

Analytical theory of light localization in one-dimensional disordered photonic crystals

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Abstract

Influence of the various types of disorder on propagation of light in one-dimensional periodic structures is studied analytically using statistical approach based on a Fokker-Planck type equation. It is shown that light localization length behaves non-monotonically as a function of disorder amplitude in all the examined models except for purely geometric disorder. This feature is explained by crossover between weak disorder regime corresponding to gradual destruction of the reflecting properties of a photonic crystal and strong disorder regime, when periodic component of the refractive index can be treated as a perturbation. The region of small disorder is shown to be universal provided that a disorder parameter is properly introduced.

Keywords: A. Photonic crystals, D. Disorder

1. Introduction

Initial progress in the area of photonic crystals pioneered by Yablonovitch [1] was based on the ideal models of photonic band structure, described in a number of textbooks [2],[3]. Though photonic crystals operated in various regions of the electromagnetic spectrum from radiowaves [4] to visible light [5] were demonstrated experimentally, wide practical application of the photonic crystal devices (lasers, waveguides and photonic circuits) is suppressed by technological imperfections unavoidable in any fabrication process. Being

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strong enough, such imperfections result in smearing or even destruction of the photonic band gaps, which is detrimental to the properties required for optoelectronic device applications. This statement is supported by many experimental studies of disordered photonic crystals carried out in recent years [6]-[20]. Of course, it is desirable to have a theory describing optical properties of the real, disordered photonic crystals quantitatively, while most of the numerous theoretical works focus on modelling [21]-[47] which does not shed enough light on general regularities.

Many results of the real and numerical experiments can be successfully explained by analytical theories developed in Refs. [48]-[56]. While some problems, like coherent backscattering of light from disordered photonic crystals, can be treated perturbatively using Green function technique [48]-[50], the problem of localization of light by such structures requires summation of all the diagrams or application of the different approaches as done in Refs. [51]-[53]. Unfortunately, the methods suggested in Refs. [51]-[52] for two- and three-dimensional disordered photonic crystals are not strict and their results can be treated only as estimations. Even a one-dimensional model of Ref. [53] restricted to the case of purely geometric disorder was not solved exactly but using a kind of heuristic assumption, namely uniform distribution of some angle variable θ . Unfortunately, this assumption is certainly wrong for the center and edges of the photonic band gap (i.e. regions of the most interest), as stated in Ref. [53]. Recently the problem of light localization in the one-dimensional case has been considered in more strict manner using the Fokker-Planck equation [54] similar to considered in Ref. [55], however some aspects have not been treated absolutely accurately. Namely, the authors considered a second-order term " η^2 " using a system of differential equations of the first order (Eq.(7) of Ref. [54]), which is equivalent to the exact finite-difference equation (17) up to the first order only. Further this quadratic term has been averaged over disorder already in the Langevin equation, a procedure which can be hardly called founded.

In this paper we present a fully analytical one-dimensional theory of light localization in photonic crystals with various types of disorder, which is free from the above mentioned shortcomings. We use generalization of the model proposed initially in Ref. [39] and applied in Ref. [56] to the density of states, which allows for both dielectric and geometric components of disorder and described in Sec. 2.1. Our approach based on a notion of angle variables and the Fokker-Planck type equation [57] provides straightforward way to calculation of the light localization length as described in Sec. 2.2. Analyt-

ical expressions obtained for different models of disorder are discussed and compared with the numerical and experimental data in Sec. 3, and then our findings are summarized in Sec. 4.

2. Theory

2.1. The models of disorder and Langevin equation

Let us consider one-dimensional two-component basically periodic photonic system disturbed by moderate amount of disorder as schematically shown in Fig. 1. We make allowance for both components of disorder, geometric (fluctuations of the layer widths $d_{1,2}$) and dielectric (fluctuations of the layer refractive indices $n_{1,2}$), and for correlations between them within a given period of a structure, but not between the different periods. Also to avoid redundant cumbersomeness we assume the average widths of the layers "A" and "B" to be equal, $\overline{d_1} = \overline{d_2} = d_0$, and consider only the case of small dielectric contrast and frequency range within the first photonic band gap. Our model involves five independent dimensionless parameters describing dielectric contrast of the structure η , disorder amplitude δ , type of disorder β, γ , and frequency position within the photonic band G ,

$$\eta = \frac{\overline{n_1} - \overline{n_2}}{\overline{n_1} + \overline{n_2}} = \frac{\overline{n_1} - \overline{n_2}}{2n_0}, \quad (1)$$

$$\delta^2 = \overline{\tau^2}, \quad (2)$$

$$\beta = \frac{\overline{\mu} + \overline{\nu}}{\pi\delta^2}, \quad (3)$$

$$\gamma^2 = \frac{\overline{\sigma^2}}{\pi^2\delta^2}, \quad (4)$$

$$G = \frac{\omega - \omega_0}{\Delta\omega/2}, \quad (5)$$

$$\omega_0 = \frac{\pi c}{n_1\overline{d_1} + n_2\overline{d_2}} = \frac{\pi c}{2n_0d_0(1 + \zeta)}, \quad (6)$$

$$\Delta\omega = \frac{4\eta\omega_0}{\pi}, \quad (7)$$

$$\sigma = \frac{\delta n_1 - \delta n_2}{n_0}, \quad (8)$$

$$\tau = \frac{\delta n_1 + \delta n_2}{2n_0} + \frac{\delta d_1 + \delta d_2}{2d_0}, \quad (9)$$

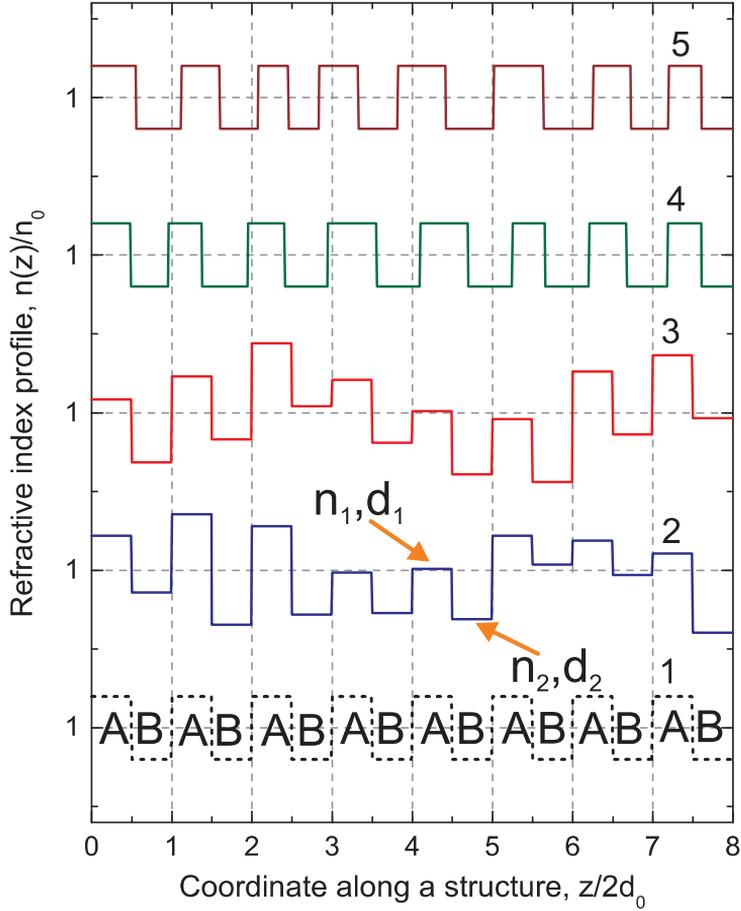


Figure 1: (color online) Schematic profiles of an ideal (curve 1) and the imperfect one-dimensional photonic crystals. Profiles 2-5 correspond to the different models of disorder described later in the text, namely 2-NU, 3-NC, 4-DU, 5-DC. A real structure is assumed to be much longer than it is schematically shown.

$$\mu = \frac{\delta n_1 \delta n_2}{n_0^2}, \quad (10)$$

$$\nu = \frac{\delta n_1 \delta d_2 + \delta n_2 \delta d_1}{2n_0 d_0}, \quad (11)$$

$$\zeta = \frac{\delta n_1 \delta d_1 + \delta n_2 \delta d_2}{2n_0 d_0}. \quad (12)$$

Here ω_0 is a central frequency of the first photonic band gap including possible disorder-induced shift described by $\bar{\zeta}$, $\Delta\omega$ is photonic band gap width in the linear in η approximation, and σ, τ, μ, ν and ζ stand for the unaveraged (specific to each period) combinations of the geometric and dielectric parameters. Since both dielectric contrast and disorder amplitude are assumed to be small, it is practical to introduce the parameter α actually playing a key role,

$$\alpha = \frac{2\eta}{\pi^2 \delta^2}. \quad (13)$$

Focusing on the universal properties of the disordered photonic crystals we assume that a structure is long enough so that self-averaging takes place, and solve one-dimensional wave equation

$$\frac{d^2 E}{dz^2} = -\frac{n^2 \omega^2}{c^2} E, \quad (14)$$

with the arbitrary initial conditions using transfer matrix

$$M_s = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad (15)$$

with the elements given explicitly in Appendix Appendix A, and acting on the two-component vectors

$$V_s = \begin{pmatrix} E(z_s) \\ \frac{c}{n_0 \omega} E'(z_s) \end{pmatrix} \quad (16)$$

located at the boundary points z_s between the periods $s - 1$ and s , so that $V_{s+1} = M_s V_s$. In terms of the logarithmical derivative $L_s = (c/n_0 \omega) E'(z_s)/E(z_s)$ evolution of the electric field along a structure can be written as

$$L_{s+1} - L_s = \frac{M_{21} + (M_{22} - M_{11})L_s - M_{12}L_s^2}{M_{11} + M_{12}L_s}, \quad (17)$$

which is a discrete counterpart of the random equation of motion, also called Langevin equation [58]. In the limiting case of small dielectric contrast and disorder amplitude it is convenient to introduce angle variable $\Psi = 2\text{arccot}L$ and deal with the following form of the Langevin equation,

$$\Psi_{s+1} - \Psi_s = \Theta_s(\Psi_{s+1}), \quad (18)$$

$$\begin{aligned} \Theta_s(\Psi) = 4\eta(G + \sin \Psi) + 2\sigma \sin \Psi - \sigma^2 \sin 2\Psi \\ + 2\pi(\tau - (\mu + \nu) \cos \Psi + \zeta - \bar{\zeta}). \end{aligned} \quad (19)$$

A few intermediate steps on the way to Eqs. (18-19) are described in Appendix A.

2.2. Disorder averaging and length of localization

In order to turn to statistical description let us study distribution function of the angle Ψ instead of spatial evolution of Ψ_s . For a given realization of disorder and initial condition $\Psi_{s_0} = \Psi^{(0)}$ these two problems are essentially the same since evolution of the initial distribution $F_{s_0}(\Psi) = \delta(\Psi - \Psi^{(0)})$ is governed by obvious equation

$$F_{s+1}(\Psi_{s+1})d\Psi_{s+1} = F_s(\Psi_s)d\Psi_s, \quad (20)$$

where Ψ_s as a function of Ψ_{s+1} is given by Eq. (18). Denoting $\Psi = \Psi_{s+1}$ and expanding r.h.s. of Eq. (20) up to the second order in Θ_s , we come to

$$F_{s+1}(\Psi) = F_s(\Psi) + \frac{\partial}{\partial \Psi} \left[\frac{\Theta_s^2}{2} \frac{\partial F_s}{\partial \Psi} - \Theta_s(\Psi) F_s(\Psi) \right]. \quad (21)$$

Since distribution function F_s depends on disorder configuration only to the left of the period s , and $\Theta_s(\Psi)$ includes fluctuations in the period s , they average independently, resulting in the following discrete version of the Fokker-Planck equation:

$$\overline{F_{s+1}} - \overline{F_s} = \frac{\partial}{\partial \Psi} \left[\frac{\overline{\Theta_s^2}}{2} \cdot \frac{\partial \overline{F_s}}{\partial \Psi} - \overline{\Theta_s(\Psi)} \cdot \overline{F_s} \right]. \quad (22)$$

Due to the notion of self-averaging $\overline{F_s(\Psi)}$ becomes independent of s far away from the initial point s_0 and we come to the stationary Fokker-Planck equation for the distribution function $F(\Psi) = \overline{F_s} = \overline{F_{s+1}}$:

$$\frac{d}{d\Psi} \left[\frac{\overline{\Theta^2}}{2} \frac{dF}{d\Psi} - \overline{\Theta(\Psi)} F(\Psi) \right] = 0. \quad (23)$$

Averaging over fluctuations in the l.h.s. of Eq. (23), using the notions given by Eqs. (1-13) and introducing the stationary flow J we obtain final form of the stationary Fokker-Planck equation,

$$(1 + \gamma^2 \sin^2 \Psi) \frac{dF}{d\Psi} - \left[\alpha(G + \sin \Psi) - \beta \cos \Psi - \frac{\gamma^2}{2} \sin 2\Psi \right] F = -J. \quad (24)$$

In principle, the value of J has to be determined from the conditions of periodicity $F(\Psi + 2\pi) = F(\Psi)$ and normalization $\int_0^{2\pi} F(\Psi) d\Psi = 1$ of the full distribution function $F(\Psi)$, however, solution of the homogeneous equation,

$$F_0(\Psi) = \frac{\exp \left[-\frac{\alpha\Lambda}{\sqrt{1+\gamma^2}} - \frac{\beta}{\gamma} \arctan(\gamma \sin \Psi) \right]}{\sqrt{1 + \gamma^2 \sin^2 \Psi}}, \quad (25)$$

$$\Lambda = \gamma^{-1} \operatorname{arctanh} \left(\frac{\gamma \cos \Psi}{\sqrt{1 + \gamma^2}} \right) - G \left(\arctan(\sqrt{1 + \gamma^2} \tan \Psi) + \pi \left[\frac{\Psi}{\pi} + \frac{1}{2} \right] \right) \quad (26)$$

shall be sufficient for our purposes ($[x]$ in Eq. (26) denotes the integer part of x).

Taking in hand explicit expression for the distribution function one can evaluate localization length ξ using a definition

$$\frac{2d_0}{\xi} = \ln \left(\frac{\overline{E_{s+1}}}{E_s} \right)^2 = \overline{\langle \ln(M_{11} + M_{12}L_s)^2 \rangle}. \quad (27)$$

Disorder averaging in the r.h.s. of Eq. (27) can be split into a part containing everything to the left of the period s (denoted by the angle brackets) and averaging on the period s (denoted by overlining). Carrying out partial averaging of Eq. (27) under the conditions of small dielectric contrast and disorder amplitude results in the following expression for the dimensionless decay rate Υ ,

$$\Upsilon = \frac{2d_0}{\pi^2 \delta^2 \xi} = 2\alpha(1 + G\langle L \rangle) - \langle 1 + 2\beta L + L^2 \rangle. \quad (28)$$

This involves averaging over the distribution function $F(\Psi)$ only, i.e. reduces to one-dimensional integration, which can be even done analytically in the particular cases discussed below.

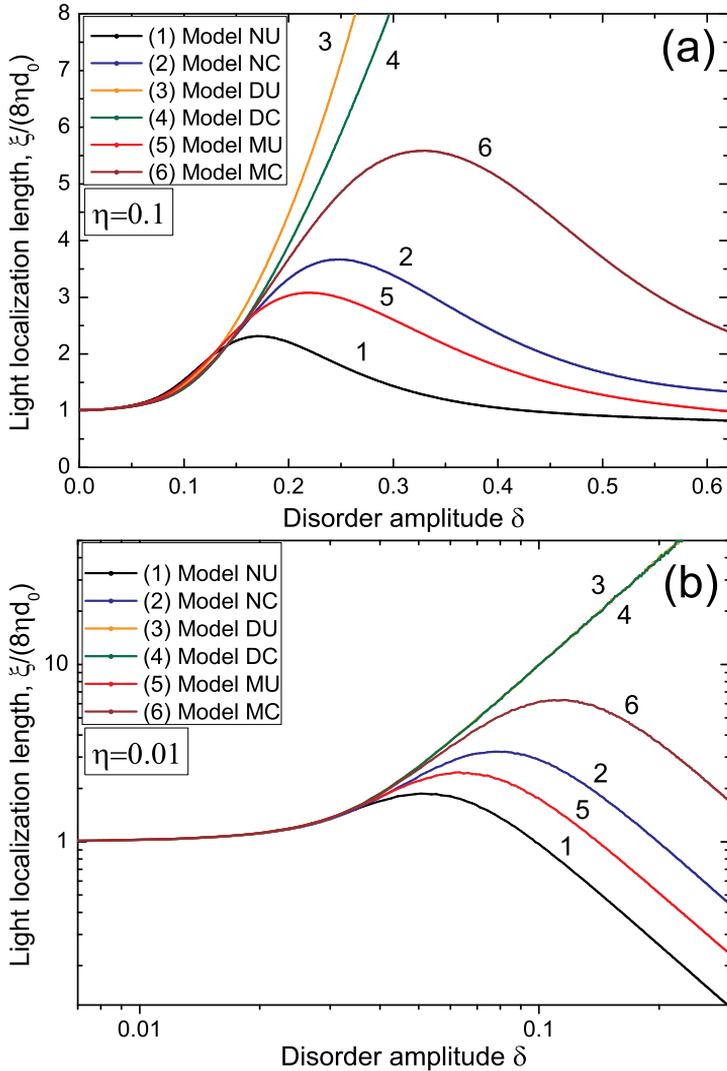


Figure 2: (color online) Normalized light localization length obtained by numerical solution of Eq. (14) is shown as a function of disorder amplitude δ for six different models of disorder described in the text, two values of η , and in different scales (linear and log-log).

3. Results and discussion

To check predictions of the theory we have performed direct numerical computation of light localization length as a function of dielectric contrast η , disorder amplitude δ and dimensionless frequency G using various models of disorder. Numerical results for the case of $G = 0$ (central frequency of the first photonic band gap) are shown in Fig. 2 for two values of dielectric contrast ($\eta = 0.1$ and $\eta = 0.01$) and six different models of disorder. The models NC and NU consider purely dielectric disorder with equal amplitudes of the fluctuations in the layers ($\overline{\delta n_1^2} = \overline{\delta n_2^2}$), fully correlated in the model NC (so that $\beta = \pi^{-1}$, $\gamma = 0$) and uncorrelated in the model NU (so that $\beta = 0$, $\gamma = 2\pi^{-1}$). The models DC and DU correspond to fully correlated and uncorrelated cases of the purely geometric disorder, so that $\beta = \gamma = 0$, and they are indistinguishable in terms of these parameters. The models MC and MU consider the case of mixed disorder with equal amplitudes of dielectric and geometric fluctuations ($\overline{\delta n_{1,2}^2}/n_0^2 = \overline{\delta d_{1,2}^2}/d_0^2$) and full correlation between the corresponding components (MC, $\beta = (2\pi)^{-1}$, $\gamma = 0$) or without any correlations (MU, $\beta = 0$, $\gamma = \sqrt{2}\pi^{-1}$). It is clearly seen that initial region of different curves in Fig. 2a is universal and can be described by positive parabolic additive while large-disorder behaviour is specific to a model, and it can be both increasing or decreasing. Interestingly, the curves 1, 5, 2, 6 on the right side of Fig. 2b are going almost in parallel but with the approximate factors of 1, 2, 4, 16. In order to explain these features let us turn to the particular cases treatable analytically to the very end, focusing primarily on the central frequency of the photonic band gap. In this case, $G = 0$, Eqs. (25,26) delivers 2π -periodic function, which is an exact solution of Eq. (24) since $J = 0$.

3.1. Purely geometric disorder

Distribution function in the case of geometric disorder (models DC, DU, $\beta = \gamma = 0$) has particularly simple form,

$$F_0(\Psi) = \frac{\exp(-\alpha \cos \Psi)}{2\pi I_0(\alpha)}. \quad (29)$$

Averaging of L and L^2 in Eq. (28) over this distribution gives the following answer,

$$\Upsilon = \frac{2\alpha I_1(\alpha)}{I_0(\alpha)} \sim \begin{cases} \alpha^2, & \alpha \ll 1 \\ 2\alpha - 1, & \alpha \gg 1 \end{cases} \quad (30)$$

which means that light localization length grows like $\sim \delta^2$ in the limit of large disorder, in agreement with numerical computations (cf. curves 3, 4 in Fig. 2b). Indeed, localization in the case of pure geometric disorder becomes possible only due to the finite value of the dielectric contrast η , therefore small- α asymptotics of Υ contains only positive powers of α . More dramatic here is a cancellation of the linear term in this series, leading to power growth of ξ with δ . In other words, purely geometric disorder can really efficiently destroy localization of light induced by periodic modulation of the refractive index.

3.2. Disorder with equal dielectric fluctuations

In the case of equal dielectric fluctuations $\delta n_1 = \delta n_2$ ($\gamma = 0$) the distribution function reads

$$F_0(\Psi) = \frac{\exp(-\alpha \cos \Psi - \beta \sin \Psi)}{2\pi I_0(\sqrt{\alpha^2 + \beta^2})}, \quad (31)$$

and the decay rate can be written as

$$\Upsilon = \frac{2\sqrt{\alpha^2 + \beta^2} I_1(\sqrt{\alpha^2 + \beta^2})}{I_0(\sqrt{\alpha^2 + \beta^2})}, \quad (32)$$

as shown in Appendix B. In the limiting case of large disorder ($\alpha \rightarrow 0$) this replicates Eq. (30) with a substitution for α by β ,

$$\Upsilon = \frac{2\beta I_1(\beta)}{I_0(\beta)} \approx \beta^2. \quad (33)$$

The latter approximation, $\Upsilon \approx \beta^2$, gives $\sim 1\%$ accuracy for the values of β less than π^{-1} , corresponding to the numerical examples discussed above. Indeed, precise asymptotics are $2d_0\xi^{-1} \approx 0.988\delta^2$ for the model NC and $2d_0\xi^{-1} \approx 0.249\delta^2$ for the model MC, which can be treated in the same way as NC but with the half value of β . Since approximately $\Upsilon \propto \beta^2$, this leads to the discussed fourfold difference between the curves 2 and 6 on the right side of Fig. 2b.

3.3. Uncorrelated dielectric component

In the general case of non-zero parameters α , β and γ the integrals involved in the r.h.s. of Eq. (28) becomes complicated and can be represented

only in terms of the formal series, which is not really practical. However, in the case of uncorrelated dielectric fluctuations $\beta = 0$ the limit of strong disorder can be treated exactly. Distribution function in this limit reduces to

$$F_0(\Psi) = \frac{1}{4K(i\gamma)\sqrt{1 + \gamma^2 \sin^2 \Psi}}, \quad (34)$$

and the inverse localization length is given by the ratio of two elliptic functions,

$$\Upsilon = 2 \left(\frac{E(i\gamma)}{K(i\gamma)} - 1 \right) \approx \gamma^2. \quad (35)$$

The latter approximation holds for $\gamma < 2\pi^{-1}$ (considered in the numerical examples) within the accuracy of 4%, and the precise asymptotics for the inverse localization length are $2d_0\xi^{-1} \approx 3.830\delta^2$ for the model NC and $2d_0\xi^{-1} \approx 1.954\delta^2$ for the model MC. These values in comparison with those presented in the preceding paragraph finally explain the difference between large disorder behaviour of the localization length observed in the numerically studied models.

Good numerical convergence suggests that expansion in γ may be efficient in the case of arbitrary α and β as well. Fortunately, corrections to the distribution function (31) of any order in γ can be written in the form

$$\frac{\delta F_n}{F_0} = \gamma^{2n} (A_n + B_n \cos \Psi) \sum_k C_k \sin^k \Psi, \quad (36)$$

suitable for analytical integration in terms of the modified Bessel functions. In the case of uncorrelated dielectric fluctuations the answer including correction of the first order in γ^2 ,

$$\Upsilon = 2\alpha\mathcal{I} + \frac{\gamma^2}{3} [3 - 4\alpha^2 + 4\alpha\mathcal{I} + (4\alpha^2 - 1)\mathcal{I}^2], \quad (37)$$

where $\mathcal{I} = I_1(\alpha)/I_0(\alpha)$, gives better than 5% agreement with the numerical curves of the models NU and MU at any amplitudes of disorder δ provided that dielectric contrast is sufficiently small. Thus, Eqs. (30), (32) and (37) provides explicit expressions for all the considered models in the linear regime, and similar expressions can be derived for other cases, say $\gamma \neq 0$ and $\beta \neq 0$.

3.4. Frequency dependence at small disorder

In the case of small disorder, $\alpha \gg 1$, the distribution function given by Eq. (25) has exponentially sharp maximum at the minimum of Λ , $\Psi^* = \pi + \arcsin G$, and the simplest approximation is $F_0(\Psi) = \delta(\Psi - \Psi^*)$, which leads to disorder free answer for the localization length,

$$\frac{2d_0}{\xi} = 4\eta\sqrt{1 - G^2}. \quad (38)$$

In order to make allowance for disorder, more refined approximation is needed, namely

$$F_0(\Psi) = P(\Psi - \Psi^*) \exp\left(-\frac{\alpha\sqrt{1 - G^2}(\Psi - \Psi^*)^2}{2(1 + \gamma^2 G^2)}\right), \quad (39)$$

where $P(x) = 1 + \sum_n P_n x^n$ takes into account higher-order terms in the main exponent, exponent with β and the denominator of Eq. (25). Carrying out evaluation of $\langle L \rangle$ and $\langle L^2 \rangle$ with such a distribution function as explained in Appendix C, we come to the following answer for light localization length including corrections of the first order in α^{-1} :

$$\frac{2d_0}{\xi} = 4\eta\sqrt{1 - G^2} - \pi^2\delta^2\frac{1 + \gamma^2 G^2}{1 - G^2}. \quad (40)$$

In the case of the central frequency of the first photonic band gap, $G = 0$, this expression reduces to

$$\frac{2d_0}{\xi} = 4\eta - \pi^2\delta^2, \quad (41)$$

proving universal (independent of the type of disorder) behaviour of the different curves in Fig. 2 in the region of moderate disorder once a disorder parameter is introduced by Eq. (2). In order to check frequency dependence of the localization length given by Eq. (40) we plot numerical results obtained in the models NC and NU and approximations given by Eqs. (38,40) in Fig. 3. As one can see from the plots, the corrections introduced by Eq. (40) slightly improve quantitative agreement in the domain of moderate disorder amplitudes and dimensionless frequencies G .

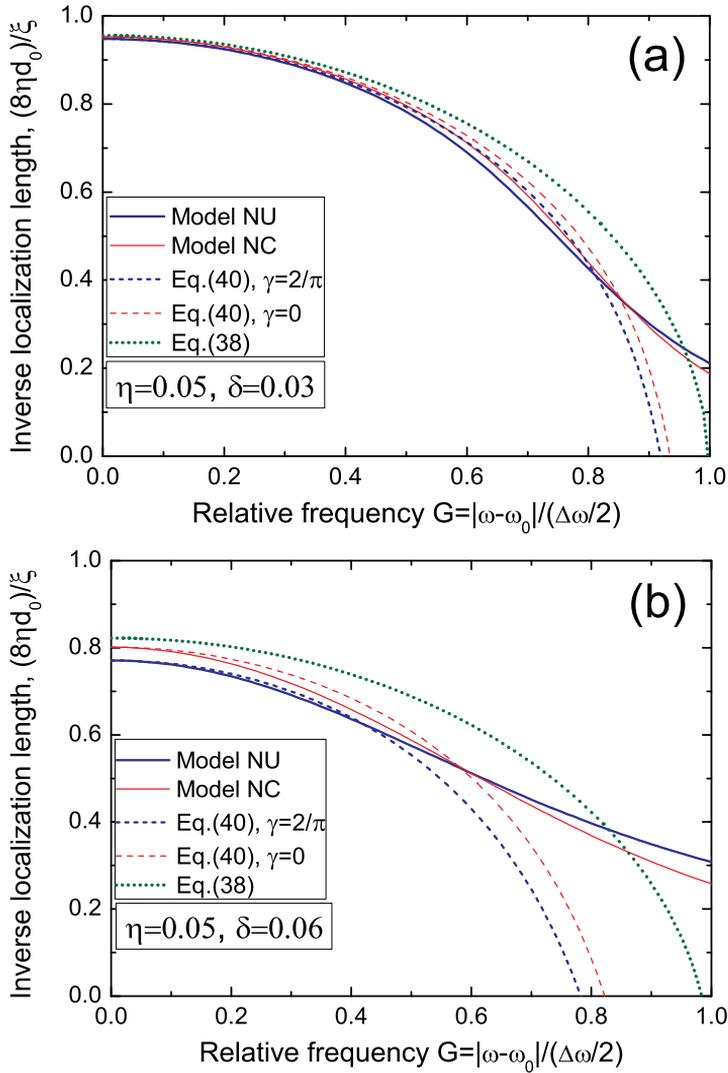


Figure 3: (color online) Inverse light localization length obtained by numerical solution of Eq. (14) is shown as a function of the dimensionless frequency G for two different models of disorder described in the text and two values of disorder amplitude $\delta = 0.03$ and $\delta = 0.06$ at the dielectric contrast $\eta = 0.05$.

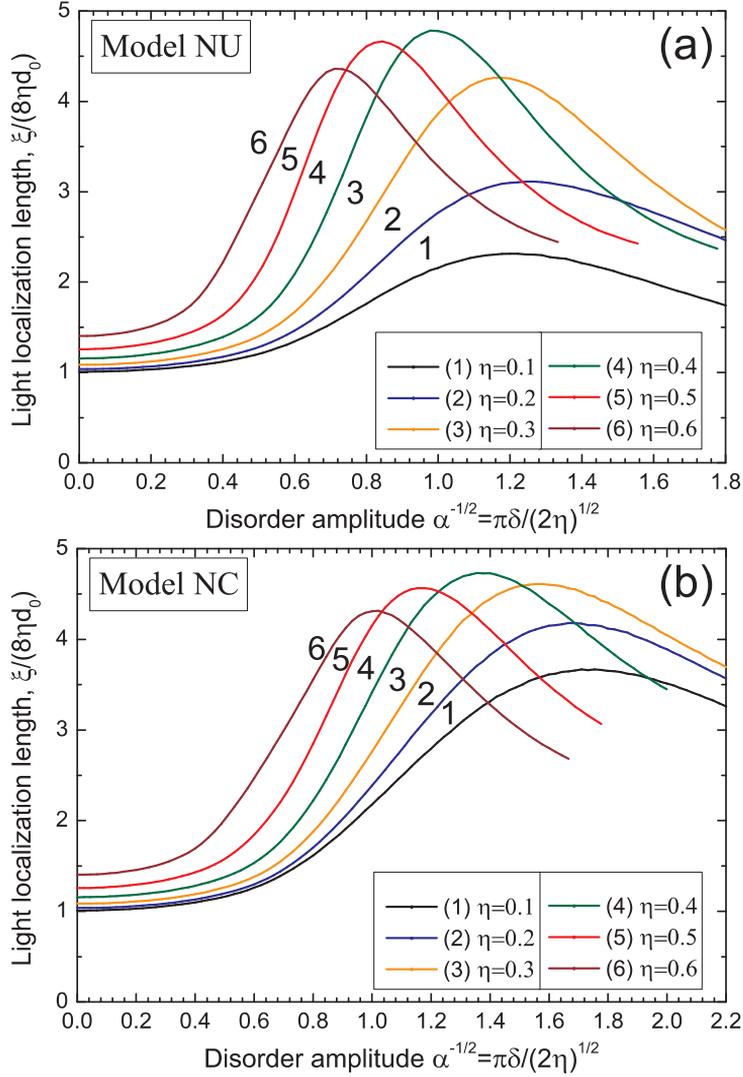


Figure 4: (color online) Normalized light localization length obtained by numerical solution of Eq. (14) is shown as a function of the reduced disorder amplitude $\alpha^{-1/2}$ for six values of the dielectric contrast η ranging from 0.1 to 0.6 and two models of disorder, (a) NU and (b) NC.

3.5. Discussion on non-small dielectric contrast and comparison with the experimental data

In the very beginning of our consideration we assumed that dielectric contrast of a photonic crystal is small, an assumption being applicable only to some specific part of the experimental systems. In order to check status of our results at higher dielectric contrasts we have analyzed numerically dependence of the localization length on the reduced disorder parameter $\alpha^{-1/2} \propto \delta$ for the values of η from 0.1 to 0.6 (i.e. up to $\bar{n}_1 = 4$ at $\bar{n}_2 = 1$, the range enclosing most of the experimentally fabricated systems). Numerical curves presented in Fig. 4 for the models NU and NC show that quantitative agreement between analytical and numerical results becomes rather poor at significant dielectric contrasts, $\eta \gtrsim 0.1$. Since such deviation occurs already at $\delta = 0$, it is not a consequence of the incorrect accounting for disorder alone but is due to an expansion started by Eq. (A.5). On the other hand, numerical curves in Fig. 4 still exhibit non-monotonic behaviour, confirming that qualitative conclusions of our analytical theory can be applied to dielectric contrast of the arbitrary magnitude. In principle, it is possible to generalize presented theory for the arbitrary magnitudes of dielectric contrast and fluctuations, however this leads to integral equation instead of the Fokker-Planck type (Eq. (23)), and currently it is not straightforward how to deal with it. We hope to address the case of non-small dielectric contrast further in future publications.

Finally, in order to check practical consistency of the presented approach let us give an example of how our analytical results can be used to describe puzzling experimental data obtained in Ref. [8]. Though experimentally studied structures were three-dimensional photonic crystals, their distinctive feature is a small dielectric contrast (less than 0.1) and extensive thickness, up to $D = 1mm$. Thanks to this, measured spectra, presented in Fig. 3 of Ref. [8], are really smooth and reproducible. However, attenuation lengths determined from those measurements are by almost an order of magnitude higher than predicted from by both one-dimensional and three-dimensional computations [8, 59]. In order to solve this discrepancy we suggest using the model DC/DU discussed in Sec. 3.1 (of course, we consider this model only as a "cartoon" on the true disorder in those samples). Using theoretical curves depicted in Fig. 3 of Ref. [8] as a reference, we are able to determine effective value of the model parameter $d_0 = 0.5\mu m$. Together with the values of dielectric contrast listed for two different samples in Table I of Ref. [8] and the only fitting parameter $\alpha = 0.2$, this allows us to obtain reasonable agreement

with the experimental spectra as shown in Fig. 5. Estimation of the localization length for two samples studied in Ref. [8] (infiltrated with ethanol and cyclohexane) using Eq. (30) gives $\xi_{eth} \approx 240\mu m$, $\xi_{cyc} \approx 70\mu m$, allowing us to estimate numerical accuracy as 20 – 30%, which is not too bad for a ”toy”, one-dimensional model of the complicated three-dimensional problem. In addition to absolute value of the localization length, our theory nicely describes smearing of the spectra, another puzzle of Ref. [8]. Thus, we have demonstrated that a mere one-dimensional analytical theory is able to give consistent description of the localization of light even in three dimensional photonic crystals.

4. Summary

We have presented a detailed study of the effect of disorder on localization of light in the one-dimensional photonic crystals. A variety of the models with short-range disorder have been considered, allowing for both independent and correlated fluctuations of the refractive index and layer widths within a period. In order to describe evolution of the electromagnetic wave along a structure under the conditions of small dielectric contrast η and disorder amplitude δ statistically we introduced slowly varying angle variable Ψ and derived the Fokker-Planck type equation for the distribution function of this variable. Then we have shown how to evaluate the light localization length using solution of the Fokker-Planck equation (24), and proved that initial region of dependence of light localization length ξ on disorder parameter δ is universal, provided that the latter is defined by Eq. (2). However, this universality disappears at $\delta \sim \sqrt{\eta}$, and subsequent behaviour is specific to a model. In most of the models, except for a particular case of the purely geometric disorder, $\xi(\delta)$ is non-monotonic, the fact being explained by crossover between weak disorder regime describable by graceful suppression of the reflecting properties of a photonic crystal and strong disorder regime, when periodic component of the refractive index can be almost fully neglected. In addition, we have checked practical consistency of the theory by comparison with the numerical computations carried out for non-small values of the dielectric contrast and the experimental data of Ref. [8] dealing with small contrast three-dimensional photonic crystals.

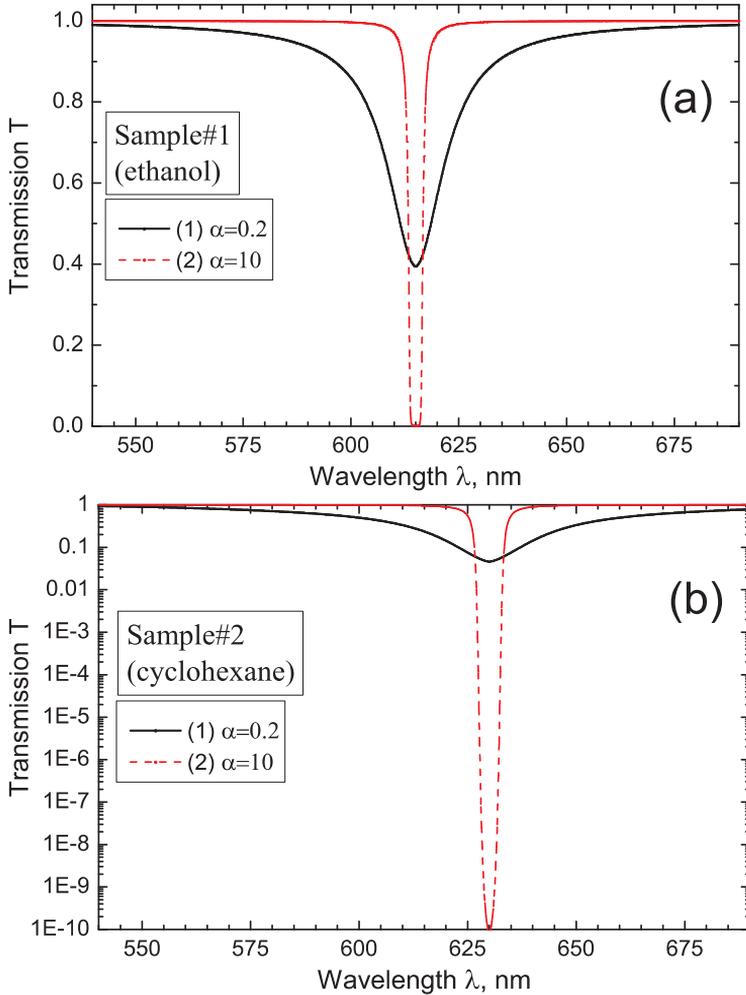


Figure 5: (color online) Transmission spectra simulated for the two samples studied in Ref. [8] (thickness $D = 300\mu m$, infiltration with (a) ethanol (b) cyclohexane) using a definition $T = \exp(-D/\xi)$ and the light localization length determined from the model DC. Dashed curves plotted for reduced amplitude of fluctuations indicate photonic band gaps of the ideal structures.

5. Acknowledgments

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Appendix A. Derivation of the Langevin equation from transfer matrix approach

Multiplication of transfer matrices of the individual layers "A" and "B" of a given period gives transfer matrix (15) with the elements

$$M_{11} = \cos \varphi_1 \cos \varphi_2 - \frac{n_1}{n_2} \sin \varphi_1 \sin \varphi_2, \quad (\text{A.1})$$

$$M_{22} = \cos \varphi_1 \cos \varphi_2 - \frac{n_2}{n_1} \sin \varphi_1 \sin \varphi_2, \quad (\text{A.2})$$

$$M_{12} = \frac{n_0}{n_1} \sin \varphi_1 \cos \varphi_2 + \frac{n_0}{n_2} \sin \varphi_2 \cos \varphi_1, \quad (\text{A.3})$$

$$M_{21} = -\frac{n_1}{n_0} \sin \varphi_1 \cos \varphi_2 - \frac{n_2}{n_0} \sin \varphi_2 \cos \varphi_1, \quad (\text{A.4})$$

where $\varphi_i = k_i d_i = n_i d_i \omega / c$. Both phases $\varphi_{1,2}$ are close to $\pi/2$ inside the first photonic band gap and can be expanded in terms of the small dielectric contrast η and amplitude of the fluctuations δ . Since any quantities proportional to the fluctuations $\delta n_{1,2}$ or $\delta d_{1,2}$ vanishes on average, this expansion must be extended at least to the terms quadratic in fluctuations and linear in η . Formally, it is convenient to represent each fluctuation as a product of δ and a random quantity of the order of unity, and put $\eta = \alpha \pi^2 \delta^2 / 2$ according to Eq. (13). Expansion of the phases

$$\varphi_i \simeq \frac{\pi}{2} \left[1 + \frac{\delta n_i}{n_0} + \frac{\delta d_i}{d_0} + \frac{\delta n_i \delta d_i}{\delta n_0 \delta d_0} - \bar{\zeta} + \frac{2G}{\pi} \eta - (-1)^i \eta \right], \quad (\text{A.5})$$

should be substituted into the foregoing expressions for the matrix elements, resulting in

$$M_{11} \simeq -1 - 2\eta - \sigma + \mu + \pi^2 \tau^2 / 2, \quad (\text{A.6})$$

$$M_{22} \simeq -1 + 2\eta + \sigma + \mu + \pi^2 \tau^2 / 2, \quad (\text{A.7})$$

$$M_{12} \simeq -2\eta G + \pi(\mu + \nu - \tau - \zeta + \bar{\zeta}), \quad (\text{A.8})$$

$$M_{21} \simeq 2\eta G + \pi(\mu + \nu + \tau + \zeta - \bar{\zeta}), \quad (\text{A.9})$$

where the notions given by Eqs. (9-12) are used. In turn, substitution of these expansions into the equation

$$\Psi_s = 2\text{arccot} \left(\frac{M_{21} - M_{11} \cot(\Psi_{s+1}/2)}{M_{12} \cot(\Psi_{s+1}/2) - M_{22}} \right) \quad (\text{A.10})$$

gives after some algebra Eqs. (18,19).

Appendix B. Localization length at $\gamma = 0$

In the case of equal dielectric fluctuations, $\gamma = 0$, it is convenient to introduce unnormalized version of the distribution function (31),

$$F_0(\Psi) = \frac{1}{2\pi} \exp \left(2\alpha \sin^2 \frac{\Psi}{2} - 2\beta \sin \frac{\Psi}{2} \cos \frac{\Psi}{2} \right). \quad (\text{B.1})$$

Mean values of the powers of logarithmical derivative $L = \cot(\Psi/2)$ can be related to normalization integral of the distribution (B.1),

$$N = e^\alpha I_0(\sqrt{\alpha^2 + \beta^2}), \quad (\text{B.2})$$

via obvious equality

$$\frac{\partial^n (N \langle L^n \rangle)}{\partial^n \alpha} = (-1)^n \frac{\partial^n N}{\partial^n \beta}. \quad (\text{B.3})$$

Applying this relation to the r.h.s of Eq. (28) multiplied by the normalization integral N , we come to the equation

$$\frac{\partial^2 (N\Upsilon)}{\partial \alpha^2} = 4 \frac{\partial N}{\partial \alpha} + (2\alpha - 1) \frac{\partial^2 N}{\partial \alpha^2} - \frac{\partial^2 N}{\partial \beta^2} + 2\beta \frac{\partial^2 N}{\partial \alpha \partial \beta} = 2e^\alpha \left[(2\alpha + 1) I_0(\kappa) + \frac{(2\alpha^2 + \beta^2)}{\kappa} \right] \quad (\text{B.4})$$

where $\kappa = \sqrt{\alpha^2 + \beta^2}$. Double integration of Eq. (B.4) results in

$$N\Upsilon = 2e^\alpha \sqrt{\alpha^2 + \beta^2} I_1(\sqrt{\alpha^2 + \beta^2}), \quad (\text{B.5})$$

which immediately gives Eq. (33).

Appendix C. Localization length at small disorder

In the case of arbitrary frequency, $G \neq 0$, full solution of Eq. (24) can be written in the form

$$F(\Psi) = JF_0(\Psi) \left[\frac{P}{P-1} \int_{-\pi/2}^{3\pi/2} \frac{d\Psi'}{\tilde{F}_0(\Psi')} - \int_{-\pi/2}^{\Psi} \frac{d\Psi'}{\tilde{F}_0(\Psi')} \right], \quad (\text{C.1})$$

where

$$P = \frac{F_0(\Psi + 2\pi)}{F_0(\Psi)} = \exp\left(\frac{2\pi\alpha G}{\sqrt{1+\gamma^2}}\right), \quad (\text{C.2})$$

$$\tilde{F}_0(\Psi) = (1 + \gamma^2 \sin^2 \Psi) F_0(\Psi), \quad (\text{C.3})$$

the flow J has to be determined from the normalization requirement, and Ψ is assumed to be inside $[-\pi/2, 3\pi/2]$ interval. While $F_0(\Psi)$ has sharp maximum at $\Psi^* = \pi + \arcsin G$ according to Eq. (39), the function $\tilde{F}_0(\Psi)$ has deep minimum at $\Psi^* = -\arcsin G$, therefore upper limit of the second integral in the r.h.s. of Eq. (C.1) can be put the same as in the first integral within the exponential accuracy, for any values of Ψ between $\pi/2$ and $3\pi/2$, so that

$$\mathcal{F}(\Psi) = \frac{JF_0(\Psi)\theta(\Psi - \frac{\pi}{2})}{P-1} \int_{-\pi/2}^{3\pi/2} \frac{d\Psi'}{\tilde{F}_0(\Psi')} + \theta\left(\frac{\pi}{2} - \Psi\right) F(\Psi), \quad (\text{C.4})$$

where $F(\Psi)$ is given by Eq. (C.1). Since any integral containing $F_0(\Psi)$ sits on $\Psi^* > \pi/2$, this distribution is essentially equivalent to Eq. (39) in the limit of small disorder, $\alpha \gg 1$, up to the exponentially small corrections. Then, in order to derive the first correction in α^{-1} to expression for the light localization length we have to evaluate Eq. (28) using the distribution function (39) with $P(x) = P_0(x) + P_1(x)$, where

$$P_0(x) = 1 + ax^2 + b\alpha x^4 + c\alpha^2 x^6, \quad (\text{C.5})$$

$$P_1(x) = \frac{\sqrt{1-G^2}(\beta - \gamma^2 G)x}{1 + \gamma^2 G^2} + \frac{[1 + \gamma^2(4 - 3G^2)]G\alpha x^3}{6(1 + \gamma^2 G^2)^2}, \quad (\text{C.6})$$

and explicit form of the coefficients a, b, c is inessential. In fact almost all the terms in Eq. (28) can be averaged just using a substitution $\langle L \rangle \rightarrow \cot(\Psi^*/2)$

except one, $\propto G\langle L \rangle$, where we have to take into account a correction

$$2\alpha G\delta\langle L \rangle = -\frac{2\alpha G\langle xP_1(x) \rangle_0}{1 + \sqrt{1 - G^2}} - \frac{\alpha G^2\langle x^2 \rangle_0}{(1 + \sqrt{1 - G^2})^2} = 2\beta \cot \frac{\Psi^*}{2} + \csc^2 \frac{\Psi^*}{2} - \frac{1 + \gamma^2 G^2}{1 - G^2} \quad (\text{C.7})$$

coming from simultaneous expansion of $\cot(\Psi/2)$ and the distribution function (39) near $\Psi = \Psi^*$ (here $\langle \dots \rangle_0$ denotes averaging over the distribution (39) with $P(x) = 1$). While the first and second terms in the r.h.s. of Eq. (C.7) cancel out, the third gives required correction in the r.h.s of Eq. (40).

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