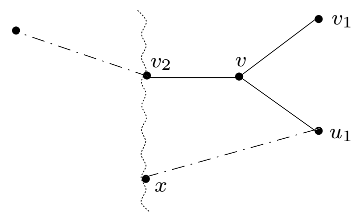
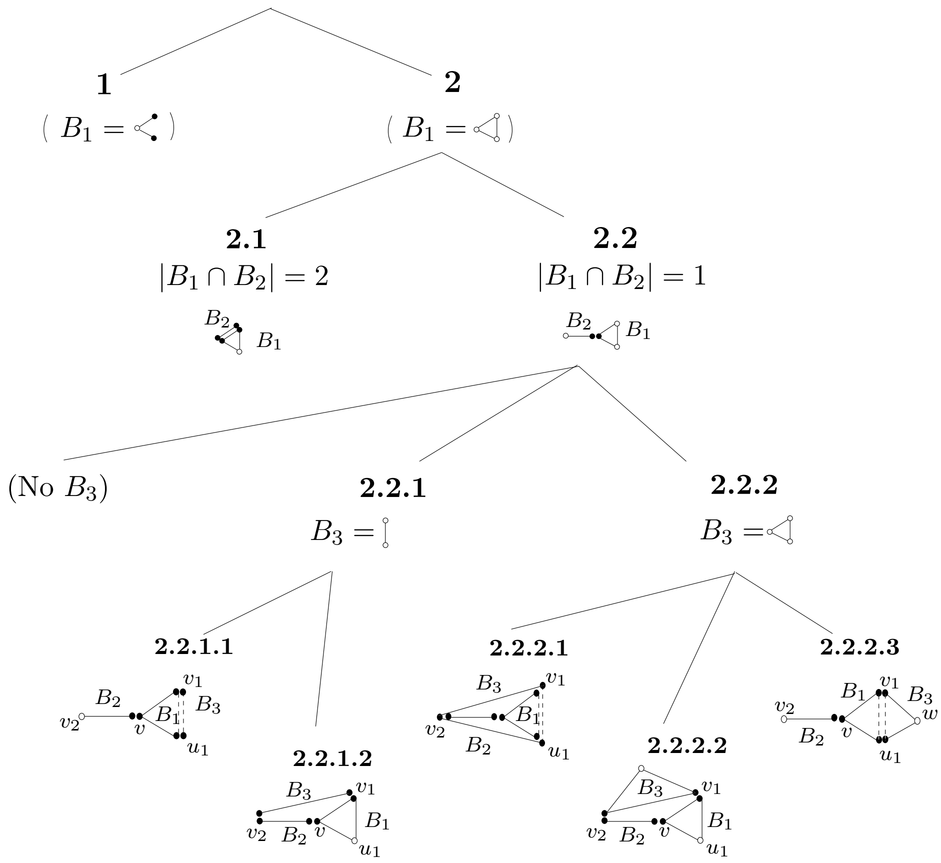


FIGURE 5.12. To the proof of Lemma 5.9, Case 1

Case 2: $B_1 \in \mathcal{B}_{\mathrm{II}}$. Consider the following cases.

Case 2.1: B_1 and B_2 have two points in common. We may assume that $v_1 = v_2$. Since $\mathrm{Val}_S(v) = 3$ and $v \perp x$, v_2 and v are joined by a double edge. By Lemma 5.8, no vertex of valence 4 may be incident to a double edge. Thus, $S \setminus \langle u_1, v, v_2\rangle \perp \langle v, v_2\rangle$ and $S \setminus u_1$ is not connected. Since valence of u_1 in S does not exceed 4, at least one connected component of $S\setminus u_1$ has more than 2 vertices (as in the proof of Lemma 5.7). Therefore, S is block-decomposable by Proposition 4.6 applied to u_1.

TABLE 5.4. To the proof of Lemma 5.9



Case 2.2: Vertex v is the only common vertex of B_1 and B_2. $\Theta_{S\setminus x}(v)$ may consist of two or three blocks. To prove the statement we need to exclude the option of three blocks. It is done in remaining part of the proof. Assume that $\Theta_{S\setminus x}(v)$ contains an additional block B_3. Clearly, B_3 is either of the first or of the second type.

Case 2.2.1: $B_3 \in \mathcal{B}_{\mathrm{I}}$. There are two ways to attach B_3 to B_1 and B_2.

Case 2.2.1.1: B_3 does not intersect B_2. In this case v_1 and u_1 are either joined by a double edge or are not joined in S at all. If they are not joined, then both v_1 and u_1 are dead ends of $\Theta_{S\setminus x}(v)$ and S is block-decomposable (as in Case 1). If they are joined by a double edge, then all three vertices v_1, u_1 and v have valence 3 in

S, so $\langle v_1, u_1, v\rangle \perp x$, and S is block-decomposable by Proposition 4.6 applied to v_2.

Case 2.2.1.2: B_3 has one point in common with each of B_1 and B_2. We can assume that B_3 consists of v_1 and v_2. Since $L_S(v_1)$ contains a connected component of order at least 3, Lemma 5.8 yields that $\mathrm{Val}_S(v_1) < 4$, so $\mathrm{Val}_S(v_1) = 3$. Hence, $S \setminus \langle \Theta_{S\setminus x}(v)\rangle \perp \langle v, v_1\rangle$. We apply Corollary 4.9 to $S = \langle S_1 = \langle v, v_1\rangle, u_1, v_2, S_2 = S \setminus \langle S_1, u_1, v_2\rangle\rangle$ with $c_1 = v$ to show that S is block-decomposable.

Case 2.2.2: $B_3 \in \mathcal{B}_{\mathrm{II}}$. There are three ways to attach B_3 to B_1 and B_2.

Case 2.2.2.1: B_3 contains all the three vertices v_1, u_1, v_2. Then all the vertices of $\Theta_{S\setminus x}(v)$ are dead ends, so S is block-decomposable by Proposition 4.6 applied to x.

Case 2.2.2.2: B_3 has one point in common with each of B_1 and B_2. We can assume that B_3 contains v_1. Then $\mathrm{Val}_S(v_1) = 4$, and $L_S(v_1)$ is connected contradicting Lemma 5.8.

Case 2.2.2.3: B_3 does not intersect B_2. Denote by w the third vertex of B_3. Let us prove first that v_2 is not joined with w in S. If they are contained in a block of the first type in $S \setminus x$, then all the vertices of $\langle v, v_1, u_1, w, v_2\rangle$ are dead ends, so S is block-decomposable by Proposition 4.6 applied to x. If w and v_2 are contained in a block of second type, then $\mathrm{Val}_S(w) = 4$, but $L_S(w)$ consists of three connected components contradicting Lemma 5.8. Therefore, $v_2 \perp w$. Observing that v_1 and u_1 are dead ends of $\Theta_{S\setminus x}(v)$, we see that they are joined only with v, w, and probably x.

Take any $y \notin \langle \Theta_{S\setminus x}(v), x\rangle$ and consider $S_1 = S \setminus y$ with a block decomposition. We will prove that x is joined with neither of v_1 and u_1. This implies that S is block-decomposable by Corollary 4.9 applied to $S = \langle S_1' = \langle v_1, u_1\rangle, v, w, S_2' = S \setminus \langle S_1', v, w\rangle\rangle$ with $c_1 = v_1$.

Suppose that $x \not\perp v_1$. Since $\mathrm{Val}_S(v) = 3$ and $v \perp \langle x, w\rangle$, v is not contained in one block with any of x and w. Thus, v_1, x and w compose a block of second type and v, v_1 is a block of first type. This implies that v, v_2, u_1 is a block of second type.

Since $v_2 \perp u_1$ in S, in order to avoid the edge (v_2, u_1) there is another block containing v_2 and u_1. Since $u_1 \not\perp w$, this block should contain w also. But then $v_2 \not\perp w$, which is already proved to be false.

By exhausting all cases we completed the proof of the lemma.

□

Corollary 5.10. Minimal non-decomposable quiver S does not contain double edges.

Proof. Let v and u be joined by a double edge. Take any non-neighbor x of v and consider $S \setminus x$ with some block decomposition. By Lemmas 5.8 and 5.9, valences of u and v do not exceed 2. Thus, they are joined with each other only and disconnected from the rest of S.

□

Proof of Theorem 5.2. Consider a quiver S satisfying assumptions of the theorem and having at least 8 vertices. By Lemmas 5.3 and 5.5, valence of any vertex of S does not exceed 4. By Lemmas 5.8 and 5.9, link of any vertex of valence 4 consists of two disjoint edges, and link of any vertex of valence 3 consists of one edge and one vertex. By Corollary 5.10, S does not contain double edges.

Now take all cycles of order 3 in S and paint all their edges and vertices in red (we assume all the remaining edges and vertices to be black). By Lemmas 5.8 and 5.9, any red edge belongs to a unique cycle of order 3. Any red vertex is contained either in four red edges, or in two red edges and at most one black edge. Notice also that due to Lemmas 5.8 and 5.9 each cycle of order 3 is cyclically oriented.

Denote by S_1 the quiver obtained by deleting all red edges from S. Let us show that S_1 is a forest. Indeed, S_1 does not contains vertices of valence 3 or more. Further, if S_1 contains a cycle C, then each vertex of this cycle is contained in two black edges, so the cycle does not contain any red vertex. This implies that no vertex of $S \setminus C$ is joined with C in S, so either S is not connected or $S = C$. In the latter case S is block-decomposable.

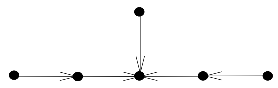
Take a block decomposition of S_1: any edge is a block. It is well defined since there are no vertices of valence 3 or more. Clearly, any red vertex is an outlet. Now consider each cycle of order 3 as a block of second type, and glue it to S_1. We obtain a block decomposition of S, which contradicts the assumptions of the theorem.

□

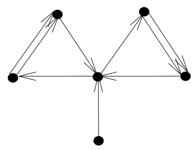
Now we are able to prove the main results of the section.

Theorem 5.11. *The only mutation-finite quivers satisfying assumptions of Theorem 5.2 are ones mutation-equivalent to one of the two quivers X_6 and E_6 shown on Figure 5.13.*

Remark 5.12. We recall that the property of quiver S to be block-decomposable is preserved by mutations. Indeed, according to [FST],



E_6



X_6

FIGURE 5.13. Minimal non-decomposable mutation-finite quivers

S is block-decomposable if and only if it corresponds to an ideal triangulation of a punctured bordered surface, and any mutation corresponds to a flip of the triangulation. Thus, any quiver mutation-equivalent to S arises from some triangulation, too. In particular, this implies that the set of non-decomposable quivers is invariant under mutations either. At the same time, the property to be *minimal* non-decomposable may not be preserved by mutations *a priori*. However, while proving Theorem 5.11, we see that minimal non-decomposable quivers are invariant anyway.

Proof of Theorem 5.11. The two quivers shown on Figure 5.13 are mutation-finite and not-decomposable (see [DO, Propositions 4 and 6]). To prove the theorem, it is sufficient to show that all other mutation-finite quivers on at most 7 vertices either are block-decomposable, or contain subquivers which are mutation-equivalent to one of X_6 and E_6. In particular, this will imply that all the quivers mutation-equivalent to X_6 or E_6 are also minimal non-decomposable.

Let S be a minimal non-decomposable mutation-finite quiver. By Theorem 5.2, $|S| \leq 7$. Since the mutation class of S is finite, multiplicities of edges of S do not exceed 2 (see Theorem 2.6). The number of quivers on at most 7 vertices with bounded multiplicities of edges is finite. This means that we can use a computer to list all quivers, choose mutation-finite ones, and check which of them are block-decomposable. However, the number of quivers in consideration is large. To reduce the time required for computations, we organize the check as follows.

First, we list all mutation classes of connected mutation-finite quivers of order 3, there are three of them (see [DO, Theorem 7]), and choose one representative in each class. They all are block-decomposable. Clearly, any connected mutation-finite quiver of order 4 contains a proper subquiver mutation-equivalent to one of these 3 quivers of order three.

Next, we add a vertex and join it with each of the 3 quivers by edges of multiplicities at most 2 in all possible ways (we use **C++** program [FST1]). For each obtained quiver we check if its mutation class is finite, and choose one representative from each finite mutation class (here we use Java applet for quivers mutations [K1]). The resulting list contains 5 quivers of order 4, they all are block-decomposable.

Continuing in the same way we get 7 finite mutation classes of order 5, again all are block-decomposable. Then we get 13 classes of order 6, exactly two of them consist of non-decomposable quivers, namely quivers mutation-equivalent to X_6 and quivers mutation-equivalent to E_6 . Since all mutation-finite quivers with at most 5 vertices are block-decomposable, all quivers mutation-equivalent X_6 or E_6 are minimal non-decomposable. Attaching a vertex to representatives of all classes of order 6, we get 15 finite mutation classes of quivers of order 7, three of them consist of non-decomposable quivers, namely classes containing E_7, or $\widetilde{E}_6$, or X_7. These three mutation classes consist of 416, 132, and 2 quivers respectively. Each quiver from the first two classes contains a subquiver mutation-equivalent to E_6, each quiver from the third class contains a subquiver mutation-equivalent to X_6. Therefore, none of them is minimal non-decomposable.

□

The following immediate corollary of Theorem 5.11 is the main tool in the classification of mutation-finite quivers.

Corollary 5.13. Every non-decomposable mutation-finite quiver contains a subquiver mutation-equivalent to E_6 or to X_6.

6. Classification of non-decomposable quivers

In this section we use Corollary 5.13 to classify all non-decomposable mutation-finite quivers.

Theorem 6.1. *A connected non-decomposable mutation-finite quiver of order greater than 2 is mutation-equivalent to one of the eleven quivers E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, X_6, X_7, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ shown on Figure 6.1.*

All these quivers have finite mutation class [DO] and are non-decomposable since each of them contains a subquiver mutation-equivalent to E_6 or X_6. We need only to prove that this list is complete.

We prove several elementary preparatory statements first.

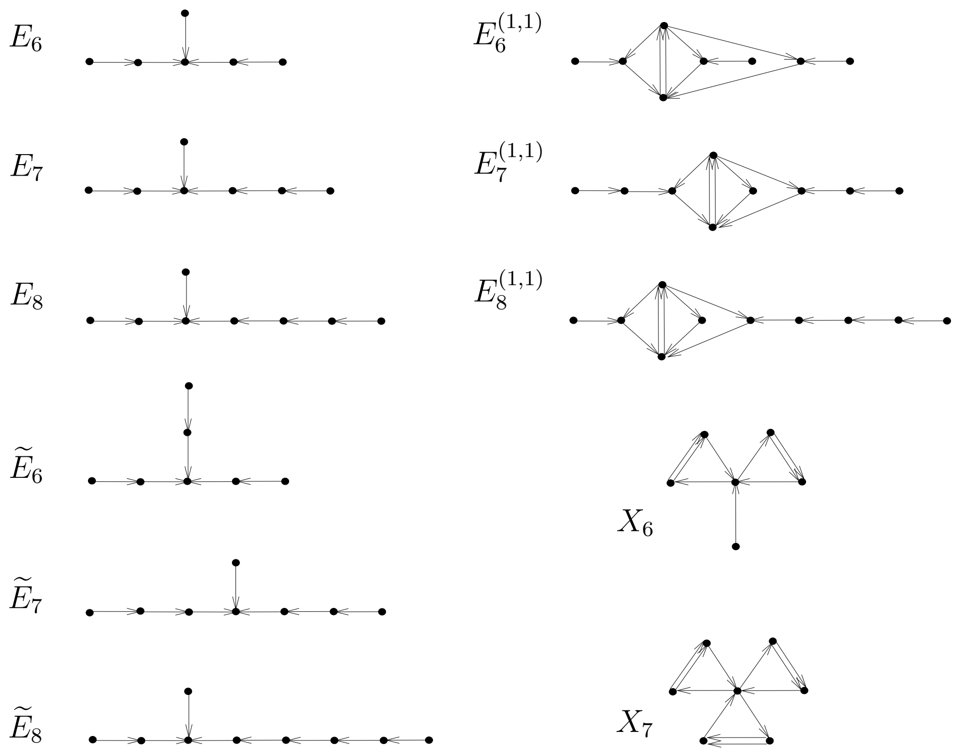


FIGURE 6.1. Non-decomposable mutation-finite quivers of order at least 3

Lemma 6.2. *Let S be a non-decomposable quiver of order $d \geq 7$ with finite mutation class. Then S contains a non-decomposable mutation-finite subquiver S_1 of order $d-1$.*

Proof. According to Corollary 5.13, S contains a subquiver S_0 mutation-equivalent to E_6 or X_6. Let S_1 be any connected subquiver of S of order $d-1$ containing S_0. Clearly, S_1 is non-decomposable, and its mutation class is finite.

□

Corollary 6.3. Suppose that for some $d \geq 7$ there are no non-decomposable mutation-finite quivers of order d. Then order of any non-decomposable mutation-finite quiver does not exceed $d-1$.

A proof of the following lemma is evident.

Lemma 6.4. *Let S_1 be a proper subquiver of S, let S_0 be a quiver mutation-equivalent to S_1. Then there exists a quiver S' which is mutation-equivalent to S and contains S_0.*

Proof of Theorem 6.1. According to Theorem 5.11, there are exactly two finite mutation classes of non-decomposable quivers of order 6, namely classes of E_6 and X_6. Due to Corollary 5.13, all other non-decomposable mutation-finite quivers have at least 7 vertices. By Lemma 6.4, in each finite mutation class of non-decomposable quivers there is a representative containing a subquiver E_6 or X_6.

Therefore, to find all finite mutation classes of non-decomposable quivers of order 7 we need to attach a vertex to E_6 and X_6 in all possible ways (i.e. by edges of multiplicity at most two, with all orientations), and to choose amongst obtained 7-vertex quivers all mutation-finite classes. This has been done by using Java applet [K1] and an elementary **C++** program [FST1] (in fact, the same algorithm was used in the proof of Theorem 5.11). In this way we get 3 finite mutation classes of quivers of order 7 with representatives X_7, E_7, and $\widetilde{E}_6$.

Now, Lemmas 6.4 and 6.2 allow us to continue the procedure. To list all finite mutation classes of non-decomposable quivers of order 8 we attach a vertex to X_7, E_7, and $\widetilde{E}_6$ in all possible ways, and choose again all mutation-finite classes. The result consists of 3 mutation classes with representatives E_8, $\widetilde{E}_7$, and $E_6^{(1,1)}$.

In the same way we analyzed the quivers of order 9 and obtained 2 mutation classes with representatives $\widetilde{E}_8$ and $E_7^{(1,1)}$. To find all finite mutation classes of non-decomposable quivers of order 10, we apply the same procedure to $\widetilde{E}_8$ and $E_7^{(1,1)}$. The result is a unique mutation class containing $E_8^{(1,1)}$.

Finally, the same procedure applied to $E_8^{(1,1)}$ gives no mutation-finite quivers at all. This implies that there are no non-decomposable mutation-finite quiver of order 11. Now Corollary 6.3 yields immediately the Theorem statement.

□

7. Minimal mutation-infinite quivers

The main goal of this section is to provide a criterion for a quiver to be mutation-finite, namely to prove Theorem 7.5. For given quiver

S, this criterion allows to check if S is mutation-finite in a polynomial in $|S|$ time.

Definition 7.1. A *minimal mutation-infinite quiver S* is a quiver that

- has infinite mutation class;
- any proper subquiver of S is mutation-finite.

Example 7.2. Any mutation-infinite quiver of order 3 is minimal. This is caused by the fact that any quiver of order at most 2 is mutation-finite.

Clearly, any minimal mutation-infinite quiver is connected. Notice that the property to be minimal mutation-infinite is not mutation invariant. Indeed, any mutation-infinite class contains quivers with arbitrary large multiplicities of edges. If $|S| > 3$, then taking a connected subquiver of S of order 3 containing an edge of multiplicity greater than 2 we get a proper subquiver of S which is mutation-infinite (see Theorem 2.6). Note also that minimal mutation-infinite quiver of order at least 4 does not contain edges of multiplicity greater than two.

We will deduce the criterion from the following lemma.

Lemma 7.3. *Any minimal mutation-infinite quiver contains at most* 10 *vertices.*

The bound provided in the lemma is sharp: there exist numerous minimal mutation-infinite quivers of order 10. We show some of them below. One source of such examples are simply-laced Dynkin diagrams of root systems of hyperbolic Kac-Moody algebras with any orientations of edges. There are two such diagrams of order 10 (e.g., see [K]), examples of corresponding quivers are shown on Fig. 7.1.

FIGURE 7.1. Minimal mutation-infinite quivers of order 10 coming from Dynkin diagrams

Remark 7.4. There is no general algorithm to determine if two infinite-mutational quivers are mutation-equivalent. However, for *acyclic* quivers (i.e., containing no oriented cycles) the following result is known (see [CK, Corollary 4]): if two acyclic quivers are mutation-equivalent, then there exists a sequence of mutations from one of them to another via acyclic quivers only. In particular, this implies that two quivers shown on Fig. 7.1 are not mutation-equivalent. Indeed, they both are trees, but it is easy to see that the only way to change the topological type of tree by mutation is to create an oriented cycle.

Another series of examples can be obtained from the quiver shown on Fig. 7.2.

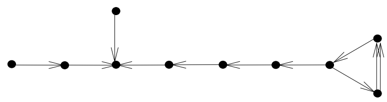


FIGURE 7.2. Minimal mutation-infinite quiver of order 10

Note that it is not clear if the quiver shown on the Fig. 7.2 is not mutation-equivalent to one of the quivers shown on Fig. 7.1.

Proof of Lemma 7.3. Let S be a minimal mutation-infinite quiver.

First, we prove a weaker statement, i.e. we show that $|S| \le 11$. In fact, this bound follows immediately from Theorems 5.2 and 6.1. Indeed, either all the proper subquivers of S are block-decomposable, or S contains a proper finite mutational non-decomposable subquiver of order $|S|-1$ (we can assume that this quiver is connected: if it is not connected but non-decomposable, it contains a non-decomposable connected component S_0, and any connected subquiver of S of order $|S|-1$ containing S_0 is non-decomposable). In the former case $|S| \le 7$ according to Theorem 5.2 (again, we emphasize that we did not require S to be mutation-finite in the assumptions of Theorem 5.2). In the latter case $|S|-1 \le 10$ due to Theorem 6.1, which proves inequality $|S| \le 11$.

Now suppose that $|S| = 11$. Then S contains a proper finite mutational non-decomposable subquiver S' of order 10. According to Theorem 6.1, S' is mutation-equivalent to $E_{10}^{(1,1)}$. The mutation class of $E_{10}^{(1,1)}$ consists of 5739 quivers, which can be easily computed using Keller's Java applet [K1]. In other words, we see that S contains one of 5739 quivers of order 10 as a proper subquiver.

Hence, we can list all minimal mutation-infinite quivers of order 11 in the following way. To each of 5739 quivers above we add one vertex in all possible ways (we can do that since the multiplicity of edge is bounded by two; the sources of the program can be found in [FST1]). For every obtained quiver we check whether all its proper subquivers of order 10 (and, therefore, all the others) are mutation-finite. However, the resulting set of the procedure above is empty: every obtained quiver has at least one mutation-infinite subquiver of order 10, so it is not minimal.

□

As a corollary of Lemma 7.3, we get the criterion for a quiver to be mutation-finite.

Theorem 7.5. *A quiver S of order at least 10 is mutation-finite if and only if all subquivers of S of order 10 are mutation-finite.*

Proof. According to Definition 7.1, every mutation-infinite quiver contains some minimal mutation-infinite quiver as a subquiver. Thus, a quiver is mutation-finite if and only if it does contain minimal mutation-infinite subquivers. By Lemma 7.3, this holds if and only if all subquivers of order at most 10 are mutation-finite. Since a subquiver of a mutation-finite quiver is also mutation-finite, the latter condition, in its turn, holds if and only if all subquivers of order 10 are mutation-finite, which completes the proof.

□

8. Growth of skew-symmetric cluster algebras

We recall the definition of growth of cluster algebra [FST, Section 11].

Definition 8.1. A cluster algebra has *polynomial growth* if the number of distinct seeds which can be obtained from a fixed initial seed by at most n mutations is bounded from above by a polynomial function of n. A cluster algebra has *exponential growth* if the number of such seeds is bounded from below by an exponentially growing function of n.

In [FST, Proposition 11.1] a complete classification of block-decomposable quivers corresponding to algebras of polynomial growth is given. It occurs that growth is polynomial if and only if the surface corresponding to a quiver is a sphere with at most three punctures and boundary components in total.

We prove the following theorem.

Theorem 8.2. *Any mutation-infinite skew-symmetric cluster algebra has exponential growth.*

Combining these two results, we see that to classify all cluster algebras of polynomial growth we need only to determine the growth of 11 exceptional mutation-finite algebras listed in Theorem 6.1. Three of them, namely E_6, E_7 and E_8 are of finite type, so they have finite number of seeds. Other three ($\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$) are of affine type, so they have linear growth (according to H. Thomas). Therefore, there are still 5 algebras for which the growth is unknown.

In other words, to complete the classification of cluster algebras by the growth rate it remains to ascertain the rates of growth in the following five cases X_6, X_7, $E_6^{(1,1)}$, $E_7^{(1,1)}$, and $E_8^{(1,1)}$.

A sequence of cluster transformations preserving the exchange matrix defines *a group-like element*. The set of all group-like elements form *generalized modular group*.

Using ideas similar to the proof of famous Tits alternative, it can be proved that in all five cases the growth rate of the generalized modular group is exponential. More precisely, studying the attracting points of some induced action it can be proved that the generalized modular group contains as a subgroup the free group of rank two.

Details on the study of growth rates in exceptional cases will be published elsewhere.

Corollary 8.3. A skew-symmetric cluster algebra of rank at least 3 has a polynomial growth if and only if

- it is associated with triangulation of either a sphere with three punctures, either a disk with two punctures, either an annulus with one puncture, or a pair of pants;
- it is one of the following exceptional affine cases: $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.

Remark 8.4. Note that by construction any cluster algebra of rank 2 is either of finite type of has a linear (i.e., polynomial) growth rate.

The rest of this section is devoted to the proof of Theorem 8.2.

Remark 8.5. It is sufficient to prove Theorem 8.2 for cluster algebras corresponding to mutation-infinite quivers of order 3. Indeed, any mutation-infinite quiver S has a mutation-equivalent quiver S' with an edge of weight at least 3. According to Theorem 2.6, any connected subquiver $S_0 \subset S$ of order 3 containing that edge is mutation-infinite. Therefore, it is enough to show that the algebra corresponding to S_0 grows exponentially.

In fact, we prove some stronger result. Denote by $S^{(n)}$ the set of quivers which can be obtained from a quiver S by at most n mutations. According to Remark 8.5, the following lemma implies Theorem 8.2.

Lemma 8.6. *Let S be a mutation-infinite quiver of order 3. Then the order $|S^{(n)}|$ grows exponentially with respect to n.*

The proof of Lemma 8.6 splits into two steps. We start by proving the following lemma. By saying that a quiver is *oriented* we mean that all the cycles are oriented.

Lemma 8.7. *For any mutation-infinite quiver S of order 3 there exists a sequence of at most 4 mutations taking S to S' such that*
(1) *S' is oriented;*
(2) *all the weights of edges of S' are greater than 1;*
(3) *an edge of maximal weight is unique.*

Proof. First, we make S oriented without empty edges. For this, we need at most 2 mutations. Indeed, if S is non-oriented without empty edges, then the mutation in a unique vertex which is neither sink nor source leads to an oriented quiver. If S has an empty edge, then by at most one mutation we put sink and source to the ends, and then, mutating in the middle vertex, we get a required quiver.

Thus, we can assume now that S is oriented without empty edges, and we have two mutations left to satisfy conditions (2) and (3). Denote the weights of S by (a, b, c) with $a \geq b \geq c$. Since S is mutation-infinite, S does not coincide with any of the three mutation-finite quivers of order 3 shown on Fig. 8.1. In particular, $a \geq b \geq 2$. We

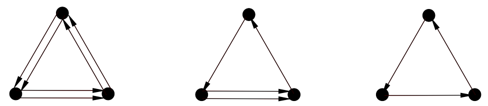


FIGURE 8.1. Oriented mutation-finite quivers of order 3 without empty edges

may assume that either $a = b$ or $c = 1$. If $c = 1$, then making a mutation preserving a and b, we obtain an oriented quiver with weights $(a, b, c' = ab - c)$. Clearly, $c' = ab - c \geq 3$, so the condition (2) holds.

Now we have a quiver satisfying the first two conditions, and one mutation left to satisfy condition (3). Again, let the weights of S be (a, a, c) with $a \geq c$. If $c = 2$, then we make a mutation changing c

to get a quiver with weights $(a, a, c' = a^2 - c)$. Since S is mutation-infinite, $a > 2 = c$, therefore $c' = a^2 - 2 > a$, so the third condition is satisfied. If $c > 2$, then mutating in any of the two other vertices, we get a quiver with weights $(a, b = ac - a, c)$. Clearly, $b = ac - a > a$ since $c > 2$.

□

The last step in proving Lemma 8.6 is Lemma 8.8. We say that a sequence of mutations is *reduced* if it does not contain two consecutive mutations in the same vertex. Note also that we differ quivers with the same weights but different orientations.

Lemma 8.8. *Let S be mutation-infinite quiver of order 3 satisfying conditions $(1) - (3)$ of Lemma 8.7, denote by (a, b, c) the weights of edges of S, $a > b \geq c$. Let S_1 and S_2 be quivers obtained from S by different reduced sequences of mutations, such that the first mutation in each sequence preserves the weight a of S. Then S_1 and S_2 are distinct.*

Clearly, Lemma 8.8 together with Lemma 8.7 imply Lemma 8.6. Before proving Lemma 8.8, we provide the following auxiliary statement.

Lemma 8.9. *Let S fit into assumptions of Lemma 8.8. Suppose that S' is obtained from S by mutation preserving the weight a. Then*
1) *the maximal weight of S' is greater than a;*
2) *S' satisfies conditions $(1) - (3)$ of Lemma 8.7.*

Proof. To prove the first statement, compute the weights of S'. If we preserve weights a and b, then weights of S' are $(a, b, c' = ab - c)$, so $c' > a$ since $b \geq 2$ and $c < a$. If we preserve weights a and c, then weights of S' are $(a, b' = ac - b, c)$, so $b' > a$ since $c \geq 2$ and $b < a$.

Now the second statement is evident.

□

The following immediate corollary of Lemma 8.9 is a partial case of Lemma 8.8.

Corollary 8.10. Let S fit into assumptions of Lemma 8.8, and let $\mu_n \dots \mu_1$ be a reduced sequence of mutations, where μ_1 preserves the maximal weight of S. Denote by S_i the quiver $\mu_i \dots \mu_1 S$. Then all the quivers $S, S_1, \dots, S_n$ are distinct.

Proof of Lemma 8.8. Suppose S_1 and S_2 coincide. We may assume that any two quivers S_1' and S_2' in the sequences of quivers from S to S_1 and S_2 respectively are distinct. Consider preceding quivers S_1' and S_2' in the sequences of quivers from S to S_1 and S_2. By Lemma 8.9, both S_1 and S_2 satisfy the following property: the edge of the maximal weight is opposite to the vertex in which the last mutation was made. Therefore, S_1' and S_2' coincide also. The contradiction proves the lemma.

□

9. Quivers of order 3

The structure of mutation classes of quivers of order 3 was described in [ABBS] and [BBH]. These papers provide complete classification of mutation classes containing quivers without oriented cycles given in different terms.

Define the *total weight* of a quiver as the sum of the weights of edges. It is proved in [ABBS] (see also [BBH, Lemma 2.1]) that if a mutation class does not contain quivers without oriented cycles, then the mutation class contains a unique (up to duality) quiver of minimal total weight, and any other mutation-equivalent quiver can be reduced to that one in a unique way.

We complete the description of mutation classes containing quivers without oriented cycles by a similar statement. We use notation from [BBH]: a quiver S of order 3 is called *cyclic* if it is an oriented cycle, and *acyclic* otherwise; S is called *cluster-cyclic* if any quiver mutation-equivalent to S is cyclic, and *cluster-acyclic* otherwise.

Theorem 9.1. *Let S be a connected cluster-acyclic quiver of order* 3*. Then*

1) *mutation class of S contains a unique (up to change of orientations of edges) quiver S_0 without oriented cycle;*

2) *the total weight of S_0 is minimal amongst all the mutation class;*

3) *any sequence of mutations decreasing total weight at each step applied to S ends in S_0.*

We use the following notation. By $(a,b,c)^-$ we mean a non-oriented cycle with weights (a,b,c). We may not fix orientations of edges since any two such quivers are mutation-equivalent under mutations in sources or sinks only. Similarly, by $(a,b,c)^+$ we mean an oriented cycle with weights (a,b,c).

Proof of Theorem 9.1. Consider a connected acyclic quiver $S_0=(a,b,c)^-$ with $a \ge b \ge c$. Denote by $\mathcal{S}$ the set of quivers satisfying conditions (1)–(3) of Lemma 8.7.

Suppose first that none of the weights is equal to 1. If $c = 0$, then the only quiver we can get by one mutation different from S_0 is $(a,b,ab)^+$, which is contained in $\mathcal{S}$. If $c \ge 2$, then we can obtain three possibly different quivers $(a,b,ab+c)^+$, $(a,ac+b,c)^+$ and $(bc+a,b,c)^+$ all belonging to $\mathcal{S}$. According to Lemma 8.9, all other quivers from mutation class of S_0 also belong to $\mathcal{S}$, which proves the first statement. Moreover, Lemmas 8.8 and 8.9 imply that for any quiver from the mutation class of S_0 belonging to $\mathcal{S}$ there is a unique reduced sequence of mutations decreasing the total weight at each step, and the minimal element is in $\mathcal{S}$ if and only if the entire mutation class is contained in $\mathcal{S}$ (this is also proved in [BBH, Lemma 2.1]). Since S_0 is the only quiver not contained in S, all the statements are proved.

Now suppose that at least one of the weights of S_0 is 1. We may assume that S_0 is mutation-infinite (there are exactly two acyclic mutation-finite quivers of order 3, namely $\widetilde{A}_2$ and A_3, and the theorem is evident for them). Our aim is to show that almost all quivers mutation-equivalent to S_0 belong to S, then looking at the remaining quivers all the statements become evident. Indeed, S_0 is of one of the following three types: $(a,1,0)^-$, $(a,1,1)^-$, or $(a,b,1)^-$, where $a \ge b \ge 2$. We list the quivers which can be obtained from that three by mutations.

The only way to change $(a,1,0)^-$ is to obtain $(a,a,1)^+$, from which we may get $(a,a,a^2-1)^+$ only which is in $\mathcal{S}$.

The quiver $(a,1,1)^-$ can be mutated into $(a+1,1,1)^+$ and $(a+1,a,1)^+$. The first one then can be mutated into the second one only, and the latter into the first one or into $(a,a+1,a^2+a-1)^+ \in \mathcal{S}$.

The quiver $(a,b,1)^-$ can be mutated either into $(a,b,ab+1)^+ \in \mathcal{S}$, or into $(a+b,b,1)^+$ or $(a+b,a,1)^+$. These two can be mutated either one into another or into quivers belonging to $\mathcal{S}$.

Therefore, in the mutation class of S_0 there is at most 3 quivers not belonging to $\mathcal{S}$, and S_0 has minimal total weight amongst them. Applying the same arguments as in the first case, we complete the proof.

□

References

[ABBS] I. Assem, M. Blais, T. Bruestle, A. Samson, *Mutation classes of skew-symmetric* 3×3 *matrices*, Comm. Alg. 36 (2008), 1209–1220.

[BGZ] M. Barot, C. Geiss, A. Zelevinsky, *Cluster algebras of finite type and positive symmetrizable matrices*, J. London Math. Soc. (2) 73 (2006), 545–564.

[BBH] A. Beineke, T. Bruestle, L. Hille, *Cluster-cyclic quivers with three vertices and the Markov equation*, Alg. and Rep. Theory 14 (2011), 97–112.

[BFZ] A.Berenstein, S. Fomin, A. Zelevinsky, *Cluster algebras* III*: Upper bounds and double Bruhat cells*, Duke Math. J. 126 (2005), 1–52.

[BR] A. Buan, I. Reiten, *Acyclic quivers of finite mutation type*, Int. Math. Res. Not. (2006), Art ID 12804, 10 pp.

[CK] P. Caldero, B. Keller, *From triangulated categories to cluster algebras* II, Ann. Sci. École Norm. Sup. (4) 39 (2006), 983–1009.

[DO] H. Derksen, T. Owen, *New graphs of finite mutation type*, Electron. J. Combin 15 (2008), #R139, 15pp.

[FST] S. Fomin, M. Shapiro, D. Thurston, *Cluster algebras and triangulated surfaces. Part* I*: Cluster complexes*, Acta Math. 201 (2008), 83–146.

[FST1] A. Felikson, M. Shapiro, P. Tumarkin, programs sources accessible at `www.math.msu.edu/~mshapiro/FiniteMutation.html`

[FZ1] S. Fomin, A. Zelevinsky, *Cluster algebras* I*: Foundations*, J. Amer. Math. Soc. 15 (2002), 497-529.

[FZ2] S. Fomin, A. Zelevinsky, *Cluster algebras* II*: Finite type classification*, Invent. Math. 154 (2003), 63-121.

[FZ3] S. Fomin, A. Zelevinsky, *Cluster algebras* IV*: Coefficients*, Compos. Math. 143 (2007), 112-164.

[K] V. Kac, *Infinite-dimensional Lie algebras*, Cambridge Univ. Press, London, 1985.

[K1] B. Keller, Quiver mutation in Java, `www.math.jussieu.fr/~keller/quivermutation`

[K2] B. Keller, *Cluster algebras, quiver representations and triangulated categories*, arXiv:0807.1960

[S1] A. Seven, *Recognizing cluster algebras of finite type*, Electron. J. Combin. 14 (2007), #R3, 35pp.

[S2] A. Seven, *Cluster algebras and semipositive symmetrizable matrices*, arXiv:0804.1456v2, to appear in Trans. Amer. Math. Soc