Census of the complex hyperbolic sporadic groups

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Abstract

The goal of this paper is to give a conjectural census of complex hyperbolic sporadic groups. We prove that only finitely many of these sporadic groups are lattices.

We also give a conjectural list of all lattices among sporadic groups, and for each group in the list we give a conjectural group presentation, as well as a list of cusps and generators for their stabilisers. We describe strong evidence for these conjectural statements, showing that their validity depends on the solution of reasonably small systems of quadratic inequalities in four variables.

1 Introduction

In [ParPau], Parker and Paupert considered symmetric triangle groups Δ in $\mathrm{SU}(2,1)$ generated by three complex reflections through angle $2\pi/p$ for $p \geqslant 3$ (the case of $p = 2$ was studied by Parker in [Par3]). By symmetric we mean that the group in question is generated by three complex reflections R_1, R_2 and R_3 with the property that there exists a regular elliptic isometry J of order 3 so that $R_{j+1} = JR_jJ^{-1}$ (where j is taken mod 3). In fact we study the group Γ generated by R_1 and J, which contains Δ with index 1 or 3.

This type of group was first studied by Mostow in [M1] (for $p = 3, 4, 5$), where an additional condition was imposed on the R_j (namely the braid relation $R_iR_jR_i = R_jR_iR_j$); these provided the first examples of non-arithmetic lattices in $\mathrm{SU}(2,1)$.

Following that, Deligne–Mostow and Mostow constructed further lattices in $\mathrm{SU}(n,1)$, $n \leqslant 9$, as monodromy groups of certain hypergeometric functions, in [DM] and [M2]. Some of these lattices turn out not to be arithmetic, although this happens only for $n \leq 3$, only one 3-dimensional example being non-arithmetic.

The lattices from [DM] in dimension 2 were known to Picard (who did not consider their arithmetic nature). These are (commensurable with) groups generated by complex reflections R_j with other values of p; see [M2] and [Sa]. Subsequently no new non-arithmetic lattices have been constructed.

In [ParPau] the last two authors stated necessary conditions for symmetric complex reflection triangle groups $\Delta = \langle R_1, R_2, R_3\rangle$ to be discrete, under the assumption that $R_1R_2R_3$ is elliptic, and R_1R_2 is elliptic or parabolic. They show that the discrete triangle groups of this kind come in three flavors: Mostow's lattices, a specific kind of subgroups of Mostow's lattices, and a third class which they called "sporadic groups"(see Section 2 for a precise definition). Our main motivation was that these new groups are candidates for new non-arithmetic lattices in $\mathrm{SU}(2,1)$. In [Pau] the third author proved that all but one of the sporadic groups are non-arithmetic (i.e.

not contained in an arithmetic lattice in $\mathrm{SU}(2,1)$) and most of them are not commensurable to the previously known lattices (the Picard, Mostow and Deligne-Mostow lattices).

It then became quite natural to use the first author's computer program (see [De1] for instance), and to go through an experimental search for Dirichlet domains for sporadic groups. The goal of the present paper is to report on the results of this search, which turned out to be quite satisfactory. We shall prove among other things that there are only finitely discrete groups among the sporadic groups (see section 7).

More specifically we prove the following (see section 7, in particular Table 3):

Theorem 1.1
- *For* $p \geqslant 7$, $\Gamma(\frac{2\pi}{p}, \sigma_1)$ *is not discrete.*
- *For* $p = 3, 5, 6, 7$, $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_1)$ *is not discrete.*
- *For* $p \geqslant 6$, $\Gamma(\frac{2\pi}{p}, \sigma_2)$ *is not discrete.*
- *For* $6 \leqslant p \leqslant 19$, $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_2)$ *is not discrete.*
- *For* $p = 4, 5, 6$, $\Gamma(\frac{2\pi}{p}, \sigma_4)$ *is not discrete.*
- *For* $p \neq 2, 3, 4, 5, 6, 8, 12$, $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_4)$ *is not discrete.*
- *For* $p \neq 2, 3, 4, 5, 6, 8, 12$, $\Gamma(\frac{2\pi}{p}, \sigma_5)$ *is not discrete.*
- $\Gamma(\frac{2\pi}{4}, \overline{\sigma}_5)$ *is not discrete.*
- $\Gamma(\frac{2\pi}{5}, \sigma_6)$ *and* $\Gamma(\frac{2\pi}{5}, \overline{\sigma}_6)$ *are not discrete.*
- *For* $p \neq 2, 3, 4, 7, 14$, $\Gamma(\frac{2\pi}{p}, \sigma_7)$ *is not discrete.*

We also give a list of groups that have a very good chance of being lattices (some cocompact, some not), but we shall not give a detailed proof that these groups really are lattices. Indeed, we have obtained outstanding evidence that Conjecture 1.1 is correct, but this evidence was obtained by doing numerical computations using floating point arithmetic, and it is conceivable (though very unlikely) that the results are flawed because of issues of precision, in a similar vein as the analysis in [De1] of the results in [M1]. Instead, we will prove Conjecture 1.1 in [DPP] by using more direct geometric methods.

We summarise the results of our computer experimentation in the following (see section 2 for notation):

Conjecture 1.1 *The following sporadic groups are non-arithmetic lattices in* $\mathrm{SU}(2,1)$*:*

- *(cocompact):* $\boldsymbol{\Gamma(\frac{2\pi}{5}, \overline{\sigma}_4)}$, $\boldsymbol{\Gamma(\frac{2\pi}{8}, \overline{\sigma}_4)}$, $\boldsymbol{\Gamma(\frac{2\pi}{12}, \overline{\sigma}_4)}$.
- *(non cocompact):* $\boldsymbol{\Gamma(\frac{2\pi}{3}, \sigma_1)}$, $\boldsymbol{\Gamma(\frac{2\pi}{3}, \sigma_5)}$, $\Gamma(\frac{2\pi}{4}, \sigma_1)$, $\boldsymbol{\Gamma(\frac{2\pi}{4}, \overline{\sigma}_4)}$, $\boldsymbol{\Gamma(\frac{2\pi}{4}, \sigma_5)}$, $\Gamma(\frac{2\pi}{6}, \sigma_1)$, $\boldsymbol{\Gamma(\frac{2\pi}{6}, \overline{\sigma}_4)}$.

Note that the only part of Conjecture 1.1 that really is conjectural is the fact that the groups in question are lattices. The fact that these groups, if lattices, cannot be arithmetic follows from the results in [ParPau] and [Pau]; the groups indicated in bold are known to not be commensurable to Deligne-Mostow-Picard lattices by [Pau] (in fact, for $\Gamma(\frac{2\pi}{4}, \overline{\sigma}_4)$ and $\Gamma(\frac{2\pi}{6}, \overline{\sigma}_4)$ this follows from non-cocompactness by the arguments in [Pau]).

Computer experiments suggest that the sporadic groups that do not appear in the statement of the Conjecture are in fact *not discrete*, apart from

$$\Gamma\left(\frac{2\pi}{3}, \overline{\sigma}_4\right),\ \Gamma\left(\frac{2\pi}{2}, \sigma_5\right),\ \Gamma\left(\frac{2\pi}{2}, \overline{\sigma}_5\right),\ \Gamma\left(\frac{2\pi}{2}, \sigma_7\right),\ \Gamma\left(\frac{2\pi}{3}, \sigma_7\right) \text{ and } \Gamma\left(\frac{2\pi}{2}, \overline{\sigma}_7\right). \tag{1.1}$$

The groups $\Gamma(\frac{2\pi}{2}, \sigma_5)$ and $\Gamma(\frac{2\pi}{2}, \overline{\sigma}_5)$ turn out to be both isomorphic to the lattice studied in [De2] (see [Par3]). The groups $\Gamma(\frac{2\pi}{2}, \sigma_7)$ and $\Gamma(\frac{2\pi}{2}, \overline{\sigma}_7)$ are isomorphic and are discrete, see [Par3]. It seems that these two groups, and also $\Gamma(\frac{2\pi}{3}, \sigma_7)$, have infinite covolume, according to our computer experiments. The group $\Gamma(\frac{2\pi}{3}, \overline{\sigma}_4)$ is understood, partly thanks to work in [ParPau]:

Theorem 1.2 $\Gamma(\frac{2\pi}{3}, \overline{\sigma}_4)$ *is a cocompact lattice in* $\mathrm{SU}(2,1)$.

The fact that this group is discrete was proved in [ParPau] (Proposition 6.4), the point being that all non-trivial Galois conjugates of the relevant Hermitian form are definite. In fact it is the only sporadic group that is contained in an arithmetic lattice, by [Pau].

In order to check that it is cocompact, one uses the same argument as in [De2]. More specifically, one needs to verify that the Dirichlet domain is cocompact, and this can be done without knowing the precise combinatorics of that polyhedron (it is enough to check study a partial Dirichlet domain, and to verify that all the 2-faces of that polyhedron are compact).

2 Sporadic groups

In this section we recall the setup and main results from [ParPau] and [Pau]. The starting point is that groups $\Gamma = \langle R_1, J\rangle$ as defined above can be parametrised up to conjugacy by $\tau = \mathrm{Tr}(R_1 J)$; we denote by $\Gamma(\psi,\tau)$ the group generated by a complex reflection R_1 through angle ψ and a regular elliptic isometry J of order 3 such that $\tau = \mathrm{Tr}(R_1 J)$. In what follows, we always suppose that $R_1 J$ is elliptic and that $R_1 R_2 = R_1 J R_1 J^{-1}$ is either elliptic or parabolic. The generators for this group were given in the following explicit form:

$$J = \begin{bmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \tag{2.1}$$

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \tau & -e^{i\psi/3}\overline{\tau}\\ 0 & e^{-i\psi/3} & 0\\ 0 & 0 & e^{-i\psi/3}\end{bmatrix} \tag{2.2}$$

These preserve the Hermitian form $\langle \mathbf{z},\mathbf{w}\rangle = \mathbf{w}^* H_\tau \mathbf{z}$ where

$$H_\tau = \begin{bmatrix} 2\sin(\psi/2) & -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau}\\ ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2) & -ie^{-i\psi/6}\tau\\ -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2)\end{bmatrix}. \tag{2.3}$$

This always produces a subgroup Γ of $\mathrm{GL}(3,\mathbb{C})$, but the signature of H_τ depends on the values of ψ and τ. For any fixed value of ψ, the parameter space for τ is described in Sections 2.4 and 2.6 of [ParPau]. When Γ preserves a Hermitian form of signature $(2,1)$ we will say that Γ is *hyperbolic*.

A basic necessary condition for a subgroup of $PU(2,1)$ to be discrete, is that all elliptic elements must have finite order. Hence, in order for the groups $\langle R_1, J\rangle$ as above to be discrete, one needs $R_1 J$ to have finite order, and $R_1 R_2$ to have finite order or be parabolic. The list of parameters where the latter necessary condition holds was obtained in [ParPau] (see also [Par3]). We recall the result in the following.

Theorem 2.1 *Let R_1 be a complex reflection of order p and J a regular elliptic isometry of order 3 in $\mathrm{PU}(2,1)$. Suppose that $R_1 J$ is elliptic, and $R_1 R_2 = R_1 J R_1 J^{-1}$ is elliptic or parabolic. If the group $\Gamma = \langle R_1, J\rangle$ is discrete then one of the following is true:*

- Γ *is one of Mostow's lattices.*
- Γ *is a subgroup of one of Mostow's lattices.*
- Γ *is one of the sporadic groups listed below.*

Mostow's lattices correspond to $\tau = e^{i\phi}$ for some angle ϕ; the subgroups of Mostow's lattices arising here correspond to $\tau = e^{2i\phi} + e^{-i\phi}$ for some angle ϕ, and sporadic groups (this can be taken as a definition) are those for which τ takes one of the 18 values $\{\sigma_1, \overline{\sigma}_1, ..., \sigma_9, \overline{\sigma}_9\}$ where the σ_i are given in the following list:

$$\begin{array}{lll} \sigma_1 := e^{i\pi/3} + e^{-i\pi/6}\,2\cos(\pi/4) & \sigma_2 := e^{i\pi/3} + e^{-i\pi/6}\,2\cos(\pi/5) & \sigma_3 := e^{i\pi/3} + e^{-i\pi/6}\,2\cos(2\pi/5)\\ \sigma_4 := e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7} & \sigma_5 := e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(2\pi/5) & \sigma_6 := e^{2\pi i/9} + e^{-\pi i/9}\,2\cos(4\pi/5)\\ \sigma_7 := e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(2\pi/7) & \sigma_8 := e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(4\pi/7) & \sigma_9 := e^{2\pi i/9} + e^{-i\pi/9}\,2\cos(6\pi/7). \end{array} \tag{2.4}$$

Therefore, for each value of $p \geqslant 3$, we have a finite number of new groups to study, the $\Gamma(2\pi/p, \sigma_i)$ and $\Gamma(2\pi/p, \overline{\sigma_i})$ which are hyperbolic. The list of sporadic groups that are hyperbolic is given in the table of Section 3.3 of [ParPau] (and we given them below in Table 3); for the sake of brevity we only recall the following:

Proposition 2.1 *For $p \geqslant 4$ and $\tau = \sigma_1, \sigma_2, \sigma_3, \overline{\sigma}_4, \sigma_5, \sigma_6, \sigma_7, \overline{\sigma}_8$ or σ_9, $\Gamma(2\pi/p, \tau)$ is hyperbolic.*

It was also shown in [ParPau] that some of the hyperbolic sporadic groups are non-discrete (see Corollary 4.2, Proposition 4.5 and Corollary 6.4 of [ParPau]):

Proposition 2.2 *For $p \geqslant 3$ and (τ or $\overline{\tau} = \sigma_3, \sigma_8$ or σ_9), $\Gamma(2\pi/p, \tau)$ is not discrete. Also, for $p \geqslant 3$, $p \neq 5$ and (τ or $\overline{\tau} = \sigma_6$), $\Gamma(2\pi/p, \tau)$ is not discrete.*

The main results of [Pau] are the following two statements:

Theorem 2.2 *For $p \geqslant 3$ and $\tau \in \{\sigma_1, \overline{\sigma}_1, ..., \sigma_9, \overline{\sigma}_9\}$, $\Gamma(2\pi/p, \tau)$ is contained in an arithmetic lattice in* $\mathrm{SU}(2,1)$ *if and only if $p = 3$ and $\tau = \overline{\sigma}_4$.*

Theorem 2.3 *The sporadic groups $\Gamma(2\pi/p, \tau)$ ($p \geqslant 3$ and $\tau \in \{\sigma_1, \overline{\sigma}_1, ..., \sigma_9, \overline{\sigma}_9\}$) fall into infinitely many distinct commensurability classes. Moreover, they are not commensurable to any Picard or Mostow lattice, except possibly when:*

- $p = 4$ or 6
- $p = 3$ and $\tau = \sigma_7$
- $p = 5$ and τ or $\overline{\tau} = \sigma_1, \sigma_2$
- $p = 7$ and $\tau = \overline{\sigma}_4$
- $p = 8$ and $\tau = \sigma_1$
- $p = 10$ and $\tau = \sigma_1, \sigma_2, \overline{\sigma}_2$
- $p = 12$ and $\tau = \sigma_1, \sigma_7$
- $p = 20$ and $\tau = \sigma_1, \sigma_2$
- $p = 24$ and $\tau = \sigma_1$

3 Dirichlet domains

Given a subgroup Γ of $\mathrm{PU}(2,1)$, the Dirichlet domain for Γ centred at p_0 is the set:

$$F_\Gamma = \left\{ x \in \mathrm{H}^2_{\mathbb{C}} : d(x, p_0) \leqslant d(x, \gamma p_0), \forall \gamma \in \Gamma \right\}.$$

A basic fact is that Γ is discrete if and only if F_Γ has nonempty interior, and in that case F_Γ is a fundamental domain for Γ modulo the action of the (finite) stabiliser of p_0 in Γ.

The simplicity of this general notion, and its somewhat canonical nature (it only depends on the choice of the centre p_0), make Dirichlet domains convenient to use in computer investigation as in [M1], [Ri], [De1] and [De2]. Note however that there is no algorithm to decide whether the set F_Γ has non-empty interior, and the procedure we describe below may never end (this is already the case in the constant curvature setting, i.e. in real hyperbolic space of dimension at least three, see for instance [EP]).

Our computer search is quite a bit more delicate than the search for fundamental domains in the setting of arithmetic groups. The recent announcement that Cartwright and Steger have been able to find presentations for the fundamental groups of all so-called *fake projective planes* mentions the use of massive computer calculations in the same vein as our work (see [CS]), but there are major differences however.

They use Dirichlet domains, but their task is facilitated by the fact that the fundamental groups of fake projective planes are known to be *arithmetic* subgroups of $\mathrm{PU}(2,1)$ (see [K] and [Y]). In particular, all the groups they consider are known to be discrete from the very beginning (which is certainly not the case for most complex hyperbolic sporadic groups). Cartwright and Steger also use the knowledge of the volumes of the corresponding fundamental domains (the list of arithmetic lattices that could possibly contain the fundamental group of a fake projective plane is brought down to a finite list by using Prasad's volume formula [Pra]). This allows one to check whether a partial Dirichlet domain

$$F_W = \left\{ x \in \mathrm{H}^2_{\mathbb{C}} : d(x, p_0) \leqslant d(x, \gamma p_0), \forall \gamma \in W \right\}$$

determined by a given finite set $W \subset \Gamma$ is actually equal to F_Γ.

For an arbitrary discrete subgroup $\Gamma \subset \mathrm{PU}(2,1)$ and an arbitrary choice of the centre p_0, the set F_Γ is a polyhedron bounded by *bisectors* (see [M1] and [G]), but it may have infinitely many faces, even if Γ is geometrically finite (see [B]).

Moreover, the combinatorics of Dirichlet domains tend to be unnecessarily complicated, and one usually expects that simpler fundamental domains can be obtained by suitable clever geometric constructions. This general idea is illustrated by Dirichlet domains for lattices in $\mathbb{R}^2$; when the group is not a rectangular lattice, i.e. not generated by two translations along orthogonal axes, the Dirichlet domain centred at any point is a hexagon (rather than a parallelogram).

In $\mathrm{H}^2_{\mathbb{C}}$, Dirichlet domains typically contain digons (pairs of vertices connected by distinct edges), see Figure 1. In particular the skeleton is not piecewise totally geodesic. One can also check that the 2-faces of a Dirichlet domain can never be contained in a totally real totally geodesic copy of $\mathrm{H}^2_{\mathbb{R}}$, which makes this notion a little bit unnatural (this was part of the motivation behind the constructions of [DFP], where fundamental domains with simpler combinatorics that those in [M1] were obtained).

4 Experimental results

4.1 The G-procedure

In order to sift through the complex hyperbolic sporadic groups, we have run the procedures explained in [De1] and [De2] in order to explore the Dirichlet domains centred at the centre of mass of the mirrors of the three generating reflexions.

In terms of the notation in Section 2, we take p_0 to be the unique fixed point in $\mathrm{H}^2_{\mathbb{C}}$ of the regular elliptic element J (this point is given either by $(1,1,1)$, $(1,\omega,\overline{\omega})$ or $(1,\overline{\omega},\omega)$ for $\omega = (-1+i\sqrt{3})/2$, depending on the parameters p and τ.

We start with the generating set $W_0 = \{R_1^{\pm1}, R_2^{\pm1}, R_3^{\pm1}\}$ for Γ, and construct an increasing sequence of sets $W_0 \subset W_1 \subset W_2 \subset \dots$ by the G-procedure (named after G. Giraud, see [De1] for the explanation of this terminology).

First define a *G-step* of the procedure by:

$$G(W) = W \cup \{\alpha^{-1}\beta : \alpha, \beta \in W \text{ yield a non-empty generic 2-face of } F_W\}$$

Here "yielding a non-empty 2-face of F_W" means that the set of points of F_W that are equidistant from p_0, αp_0 and βp_0 has dimension two (i.e. it has non-empty interior in the corresponding intersection of two bisectors). "Generic" means that this 2-face is not contained in a complex geodesic (see [De1]).

Definition 4.1 *The set W is called G-closed if $G(W) = W$.*

The sequence W_k is defined inductively by

$$W_{k+1} = G(W_k).$$

The hope is that this sequence stabilises to a G-closed set $W = W_N$ after a finite number of steps. In particular, this procedure is probably suitable only to the search for lattices (not for discrete groups with infinite covolume).

4.2 Issues of precision

The determination of the sequence of sets W_k described in Section 4.1 depends on being able to determine the precise list of all nonempty 2-faces of the polyhedron W, for a given finite set $W \subset \Gamma$. The difficult part is to prove that two bisectors really yield a subset of F_W of dimension smaller than 2, when they appear to do so numerically.

Recall that the polyhedron F_W is described by a (possibly large) set of quadratic inequalities in 4 variables (the real and imaginary parts of the ball coordinates, for instance), where the coefficients of the quadratic polynomials are obtained from matrices which are possibly very long words in the generators R_1, R_2, R_3.

The computation of these matrices can be done without loss of precision, since it can be reduced to arithmetic in the relevant number field (see Section 2.5 of [ParPau]).

It is not clear how to solve the corresponding system of quadratic inequalities. In order to save computational time, and for the lack of having better methods, we have chosen to do all the computations numerically, with a fixed (somewhat rough) precision, essentially the same way as described in [De2]. We now briefly summarise what our computer program does.

For a given (coequidistant) bisector intersection B, we need a method to test whether $B \cap F_W$ has dimension two. In order to do this, We work in spinal coordinates (see [De1]), and fit the disk B into a rectangular $N \times N$ grid. The 2-face is declared non-empty whenever we find more than one point in a given horizontal and in a given vertical line in the grid. For the default version of the program, we take $N = 1000$.

In particular, the above description suggests that whenever the polyhedron F_W becomes small enough, our program will not find any 2-face whatsoever. If this happens at some stage k, the program will consider W_k as being G-closed and stop.

When fed a group that has infinite covolume, one expects that the program would often run forever, since in that case Dirichlet domains have tend to infinitely many faces. In practice, after a certain number of steps, the sets W_k are too large for the computer's capacity, and the program will crash.

For the groups we have tested (namely all sporadic groups with $p \leqslant 24$), we have found these three behaviours:

A: The program finds a G-closed set $W_N = G(W_N)$, and the set of numerically non-empty two-faces is non-empty.

B: The program finds a set W_N for which it does not find any nonempty 2-face whatsoever (in particular W_N is Giraud closed, so the program stops).

C: The program exceeds its capacity in memory and crashes.

As a working hypothesis, we shall interpret Behaviour B as meaning that the group is not discrete, and Behaviour C meaning that the group has infinite covolume (the latter behaviour is of course also conceivable when the group is actually not discrete, or when we make a bad choice of the centre of the Dirichlet domain).

4.3 Census of sporadic groups generated by reflections of small order

The computer program available on the first author's webpage at

`http://www-fourier.ujf-grenoble.fr/~deraux/java`

was run for all sporadic groups (see Section 2) with $2 \leqslant p \leqslant 24$.

The groups with $p = 2$ were analysed by Parker in [Par3], and our program confirms his results; in that case τ and $\overline{\tau}$ give the same groups, and only $\tau = \sigma_5$ or σ_7 appear to be discrete. Both exhibit behaviour A, but the first one gives a compact polyhedron; as mentioned in the introduction, this lattice is actually the same as the $(4,4,4;5)$-triangle group, i.e. the group that is studied in [De2], see [Par3] and [Sc]). The Giraud-closed polyhedron obtained for σ_7 has infinite volume.

For $3 \leqslant p \leqslant 24$, there are few groups that exhibit Behaviour A (as defined in Section 4.2), namely: all groups with $\tau = \overline{\sigma}_4$, those with $\tau = \sigma_1$, $p = 3, 4, 5, 6$, and finally those with $\tau = \sigma_5$, $p = 3, 4$ or 5.

Pictures of the (isometry classes of) 3-faces of the Dirichlet domain for $\Gamma(2\pi/3, \overline{\sigma}_4)$ are given in Figure 1. We chose to display the faces for that specific group because its combinatorics are particularly simple among all sporadic groups (Dirichlet domains for sporadic lattices can have about a hundred faces).

In case of Behaviour A, the program provides a list of faces for the polyhedron F_W, and checks whether it has side-pairings in the sense of the Poincaré polyhedron theorem (once again, we choose to check this only numerically).

There is a minor issue of ambiguity between the side pairings, due to the fact that most groups $\Gamma(\frac{2\pi}{p}, \tau)$ actually contain J, which means that the centre of the Dirichlet domain has non-trivial stabiliser. Possibly after adjusting the side-pairings by pre-composing them with J or J^{-1}, all the groups exhibiting Behaviour A turn out to have side-pairings (or at least they appear to, numerically).

Another way to take care of the issue of non-trivial stabiliser for the centre of the Dirichlet domain is of course simply to change the centre (within reasonably small distance to the centre of mass of the mirrors, since we want the side-pairings obtained from the Dirichlet domain to be related in simple terms to the original generating reflections).

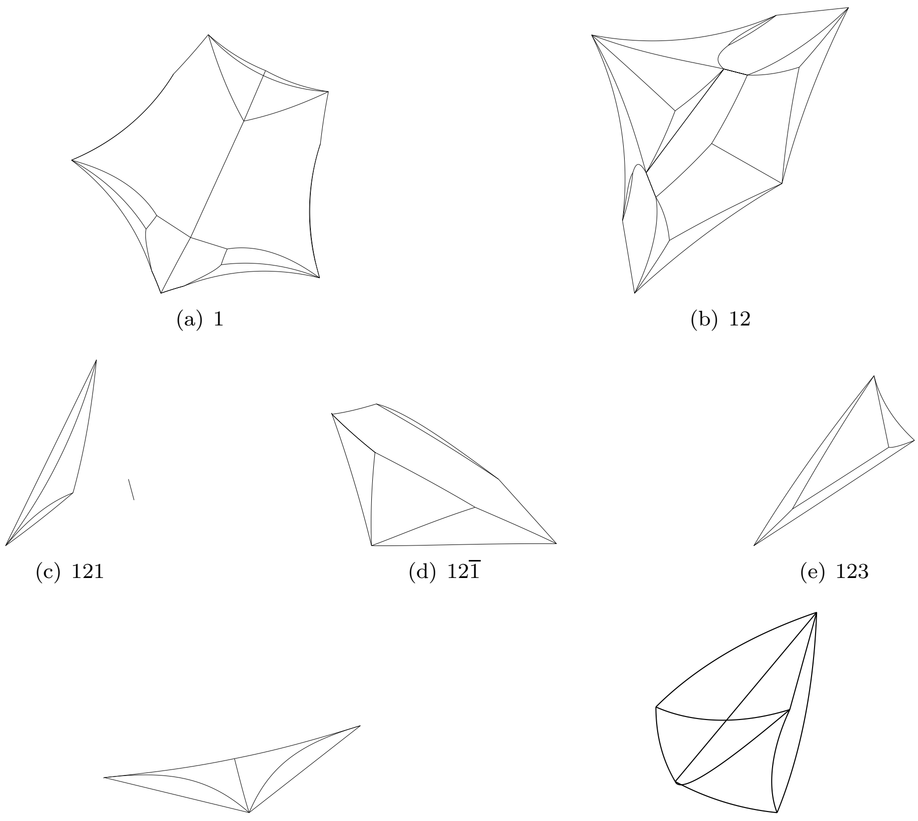

Figure 1: Faces of the Dirichlet domain for $\Gamma(2\pi/3, \overline{\sigma}_4)$, drawn in spinal coordinates.

In either case, either after adjusting the side-pairings by elements of the stabiliser, or after changing the centre, we are in a position to check the cycle conditions of the Poincaré polyhedron theorem. The general philosophy that grew out of [De1] (see also [M1], or even [Pic]) is that the only cycle conditions that need to be checked are those for complex totally geodesic 2-faces, where the cycle transformations are simply complex reflections.

Our program goes through all these complex 2-faces, and computes the rotation angle of the cycle transformations (as well as the total angle inside the polyhedron along the cycle).

Table 1 gives the list of sporadic groups that exhibit Behaviour A and whose complex cycles rotate by an integer part of 2π (possibly after changing the centre of the Dirichlet domain).

τ	p
σ_1	$p = 3, 4, 6$
$\overline{\sigma}_4$	$p = 3, 4, 5, 6, 8, 12$
σ_5	$p = 3, 4$

Table 1: Sporadic groups with $3 \leqslant p \leqslant 24$ whose Dirichlet domain satisfies the hypotheses of the Poincaré polyhedron theorem, at least numerically. For $\tau = \overline{\sigma}_4$, $p = 8$, one needs to use a centre for the Dirichlet domain other than the centre of mass of the mirrors of the three reflections.

For groups that exhibit behaviour A but whose cycle transformations rotate by angles that are not integer parts of 2π, all one can quickly say is that the G-closed polyhedron cannot be a fundamental domain for their action (even modulo the stabiliser of p_0), but the group may still be a lattice. This issue is related to the question of whether the integrality condition of [DM] is close to being necessary and sufficient for the corresponding reflection group to be a lattice (see the analysis in [M3]).

There is of course a natural refinement of the procedure described in Section 4.1 to handle this case. Suppose a given cycle transformation g rotates by an angle α, and $2\pi/\alpha$ is not an integer. If that number is not rational, the group is not discrete (the irrationality can of course be difficult to actually prove). If $\alpha = 2\pi m/n$ for $m, n \in \mathbb{Z}$, then some power $h = g^k$ rotates by an angle $2\pi/n$, and it is natural to replace the G-closed set of group elements W by

$$W \cup hWh^{-1}. \tag{4.1}$$

One then starts over with the G-procedure as described in Section 4.1, starting from $W_0 = W \cup hWh^{-1}$.

The groups with problematic rotation angles are

$$\Gamma\left(\frac{2\pi}{5}, \sigma_1\right), \Gamma\left(\frac{2\pi}{5}, \sigma_5\right),$$

and all groups with $\tau = \overline{\sigma}_4$, $p \neq 3, 4, 5, 6, 8, 12$. The ones with $\tau = \overline{\sigma}_4$ are known not to be discrete, see Theorem 1.1.

The groups $\Gamma(\frac{2\pi}{5}, \sigma_1)$ and $\Gamma(\frac{2\pi}{5}, \sigma_5)$ do not seem to be discrete. Indeed, their Giraud-closed sets have problematic rotation angles, see Table 2. In both cases, after implementing the refinement of (4.1), the G-

Group	cycle transformation	angle
$\Gamma(\frac{2\pi}{5}, \sigma_1)$	$(R_1R_2)^2$	$4\pi/5$
$\Gamma(\frac{2\pi}{5}, \sigma_5)$	$((R_1J)^5R_2^{-1})^2$	$4\pi/15$

Table 2: Some problematic rotation angles in Giraud-closed polyhedra.

procedure exhibits behaviour B.

5 Group presentations

From the geometry of the Dirichlet domains for sporadic lattices, one can infer explicit group presentations. Indeed one knows that the side-pairings generate the group, and the relations are normally generated by the cycle transformations, see [EP] for instance.

Given that there are many faces, it is of course quite prohibitive to write down such a presentation by hand. It is reasonably easy however to have a computer do this. Our program produces files that can be passed to GAP in order to simplify the presentations (it is quite painful, even though not impossible, to do these simplifications by hand). It turns out that the presentations coming from the Dirichlet domains can all be reduced to quite a simple form.

Note that the results of this section are just as conjectural as the statement of Conjecture 1.1, since they depend on the accuracy of the combinatorics of the Dirichlet domains.

The groups with $\tau = \sigma_1$, $\overline{\sigma}_4$ are generated by R_1, R_2 and R_3, that is J can be expressed as a product of the R_j's. For $\tau = \sigma_5$ this not the case, and $\langle R_1, R_2, R_3\rangle$ has index 3 in $\langle J, R_1\rangle$.

$$\Gamma(\tfrac{2\pi}{3},\sigma_1): \quad \begin{gathered} J = 12312312 = 23123123 = 31231231;\\ 1^3 = Id; \quad (123)^8 = Id;\\ (12)^3 = (21)^3; \quad [1(23\overline{2})]^2 = [(23\overline{2})1]^2; \quad 1(232\overline{32})1 = (232\overline{32})1(232\overline{32}). \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{4},\sigma_1): \quad \begin{gathered} J = 12312312 = 23123123 = 31231231;\\ 1^4 = Id; \quad (123)^8 = Id; \quad (12)^{12};\\ (12)^3 = (21)^3; \quad [1(23\overline{2})]^2 = [(23\overline{2})1]^2; \quad 1(232\overline{32})1 = (232\overline{32})1(232\overline{32}). \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{6},\sigma_1): \quad \begin{gathered} J = 12312312 = 23123123 = 31231231;\\ 1^6 = Id; \quad (123)^8 = Id; \quad (12)^6; \quad [1(23\overline{2})]^{12} = Id;\\ (12)^3 = (21)^3; \quad [1(23\overline{2})]^2 = [(23\overline{2})1]^2; \quad 1(232\overline{32})1 = (232\overline{32})1(232\overline{32}); \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{3},\overline{\sigma}_4): \quad \begin{gathered} J^{-1} = 1231231 = 2312312 = 3123123;\\ 1^3 = Id; \quad (123)^7 = Id;\\ (12)^2 = (21)^2; \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{4},\overline{\sigma}_4): \quad \begin{gathered} J^{-1} = 1231231 = 2312312 = 3123123;\\ 1^4 = Id; \quad (123)^7 = Id;\\ (12)^2 = (21)^2; \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{5},\overline{\sigma}_4): \quad \begin{gathered} J^{-1} = 1231231 = 2312312 = 3123123;\\ 1^5 = Id; \quad (123)^7 = Id; \quad (12)^{20};\\ (12)^2 = (21)^2; \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{6},\overline{\sigma}_4): \quad \begin{gathered} J^{-1} = 1231231 = 2312312 = 3123123;\\ 1^6 = Id; \quad (123)^7 = Id; \quad (12)^{12};\\ (12)^2 = (21)^2; \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{8},\overline{\sigma}_4): \quad \begin{gathered} J^{-1} = 1231231 = 2312312 = 3123123;\\ 1^8 = Id; \quad (123)^7 = Id; \quad (12)^8; \quad [1(23\overline{2})]^{24};\\ (12)^2 = (21)^2; \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{12},\overline{\sigma}_4): \quad \begin{gathered} J^{-1} = 1231231 = 2312312 = 3123123;\\ 1^{12} = Id; \quad (123)^7 = Id; \quad (12)^6; \quad [1(23\overline{2})]^{12};\\ (12)^2 = (21)^2; \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{3},\sigma_5): \quad \begin{gathered} J^3 = Id; \quad J1J^{-1} = 2; \quad J2J^{-1} = 3; \quad J3J^{-1} = 1;\\ 1^3 = Id; \quad (123)^{10};\\ (12)^2 = (21)^2; \quad 1(23\overline{2})1(23\overline{2})1 = (23\overline{2})1(23\overline{2})1(23\overline{2}). \end{gathered}$$

$$\Gamma(\tfrac{2\pi}{4},\sigma_5): \quad \begin{gathered} J^3 = Id; \quad J1J^{-1} = 2; \quad J2J^{-1} = 3; \quad J3J^{-1} = 1;\\ 1^4 = Id; \quad (123)^{10}; \quad (1\overline{3}23\overline{1}23\overline{2})^{12};\\ (12)^2 = (21)^2; \quad 1(23\overline{2})1(23\overline{2})1 = (23\overline{2})1(23\overline{2})1(23\overline{2}). \end{gathered}$$

6 Description of the cusps of the non-compact examples

The geometry of the Dirichlet domains for sporadic lattices gives information about the isotropy groups of any vertex. Rather than giving a whole list, we gather information about the cusps in the Dirichlet domain and in $M = \Gamma \setminus H^2_{\mathbb{C}}$, by giving the number of cusps, as well as generators and relations for their stabilisers. All of the relations follow directly from the conjectural presentations given above. Some follow directly but others with slightly more work.

Once again, the results of this section are conjectural (they depend on the accuracy of the combinatorics of the Dirichlet domains).

p	τ	# cusps	# cusps in M	Generators	Relations	
3	σ_1	3	1	1, 2	$1^3 = 2^3 = Id,$	$(12)^3 = (21)^3$
4	σ_1	6	1	1, $23\overline{2}$	$1^4 = (23\overline{2})^4 = Id,$	$[1(23\overline{2})]^2 = [(23\overline{2})1]^2$
6	σ_1	6	2	1, $232\overline{3}\,\overline{2}$	$1^6 = (232\overline{3}\overline{2})^6 = Id,$	$1(232\overline{3}\overline{2})1 = (232\overline{3}\overline{2})1(232\overline{3}\overline{2})$
				1, $\overline{3}\,\overline{2}323$	$1^6 = (\overline{3}\overline{2}323)^6 = Id,$	$1(\overline{3}\overline{2}323)1 = (\overline{3}\overline{2}323)1(\overline{3}\overline{2}323)$
4	$\overline{\sigma}_4$	3	1	1, 2	$1^4 = 2^4 = Id,$	$(12)^2 = (21)^2$
6	$\overline{\sigma}_4$	6	1	1, $23\overline{2}$	$1^6 = (23\overline{2})^6 = Id,$	$1(23\overline{2})1 = (23\overline{2})1(23\overline{2})$
3	σ_5	3	1	$23\overline{2}$, $(1J)^5$	$(23\overline{2})^3 = [(IJ)^5]^6 = Id,$	$[(23\overline{2})(1J)^{-5}]^2 = [(1J)^{-5}(23\overline{2})]^2$
4	σ_5	3	1	1, 2	$1^4 = 2^4 = Id,$	$(12)^2 = (21)^2$

7 Non-discreteness results

In this section we prove some restrictions on the parameters for the group $\Gamma(2\pi/p, \tau)$ to be discrete. The main tools are Knapp's theorem and Jørgensen's inequality. The former gives a necessary and sufficient condition for a two-generator subgroup of PU(1, 1) to be discrete, assuming both generators as well as their product are elliptic. The reference for Knapp's theorem is [Kna], see also [KS]. The full list of possible rotation angles for A, B and AB will not be needed here. In fact we shall only use the following special case of Knapp's theorem, that applies to isosceles triangles.

Theorem 7.1 *(Knapp) Consider a triangle in $\mathrm{H}^2_{\mathbb{R}}$ with angles α, α, β, and let Δ be the group generated by the reflections in its sides. If Δ is discrete then one of the following holds:*

- $\alpha = \frac{\pi}{q}$ *and* ($\beta = \frac{2\pi}{r}$ *or* $\frac{4\pi}{q}$) *with* $q, r \in \mathbb{N}^*$
- $\alpha = \frac{2\pi}{r}$ *and* $\beta = \frac{2\pi}{r}$ *with* $r \in \mathbb{N}^*$.

Remark 7.1 *In a few cases, we also use the spherical geometry version of Knapp's theorem, which is a result of Schwarz (see [ParPau]).*

Basic hyperbolic trigonometry gives a relationship between the angles and the length of the base of the triangle. Indeed, if the length of the base is 2δ, then

$$\cosh\delta \ \sin\alpha = \cos\frac{\beta}{2} \tag{7.1}$$

This gives a practical computational way to check whether the conditions of Knapp's theorem hold.

Note also that the statement of Knapp's theorem implies that if $\alpha = \pi/q$ for $q \in \mathbb{N}^*$, and if the angle β is larger than $2\pi/3$, then the group cannot be discrete. In view of formula (7.1), the latter statement is that same as one would obtain from Jørgensen's inequality (see [JKP]):

Theorem 7.2 *(Jiang-Kamiya-Parker) Let A be a complex reflection through angle $2\alpha = \frac{2\pi}{q}$ with $q \in \mathbb{N}^*$, with mirror the complex line L_A. Let $B \in \mathrm{PU}(2,1)$ be such that $B(L_A)$ and L_A are ultraparallel, and denote their distance by 2δ. If*

$$|\cosh\delta\ \sin\alpha| < \frac{1}{2} \tag{7.2}$$

then $\langle A, B\rangle$ is non-discrete.

In certain cases we need to deal with groups generated by vertical Heisenberg translations. In this case we need results that generalise the above version of Jørgensen's inequality and Knapp's theorem. These results are complex hyperbolic versions of Shimizu's lemma, Proposition 5.2 of [Par1] and a lemma of Beardon, Theorem 3.1 of [Par2]. We combine them in the following statement which is equivalent to the statements given in [Par1] and [Par2].

Theorem 7.3 *(Parker) Let $A \in \mathrm{SU}(2,1)$ be a parabolic map conjugate to a vertical Heisenberg translation with fixed point z_A. Let $B \in \mathrm{SU}(2,1)$ be a map not fixing z_A. If $\langle A, B\rangle$ is discrete then either $\mathrm{tr}(ABAB^{-1}) \leq -1$ or $\mathrm{tr}(ABAB^{-1}) = 3 - 4\cos^2(\pi/r)$ for some $r \in \mathbb{N}$ with $r \geq 3$. In particular, if*

$$2 < \mathrm{tr}(ABAB^{-1}) < 3 \tag{7.3}$$

then $\langle A, B\rangle$ is non-discrete.

7.1 Summary of results

τ	p with H_τ hyperbolic	p where non-discrete	Result used
σ_1	$[3,\infty)$	$7, 8, [10,\infty)$	Proposition 7.4
		9	Proposition 7.9
$\overline{\sigma}_1$	$[3,7]$	$5, 7$	Proposition 7.5
		$3, 6, 7$	Proposition 7.10
σ_2	$[3,\infty)$	$[6,9], [11,\infty)$	Proposition 7.6
		10	Proposition 7.11
$\overline{\sigma}_2$	$[3,19]$	$[6,9], [11,19]$	Proposition 7.7
		10	Proposition 7.11
σ_3	$[3,\infty)$	$[3,\infty)$	Proposition 4.5 of [ParPau]
$\overline{\sigma}_3$	$[3,6]$	$[3,6]$	Proposition 4.5 of [ParPau]
σ_4	$[4,6]$	$[4,6]$	Proposition 7.3
$\overline{\sigma}_4$	$[3,\infty)$	$7, [9,11], [13,\infty)$	Proposition 7.1
σ_5	$[2,\infty)$	$7, [9,11], [13,\infty$	Proposition 7.2
$\overline{\sigma}_5$	$\{2, 4\}$	4	Proposition 7.3
σ_6	$[3,\infty)$	$3, 4, [6\infty)$	Proposition 4.5 of [ParPau]
		5	Proposition 7.3
$\overline{\sigma}_6$	$[3,29]$	$3, 4, [6\infty)$	Proposition 4.5 of [ParPau]
		5	Proposition 7.3
σ_7	$[2,\infty)$	$5, 6, [8,13], [15,\infty)$	Proposition 7.8
$\overline{\sigma}_7$	$\{2\}$		
σ_8	$[4,41]$	$[4,41]$	Corollary 4.2 of [ParPau]
$\overline{\sigma}_8$	$[4,\infty)$	$[4,\infty)$	Corollary 4.2 of [ParPau]
σ_9	$[3,\infty)$	$[3,\infty)$	Corollary 4.2 of [ParPau]
$\overline{\sigma}_9$	$[4,8]$	$[4,8]$	Corollary 4.2 of [ParPau]

Table 3: Values of the parameter where Knapp or Jørgensen show non-discreteness.

Table 3 summarises our result on non-discreteness of sporadic groups. The second column gives the values of p for which the Hermitian form has signature $(2,1)$ (taken from [ParPau]). The third and fourth columns give values of p for which a well chosen subgroup fails the Knapp test or the Jørgensen test (and hence the group is not discrete). If this was done in [ParPau] we give the reference. For some values of τ we apply Knapp and Jørgensen to two different complex reflections in the group (in which case the results are listed on two separate lines).

7.2 Using Knapp and Jørgensen with powers of R_1R_2

7.2.1 The general set up

Recall from [ParPau] that for any sporadic value τ, there is a positive rational number r/s so that

$$|\tau|^2 = 2 + 2\cos(r\pi/s), \tag{7.4}$$

which corresponds to the fact that R_1R_2 should have finite order. The values of these r and s are clearly the same for σ_j and $\overline{\sigma}_j$, and are given by

τ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9
r/s	1/3	1/5	3/5	1/2	1/2	1/2	1/7	5/7	3/7

(7.5)

Straightforward calculation shows

$$R_1R_2 = \begin{bmatrix} e^{2i\pi/3p}(1-|\tau^2|) & e^{4i\pi/3p}\tau & \tau^2-\overline{\tau} \\ -\overline{\tau} & e^{2i\pi/3p} & e^{-2i\pi/3p}\tau \\ 0 & 0 & e^{-4i\pi/3p} \end{bmatrix}$$

which has eigenvalues $-e^{2i\pi/3p}e^{ri\pi/s}$, $-e^{2i\pi/3p}e^{-ri\pi/s}$, $e^{-4i\pi/3p}$. Therefore $(R_1R_2)^s$ has a repeated eigenvalue. An $e^{-4i\pi/3p}$-eigenvector of R_1R_2 is given by

$$\mathbf{p}_{12} = \begin{bmatrix} e^{-2i\pi/3p}\tau^2 + e^{4i\pi/3p}\overline{\tau} - e^{-2i\pi/3p}\overline{\tau} \\ e^{2i\pi/3p}\overline{\tau}^2 + e^{-4i\pi/3p}\tau - e^{2i\pi/3p}\tau \\ 2\cos(2\pi/p) + 2\cos(2r\pi/s) \end{bmatrix}.$$

For most values of p and τ this vector is negative, in which case its orthogonal complement with respect to H gives a complex line in the ball. Hence (in most cases) it is a complex reflection, and one checks easily that it commutes with both R_1 and R_2.

Likewise, for most values of p, τ, $(R_2R_3)^s$ is a complex reflection that commutes with R_2 and R_3, and it fixes a complex line whose polar vector is $\mathbf{p}_{23} = J(\mathbf{p}_{12})$. If the distance between these two lines is $2\delta_p$ then

$$\cosh^2(\delta_p) = \frac{\langle \mathbf{p}_{12}, \mathbf{p}_{23}\rangle\langle \mathbf{p}_{23}, \mathbf{p}_{12}\rangle}{\langle \mathbf{p}_{12}, \mathbf{p}_{12}\rangle\langle \mathbf{p}_{23}, \mathbf{p}_{23}\rangle} = \frac{\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right|^2}{\big(2\cos(2\pi/p) + 2\cos(r\pi/s)\big)^2}. \tag{7.6}$$

The eigenvalues of $(R_1R_2)^s$ are $(-1)^{r+1}e^{2s\pi/3p}$, $(-1)^{r-1}e^{2s\pi/3p}$, $e^{-4s\pi/3p}$. Therefore the rotation angle of $(R_1R_2)^s$ is $(r-1)\pi + 2s\pi/p$. This may or may not be of the form $2\pi/c$. When it is not, we can find a positive integer k so that $(R_1R_2)^{sk}$ is a complex reflection whose angle has the form $2\pi/c$. We define $2\alpha_p$ to be the smallest positive rotation angle among all powers of $(R_1R_2)^s$.

In order to prove non-discreteness using the Jørgensen inequality, we now need to find values of p such that

$$\cosh\delta_p \sin\alpha_p = \frac{\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right|\sin\alpha_p}{\left|2\cos(2\pi/p) + 2\cos(r\pi/s)\right|} < \frac{1}{2}. \tag{7.7}$$

In order to prove non-discreteness using Knapp's theorem we must find values of p for which

$$\cosh\delta_p \sin\alpha_p = \frac{\left|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\right|\sin\alpha_p}{\left|2\cos(2\pi/p) + 2\cos(r\pi/s)\right|} \neq \cos(\pi/q) \text{ or } \cos(2\alpha_p) \tag{7.8}$$

for a natural number q.

Since $(R_1R_2)^s$ is a complex reflection that rotates through angle $(r-1)\pi + 2s\pi/p$, we can apply the test of Jørgensen's inequality simply to $(R_1R_2)^s$. As $\big|\sin\big((r-1)\pi + 2s\pi/p\big)\big| = \big|2\sin(2s\pi/p)\big|$ this involves finding values of p for which

$$\Big|\cosh(\delta_p)\sin(2s\pi/p)\Big| = \frac{\big|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\big|\;|\sin(2s\pi/p)|}{|2\cos(2\pi/p) + 2\cos(2r\pi/s)|} < \frac{1}{2}. \tag{7.9}$$

For fixed r and s, as p tends to infinity, the left hand side tends to zero. This shows at once that there can be only finitely many discrete groups among all sporadic groups; the rest of the paper is devoted to the proof of Theorem 1.1, which is a vast refinement of that statement.

In the next few sections, we shall apply Knapp or Jørgensen to various powers of R_1R_2 (other elements in the group as well) in order to get the better non-discreteness results.

7.2.2 Cases where $|\tau|^2 = 2$

From (7.4), for any τ with $|\tau|^2 = 2$, we have $r/s = 1/2$ and so

$$\cosh\delta_p = \frac{\big|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\big|}{\big|2\cos(2\pi/p)\big|}.$$

This happens for $\tau = \sigma_4$, $\overline{\sigma}_4$, σ_5, $\overline{\sigma}_5$, σ_6 or $\overline{\sigma}_6$. For all these values we have

$$(R_1R_2)^2 = \begin{bmatrix} -e^{4i\pi/3p} & 0 & e^{2i\pi/3p}\overline{\tau} + e^{-4i\pi/3p}\tau^2 - e^{-4i\pi/3p}\overline{\tau} \\ 0 & -e^{4i\pi/3p} & e^{-2i\pi/p}\tau + \overline{\tau}^2 - \tau \\ 0 & 0 & e^{-8i\pi/3p} \end{bmatrix},$$

which is a complex reflection commuting with both R_1 and R_2, and whose rotation angle is $(p-4)\pi/p$. Note that $(p-4)\pi/p = 2\pi/c$ for some $c \in \mathbb{Z}\cup\{\infty\}$ if and only if p and c are as given in the following table:

p	2	3	4	5	6	8	12
c	-2	-6	∞	10	6	4	3

(When $p = 4$, and hence $c = \infty$, we find that $(R_1R_2)^2$ is parabolic.) For other values of p, by choosing an appropriate power k, we can arrange that $(R_1R_2)^{2k}$ rotates by a smaller angle than $(R_1R_2)^2$:

Lemma 7.1 *Let $(R_1R_2)^2$ be as above. There exists $k \in \mathbb{Z}$ so that $(R_1R_2)^{2k}$ has rotation angle $2\alpha_p$ where*

$$\alpha_p = \frac{\gcd(p-4, 2p)\pi}{2p}.$$

In particular

- *If $p \equiv 1$ (mod 2), then $\gcd(p-4, 2p) = 1$, and so $\alpha_p = \frac{\pi}{2p}$.*
- *If $p \equiv 2$ (mod 4), then $\gcd(p-4, 2p) = 2$, and so $\alpha_p = \frac{\pi}{p}$.*
- *If $p \equiv 4$ (mod 8), then $\gcd(p-4, 2p) = 8$, and so $\alpha_p = \frac{4\pi}{p}$.*
- *If $p \equiv 0$ (mod 8), then $\gcd(p-4, 2p) = 4$, and so $\alpha_p = \frac{2\pi}{p}$.*

Proof. We want to find k so that $k(p-4)\pi/p$ reduced modulo 2π is "minimal". More precisely, we write this as

$$k(p-4)\pi/p - 2\pi l = 2\alpha_p$$

for $k \in \mathbb{N}^*$, $l \in \mathbb{N}$, and we want to find α_p of the form π/c for some $c \in \mathbb{N}$. The optimal value of k depends on arithmetic properties of p. Let $d = \gcd(p-4, 2p)$ then we can find integers k and l so that $k(p-4) - l(2p) = d$. This means that $k(p-4)\pi/p - 2\pi l = d\pi/p$ and so $\alpha_p = d\pi/2p$. This proves the first assertion.

If we write $(p-4) = ad$ and $2p = bd$ then, eliminating p, we have $2ad + 8 = bd$ and so $d = 1$, 2, 4 or 8. It is easy to check which values of p correspond to which value of d. □

In the case where $c = \infty$ the map $(R_1R_2)^2$ is parabolic. Up to multiplying by a cube root of unity, we have

$$\mathrm{tr}\Big((R_1R_2)^2J(R_1R_2)^2J^{-1}\Big) = \mathrm{tr}\Big((R_1R_2)^2(R_2R_3)^2\Big) = 3 - \big|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\big|^2.$$

Thus applying Theorem 7.3 with $A = (R_1R_2)^2$ and $J = B$ we see can prove non-discreteness by showing that

$$\big|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\big| < 1 \quad \text{or} \quad \big|\overline{\tau}^2 + e^{-2i\pi/p}\tau - \tau\big| \neq 2\cos(\pi/r) \tag{7.10}$$

with r a natural number at least 3.

Checking (7.7), (7.8) or (7.10) is best done by a computer.

Proposition 7.1 *Let $\tau = \overline{\sigma}_4 = (-1 - i\sqrt{7})/2$ and so $r/s = 1/2$. Then:*

- *If p is odd then (7.7) holds for $p \geqslant 7$;*
- *If $p \equiv 2 \pmod 4$ then (7.7) holds for $p \geqslant 10$;*
- *If $p \equiv 4 \pmod 8$ then (7.8) holds for $p = 20$ and (7.7) holds for $p \geqslant 28$;*
- *If $p \equiv 0 \pmod 8$ then (7.7) holds for $p \geqslant 16$.*

Thus for all the values of p given above $\langle (R_1R_2)^2, J\rangle$ and hence $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_4)$ is not discrete.

Proof. For the sake of concreteness, we list some of the values in the following table

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
7	$\pi/14$	$0.4257\ldots$	(7.7)
9	$\pi/18$	$0.2650\ldots$	(7.7)
10	$\pi/10$	$0.4423\ldots$	(7.7)
14	$\pi/14$	$0.2774\ldots$	(7.7)
20	$\pi/5$	$0.6748\ldots$	(7.8)
28	$\pi/7$	$0.4754\ldots$	(7.7)
16	$\pi/8$	$0.4601\ldots$	(7.7)
24	$\pi/12$	$0.2889\ldots$	(7.7)

□

Proposition 7.2 *Let $\tau = \sigma_5 = e^{2i\pi/9} + e^{-i\pi/9}2\cos(2\pi/5)$ and so $r/s = 1/2$. Then:*

- *If p is odd then (7.7) holds when $p \geqslant 7$;*
- *If $p \equiv 2 \pmod 4$ then (7.7) holds when $p \geqslant 10$;*
- *If $p \equiv 4 \pmod 8$ then (7.8) holds when $p = 20$ and (7.7) holds when $p \geqslant 28$;*
- *If $p \equiv 0 \pmod 8$ then (7.7) holds when $p \geqslant 16$.*

Thus for all the values of p given above $\langle (R_1R_2)^2, J\rangle$ and hence $\Gamma(\frac{2\pi}{p}, \sigma_5)$ is not discrete.

Proof. Some values are given in the following table:

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
7	$\pi/14$	$0.4977\ldots$	(7.7)
9	$\pi/18$	$0.3011\ldots$	(7.7)
10	$\pi/10$	$0.4974\ldots$	(7.7)
14	$\pi/14$	$0.3032\ldots$	(7.7)
20	$\pi/5$	$0.7202\ldots$	(7.8)
28	$\pi/7$	$0.4988\ldots$	(7.7)
16	$\pi/8$	$0.4980\ldots$	(7.7)
24	$\pi/12$	$0.3053\ldots$	(7.7)

□

Recall from [ParPau] that $\Gamma(\frac{2\pi}{p},\sigma_4)$ has signature $(2,1)$ exactly when $4 \leqslant p \leqslant 6$; that $\Gamma(\frac{2\pi}{p},\overline{\sigma}_5)$ has signature $(2,1)$ exactly when $p=2$ or 4, and that $\Gamma(\frac{2\pi}{p},\sigma_6)$ and $\Gamma(\frac{2\pi}{p},\overline{\sigma}_6)$ are not discrete except possibly when $p=5$. Hence for each of these values of τ we only have finitely many things to check. We gather these cases into a single result.

Proposition 7.3

- *If $\tau=\sigma_4=(-1+i\sqrt{7})/2$, and so $r/s=1/2$, and $p=4$ then* (7.10) *holds;*
- *If $\tau=\sigma_4=(-1+i\sqrt{7})/2$, and so $r/s=1/2$, and $p=5$ then* (7.7) *holds;*
- *If $\tau=\sigma_4=(-1+i\sqrt{7})/2$, and so $r/s=1/2$, and $p=6$ then* (7.8) *holds;*
- *If $\tau=\overline{\sigma}_5=e^{-2i\pi/9}+e^{i\pi/9}2\cos(2\pi/5)$, and so $r/s=1/2$, and $p=4$ then* (7.10) *holds;*
- *If $\tau=\sigma_6=e^{2i\pi/9}+e^{-i\pi/9}2\cos(4\pi/5)$, and so $r/s=1/2$, and $p=5$ then* (7.8) *holds;*
- *If $\tau=\overline{\sigma}_6=e^{-2i\pi/9}+e^{i\pi/9}2\cos(4\pi/5)$, and so $r/s=1/2$, and $p=5$ then* (7.8) *holds.*

Thus for these values of τ and p then $\langle (R_1R_2)^2, J\rangle$ and hence $\Gamma(\frac{2\pi}{p},\tau)$ is not discrete.

Proof. Suppose $\tau=\sigma_4=(-1+i\sqrt{7})/2$. If $p=4$ we have

$$\left|\overline{\tau}^2+e^{-2i\pi/p}\tau-\tau\right|=\sqrt{3-\sqrt{7}}=0.595\ldots$$

If $p=5$ then $\alpha_p=\pi/10$ and $\cosh(\delta_p)\sin(\alpha_p)=0.445\ldots$. If $p=6$ then $\alpha_p=\pi/6$ and $\cosh(\delta_p)\sin(\alpha_p)=0.550\ldots\in\big(\cos(\pi/3),\cos(\pi/4)\big)$.

If $\tau=\overline{\sigma}_5=e^{-2i\pi/9}+e^{i\pi/9}2\cos(2\pi/5)$ and $p=4$ then

$$\left|\overline{\tau}^2+e^{-2i\pi/p}\tau-\tau\right|=\sqrt{\frac{7+\sqrt{5}-3\sqrt{3}-\sqrt{15}}{2}}=0.289\ldots.$$

If $\tau=\sigma_6=e^{2i\pi/9}+e^{-i\pi/9}2\cos(4\pi/5)$ and $p=5$ then $\cosh(\delta_p)\sin(\alpha_p)=0.937\ldots\in\big(\cos(\pi/8),\cos(\pi/9)\big)$.

If $\tau=\overline{\sigma}_6=e^{-2i\pi/9}+e^{i\pi/9}2\cos(4\pi/5)$ and $p=5$ then $\cosh(\delta_p)\sin(\alpha_p)=0.750\ldots\in\big(\cos(\pi/4),\cos(\pi/5)\big)$. □

7.2.3 Cases where $|\tau|^2=3$

We now consider the case $|\tau|^2=3$, which happens for $\tau=\sigma_1$ or $\overline{\sigma}_1$. In this case $r/s=1/3$ and

$$(R_1R_2)^3=\begin{bmatrix} e^{2i\pi/p} & 0 & (e^{-2i\pi/p}-1)\big(e^{-2i\pi/3p}\tau^2+e^{4i\pi/3p}\overline{\tau}-e^{-2i\pi/3p}\overline{\tau}\big)\\ 0 & e^{2i\pi/p} & (e^{-2i\pi/p}-1)\big(e^{2i\pi/3p}\tau^2+e^{-4i\pi/p}\tau-e^{2i\pi/3p}\tau\big)\\ 0 & 0 & e^{-4i\pi/p}\end{bmatrix}.$$

This is a complex reflection commuting with both R_1 and R_2, with angle $6\pi/p$. As above, we want to check whether (7.7) holds for α_p the smallest possible rotation angle of powers of $(R_1R_2)^3$.

- If $p\equiv 1$ or 2 (mod 3), then we can find $k,l\in\mathbb{N}$ so that $6k\pi/p-2\pi l=2\pi/p$. Hence $\alpha_p=\pi/p$.
- If 3 divides p then $6\pi/p$ is already in the form $2\pi/c$, hence $\alpha_p=3\pi/p$.

Proposition 7.4 *Let $\tau=\sigma_1=e^{i\pi/3}+e^{-i\pi/6}2\cos(\pi/4)$ and so $r/s=1/3$. Then*

- *If $p\equiv 1$ or 2 (mod 3) then* (7.8) *holds when $p=7$ and* (7.7) *holds when $p\geqslant 8$;*
- *If p is divisible by 3 then* (7.8) *holds when $p=12$, 15 or 18 and* (7.7) *holds when $p\geqslant 21$.*

Thus for all the values of p given above $\langle (R_1R_2)^3, J\rangle$ and hence $\Gamma(\frac{2\pi}{p},\sigma_1)$ is not discrete.

Proof. Some values are given in the following table

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
7	$\pi/7$	$0.6510\ldots$	(7.8)
8	$\pi/8$	$0.4969\ldots$	(7.7)
12	$\pi/4$	$0.8134\ldots$	(7.8)
15	$\pi/5$	$0.6510\ldots$	(7.8)
18	$\pi/6$	$0.5416\ldots$	(7.8)
21	$\pi/7$	$0.4631\ldots$	(7.7)

□

From [ParPau] we know that if $\tau = \overline{\sigma}_1 = e^{-i\pi/3} + e^{i\pi/6} 2\cos(\pi/4)$ then the only values of p that give signature $(2,1)$ are those with $3 \leqslant p \leqslant 7$.

Proposition 7.5 *Let $\tau = \overline{\sigma}_1 = e^{-i\pi/3} + e^{i\pi/6} 2\cos(\pi/4)$ and so $r/s = 1/3$.*

- *If $p = 5$ then (7.8) holds;*
- *If $p = 7$ then (7.7) holds.*

Thus for $p = 5$ or 7 we see that $\langle (R_1R_2)^3, J\rangle$ and hence $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_1)$ is not discrete.

7.2.4 Cases where $|\tau|^2 = 2 + 2\cos(\pi/5)$

This happens for $\tau = \sigma_2$ or $\overline{\sigma}_2$. In that case $r/s = 1/5$ and $(R_1R_2)^5$ is a complex reflection with eigenvalues $e^{10i\pi/3p}$, $e^{10i\pi/3p}$, $e^{-20i\pi/3p}$, thus it has rotation angle $10\pi/p$.

- If p is not divisible by 5, then we can find $k, l \in \mathbb{N}$ such that $10k\pi/p - 2\pi l = 2\pi/p$. Hence $\alpha_p = \pi/p$.
- If p is divisible by 5, then $10\pi/p$ is already in the form $2\pi/c$, hence $\alpha_p = 5\pi/p$.

Proposition 7.6 *Let $\tau = \sigma_2 = e^{i\pi/3} + e^{-i\pi/6} 2\cos(\pi/5)$ and so $r/s = 1/5$.*

- *If p is not divisible by 5 then (7.8) holds when $p = 6$ or 7 and (7.7) holds when $p \geqslant 8$;*
- *If p is divisible by 5 then (7.8) holds when $15 \leqslant p \leqslant 30$ and (7.7) holds when $p \geqslant 35$.*

Thus for for these values of p we see that $\langle (R_1R_2)^5, J\rangle$, and hence, $\Gamma(\frac{2\pi}{p}, \sigma_2)$ is not discrete.

Proof. Some values are given in the following table

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
6	$\pi/6$	$0.631\ldots$	(7.8)
7	$\pi/7$	$0.516\ldots$	(7.8)
8	$\pi/8$	$0.438\ldots$	(7.7)
15	$\pi/3$	$0.908\ldots$	(7.8)
20	$\pi/4$	$0.729\ldots$	(7.8)
25	$\pi/5$	$0.601\ldots$	(7.8)
30	$\pi/6$	$0.508\ldots$	(7.8)
35	$\pi/7$	$0.440\ldots$	(7.7)

□.

From [ParPau] we know that if $\tau = \overline{\sigma}_2 = e^{-i\pi/3} + e^{i\pi/6} 2\cos(\pi/5)$ then the Hermitian form has signature $(2,1)$ only when $3 \leq p \leq 19$.

Proposition 7.7 *Let $\tau = \overline{\sigma}_2 = e^{-i\pi/3} + e^{i\pi/6} 2\cos(\pi/5)$ and so $r/s = 1/5$.*

- *If p is not divisible by* 5 *then* (7.8) *holds when* $p = 6$ *and* (7.7) *holds when* $7 \leqslant p \leqslant 19$*;*
- *If p is divisible by* 5 *then* (7.8) *holds when* $p = 15$.

Thus for for these values of p we see that $\langle (R_1R_2)^5, J\rangle$*, and hence* $\Gamma(\frac{2\pi}{p}, \overline{\sigma}_2)$*, is not discrete.*

Proof. Some values are given below

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
6	$\pi/6$	0.5660	(7.8)
7	$\pi/7$	0.4713	(7.7)
15	$\pi/5$	0.8718	(7.8)

□

7.2.5 Cases where $|\tau|^2 = 2 + 2\cos(\pi/7)$

This happens for $\tau = \sigma_7$ or $\overline{\sigma}_7$. In this case $r/s = 1/7$. The only group with $\tau = \overline{\sigma}_7$ and signature $(2,1)$ is $p = 2$. This group is a relabelling of the group with $\tau = \sigma_7$ and $p = 2$. It is discrete. So for the remainder of this section we consider the case when $\tau = \sigma_7 = e^{2i\pi/9} + e^{-i\pi/9}2\cos(2\pi/7)$.

Then $(R_1R_2)^7$ is a complex reflection with eigenvalues $e^{14i\pi/3p}$, $e^{14i\pi/3p}$, $e^{-28i\pi/3p}$; thus it has rotation angle $14\pi/p$.

- If p is not divisible by 7, then we can find $k, l \in \mathbb{N}$ so that $14k\pi/p - 2\pi l = 2\pi/p$. Hence $\alpha_p = \pi/p$.
- If p is divisible by 7, then $14\pi/p$ is already in the form $2\pi/c$, hence $\alpha_p = 7\pi/p$.

Proposition 7.8 *Let* $\tau = \sigma_7 = e^{2i\pi/9} + e^{-i\pi/9}2\cos(2\pi/7)$ *and so* $r/s = 1/7$.

- *If p is not divisible by 7 then* (7.8) *holds for* $p = 5$ *or* 6 *and* (7.7) *holds when* $p \geqslant 8$*;*
- *If p is divisible by* 7 *then* (7.8) *holds for* $21 \leqslant p \leqslant 42$ *and* (7.7) *holds when* $p \geqslant 49$*; with* $p \geqslant 49$.

Thus for for these values of p we see that $\langle (R_1R_2)^7, J\rangle$*, and hence* $\Gamma(\frac{2\pi}{p}, \sigma_7)$*, is not discrete.*

Proof. Some values are given below

p	α_p	$\cosh(\delta_p)\sin(\alpha_p)$	
5	$\pi/5$	0.929...	(7.8)
6	$\pi/6$	0.702...	(7.8)
8	$\pi/8$	0.476...	(7.7)
21	$\pi/3$	0.921...	(7.8)
28	$\pi/4$	0.739...	(7.8)
35	$\pi/5$	0.608...	(7.8)
42	$\pi/6$	0.514...	(7.8)
49	$\pi/7$	0.444...	(7.7)

□

7.3 Using Knapp and Jørgensen with powers of $R_1R_2R_3R_2^{-1}$

7.3.1 The general set up

Painful calculations show

$$R_1R_2R_3R_2^{-1} = \begin{bmatrix} e^{2i\pi/3p}(1-|\tau^2-\overline{\tau}|^2) & e^{-2i\pi/3p}(\tau-(\tau^2-\overline{\tau})\overline{\tau}) & -\tau^2+(\tau^2-\overline{\tau})(|\tau|^2-e^{2i\pi/p}) \\ \tau(\tau-\overline{\tau}^2) & e^{-4i\pi/3p}(1-|\tau|^2) & e^{-2i\pi/3p}\tau(|\tau|^2-1+e^{2i\pi/p}) \\ e^{-2i\pi/3p}(\tau-\overline{\tau}^2) & -e^{-2i\pi/p}\overline{\tau} & e^{2i\pi/3p}+e^{-4i\pi/3p}|\tau|^2 \end{bmatrix},$$

hence $\mathrm{tr}(R_1R_2R_3R_2^{-1}) = e^{2i\pi/3p}(2-|\tau^2-\overline{\tau}|^2)+e^{-4i\pi/3p}$. An $e^{-4i\pi/3p}$ eigenvector of $R_1R_2R_3R_2^{-1}$ is given by

$$\mathbf{p}_{1232} = \begin{bmatrix} e^{-4i\pi/3p}\big(\tau(1-e^{2i\pi/p})-(\tau^2-\overline{\tau})\overline{\tau}\big) \\ |\tau|^2(1-e^{-2i\pi/p})-\overline{\tau}(\overline{\tau}^2-\tau)-|1-e^{2i\pi/p}|^2 \\ e^{-2i\pi/p}\big(\overline{\tau}(1-e^{-2i\pi/p})-(\overline{\tau}^2-\tau)\tau\big) \end{bmatrix}.$$

Suppose that $|\tau^2-\overline{\tau}|^2 = 2+2\cos(r'\pi/s')$. Then $(R_1R_2R_3R_2^{-1})^s$ is a complex reflection. The values of r' and s' are clearly the same for σ_j and $\overline{\sigma}_j$. They are (see [ParPau]):

τ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
r'/s'	1/2	1/3	1/3	2/3	2/5	4/5

(7.11)

Let $2\delta'_p$ denote the distance from its mirror to the image of its mirror under J (with polar vector $\mathbf{p}_{2313} = J(\mathbf{p}_{1232})$). Then:

$$\cosh(\delta'_p) = \frac{|\langle \mathbf{p}_{2313}, \mathbf{p}_{1232}\rangle|}{|\langle \mathbf{p}_{1232}, \mathbf{p}_{1232}\rangle|} = \frac{\big|(1-e^{2i\pi/p})\tau+|\tau|^2(\overline{\tau}^2-2\tau)+e^{-2i\pi/p}\overline{\tau}^2\big|}{2\cos(2\pi/p)+2\cos(r'\pi/s')}.$$

Let α'_p be the smallest non-zero angle that a power of $(R_1R_2R_3R_2^{-1})^s$ rotates by. Let δ'_p, r' and s' be as above. In order to prove non-discreteness using the Jørgensen inequality, we need to find values of p such that

$$\cosh\delta'_p\sin\alpha'_p = \frac{\big|(1-e^{2i\pi/p})\tau+|\tau|^2(\overline{\tau}^2-2\tau)+e^{-2i\pi/p}\overline{\tau}^2\big|\sin\alpha'_p}{\big|2\cos(2\pi/p)+2\cos(r'\pi/s')\big|} < \frac{1}{2}. \tag{7.12}$$

In order to prove non-discreteness using Knapp's theorem we must find values of p for which

$$\cosh\delta'_p\sin\alpha'_p = \frac{\big|(1-e^{2i\pi/p})\tau+|\tau|^2(\overline{\tau}^2-2\tau)+e^{-2i\pi/p}\overline{\tau}^2\big|\sin\alpha'_p}{\big|2\cos(2\pi/p)+2\cos(r'\pi/s')\big|} \neq \cos(\pi/q) \text{ or } \cos(2\alpha_p) \tag{7.13}$$

for a natural number q.

7.4 When $|\tau^2-\overline{\tau}|^2 = 2$

In this case $\tau = \sigma_1$ or $\overline{\sigma}_1$, and $r'/s' = 1/2$. Moreover, $(R_1R_2R_3R_2^{-1})^2$ is a complex reflection with angle $(p-4)\pi/p$. So we proceed as in the section with $|\tau|^2 = 2$. In particular, α'_p is given by Lemma 7.1.

Using Proposition 7.4, we already know that when $p = 7$, 8 or $p \geq 10$ then $\Gamma(\frac{2\pi}{p},\sigma_1)$ is not discrete. Therefore, we restrict our attention to $p \leq 9$.

Proposition 7.9 *Let $\tau = \sigma_1 = e^{i\pi/3}+e^{-i\pi/6}2\cos(\pi/4)$ and so $r'/s' = 1/2$. Then (7.13) holds for $p = 9$. Thus $\langle (R_1R_2R_3R_1^{-1})^2, J\rangle$, and hence also $\Gamma(\frac{2\pi}{p},\sigma_1)$, is not discrete.*

Proof. In this case $\alpha'_p = \pi/18$ and $\cosh(\delta'_p)\sin(\alpha'_p) = 0.686\ldots \in \big(\cos(\pi/3),\cos(\pi/4)\big)$. □

For $\overline{\sigma}_1$, recall from [ParPau] that $\Gamma(\frac{2\pi}{p},\overline{\sigma}_1)$ has signature $(2,1)$ exactly when $3 \leqslant p \leqslant 7$.

Proposition 7.10 *Let $\tau = \overline{\sigma}_1 = e^{-i\pi/3}+e^{+i\pi/6}2\cos(\pi/4)$ and so $r'/s' = 1/2$. Then*

- *If $p = 3$ or $p = 6$ then (7.13) holds;*
- *If $p = 7$ then (7.12) holds.*

Thus for $p = 3$, 6 or 7 the group $\langle (R_1R_2R_3R_1^{-1})^2, J\rangle$, and hence also $\Gamma(\frac{2\pi}{p},\overline{\sigma}_1)$, is not discrete.

Proof. The values of $\cosh(\delta'_p)\sin(\alpha'_p)$ are:

p	α_p	$\cosh(\delta'_p)\sin(\alpha'_p)$	
3	$\pi/6$	$0.982\ldots$	(7.13)
7	$\pi/14$	$0.269\ldots$	(7.12)
6	$\pi/6$	$0.859\ldots$	(7.13)

□

7.5 Cases where $|\tau^2 - \overline{\tau}|^2 = 3$

We only consider the case $\tau = \sigma_2$ or $\overline{\sigma}_2$ (since σ_3 or $\overline{\sigma}_3$ were already handled in [ParPau]). In this case $r'/s' = 1/3$. Then $(R_1 R_2 R_3 R_1^{-1})^3$ is a complex reflection with angle $6\pi/p$. So we proceed as in the section with $|\tau|^2 = 3$. Namely,

- if p is not divisible by 3, some power gives an angle $\alpha_p = \pi/p$;
- if p is divisible by 3, some power gives $\alpha_p = 3\pi/p$.

In order to use Jørgensen, we check whether $\cosh \delta'_p \sin \alpha_p < \frac{1}{2}$.

For $\tau = \sigma_2$, using Propositions 7.6 and 7.7 we only need to consider the cases where $p \leq 5$ or $p = 10$. This method yields nothing new for $p \leq 5$.

Proposition 7.11 *Let* $p = 10$.

- *If* $\tau = \sigma_2 = e^{i\pi/3} + e^{-i\pi/6} 2\cos(\pi/5)$, *and so* $r'/s' = 1/3$, *then* (7.13) *holds;*
- *If* $\tau = \overline{\sigma}_2 = e^{-i\pi/3} + e^{i\pi/6} 2\cos(\pi/5)$, *and so* $r'/s' = 1/3$, *then* (7.12) *holds.*

Thus for $p = 10$ *and* $\tau = \sigma_2$ *or* $\overline{\sigma}_2$, *the group* $\langle (R_1 R_2 R_3 R_2^{-1})^3, J \rangle$ *is not discrete. Hence* $\Gamma(\frac{2\pi}{10}, \sigma_2)$ *and* $\Gamma(\frac{2\pi}{10}, \overline{\sigma}_2)$ *are not discrete.*

Proof. When $p = 10$ and $\tau = \sigma_2$ we have

$$\cosh(\delta'_p) \sin(\alpha'_p) = 0.6181\ldots \in \big(\cos(\pi/3), \cos(\pi/4)\big).$$

When $p = 10$ and $\tau = \overline{\sigma}_2$ we have $\cosh(\delta'_p)\sin(\alpha'_p) = 0.3871\ldots < 1/2$. □

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