Nonparametric predictive inference for system failure time based on bounds for the signature

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Abstract

System signatures provide a powerful framework for reliability assessment for systems consisting of exchangeable components. The use of signatures in nonparametric predictive inference has been presented and leads to lower and upper survival functions for the system failure time, given failure times of tested components. However, deriving the system signature is computationally complex. This paper presents how limited information about the signature can be used to derive bounds on such lower and upper survival functions and related inferences. If such bounds are sufficiently decisive they also indicate that more detailed computation of the system signature is not required.

Key words: bounds; exchangeable components; lower and upper survival functions; nonparametric predictive inference; system signature.

1 Introduction

System signatures are a powerful tool for quantifying reliability of coherent systems consisting of exchangeable components [16]. Consider a system consisting of m exchangeable components, it could be said that such components are all 'of the same type'. Throughout this paper it is assumed that the system is coherent. Let the random failure time of the system be T_S, and let $T_{j:m}$ be the j-th order statistic of the m random component failure

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times for $j = 1, \ldots, m$, with $T_{1:m} \leq T_{2:m} \leq \ldots \leq T_{m:m}$. The system's signature is the m-vector q with j-th component $q_j = P(T_S = T_{j:m})$, the probability that the system fails at the moment of the j-th component failure. Assume that $\sum_{j=1}^{m} q_j = 1$, so the system functions if all components function, has failed if all components have failed, and system failure only occurs at times of component failures. The survival function of the system failure time is

$$P(T_S > t) = \sum_{j=1}^{m} q_j P(T_{j:m} > t) \qquad (1)$$

Recently, the use of signatures for nonparametric predictive inference (NPI) for system reliability has been presented [6]. In NPI for system reliability, lower and upper survival functions are derived for the system's failure time, these reflect the limited knowledge about reliability of the components, using only the information from component tests. A brief introduction to NPI and overview of the results in [6] is given in Section 2.

Derivation of the signature is not straightforward, even for relatively basic systems. But for specific inferences it may not be necessary to compute the exact signature. If computation of signatures is stopped before the exact signature is derived, one typically has bounds for the probabilities q_j. We explore the use of such bounds in NPI, leading to lower and upper bounds for the NPI lower and upper survival functions. For specific inferences, these bounds based on partially known signatures may already be conclusive, meaning that no further computation is needed. The basic results for the use of such bounds in NPI are presented in Section 3, together with explanation of the possible use of information on signatures for subsystems and comparison of the failure times of two systems. Examples in Section 4 illustrate the results in this paper. Section 5 presents some concluding remarks.

2 Using signatures in NPI

Nonparametric predictive inference (NPI) is a statistical method to learn from data in the absence of prior knowledge and using only few modelling assumptions [5]. It provides a solution to some explicit goals for objective (Bayesian) inference, for example the empirical and logical norms as formulated by Williamson [4, 17], it is exactly calibrated from frequentist statistics point of view [15], and it has strong consistency properties within theory of interval probability [2]. NPI is based on Hill's assumption $A_{(n)}$ [14] which gives direct probabilities [11] for one or more real-valued future random quantities, based on observations of n related random quantities. These probabilities are such that all orderings of the future random quantities among the observed random quantities are equally likely; for more details we refer to Coolen [5]. NPI is a framework of statistical theory and methods that use $A_{(n)}$-based lower and upper probabilities [9, 10]. An informal way to interpret lower and upper probabilities is as follows: A lower probability $\underline{P}(E)$ for an event E reflects the evidence in available information in favour of the event E, the corresponding upper probability $\overline{P}(E)$ for this event reflects the evidence in available information against this event. These are logically linked by the conjugacy property $\underline{P}(E) = 1 - \overline{P}(E^c)$, where E^c is the complementary event to E [9].

Suppose that in a test of n components, exchangeable with those in the system considered, the observed failure times were $t_1 < t_2 < \ldots < t_n$. For ease of notation, define $t_0 = 0$ and $t_{n+1} = \infty$. These n observations partition the non-negative real-line into $n+1$ intervals $I_i = (t_{i-1}, t_i)$ for $i = 1, \ldots, n+1$. Consider reliability of a system with m components, so interest is in the m failure times of those components, say $T_1, \ldots, T_m$. The test data and the future observations $T_1, \ldots, T_m$ are linked via repeated use of the assumption $A_{(n)}$ [5, 8]. The order statistics of the m future observations $T_1, \ldots, T_m$ are denoted by $T_{1:m} \leq T_{2:m} \leq \ldots \leq T_{m:m}$. The following probabilities hold for $T_{j:m}$, for $j = 1, \ldots, m$ and for $i = 1, \ldots, n+1$ [8]

$$P(T_{j:m} \in I_i) = \binom{i+j-2}{i-1}\binom{n-i+1+m-j}{n-i+1}\binom{n+m}{n}^{-1}$$

These probabilities lead to the following NPI lower and upper survival functions for $T_{j:m}$, which are the sharpest bounds for the probability of the event $T_{j:m} > t$ that can be justified without further assumptions. The NPI lower survival function for $T_{j:m}$ is, for $t \in (t_{i-1}, t_i]$

$$\underline{S}_{T_{j:m}}(t) = \underline{P}(T_{j:m} > t) = \sum_{l=i+1}^{n+1} P(T_{j:m} \in I_l)$$

and the NPI upper survival function is, for $t \in [t_{i-1}, t_i)$

$$\overline{S}_{T_{j:m}}(t) = \overline{P}(T_{j:m} > t) = \sum_{l=i}^{n+1} P(T_{j:m} \in I_l)$$

At observed failure times t_i, these NPI lower and upper survival functions are equal, that is $\underline{S}_{T_{j:m}}(t_i) = \overline{S}_{T_{j:m}}(t_i)$ for $i = 1, \ldots, n$, while $\underline{S}_{T_{j:m}}(0) = \overline{S}_{T_{j:m}}(0) = 1$. For $t > t_n$, $\underline{S}_{T_{j:m}}(t) = 0$ and $\overline{S}_{T_{j:m}}(t) = \prod_{l=j}^{m} \frac{l}{n+l} > 0$. This reflects that there is no evidence in favour of such components, and hence the system, surviving past time t_n (reflected by the lower survival function being zero), but the evidence against this is limited as there are only n observations (reflected by the upper survival function being a positive decreasing function of n). The NPI lower and upper survival functions for the failure time T_S of a system with signature q are [6]

$$\underline{S}_{T_S}(t) = \underline{P}(T_S > t) = \sum_{j=1}^{m} q_j \underline{S}_{T_{j:m}}(t) \tag{2}$$

$$\overline{S}_{T_S}(t) = \overline{P}(T_S > t) = \sum_{j=1}^{m} q_j \overline{S}_{T_{j:m}}(t) \tag{3}$$

While this is a straightforward generalization of (1), the derivation involves m optimisation problems which take on the optima simultaneously [6]. For more information about the statistical framework of NPI, the theory of imprecise probability and applications in the area of reliability, the reader is referred to [5, 9, 10][1].

3 Partially known signatures

Computation of the system signature is a complex problem due to the fact that $m!$ orderings in which the m components can fail must be considered [3, 16]. Explicit expressions for the signature of some specific system structures are available [12], but general algorithms to compute signatures have not received much attention in the literature, with the noticeable exception of a logical approach presented by Boland [3] which uses the concept of minimal ordered cut sets, reducing the total number of orderings that need to be counted by grouping together orderings which share the same minimal ordered cut set. However, as any computational method has to deal with the very large number of orderings, it is interesting to consider if one really needs to know the exact signature for a specific inference on the system's reliability. It is likely that any method for computing the signature, if ended before the exact signature has been derived, will provide bounds for the probabilities q_j of the signature. In this paper the use of bounds on q_j is explored in NPI. The method presented can be applied throughout the process of computation of the signature and can indicate when further computation is not required.

Assume that bounds on the elements of signature $q = (q_1, \ldots, q_m)$ have been derived, with $0 \leq \underline{q}_j \leq q_j \leq \overline{q}_j \leq 1$. Assume $\sum_{j=1}^m \underline{q}_j \leq 1$ and $\sum_{j=1}^m \overline{q}_j \geq 1$, so at least one signature (with elements summing to one) exists between these bounds. We also assume, for all $j = 1, \ldots, m$

$$\underline{q}_j \geq 1 - \sum_{\substack{l=1 \\ l \neq j}}^{m} \overline{q}_l \quad \text{and} \quad \overline{q}_j \leq 1 - \sum_{\substack{l=1 \\ l \neq j}}^{m} \underline{q}_l \tag{4}$$

If these inequalities are not satisfied then $\underline{q}_j$ can be increased or $\overline{q}_j$ decreased, to the value which gives equality in the corresponding inequality, without any change to the signatures q whose elements are all within these bounds.

Suppose that we want to derive the NPI lower and upper survival functions (2) and (3) based on the observed failure times of n tested components which are exchangeable with those in the system. If the exact system signature is not known, but bounds $\underline{q}_j$ and $\overline{q}_j$ are available for each probability q_j, then these can be used to derive lower and upper bounds for these NPI lower and upper survival functions. These will be the tightest possible

[1] See also www.npi-statistics.com and www.sipta.org

bounds corresponding to these bounds for the elements of the signature. Because $\underline{S}_{T_{j:m}}(t)$ and $\overline{S}_{T_{j:m}}(t)$ are increasing functions of j, for all $t > 0$, it is clear that we can derive two signatures with all their elements within the bounds and such that one of them provides the maximum lower bound for both $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$ and the other provides the minimum upper bound for both $\underline{S}_{T_S}(t)$ and $\overline{S}_{T_S}(t)$, for all $t > 0$. This corresponds to the link between the stochastic ordering of random failure times of systems and the stochastic ordering of their signatures [16]. It is logical to call the signature within these bounds which provides the maximum lower bound for the NPI lower and upper survival functions, the 'pessimistic signature', and denote it by q^p. Similarly, we call the one which provides the minimum upper bound for the NPI lower and upper survival functions, the 'optimistic signature', and denote it by q^o. These terms follow the logical interpretation of 'pessimistic' and 'optimistic' in terms of survival of the system and the lack of knowledge of the actual NPI lower and upper survival functions as the exact signature is not known.

The pessimistic signature puts the probability mass that is flexible according to the given bounds $\underline{q}_j$ and $\overline{q}_j$ as far to the left as possible, so to elements with lower values of j, hence making earlier system failure more likely. The optimistic signature puts this probability mass as far to the right as possible, so to elements with higher values of j, hence making later system failure more likely. Algorithms to derive q^p and q^o are easy to implement, and lead to

$$\begin{aligned} q^p &= (\overline{q}_1, \ldots, \overline{q}_{j_p-1}, 1 - \sum_{j=1}^{j_p-1} \overline{q}_j - \sum_{j=j_p+1}^{m} \underline{q}_j, \underline{q}_{j_p+1}, \ldots, \underline{q}_m) \\ q^o &= (\underline{q}_1, \ldots, \underline{q}_{j_o-1}, 1 - \sum_{j=1}^{j_o-1} \underline{q}_j - \sum_{j=j_o+1}^{m} \overline{q}_j, \overline{q}_{j_o+1}, \ldots, \overline{q}_m) \end{aligned}$$

for some $j_p, j_o \in \{1, \ldots, m\}$. The assumptions (4) ensure that the j_p, j_o are unique and $q^p_{j_p} \in [\underline{q}_{j_p}, \overline{q}_{j_p}]$ and $q^o_{j_o} \in [\underline{q}_{j_o}, \overline{q}_{j_o}]$.

The lower and upper bounds for the NPI lower and upper survival functions for T_S follow immediately from (2), (3) and the pessimistic and optimistic signatures q^p and q^0,

$$\underline{S}^p_{T_S}(t) = \sum_{j=1}^{m} q^p_j \underline{S}_{T_{j:m}}(t) \quad (5)$$

$$\underline{S}^o_{T_S}(t) = \sum_{j=1}^{m} q^o_j \underline{S}_{T_{j:m}}(t) \quad (6)$$

and the lower and upper bounds for the NPI upper survival function for T_S are

$$\overline{S}^p_{T_S}(t) = \sum_{j=1}^{m} q^p_j \overline{S}_{T_{j:m}}(t) \tag{7}$$

$$\overline{S}^o_{T_S}(t) = \sum_{j=1}^{m} q^o_j \overline{S}_{T_{j:m}}(t) \tag{8}$$

These are the sharpest bounds for the NPI lower and upper survival functions for T_S corresponding to the bounds $\underline{q}_j$ and $\overline{q}_j$ for q_j, for $j = 1, \ldots, m$.

Due to the construction of these bounds, it is clear that they can actually be attained. So, when the real signature q is only known up to such bounds for its individual elements, it follows that the NPI lower and upper survival functions for T_S are between their respective bounds, and nothing more can be deduced without additional assumptions or indeed without further computation of the signature. Further computation which falls short of deriving the exact signature will lead to new bounds for the NPI lower and upper survival functions which are within the corresponding earlier bounds. This may be useful for deciding if further computation is required for a specific inferential problem. For example, if one is interested in the system's reliability at time t^* and requires a minimum probability of p^* for the system to function at time t^*, then $\underline{S}^p_{T_S}(t^*) \geq p^*$ would imply that the reliability requirement is certainly met without need for further computation of the signature. Similarly, if $\overline{S}^o_{T_S}(t^*) < p^*$ then the reliability requirement is certainly not met. In the other situations one cannot draw a firm conclusion about whether or not the reliability requirement is met and one may want to continue computation of the system signature. Even with the exact signature it is possible that no firm conclusion can be drawn, namely if $\underline{S}_{T_S}(t^*) < p^* \leq \overline{S}_{T_S}(t^*)$. In such a case one would either require more test data or use additional information, insights or assumptions in order to reach a conclusion. We consider it an advantage of the use of lower and upper probabilities [9] that such situations can occur, as they reflect the limits to the amount of information in test results. The use of these lower and upper bounds at different levels of computation of the system signature, so with increasingly accurate bounds, will be illustrated in Example 1 in Section 4. In all examples we will concentrate on the optimal lower bound for the NPI lower survival function and the optimal upper bound for the NPI upper survival function, which are likely to be of most relevance for inferences.

It may be possible to derive a system's signature by combining signatures of its subsystems. Gaofeng et al [13] present such algorithms for a system consisting of two subsystems in parallel or series configuration, with all components in the system exchangeable. Of course, this combination can be applied repeatedly to derive the system's signature for quite complicated systems, as long as they can be built up by a sequence of pairwise combinations of subsystems, either in series or parallel configuration. For the NPI approach, bounds for the signatures of two subsystems in parallel or series configuration can be used to derive bounds for the full system's signature, using the same algorithms. The reason for this is the assumption that the system is coherent, which implies that a decrease (increase) in reliability of a component can never lead to increased (decreased) reliability of the system, therefore a decrease (increase) in reliability of a subsystem can never lead to increased (decreased) reliability of the system. The pessimistic signatures for the two subsystems can be combined to give the pessimistic signature for the full system, and combining the optimistic signatures for the two subsystems leads to the optimistic signature for the full system. For the formulae for such combinations and the algorithms to compute them the reader is refered to [13]; these are used with the optimistic and pessimistic signatures in Example 2 in Section 4 to illustrates this approach.

In addition to the survival of a system consisting of exchangeable components, the pessimistic and optimistic signatures can be used to derive optimal bounds for other inferences. For example, Coolen and Al-nefaiee [6] considered the comparison of the failure times of two coherent systems, each consisting of exchangeable components. It is assumed that the failure times of the components in the different systems are fully independent, so any information about components' failure times of one system does not affect (lower and upper) probabilities involving only failure times of components of the other system. Due to the monotonicity of this comparison with regard to the systems' signatures, such comparison with exactly known signatures [6] can be generalized to partially known signatures. Let the signatures of systems A and B be q^a and q^b and their failure times T^a and T^b, and assume that these systems have m_a and m_b components and that n_a and n_b components exchangeble with those in the respective system were tested, with failure times $t_1^a < t_2^a < \ldots < t_{n_a}^a$ and $t_1^b < t_2^b < \ldots < t_{n_b}^b$. Let $t_0^a = t_0^b = 0$ and $t_{n_a+1}^a = t_{n_b+1}^b = \infty$. If the exact signatures are known, NPI lower and upper probabilities for the event $T^a \leq T^b + \delta$ are [6]

$$\underline{P}(T^a \leq T^b + \delta) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^a q_j^b \underline{P}(T_{i:m_a}^a \leq T_{j:m_b}^b + \delta)$$

where

$$\underline{P}(T_{i:m_a}^a \leq T_{j:m_b}^b + \delta) = \sum_{l=1}^{n_a} P_l^{a,i} \underline{P}(T_{j:m_b}^b + \delta \geq t_l^a)$$

with $P_l^{a,i} = P(T_{i:m_a}^a \in (t_{l-1}^a, t_l^a))$. Let $v_{l,\delta} \in \{1, \ldots, n_b+1\}$ be such that $t_{v_{l,\delta}-1}^b < t_l^a - \delta < t_{v_{l,\delta}}^b$, then

$$\underline{P}(T_{j:m_b}^b + \delta \geq t_l^a) = \sum_{v=v_{l,\delta}+1}^{n_b+1} P(T_{j:m_b}^b \in (t_{v-1}^b, t_v^b))$$

The corresponding NPI upper probability is

$$\overline{P}(T^a \le T^b + \delta) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^a q_j^b \overline{P}(T^a_{i:m_a} \le T^b_{j:m_b} + \delta)$$

where

$$\overline{P}(T^a_{i:m_a} \le T^b_{j:m_b} + \delta) = \sum_{l=1}^{n_a+1} P_l^{a,i} \overline{P}(T^b_{j:m_b} + \delta \ge t^a_{l-1})$$

and

$$\overline{P}(T^b_{j:m_b} + \delta \ge t^a_{l-1}) = \sum_{v=v_{l,\delta}}^{n_b+1} P(T^b_{j:m_b} \in (t^b_{v-1}, t^b_v))$$

If the exact signatures are not available but instead bounds $\underline{q}^a$ and $\overline{q}^a$ for q^a and $\underline{q}^b$ and $\overline{q}^b$ for q^b have been derived, which are assumed to satisfy (4), then the optimal lower bound for the NPI lower probability for the event $T^a \le T^b + \delta$ is derived using the optimistic signature $q^{a,o}$ for System A and the pessimistic signature $q^{b,p}$ for System B

$$\underline{P}^l(T^a \le T^b + \delta) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^{a,o} q_j^{b,p} \underline{P}(T^a_{i:m_a} \le T^b_{j:m_b} + \delta)$$

The optimal upper bound for the NPI upper probability for $T^a \le T^b + \delta$ is derived using the pessimistic signature $q^{a,p}$ for System A and the optimistic signature $q^{b,o}$ for System B

$$\overline{P}^u(^a \le T^b + \delta) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^{a,p} q_j^{b,o} \underline{P}(T^a_{i:m_a} \le T^b_{j:m_b} + \delta)$$

These bounds follow from the monotonicity of these NPI lower and upper probabilities with regard to the signatures. The lower bound for the NPI lower probability for this event corresponds to maximum optimism about the lifetime of System A and maximum pessimism about the lifetime of System B, which is fully in line with intuition, and of course the other way around for the upper bound for the NPI upper probability. The upper bound for the NPI lower probability and the lower bound for the NPI upper probability are of course derived by taking the alternative optimistic or pessimistic signatures, but these are less likely to be of interest. This is illustrated in Example 3 in Section 4.

4 Examples

Example 1.

For the system in Figure 1, assume that all 7 components are exchangeable, in the sense that every ordering of their failure times is equally likely. Computing the signature for this system involves determining for all of the $7! = 5040$ orderings of the failure times of the components, at which of these ordered times the system fails. Of course, all $6! = 720$ orderings with failure of Component 1 occurring first lead to immediate failure, from which we can conclude the lower bound $\underline{q}_1 = 0.143$. It is easy to see that no other component's failure will lead to immediate system failure if it is the first to fail, hence also the upper bound $\overline{q}_1 = 0.143$. In addition, it is easy to see that the system cannot function with at most two functioning components, this leads to the upper bounds $\overline{q}_6 = \overline{q}_7 = 0$. This information, using conditions (4) but without further computation, can be reflected by $\underline{q} = (0.143, 0, 0, 0, 0, 0, 0)$ and $\overline{q} = (0.143, 0.857, 0.857, 0.857, 0.857, 0, 0)$. The corresponding pessimistic and optimistic signatures are $q^p = (0.143, 0.857, 0, 0, 0, 0, 0)$ and $q^o = (0.143, 0, 0, 0, 0.857, 0, 0)$.

Computation of signatures by counting orderings typically leads to information in the form of lower bounds $\underline{q}_j$ for individual elements of the signature. To illustrate the method presented in this paper further, Table 1 provides, in addition to the first case just mentioned, three more combinations of lower and upper bounds for this system's signature as occurred at different stages of its computation [3], with increasing amount of information in Cases 1 to 4. For each case the pessimistic and optimistic signatures are also presented in this table. Test component failure times were simulated for this example, with $n = 100$ observations taken from the Weibull distribution with shape parameter 3 and scale parameter 1. The corresponding lower bounds for the NPI lower survival function, $\underline{S}^p_{T_S}(t)$ as given in Equation (5), and the upper bounds for the NPI upper survival function, $\overline{S}^o_{T_S}(t)$ as given in Equation (8), are presented in the plots in Figure 2, where in each plot also the NPI lower and upper survival functions are presented based on the exact signature, which is $q = (1/5040) \times (720, 1200, 1392, 1440, 288, 0, 0) = (0.143, 0.238, 0.276, 0.286, 0.057, 0, 0)$. These plots illustrate the use of the bounds as presented in this paper, and also show that the lower bound of the NPI lower survival function moves up if more details about the signature become known, in which case the upper bound for the NPI upper survival function moves down. This figure illustrates nicely the effect of increasing knowledge of the signature, reflected by bounds for the NPI lower and upper survival functions which become closer together. Of course, with more information about the signature, going from Case 1 to Case 4, the resulting bounds are fully nested with those from Case 1 being the widest and containing the bounds from the other three cases, and so on.

As possible use of these bounds in order to determine when no further computation for the signature is needed, suppose that there is a reliability requirement that the system's failure time should exceed 0.5 with probability at least 0.8. With the bounds for the signature in Case 1, the upper bound for the NPI upper survival function at 0.5 is greater than 0.8 and the corresponding lower bound for the NPI lower survival function is less than 0.8, but for the bounds in Case 2, based on some additional computations, the upper bound for the NPI upper survival function at 0.5 is less than 0.8, so it is clear that the reliability requirement cannot be met and hence that no further computation of the signature is needed. Similarly, if one only requires that the system's failure time should exceed 0.5 with probability at least 0.3 then one needs no more computation once the bounds in Case 4 have been derived, as the corresponding lower bound for the NPI lower survival function at 0.5 exceeds 0.3 hence this reliability requirement is certainly met.

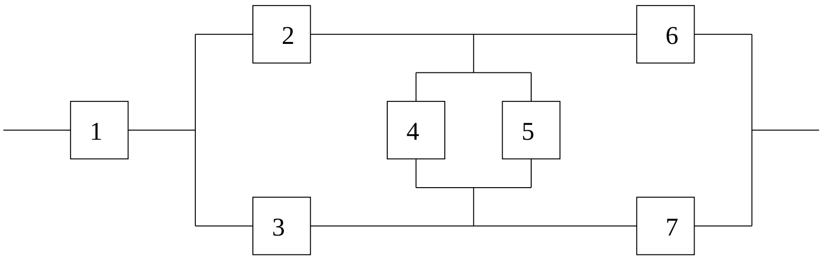

Figure 1: A system with 7 components (Exs. 1,3)

Case 1	$\underline{q}$	(0.143, 0, 0, 0, 0, 0, 0)
	$\overline{q}$	(0.143, 0.857, 0.857, 0.857, 0.857, 0, 0)
	q^p	(0.143, 0.857, 0, 0, 0, 0, 0)
	q^o	(0.143, 0, 0, 0, 0.857, 0, 0)
Case 2	$\underline{q}$	(0.143, 0.143, 0, 0, 0, 0, 0)
	$\overline{q}$	(0.143, 0.857, 0.714, 0.714, 0.714, 0, 0)
	q^p	(0.143, 0.857, 0, 0, 0, 0, 0)
	q^o	(0.143, 0.143, 0, 0, 0.714, 0, 0)
Case 3	$\underline{q}$	(0.143, 0.143, 0.076, 0, 0, 0, 0))
	$\overline{q}$	(0.143, 0.781, 0.714, 0.638, 0.638, 0, 0)
	q^p	(0.143, 0.781, 0.076, 0, 0, 0, 0)
	q^o	(0.143, 0.143, 0.076, 0, 0.638, 0, 0)
Case 4	$\underline{q}$	(0.143, 0.143, 0.152, 0.157, 0, 0, 0)
	$\overline{q}$	(0.143, 0.548, 0.557, 0.562, 0.405, 0, 0)
	q^p	(0.143, 0.548, 0.152, 0.157, 0, 0, 0)
	q^o	(0.143, 0.143, 0.152, 0.157, 0.405, 0, 0)

Table 1: Bounds, pessimistic and optimistic signatures (Ex. 1)

Example 2.

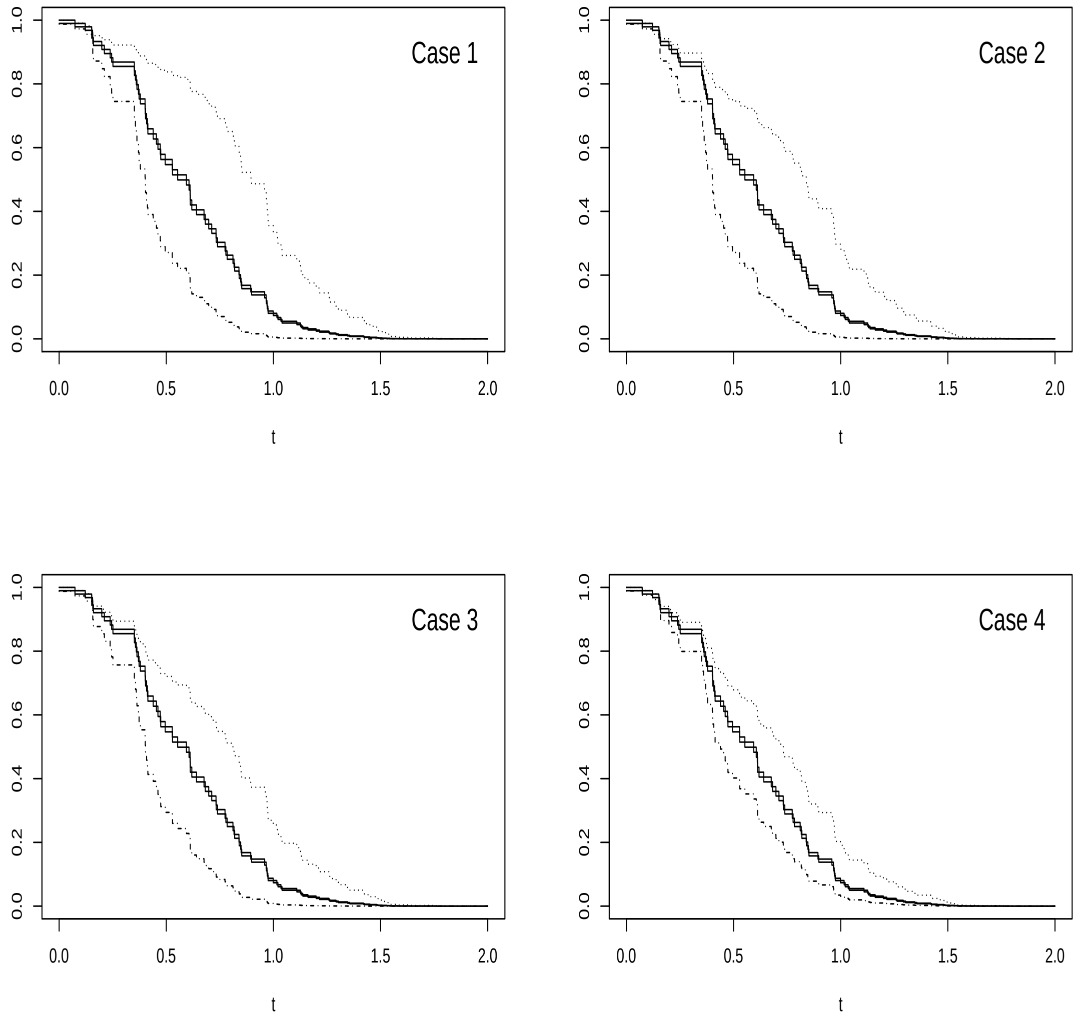

Figure 2: NPI lower and upper survival functions (Ex. 1)

Figure 3 shows a coherent system consisting of 17 exchangeable components, which consists of two subsystems in parallel configuration. Subsystem A is the same system, consisting of 7 components (numbered 1-7), as considered in Example 1. Subsystem B consists of 10 components (numbered 8-17). While the exact signature for this full system can be obtained by using the given signature for Subsystem A together with repeated use of the algorithm presented by Gaofeng et al [13] for Subsystem B and for the combination of the two subsystems, we assume, in order to illustrate the use of the bounds on signatures presented in this paper, that the signatures of subsystems A and B have only been derived partially, with the bounds and corresponding pessimistic and optimistic signatures as presented in Table 2.

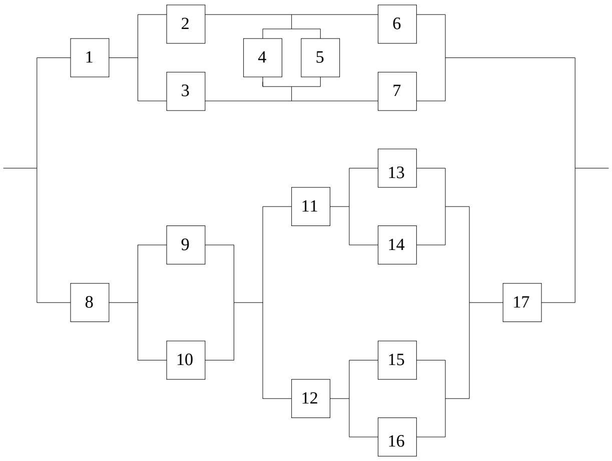

Figure 3: Two subsystems in parallel (Ex. 2)

A	$\underline{q}^a$	(0.143, 0.143, 0.152, 0.157, 0.100, 0, 0)
	$\overline{q}^a$	(0.143, 0.448, 0.457, 0.462, 0.405, 0, 0)
	$q^{a,p}$	(0.143, 0.448, 0.152, 0.157, 0.100, 0, 0)
	$q^{a,o}$	(0.143, 0.143, 0.152, 0.157, 0.405, 0, 0)
B	$\underline{q}^b$	(0.200, 0.222, 0.072, 0.100, 0.046, 0.013, 0, 0, 0, 0)
	$\overline{q}^b$	(0.200, 0.222, 0.419, 0.447, 0.393, 0.360, 0, 0, 0, 0)
	$q^{b,p}$	(0.200, 0.222, 0.419, 0.100, 0.046, 0.013, 0, 0, 0, 0)
	$q^{b,o}$	(0.200, 0.222, 0.072, 0.100, 0.046, 0.360, 0, 0, 0, 0)

Table 2: Bounds, pessimistic and optimistic signatures for subsystems A and B (Ex. 2)

The pessimistic signature for the full 17-component system is derived by application of the algorithm presented by Gaofeng et al [13] with the use of the pessimistic signatures $q^{a,p}$ and $q^{b,p}$, which leads to

$$\begin{aligned} q^p &= (0, 0.015, 0.050, 0.099, 0.161, 0.158, 0.136, 0.109, \\ &\quad 0.084, 0.064, 0.048, 0.035, 0.023, 0.013, 0.005, 0, 0) \end{aligned}$$

Applying the same algorithm with the optimistic signatures $q^{a,o}$ and $q^{b,o}$ leads to

$$\begin{aligned} q^o &= (0, 0.015, 0.031, 0.040, 0.046, 0.051, 0.061, 0.078, \\ &\quad 0.106, 0.128, 0.164, 0.128, 0.084, 0.047, 0.021, 0, 0) \end{aligned}$$

In Figure 4, the left plot presents the lower bound for the NPI lower survival function and the upper bound for the NPI upper survival function, both for the failure time of the full system and based on $n = 10$ failure times of tested components which are exchangeable with those in the system. These failure times were actually simulated from the Weibull distribution with shape parameter 2 and scale parameter 1, but this is not of much relevance as we use no information or assumptions about any underlying probability distribution for the failure times. The right plot in Figure 4 is included for comparison with the following situation: Suppose that one would apply the NPI method presented in this paper directly to each subsystem individually, using the bounds given in Table 2, but neglecting the fact that all components in both subsystems are exchangeable. Making this mistake, one could continue by calculating bounds for the full system's survival function following the standard way for simple parallel systems (effectively using '$1-(1-S_a)(1-S_b)$', with self-explanatory notation). The resulting lower and upper survival functions are greater than (or equal to) the correctly derived bounds for the NPI lower and upper survival function, because for the correct method the dependence of the components in both systems is taken into account [5, 6]. An intuitive explanation is as follows: The parallel system will only fail if both subsystems fail, and if one subsystem is known to fail this contains some information that suggests that the components are not very reliable, which as a consequence increases the (lower and upper) probability that the second subsystem also fails (when compared to the situation with the wrongly assumed independence between the two subsystems). So, in addition to illustrating the use of the algorithm by Gaofeng et al [13] in case signatures are only partially known, this example also shows the importance of taking the dependence of the exchangeable components, due to the limited information about their reliability from the test results, carefully into account, as is done by the NPI approach.

Example 3.
Consider the systems of Figures 5 and 1, called System A and System B, respectively. Assume that each system consists of exchangeable components but these are different for the two systems, assuming independence of the failure times of components in the different systems. Assume that bounds $\underline{q}^a$ and $\overline{q}^a$ are available for the signature of System A, and bounds $\underline{q}^b$ and $\overline{q}^b$ for the signature of System B as given in Table 3, which also presents the pessimistic and optimistic signatures corresponding to these bounds. Assume further that $n_a = n_b = 30$ components exchangeable with those of each type in the respective system have been tested, leading to the failure times in Table 4. The optimal lower bound for the NPI lower probability and the optimal upper bound for the NPI upper probability for the event $T_S^a \leq T_S^b + \delta$ are presented in Figure 6 as functions of δ. These bounds tend to be the more relevant ones for reliability inferences, as briefly discussed in Example 1. This figure also gives the NPI lower and upper probabilities for this event corresponding to the exact signatures [6], which for System B was given in Example 1 and for System A is equal to

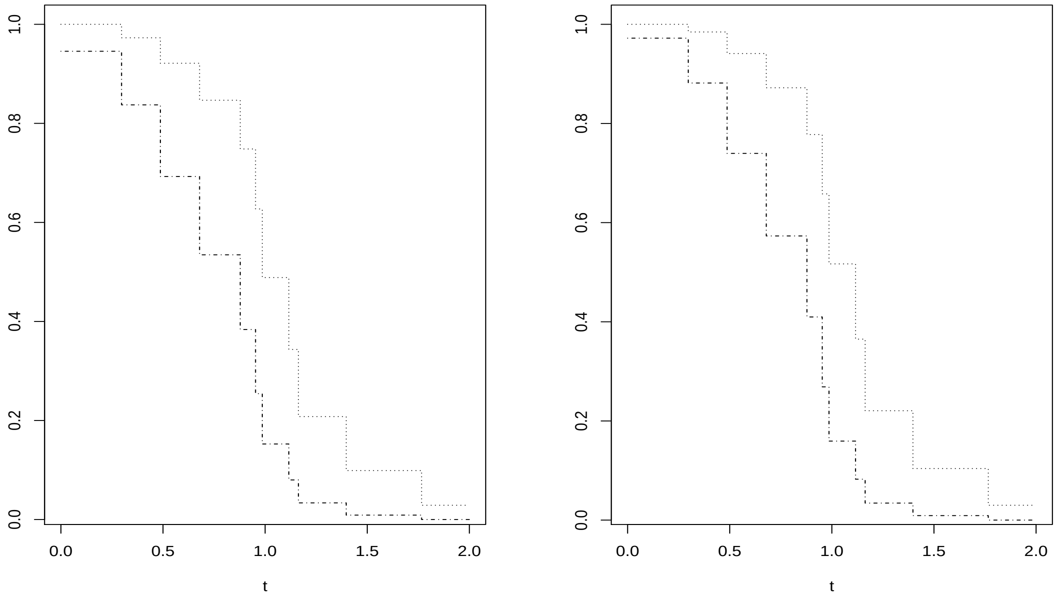

Figure 4: Bounds on NPI lower and upper survival functions (Left); similar but resulting from wrongly assumed independence of subsystems (Right) (Ex. 2)

$q^a = (1/720) \times (0, 96, 192, 336, 96, 0) = (0, 0.133, 0.267, 0.467, 0.133, 0)$. Figure 6 presents the optimal bounds for the NPI lower and upper probabilities for the event $T_S^A < T_S^B + \delta$, as function of δ, together with these actual NPI lower and upper probabilities corresponding to exactly known signatures for both systems. This figure gives a good impression of the actual difference between the failure times of these two systems, where it should be remarked that the bounds based on the partial information are still relatively wide compared to the NPI lower and upper probabilities based on the exact signatures, which shows by considering the vertical distances between the functions at specific values of δ.

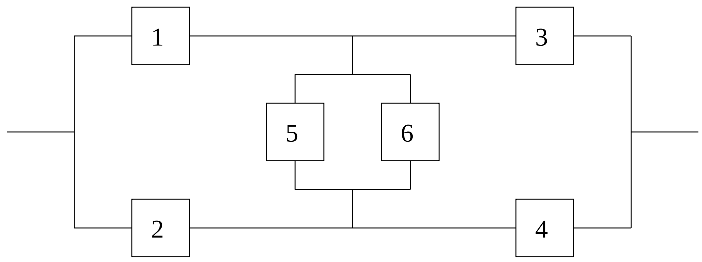

Figure 5: System A (Ex. 3)

System A	$\underline{q}^a$	(0, 0.133, 0.267, 0.044, 0, 0)
	$\overline{q}^a$	(0, 0.133, 0.267, 0.600, 0.556, 0)
	$q^{a,p}$	(0, 0.133, 0.267, 0.600, 0, 0)
	$q^{a,o}$	(0, 0.133, 0.267, 0.044, 0.556, 0)
System B	$\underline{q}^b$	(0.143, 0.143, 0.152, 0.157, 0.100, 0, 0)
	$\overline{q}^b$	(0.143, 0.448, 0.457, 0.452, 0.405, 0, 0)
	$q^{b,p}$	(0.143, 0.448, 0.152, 0.157, 0.100, 0, 0)
	$q^{b,o}$	(0.143, 0.143, 0.152, 0.157, 0.405, 0, 0)

Table 3: Bounds, pessimistic and optimistic signatures (Ex. 3)

System A			System B		
0.223	0.747	0.994	0.154	0.585	1.076
0.265	0.798	1.008	0.155	0.598	1.169
0.372	0.807	1.073	0.347	0.642	1.239
0.419	0.824	1.115	0.402	0.692	1.248
0.564	0.850	1.167	0.483	0.738	1.327
0.630	0.887	1.182	0.512	0.822	1.421
0.675	0.914	1.275	0.513	0.843	1.569
0.685	0.921	1.397	0.548	0.848	1.643
0.709	0.981	1.400	0.563	0.863	1.735
0.727	0.987	1.425	0.574	0.938	2.565

Table 4: Component failure times (Ex. 3)

5 Concluding remarks

all 7 components are exchangeable, in the sense that every ordering of their failure times is equally likely. - add comment as suggested by ref

The concept of system signature and its use for reliability quantification has received increasing attention in the literature in recent years. However, computation of the signature has received relatively little attention and is complex for most systems. In this paper, it is illustrated how one can base reliability inferences on a partially known signature, assuming that bounds for the probabilities in the signature are available. Such bounds may typically result from computations that are based on counting all the orderings, where any further computations lead to sharpening of the bounds. Recently, interesting results have been presented by Gaofeng et al [13], who show how signatures for subsystems can be combined

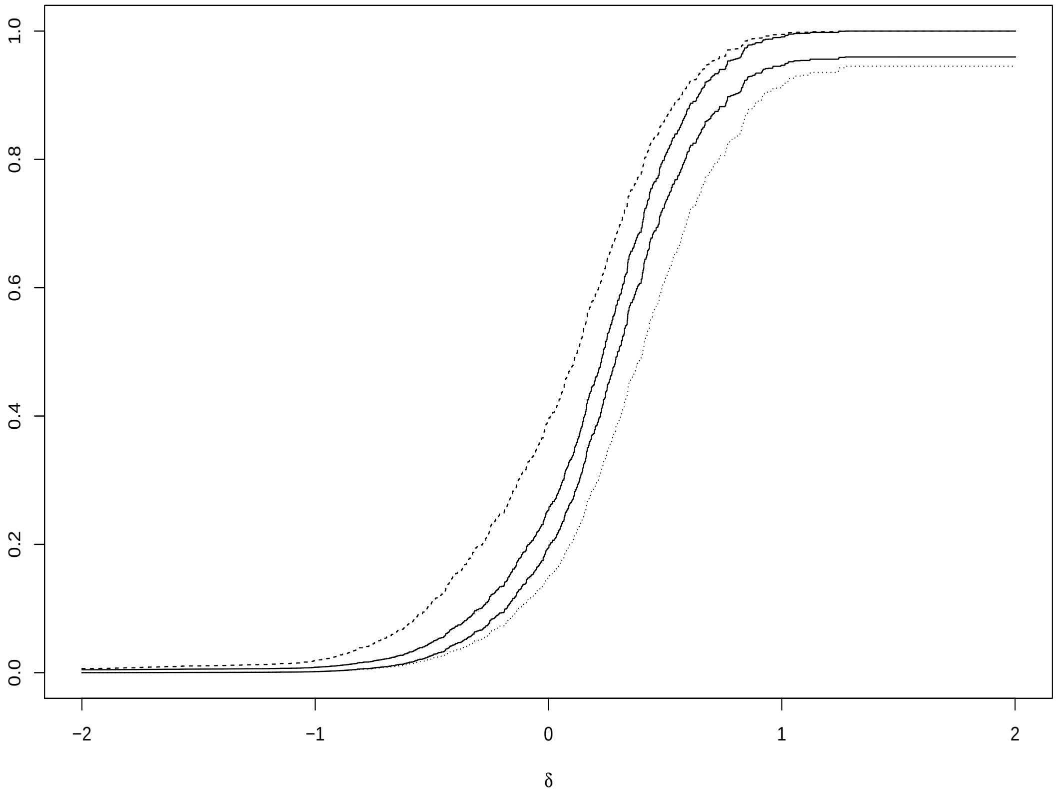

Figure 6: NPI lower and upper probabilities for $T_S^A < T_S^B + \delta$ and their bounds (Ex. 3).

to derive a system's signature in case of two subsystems in series or parallel configuration. Their approach can also be used with partially known signatures, as illustrated in this paper. An interesting topic for further research is whether such results can also be derived for subsystems that are in different configurations.

The bounds on signatures considered in this paper could themselves be interpreted as imprecise probabilities [9, 10]. For the inferences considered in this paper, the bounds corresponded to logical and well-identifiable signatures within the bounds, called the optimistic and pessimistic signatures. Of course, one may be interested in other inferential problems for which this nice monotonicity with regard to the signature does not hold. For example, if one would be interested in the system failing in its second year of operation then the bounds would be less easy to derive. One could still apply the ideas presented in this paper, but deriving the bounds for the inferences that correspond to the bounds for the signature would need to be formulated as constrained optimisation problems that may require numerical solution methods.

As indicated in the examples, the lower bound for the NPI lower survival function and the upper bound for the NPI upper survival function are most likely to be of main interest. However, the two other bounds presented can also be useful, particularly as the lower and upper bounds for the NPI lower survival function provide a clear indication of the accuracy with which, at any specific stage of computation, the real NPI lower survival function can be approximated (and similar of course for the NPI upper survival function). This may also be useful to provide an indication of the value of additional calculations to derive the signature.

An interesting further question is whether it is possible to learn about the system signature from failure observations. Recently, Aslett [1] has made interesting contributions to Bayesian learning of the system signature when only data for the whole system are available. This is important for 'black-box' systems, where it is not possible to construct the signature on the basis of available information. In such cases, system failure data can enable learning about some aspects of the system signature and hence of the actual structure of the system.

A main restriction for the use of the signature is the fact that it can only be applied to systems with exchangeable components. This means that all components have to be 'of the same type' and, beyond that, they all should have exchangeable roles in the system as it is their failure times that are explicitly assumed to be exchangeable. This is very restrictive for real-world systems, and it leads to some question marks about the applicability of methods using the signature for most systems of practical interest. It for example also avoids the use of signature-based methods for inferences and decision support in case of maintenance or replacement of individual components, as after such an action the component involved will typically have a changed future lifetime which therefore would typically no longer by exchangeable with the lifetimes of the other components in the system. Recently, an alternative concept entitled 'survival signature' has been presented [7], which is closely related to the signature for systems with exchangeable components but can be, quite straigthforwardly, generalized to systems with multiple types of components. Further research on this topic, including its use within the NPI framework, is ongoing and the authors hope to report on it in the near future.

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