

## Non-homogeneous random walks with non-integrable increments and heavy-tailed random walks on strips

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### Abstract

We study asymptotic properties of spatially non-homogeneous random walks with non-integrable increments, including transience, almost-sure bounds, and existence and non-existence of moments for first-passage and last-exit times. In our proofs we also make use of estimates for hitting probabilities and large deviations bounds. Our results are more general than existing results in the literature, which consider only the case of sums of independent (typically, identically distributed) random variables. We do not assume the Markov property. Existing results that we generalize include a circle of ideas related to the Marcinkiewicz–Zygmund strong law of large numbers, as well as more recent work of Kesten and Maller. Our proofs are robust and use martingale methods. We demonstrate the benefit of the generality of our results by applications to some non-classical models, including random walks with heavy-tailed increments on two-dimensional strips, which include, for instance, certain generalized risk processes.

**Keywords:** Heavy-tailed random walks; non-homogeneous random walks; transience; rate of escape; passage times; last exit times; semimartingales; random walks on strips; random walks with internal degrees of freedom; risk process.

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## 1 Introduction

There is an extensive and rich theory of sums of independent, identically distributed (i.i.d.) random variables (classical ‘random walks’): see for instance the books of Kallenberg [17, Chapter 9], Loève [27, §26.2], or Stout [38, §3.2]. When the summands are integrable, the (first-order) asymptotic behaviour is governed by the mean. Completely different phenomena occur when the mean does not exist: see classical references such as [4, 8, 11] or more recent work such as [6, 15, 24]. In this paper we study an extension of this problem to general stochastic processes with non-integrable increments to include, for example, spatially non-homogeneous random walks.

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Let  $(X_t)_{t \in \mathbb{Z}^+}$  be a stochastic process on  $\mathbb{R}$  adapted to the filtration  $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$ . (Throughout the paper we set  $\mathbb{Z}^+ := \{0, 1, 2, \dots\}$  and  $\mathbb{N} := \{1, 2, \dots\}$ .) We will be concerned with the asymptotic behaviour of  $X_t$  given ‘heavy-tailed’ conditions on its increments. As we present our general results, it is helpful to keep in mind the classical independent-increments case, where  $X_t = S_t$  given by  $S_0 := 0$  and, for  $t \in \mathbb{N}$ ,  $S_t := \sum_{s=1}^t \zeta_s$  for a sequence of independent (often, i.i.d.)  $\mathbb{R}$ -valued random variables  $\zeta_1, \zeta_2, \dots$ . Thus we start with a brief summary of some known results in that setting. Many of the results that we discuss for random walks have analogues for suitable Lévy processes: see e.g. the book of Sato [35], particularly Sections 37 and 48.

A classical result of Kesten [18, Corollary 3] states that if  $\zeta_1, \zeta_2, \dots$  are i.i.d. random variables with  $\mathbb{E}|\zeta_1| = \infty$ , then as  $t \rightarrow \infty$ ,  $t^{-1}S_t$  either: (i) tends to  $+\infty$  a.s.; (ii) tends to  $-\infty$  a.s.; or (iii) satisfies

$$-\infty = \liminf_{t \rightarrow \infty} t^{-1}S_t < \limsup_{t \rightarrow \infty} t^{-1}S_t = +\infty, \text{ a.s.} \tag{1.1}$$

Erickson [8] gives criteria for classifying such behaviour. Other classical results deal with the growth rate of the upper envelope of  $S_t$ , i.e., determining sequences  $a_t$  for which  $|S_t| \geq a_t$  infinitely often (or not), or  $S_t \geq a_t$  infinitely often; here we mention the work of Feller [11], as well as results related to the Marcinkiewicz–Zygmund strong law of large numbers (see e.g. [20, Theorem 1]). The lower envelope behaviour, i.e., when  $|S_t| \geq a_t$  all but finitely often, is considered by Griffin [13] (particularly Theorem 3.5); see also Pruitt [32].

Note that (1.1) can hold and  $S_t$  be transient (with respect to bounded sets); Loève [27, §26.2] gives the example of a symmetric stable random walk without a mean. The general criterion for deciding between transience and recurrence is due to Chung and Fuchs (see e.g. [17, Theorem 9.4] or [27, §26.2]), and is rather subtle: Shepp showed [37] that there exist distributions for  $\zeta_1$  with *arbitrarily heavy* tails but for which  $S_t$  is still recurrent. By assuming additional regularity for the distribution of  $\zeta_1$ , one can obtain more tractable criteria for recurrence; Shepp gives a criterion when the distribution of  $\zeta_1$  is symmetric [36, Theorem 5].

In the present paper we extend aspects of this classical theory to a much more general setting, in which  $X_t$  is an  $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$ -adapted process whose increments satisfy certain moment or tail conditions. Our primary interest is the case of one-sided transience, when  $X_t \rightarrow +\infty$  a.s. or  $X_t \rightarrow -\infty$  a.s. We give criteria classifying such behaviour, and quantify the rate of escape via almost-sure bounds. We also quantify the transience by studying the existence and non-existence of moments for *first passage times* and *last exit times*; in the setting of  $X_t = S_t$  a sum of i.i.d. random variables, corresponding sharp results are given by Kesten and Maller [19]. We state our results for this model in Section 2.

Our proofs are robust and are based on semimartingale ideas, and so are quite different from the arguments used for the i.i.d. case. Semimartingale techniques are by now well established for stochastic systems that are ‘near-critical’ in some sense and whose increments have at least one moment; see for example [2, 10, 25, 26, 29, 30]. One contribution of the present paper is to show that essentially similar methods are equally powerful in the heavy-tailed setting. While not as sharp as the results available in the i.i.d. case, our results are considerably more general, and our proofs are relatively short, and based on some intuitively appealing ideas.

We give applications of our general results to Markov chains on *strips* of the form  $\mathcal{A} \times \mathbb{Z}$  for a countable (finite or infinite) set  $\mathcal{A}$ . Random walks on strips or half strips  $(\mathcal{A} \times \mathbb{Z}^+)$  have received attention in the literature (see [9, 10, 28] and references therein), motivated by various applied problems, including queuing theory; they can also be viewed as random walks with *internal degrees of freedom*, which were introduced by

Sinai as a tool for studying the Lorentz gas (see e.g. [23]). We are concerned with the case in which the  $\mathbb{Z}$ -components of the increments of the walk have *heavy tails*; the previous literature has considered only the light-tailed setting (typically, assuming uniformly bounded increments). The heavy-tailed setting leads to new phenomena, including a phase transition governed by the recurrence properties of the projection onto  $\mathcal{A}$  of the process.

We describe the strip model and corresponding results in detail in Section 3.1; to finish this section we give one additional source of motivation, arising from *risk theory*, and outline the main features of our results. A special case of our strip model can be viewed as an insurance or portfolio model in the presence of rare catastrophes. In the Markov chain  $(U_t, V_t)$  on  $\mathcal{A} \times \mathbb{Z}$ ,  $V_t \in \mathbb{Z}$  is the total revenue of the insurance company, or the total value of the portfolio, after  $t$  time units (days, say). The other variable,  $U_t \in \mathcal{A}$ , represents the current ‘state of the market’, with  $U_t = 0$  (say) corresponding to a catastrophe. Suppose that  $\mathbb{E}[V_{t+1} - V_t \mid U_t = \ell] = \mu_\ell > 0$  is well-defined for  $\ell \neq 0$ ;  $\mu_\ell$  is the average daily profit, which, in the insurance model, is determined by insurance premiums and the daily pay-out rate under usual conditions. On the other hand, when  $U_t = 0$ , we assume  $V_t$  decreases by a *non-integrable* amount, representing the catastrophic crash. Catastrophes are rare, so we assume that the time between successive visits to  $U_t = 0$  is itself non-integrable. Under what conditions is eventual ruin assured? This model extends the standard *risk process* of insurance theory: see e.g. [33, §3.5.1].

Our results show a crucial distinction between two possible scenarios, depending on whether the *induced Markov chain*  $U_t$  is positive- or null-recurrent ( $U_t$  is itself a Markov chain under the conditions that we impose). If  $U_t$  is positive-recurrent, the boundary state  $0 \in \mathcal{A}$  dominates the asymptotics, and  $V_t \rightarrow -\infty$ . The case where  $U_t$  is null-recurrent is more subtle, and we give conditions for  $V_t \rightarrow -\infty$  or  $V_t \rightarrow +\infty$  depending on the tails of the increments of  $V_t$  at  $U_t = 0$  and the tails of the return times of  $U_t$  to state 0. We also quantify the rate of transience, giving rates at which  $V_t$  tends to  $\pm\infty$ . In the context of the risk model, our results confirm the expectation that pricing is problematic in such genuinely heavy-tailed risk situations: in certain conditions, the insurance company cannot stabilize the situation however large  $\mu_\ell$ ,  $\ell \neq 0$  may be (i.e., however much premium it charges); we refer to Section 3.1 for precise statements.

## 2 Main results

We write  $\Delta_t := X_{t+1} - X_t$ ,  $t \in \mathbb{Z}^+$ , for the increments of  $X_t$ . For any real number  $x$ , we write  $x^+ := x\mathbb{1}\{x > 0\}$  and  $x^- := -x\mathbb{1}\{x < 0\}$ , where ‘ $\mathbb{1}$ ’ denotes the indicator function; thus  $x = x^+ - x^-$ .

For definiteness, we take  $X_0 = 0$  throughout. In most of our results, we impose ‘heavy tail’ conditions on either  $\Delta_t^+$  or  $\Delta_t^-$ ; typically these conditions are one-sided (i.e., inequalities). The following basic result shows that, under the conditions of most of our theorems, the process  $X_t$  has non-trivial asymptotic behaviour. The proofs of this and of the other results in this section are given in Section 4.

**Proposition 2.1.** *Suppose that either (i) there exist  $\gamma > 0$ ,  $c > 0$ , and  $x_0 < \infty$  for which  $\mathbb{P}[\Delta_t^+ > x \mid \mathcal{F}_t] \geq cx^{-\gamma}$ , a.s., for all  $x \geq x_0$  and all  $t$ ; or (ii) there exist  $\gamma \in (0, 1)$ ,  $c > 0$ , and  $x_0 < \infty$  for which  $\mathbb{E}[\Delta_t^+ \mathbb{1}\{\Delta_t^+ \leq x\} \mid \mathcal{F}_t] \geq cx^{1-\gamma}$ , a.s., for all  $x \geq x_0$  and all  $t$ ; or either (i) or (ii) holds with  $\Delta_t^-$  instead of  $\Delta_t^+$ . Then*

$$\limsup_{t \rightarrow \infty} |X_t| = \infty, \text{ a.s.} \tag{2.1}$$

In the i.i.d. case where  $X_t = S_t = \sum_{s=1}^t \zeta_s$  and  $\mathbb{E}|\zeta_1| = \infty$ , (2.1) follows from the result of Kesten [18, Corollary 3] mentioned above, and (2.1) also holds automatically

if  $X_t$  is an irreducible time-homogeneous Markov chain on a locally finite unbounded subset of  $\mathbb{R}$ .

Our first main result gives conditions under which  $X_t$  is transient to the right, i.e.,  $X_t \rightarrow +\infty$  a.s. as  $t \rightarrow \infty$  (or transient to the left, by considering  $-X_t$ ). Together with our Theorem 2.6 below on the rate of escape, Theorem 2.2 can be viewed as an analogue of Erickson's [8] result in the case of a sum of i.i.d. random variables; in the i.i.d. case the conclusion of Theorem 2.2 follows from [8, Corollary 1]. The results of [8] show that the conditions in Theorem 2.2 are close to optimal (see also Remark 2.3 and the comments in Section 6).

**Theorem 2.2.** *Let  $\alpha \in (0, 1)$  and  $\beta > \alpha$ . Suppose that there exist  $C < \infty$ ,  $c > 0$ , and  $x_0 < \infty$  for which, for all  $t$ ,*

$$\mathbb{E}[(\Delta_t^-)^\beta \mid \mathcal{F}_t] \leq C, \text{ a.s.}, \tag{2.2}$$

and, for all  $x \geq x_0$  and all  $t$ ,

$$\mathbb{E}[\Delta_t^+ \mathbf{1}\{\Delta_t^+ \leq x\} \mid \mathcal{F}_t] \geq cx^{1-\alpha}, \text{ a.s.} \tag{2.3}$$

Then  $X_t \rightarrow +\infty$  a.s. as  $t \rightarrow \infty$ .

**Remark 2.3.** *Condition (2.3) is natural. For  $\gamma \leq 1$ ,  $(\Delta_t^+)^\gamma \geq x^{\gamma-1} \Delta_t^+ \mathbf{1}\{\Delta_t^+ \leq x\}$  for any  $x > 0$ , so (2.3) implies that  $\mathbb{E}[(\Delta_t^+)^\gamma \mid \mathcal{F}_t] = \infty$  for any  $\gamma > \alpha$ . A counterexample due to K.L. Chung (see the Mathematical Reviews entry for [7]; also Baum [3]) shows that (2.3) cannot be replaced by a condition on the moments of the increments, even in the case of a sum of i.i.d. random variables. Chung's example has, for  $\alpha \in (0, 1)$  and  $\beta > \alpha$ ,  $\mathbb{E}[(\zeta_1^-)^\beta] < \infty$  and  $\mathbb{E}[(\zeta_1^+)^\alpha] = \infty$ , but  $\mathbb{E}[\zeta_1^+ \mathbf{1}\{\zeta_1^+ \leq x\}] = o(x^{1-\alpha})$  along a subsequence, so (2.3) does not hold. For  $X_t = S_t$  as in Chung's example,  $\liminf_{t \rightarrow \infty} X_t = -\infty$ , a.s.*

Our next two results deal with the growth rate of  $X_t$ , and provide almost-sure bounds. First we have the following upper bounds.

**Theorem 2.4.** *Suppose that there exist  $\theta \in (0, 1]$ ,  $\phi \in \mathbb{R}$ ,  $x_0 < \infty$  and  $C < \infty$  such that, for all  $x \geq x_0$  and all  $t$ ,*

$$\mathbb{P}[\Delta_t^+ \geq x \mid \mathcal{F}_t] \leq Cx^{-\theta}(\log x)^\phi, \text{ a.s.} \tag{2.4}$$

(i) *If  $\theta \in (0, 1)$ , then, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t \in \mathbb{Z}^+$ ,*

$$X_t \leq t^{1/\theta}(\log t)^{\frac{\phi+2}{\theta}+\varepsilon}.$$

(ii) *If  $\theta = 1$ , then, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t \in \mathbb{Z}^+$ ,*

$$X_t \leq t(\log t)^{(1+\phi)^++1+\varepsilon}.$$

**Remark 2.5.** *In the case of a sum of independent random variables, Theorem 2.4 is slightly weaker than optimal. Suppose that  $\zeta_1, \zeta_2, \dots$  are independent, and that for some  $\theta \in (0, 1)$  and  $\phi \in \mathbb{R}$ ,*

$$\sup_{k \in \mathbb{N}} \limsup_{x \rightarrow \infty} (x^\theta (\log x)^{-\phi} \mathbb{P}[|\zeta_k| \geq x]) < \infty.$$

Then, with  $S_t = \sum_{s=1}^t \zeta_s$ , for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t \in \mathbb{Z}^+$ ,

$$|S_t| \leq t^{1/\theta}(\log t)^{\frac{\phi+1}{\theta}+\varepsilon}. \tag{2.5}$$

The bound (2.5) belongs to a family of classical results with a long history; the case  $\phi = 0$  is due to Lévy and Marcinkiewicz (quoted by Feller [11, p. 257]), and the general case of (2.5) follows for example from a result of Loève [27, p. 253]. Under the additional condition that the summands are identically distributed, sharp results are given by Feller [11, Theorem 2]; for a recent reference, see [24]. Related results in the i.i.d. case are also given by Chow and Zhang [5] (see also [20, Theorem 2]).

The next result shows that if we impose a variant of the condition (2.3) in Theorem 2.2, not only does  $X_t \rightarrow +\infty$ , a.s., but it does so at a particular rate of escape.

**Theorem 2.6.** *Let  $\alpha \in (0, 1)$  and  $\beta > \alpha$ . Suppose that there exist  $C < \infty$ ,  $c > 0$ , and  $x_0 < \infty$  for which (2.2) holds, and*

$$\mathbb{P}[\Delta_t^+ > x \mid \mathcal{F}_t] \geq cx^{-\alpha}, \text{ a.s.}, \tag{2.6}$$

for all  $x \geq x_0$  and all  $t$ . Then for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t \in \mathbb{Z}^+$ ,

$$X_t \geq t^{1/\alpha}(\log t)^{-(1/\alpha)-\varepsilon}.$$

**Remark 2.7.** *Note that (2.6) implies that, a.s.,*

$$\mathbb{E}[(\Delta_t^+)^{\alpha} \mid \mathcal{F}_t] = \int_0^{\infty} \mathbb{P}[\Delta_t^+ > y^{1/\alpha} \mid \mathcal{F}_t] dy \geq c \int_{x_0}^{\infty} y^{-1} dy = \infty.$$

Conditions (2.3) and (2.6) are closely related, but neither implies the other. However, if one replaces the inequalities by equalities, the former implies the latter: more generally, see Lemma 6.1 in the Appendix. In the case where  $X_t = S_t$  is a sum of i.i.d. random variables, a weaker version of Theorem 2.6 was obtained by Derman and Robbins [7] and stated in a stronger form by Stout [38, Theorem 3.2.6]; although Stout’s statement is still weaker than our Theorem 2.6, his proof gives essentially the same result (in the i.i.d. case). Also relevant in the i.i.d. case is a result of Chow and Zhang [5, Theorem 1]. Chung’s counterexample (see Remark 2.3) shows that the condition (2.6) cannot be replaced by a moments condition, for instance.

Theorems 2.4 and 2.6 have the following immediate corollary.

**Corollary 2.8.** *Let  $\alpha \in (0, 1)$  and  $\beta > \alpha$ . Suppose that (2.2) holds for some  $C < \infty$  and all  $t$ , and that, uniformly in  $t$  and  $\omega$ ,*

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[\Delta_t^+ > x \mid \mathcal{F}_t]}{\log x} = -\alpha, \text{ a.s.}$$

Then

$$\lim_{t \rightarrow \infty} \frac{\log X_t}{\log t} = \frac{1}{\alpha}, \text{ a.s.}$$

*Proof.* Note that the uniformity in the condition in the corollary ensures that for any  $\varepsilon > 0$  there exists  $x_0 < \infty$  such that, for all  $x \geq x_0$  and all  $t$ ,

$$x^{-\alpha-\varepsilon} \leq \mathbb{P}[\Delta_t^+ > x \mid \mathcal{F}_t] \leq x^{-\alpha+\varepsilon}, \text{ a.s.}$$

Theorem 2.4 with the upper bound in the last display and (2.2) then shows that for any  $\varepsilon > 0$ , a.s.,  $X_t \leq t^{(1/\alpha)+\varepsilon}$  for all but finitely many  $t$ . On the other hand, Theorem 2.6 with the lower bound in the last display and (2.2) shows that for any  $\varepsilon > 0$ , a.s.,  $X_t \geq t^{(1/\alpha)-\varepsilon}$  for all but finitely many  $t$ . Since  $\varepsilon > 0$  was arbitrary, the result follows.  $\square$

For any  $x \in \mathbb{R}$ , write

$$\tau_x := \min\{t \in \mathbb{Z}^+ : X_t \geq x\}, \tag{2.7}$$

for the *first passage time* into the half-line  $[x, \infty)$ ; here and throughout the paper we adopt the usual convention that  $\min \emptyset := \infty$ . Under the conditions of Theorem 2.2,  $X_t \rightarrow +\infty$ , a.s., so that  $\tau_x < \infty$  a.s., for all  $x \in \mathbb{R}$ . It is natural to study the *tails* or *moments* of the random variable  $\tau_x$  in order to quantify the transience in a precise sense. In the i.i.d. case for  $X_t = S_t$ , sharp results on the existence or non-existence of moments for  $\tau_x$  are given by Kesten and Maller [19, Theorem 2.1]; see [19] for references to earlier work. In our more general setting, we have the following two results.

**Theorem 2.9.** *Let  $\alpha \in (0, 1)$  and  $\beta > \alpha$ . Suppose that there exist  $c > 0$ ,  $C < \infty$ , and  $x_0 < \infty$  for which (2.2) holds for all  $t$  and (2.3) holds for all  $x \geq x_0$  and all  $t$ . Then for any  $x \in \mathbb{R}$  and any  $p \in [0, \beta/\alpha)$ ,  $\mathbb{E}[\tau_x^p] < \infty$ .*

**Theorem 2.10.** *Let  $\alpha \in (0, 1]$  and  $\beta > 0$ . Suppose that, for some  $C < \infty$ ,  $\mathbb{E}[(\Delta_t^+)^{\alpha} | \mathcal{F}_t] \leq C$  a.s. for all  $t$ , and  $\mathbb{E}[(\Delta_t^-)^{\beta} | \mathcal{F}_t] = \infty$  a.s. for all  $t$ . Then, for any  $x > 0$ ,  $\mathbb{E}[\tau_x^{\beta/\alpha}] = \infty$ .*

Note that in Theorem 2.9,  $\beta/\alpha > 1$ , so in particular  $\mathbb{E}[\tau_x] < \infty$  for any  $x \in \mathbb{R}$ . The results of Kesten and Maller [19] in the i.i.d. case show that the conditions in Theorems 2.9 and 2.10 are not far from optimal: see also the comments in Section 6.

Our final results for this section concern *last exit times*. For  $x \in \mathbb{R}$ , let

$$\lambda_x := \max\{t \in \mathbb{Z}^+ : X_t \leq x\}, \tag{2.8}$$

the last time (if finite) at which  $X_t \in (-\infty, x]$ . Again, if  $X_t \rightarrow +\infty$  a.s. (such as under the conditions of Theorem 2.2) then  $\lambda_x < \infty$  a.s. for all  $x \in \mathbb{R}$ , and the moments of the random variables  $\lambda_x$  provide a quantitative characterization of the transience. Again, in the i.i.d. case sharp results are given by Kesten and Maller [19, Theorem 2.1].

**Theorem 2.11.** *Let  $\alpha \in (0, 1)$  and  $\beta > \alpha$ . Suppose that there exist  $c > 0$ ,  $C < \infty$ , and  $x_0 < \infty$  for which (2.2) holds for all  $t$  and (2.3) holds for all  $x \geq x_0$  and all  $t$ . Then for any  $x \in \mathbb{R}$  and any  $p \in [0, (\beta/\alpha) - 1)$ ,  $\mathbb{E}[\lambda_x^p] < \infty$ .*

**Theorem 2.12.** *Let  $\alpha \in (0, 1]$  and  $\beta > \alpha$ . Suppose that there exist  $c > 0$ ,  $C < \infty$ , and  $x_0 < \infty$  such that  $\mathbb{E}[(\Delta_t^+)^{\alpha} | \mathcal{F}_t] \leq C$  a.s. for all  $t$ , and, for all  $x \geq x_0$  and all  $t$ ,  $\mathbb{P}[\Delta_t^- > x | \mathcal{F}_t] \geq cx^{-\beta}$  a.s. Then for any  $x \in \mathbb{R}$  and any  $p > (\beta/\alpha) - 1$ ,  $\mathbb{E}[\lambda_x^p] = \infty$ .*

The rest of the paper is organized as follows. In Section 3 we give applications of our results from Section 2 to some non-classical models, including Markov chains on strips with heavy-tailed increments. In Section 4 we prove our general results from Section 2, and then in Section 5 we prove the results on applications given in Section 3. Finally, in Section 6, we make some additional remarks on some of the conditions in our theorems and their relationship to conditions in the literature on sums of i.i.d. random variables.

Finally, we make a note on notation. We reserve the standard Landau  $O(\cdot)$ ,  $o(\cdot)$  notation for situations in which the implicit constants are non-random, i.e., the implicit inequalities are uniform in probability space elements  $\omega$  (in some set of probability 1). So, for example,  $Z_t = O(a_t)$ , a.s., if and only if there exist some finite absolute constants  $C_0$  and  $t_0$  for which  $Z_t \leq C_0 a_t$ , a.s., for all  $t \geq t_0$ . In situations where it is convenient to extend the notation to allow  $C_0 = C_0(\omega)$  or  $t_0 = t_0(\omega)$  to be *random*, we augment the notation and write  $O_{\omega}(\cdot)$ ,  $o_{\omega}(\cdot)$  to make the distinction clear.

### 3 Applications

#### 3.1 Heavy-tailed random walks on strips

In this section we describe an application of the one-dimensional results of Section 2 to a higher-dimensional model. The model we consider will be a random walk on a *strip*. Such models are of interest in various contexts: see [9] for a selection of references, including applications to communications systems, queueing models, and random walks with internal degrees of freedom.

Denote by  $\mathcal{S}_k := \{0, 1, \dots, k - 1\} \times \mathbb{Z}$  the strip of *width*  $k$ , and by  $\mathcal{S}_{\infty} := \mathbb{Z}^+ \times \mathbb{Z}$  the infinite-width strip.

Starting with early work of Malyshev [28], random walks on *finite*-width strips  $\mathcal{S}_k$  (or half-strips  $\{0, 1, \dots, k - 1\} \times \mathbb{Z}^+$ ) have received some attention in the literature; see [9]

and [10, §3.1]. The random walks in *periodic environments* described by Key [21, §9] are essentially random walks on strips; what we call strips are also known as *ladders*, see e.g. [31]. In these previous studies, the increments of the walk have been integrable. In the present paper we are primarily interested in the case of an *infinite-width* strip with *non-integrable* increments for the random walk, which can give rise to very different and rather subtle phenomena. The model and results that we describe in this section can be stated in more generality in terms of random walks with a distinguished subset of the state space: for ease of exposition, we defer the more general description to Section 3.2.

We consider a Markov chain  $(U_t, V_t)$  on  $\mathcal{S}_k$  or  $\mathcal{S}_\infty$ ; the first coordinate of the chain describes which *line* the chain is currently on, while the second coordinate describes the location on the given line. The transition probabilities are given by

$$\mathbb{P}[(U_{t+1}, V_{t+1}) = (\ell', x + d) \mid U_t = \ell, V_t = x] = \phi(\ell, \ell'; d), \tag{3.1}$$

where  $\phi$  satisfies the obvious conditions; the right-hand side of (3.1) does not depend on  $x$ , so the transition law is spatially homogeneous in the second coordinate. In [9, 10] the transition law has the same partial homogeneity as expressed by (3.1); in addition, [9, 10] make an assumption of a uniform one-sided bound on the increments, appropriate for the half-strip problem. The translation invariance condition (3.1) is also standard in the literature on random walks with internal degrees of freedom: see e.g. [23].

A consequence of (3.1) is that

$$\mathbb{P}[U_{t+1} = \ell' \mid U_t = \ell] = \sum_{d \in \mathbb{Z}} \phi(\ell, \ell'; d) =: q_{\ell, \ell'}.$$

Thus the projection  $(U_t)_{t \in \mathbb{Z}^+}$  is itself a Markov chain, which records the current line that the random walk is on; this Markov chain has transition probabilities  $q_{\ell, \ell'}$ . In the terminology of [10, §3.1],  $U_t$  is the *induced* chain.

We remark that  $W_t := (U_t, V_t - V_{t-1})$  also describes a Markov chain, with transitions  $\mathbb{P}[W_{t+1} = (\ell', d') \mid W_t = (\ell, d)] = \phi(\ell, \ell'; d')$ ; one may write  $V_t = V_0 + \sum_{s=1}^t v(W_s)$  where  $v(\ell, d) = d$ , so that  $V_t$  may be represented as an *additive functional* of the Markov chain  $W_t$ . Additive functionals of Markov chains have been extensively studied, primarily in the case in which the underlying chain is ergodic: see e.g. [16, 22, 34].

The primary assumption in this section is the following.

- (B1) Suppose that the transition probabilities of  $(U_t, V_t)$  are given by (3.1). Moreover, suppose that  $U_t$  is an irreducible Markov chain and that  $U_t$  is recurrent.

Of course, in the finite-width setting, irreducibility of  $U_t$  automatically implies recurrence (in fact, positive-recurrence), so the recurrence part of assumption (B1) is only non-trivial in the infinite-width setting, when  $U_t \in \mathbb{Z}^+$ .

**Remark 3.1.** *The structure of the strip is unimportant for our results. In fact, our results extend to any appropriate model on  $\mathcal{A} \times \mathbb{Z}$  for any countable set  $\mathcal{A}$ , provided the induced chain on  $\mathcal{A}$  is recurrent; more generally, see Section 3.2. Regarded in this way, this framework also contains the correlated or persistent random walk (see e.g. [12]) in which  $\mathcal{A} = \{\pm 1\}$  is a set of directions.*

Suppose for now that the Markov chain  $(U_t)_{t \in \mathbb{Z}^+}$  has a unique stationary distribution  $(\pi_\ell)_{\ell \in \{0, \dots, k-1\}}$  with  $\pi_\ell > 0$  for all  $\ell$ . In the case where the in-line jump distributions each have a finite mean  $\mu_\ell = \mathbb{E}[V_{t+1} - V_t \mid U_t = \ell]$ , the recurrence classification of the random walk on a strip depends on  $\sum \pi_\ell \mu_\ell$ : see [34] for a result along these lines for a broader class of additive functionals of Markov chains. In the case of a *half-strip*, the additive

functional representation is not directly available, and recurrence/transience results are given in [10, §3.1]; an earlier result was obtained by Falin [9].

Here we are interested in the very different situation, in either the finite-width or infinite-width case, in which at least one of the means  $\mu_\ell$  is not defined. We take the 0-line (the ‘boundary’) to be a distinguished line with heavy tails with exponent  $\alpha$  to the right, say; the other lines (the ‘bulk’) may also have heavy tails (with exponent  $\beta$  to the left, say). Under what conditions does the boundary dominate? Or the bulk? The results that we present below give conditions under which  $V_t \rightarrow +\infty$  or  $V_t \rightarrow -\infty$ .

Our main interest in this section is the infinite-width case, for which the embedded process  $U_t$  need not be positive-recurrent: clearly the recurrence properties of  $U_t$  are crucial. Let  $\nu := \min\{t \in \mathbb{N} : U_t = 0\}$  denote the time of the first return to the 0-line. Then under (B1),  $U_t$  is positive-recurrent if  $\mathbb{E}[\nu] < \infty$  but null-recurrent if  $\mathbb{E}[\nu] = \infty$ .

A basic example to bear in mind is the case in which when  $U_t = 0$ ,  $V_t$  jumps only in the positive direction with increments of tail exponent  $\alpha \in (0, 1)$ , while if  $U_t \neq 0$ ,  $V_t$  jumps in the negative direction with increments of tail exponent  $\beta$ . We give results that show  $V_t \rightarrow -\infty$  or  $V_t \rightarrow +\infty$  depending on the relationship between  $\alpha$ ,  $\beta$ , and  $\gamma$ , the tail exponent of  $\nu$ ; we also quantify the rate of escape of  $V_t$ .

To simplify our statements, we introduce some more notation. For  $x \geq 0$ ,

$$\mathbb{P}[(V_{t+1} - V_t)^+ > x \mid U_t = \ell, V_t = z] = \sum_{y>x} \sum_{\ell'} \phi(\ell, \ell'; y) =: T_\ell^+(x),$$

which depends only on  $\ell$  and  $x$ , and not on  $z$  or  $t$ . Similarly, let

$$T_\ell^-(x) := \mathbb{P}[(V_{t+1} - V_t)^- > x \mid U_t = \ell, V_t = z], \text{ and}$$

$$M_\ell^\pm(\beta) := \mathbb{E}[(V_{t+1} - V_t)^\pm]^\beta \mid U_t = \ell, V_t = z].$$

First we consider the case where  $U_t$  is positive-recurrent. For example, suppose that  $|\mu_\ell| < \infty$  for all  $\ell \neq 0$ , but that on line 0 the mean of  $V_t$  is undefined. In this case we show that, in contrast to the case in which all the  $\mu_\ell$  are finite, this single line dominates the asymptotic behaviour of the process. The intuition in this case is that the process spends a positive fraction of its time in line 0, and so the long jumps from line 0 dominate.

**Theorem 3.2.** *Suppose that (B1) holds and that  $U_t$  is positive-recurrent. Suppose that there exist  $\alpha \in (0, 1)$ ,  $\beta > \alpha$ , and  $C < \infty$  such that (i)  $M_\ell^-(\beta) \leq C$  for all  $\ell$ ; (ii)*

$$\lim_{x \rightarrow \infty} \frac{\log T_0^+(x)}{\log x} = -\alpha; \tag{3.2}$$

*and (iii)  $M_\ell^+(\beta) \leq C$  for all  $\ell \neq 0$ . Then  $V_t \rightarrow +\infty$  a.s. as  $t \rightarrow \infty$ , and, moreover,*

$$\lim_{t \rightarrow \infty} \frac{\log V_t}{\log t} = \frac{1}{\alpha}, \text{ a.s.}$$

Under conditions related in spirit to those in Theorem 3.2, including ergodicity of  $U_t$  and heavy tails for the increments of  $V_t$ , certain results on convergence to stable laws are obtained by Jara *et al.* [16].

In the case where  $U_t$  is null-recurrent, the intuition changes, since the process spends only a vanishing fraction of its time in line 0. In this case the tail of  $\nu$  becomes crucial, and the effects of either the boundary or the bulk may dominate, as shown by the contrast between the next two theorems.



**Theorem 3.3.** *Suppose that (B1) holds,  $U_t$  is null-recurrent, and, for some  $\gamma \in (0, 1]$ ,*

$$\lim_{t \rightarrow \infty} \frac{\log \mathbb{P}[\nu > t]}{\log t} = -\gamma. \tag{3.3}$$

*Suppose that there exist  $\alpha \in (0, 1)$ ,  $\beta > 0$ , and  $C < \infty$  such that (i)  $M_\ell^-(\beta) \leq C$  for all  $\ell$ ; (ii) (3.2) holds; and (iii)  $M_\ell^+(\beta) \leq C$  for all  $\ell \neq 0$ . Then if  $\alpha < \gamma(\beta \wedge 1)$ ,  $V_t \rightarrow +\infty$  a.s. as  $t \rightarrow \infty$ , and, moreover,*

$$\lim_{t \rightarrow \infty} \frac{\log V_t}{\log t} = \frac{\gamma}{\alpha}, \text{ a.s.}$$

**Theorem 3.4.** *Suppose that (B1) holds,  $U_t$  is null-recurrent, and, for some  $\gamma \in (0, 1)$ , (3.3) holds. Suppose that there exist  $\alpha, \beta \in (0, 1)$ ,  $\delta > 0$ , and  $C < \infty$  such that (i)  $M_0^+(\alpha) + M_0^-(\alpha) \leq C$ ; (ii) uniformly for all  $\ell \neq 0$ ,*

$$\lim_{x \rightarrow \infty} \frac{\log T_\ell^-(x)}{\log x} = -\beta;$$

*and (iii)  $M_\ell^+(\beta + \delta) \leq C$  for all  $\ell \neq 0$ . Then if  $\alpha > \gamma\beta$ ,  $V_t \rightarrow -\infty$  a.s. as  $t \rightarrow \infty$ , and, moreover,*

$$\lim_{t \rightarrow \infty} \frac{\log |V_t|}{\log t} = \frac{1}{\beta}, \text{ a.s.}$$

**Remark 3.5.** *In the present paper we do not address the behaviour of first passage or last exit times for the random walk on a strip: we leave this as an open problem.*

The next result demonstrates how, via a concrete family of examples, one may achieve the condition (3.3). To do this, we take  $U_t$  to have asymptotically zero drift, specifically,  $\mathbb{E}[U_{t+1} - U_t \mid U_t = x]$  to be of order  $1/x$ . Fundamental work of Lamperti [25, 26] showed that such processes are near-critical from the point of view of recurrence classification. We prove Proposition 3.6 using results from [1, 2], which generalize Lamperti’s work [26].

**Proposition 3.6.** *Let  $\gamma \in (0, 1]$ . Suppose that there exist  $C < \infty$  and  $\sigma^2 \in (0, \infty)$  such that the following hold for all  $x \in \mathbb{Z}^+$ :*

$$\begin{aligned} \mathbb{P}[|U_{t+1} - U_t| \geq C \mid U_t = x] &= 0; \\ \mathbb{E}[(U_{t+1} - U_t)^2 \mid U_t = x] &= \sigma^2 + o(1); \\ \mathbb{E}[U_{t+1} - U_t \mid U_t = x] &= \left(\frac{1}{2} - \gamma\right) \frac{\sigma^2}{x} + o(1/x). \end{aligned}$$

*Then (3.3) holds for this  $\gamma \in (0, 1]$ .*

As an example, one may take  $U_t$  to be a simple symmetric random walk on  $\mathbb{Z}^+$  with reflection at 0; in that case,  $\gamma = 1/2$ .

### 3.2 Non-homogeneous random walk with a distinguished subset of the state space

In this section we describe a model that generalizes the strip model described in Section 3.1 (see Section 5.4 for details of the relationship), and whose study can, in important aspects, be reduced to the study of the one-dimensional model of Section 2. For this section, unlike Section 3.1, we do not assume the Markov property.

We consider a stochastic process  $(Y_t)_{t \in \mathbb{Z}^+}$  adapted to a filtration  $(\mathcal{G}_t)_{t \in \mathbb{Z}^+}$  and taking values in a subset  $\mathcal{S}$  of  $\mathbb{R}$  with  $\sup \mathcal{S} = +\infty$  and  $\inf \mathcal{S} = -\infty$ . We assume that there is a distinguished subset  $\mathcal{C} \subset \mathcal{S}$  of the state space. Roughly speaking, the process will

jump out of the set  $\mathcal{C}$  with heavier tails than in the remainder of the state space. For convenience we assume  $0 \in \mathcal{C}$  and  $Y_0 = 0$  a.s., although this is inessential for our results.

Define  $\sigma_0 := 0$  and, for  $n \in \mathbb{N}$ ,  $\sigma_n := \min\{t > \sigma_{n-1} : Y_t \in \mathcal{C}\}$ . We assume that  $\mathcal{S}$  and  $\mathcal{C}$  are sufficiently regular that the  $\sigma_n$  are stopping times:

(C1) Suppose that for all  $n$ ,  $\sigma_n$  is a  $(\mathcal{G}_t)_{t \in \mathbb{Z}^+}$  stopping time, and  $\mathbb{P}[\sigma_{n+1} < \infty \mid \mathcal{G}_{\sigma_n}] = 1$ .

If  $\mathcal{S}$  is countable, then the stopping-time property in (C1) holds automatically with  $\mathcal{G}_n = \sigma(Y_0, Y_1, \dots, Y_n)$  the natural filtration; in more generality, it suffices that  $\mathcal{C}$  be a measurable set, see e.g. [17, Lemma 7.6]. In (C1) we make the further assumption that the  $\sigma_n$  are all finite, which amounts to a notion of *recurrence* for  $\mathcal{C}$ .

For  $n \in \mathbb{Z}^+$ , take  $\nu_n := \sigma_{n+1} - \sigma_n$ , so that  $\nu_0 = \sigma_1$  is the first passage time into  $\mathcal{C}$  and  $\nu_1, \nu_2, \dots$  are the durations of the subsequent excursions from  $\mathcal{C}$ . Note that since  $\nu_n \geq 1$ ,  $\sigma_n \geq n$  and  $\sigma_n$  is increasing in  $n$ , so in particular  $\sigma_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Assumption (C1) implies that  $\nu_n < \infty$  for all  $n$ , a.s.

Write  $D_t := Y_{t+1} - Y_t$  for the increments of  $Y_t$ . Our first result covers the case where the average duration of the excursions from  $\mathcal{C}$  is uniformly finite. We assume:

(C2) Suppose that there exists  $B < \infty$  such that  $\mathbb{E}[\nu_n \mid \mathcal{G}_{\sigma_n}] \leq B$ , a.s., for all  $n$ .

**Theorem 3.7.** *Suppose that (C1) and (C2) hold. Suppose there exist  $\alpha \in (0, 1)$ ,  $\beta > \alpha$ , and  $C < \infty$  so that (i)  $\mathbb{E}[(D_t^-)^\beta \mid \mathcal{G}_t] \leq C$  a.s.; (ii) on  $\{Y_t \in \mathcal{C}\}$ , uniformly in  $t$  and  $\omega$ ,*

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[D_t^+ > x \mid \mathcal{G}_t]}{\log x} = -\alpha, \text{ a.s.}; \tag{3.4}$$

*and (iii) on  $\{Y_t \notin \mathcal{C}\}$ ,  $\mathbb{E}[(D_t^+)^\beta \mid \mathcal{G}_t] \leq C$  a.s. Then  $Y_t \rightarrow +\infty$  a.s., and, moreover,*

$$\lim_{t \rightarrow \infty} \frac{\log Y_t}{\log t} = \frac{1}{\alpha}, \text{ a.s.}$$

In the case where the  $\nu_n$  may not have a finite mean, we need to impose a mild additional regularity condition on the tails of  $\nu_n$ . Specifically, we assume:

(C3) Suppose that for some  $\gamma \in (0, 1]$ , uniformly in  $n$  and  $\omega$ ,

$$\lim_{t \rightarrow \infty} \frac{\log \mathbb{P}[\nu_n > t \mid \mathcal{G}_{\sigma_n}]}{\log t} = -\gamma, \text{ a.s.}$$

The next result gives conditions for the influence of  $\mathcal{C}$  to dominate.

**Theorem 3.8.** *Suppose that (C1) and (C3) hold. Suppose that there exist  $\alpha \in (0, 1)$ ,  $\beta > 0$ , and  $C < \infty$  such that (i)  $\mathbb{E}[(D_t^-)^\beta \mid \mathcal{G}_t] \leq C$  a.s.; (ii) on  $\{Y_t \in \mathcal{C}\}$ , (3.4) holds; and (iii) on  $\{Y_t \notin \mathcal{C}\}$ ,  $\mathbb{E}[(D_t^+)^\beta \mid \mathcal{G}_t] \leq C$  a.s. Then if  $\alpha < \gamma(\beta \wedge 1)$ ,  $Y_t \rightarrow +\infty$  a.s., and*

$$\lim_{t \rightarrow \infty} \frac{\log Y_t}{\log t} = \frac{\gamma}{\alpha}, \text{ a.s.}$$

The next result gives conditions for the influence of  $\mathcal{S} \setminus \mathcal{C}$  to dominate.

**Theorem 3.9.** *Suppose that (C1) and (C3) hold and that  $\gamma \in (0, 1)$ . Suppose that there exist  $\alpha, \beta \in (0, 1)$ ,  $\delta > 0$ , and  $C < \infty$  such that (i) on  $\{Y_t \in \mathcal{C}\}$ ,  $\mathbb{E}[|D_t|^\alpha \mid \mathcal{G}_t] \leq C$  a.s.; (ii) on  $\{Y_t \notin \mathcal{C}\}$ , uniformly in  $t$  and  $\omega$ ,*

$$\lim_{x \rightarrow \infty} \frac{\log \mathbb{P}[D_t^- > x \mid \mathcal{G}_t]}{\log x} = -\beta, \text{ a.s.};$$

*and (iii) on  $\{Y_t \notin \mathcal{C}\}$ ,  $\mathbb{E}[(D_t^+)^{\beta+\delta} \mid \mathcal{G}_t] \leq C$  a.s. Then if  $\alpha > \gamma\beta$ ,  $Y_t \rightarrow -\infty$  a.s., and*

$$\lim_{t \rightarrow \infty} \frac{\log |Y_t|}{\log t} = \frac{1}{\beta}, \text{ a.s.}$$

## 4 Proofs for Section 2

### 4.1 Overview

Our proofs are based on some semimartingale (or Lyapunov function) ideas. That is, for appropriate choices of Lyapunov function  $f : \mathbb{R} \rightarrow [0, \infty)$  we study the process  $f(X_t)$ ; typically we require that  $f(X_t)$  satisfy variations of Foster–Lyapunov style drift conditions. The Lyapunov functions that we study are of two basic kinds: either  $f(x) \rightarrow 0$  or  $f(x) \rightarrow \infty$  as  $x \rightarrow \pm\infty$ . These functions allow us to study different properties of the process  $X_t$ . The technical details of the proofs consist of two main components: first proving that  $f(X_t)$  satisfies a suitable drift condition, and then using semimartingale ideas to extract information about the asymptotic behaviour of  $X_t$  itself. For example, if  $f(X_t)$  satisfies a local submartingale/supermartingale condition, we can estimate hitting probabilities for  $X_t$  via stopping-time arguments. Verification of drift conditions for  $f(X_t)$  usually entails some Taylor’s formula expansions as well as some careful truncation ideas to deal with the heavy tails.

The remainder of this section is arranged as follows. In Section 4.2 we give some fundamental semimartingale results that will form part of our toolbox, largely taken from [2, 29]. In Section 4.3 we introduce our Lyapunov functions and, in a series of lemmas, undertake the technical estimates that we need to apply our semimartingale methods. Finally, in Section 4.4 we complete the proofs of the theorems.

### 4.2 Preliminaries

In this section we state some useful results from the literature that we will need. We will use the following result on existence of passage-time moments for one-dimensional stochastic processes, which is a direct consequence of Theorem 1 of [2].

**Lemma 4.1.** *Let  $(Z_t)_{t \in \mathbb{Z}^+}$  be an  $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$ -adapted process on  $[0, \infty)$ . For  $z > 0$ , let  $\sigma_z := \min\{t \in \mathbb{Z}^+ : Z_t \leq z\}$ . Suppose that there exist  $C \in (0, \infty)$  and  $\eta \in [0, 1)$  for which*

$$\mathbb{E}[Z_{t+1} - Z_t \mid \mathcal{F}_t] \leq -CZ_t^\eta, \text{ a.s.,}$$

*on  $\{t < \sigma_z\}$ . Then for any  $p \in [0, 1/(1 - \eta)]$ ,  $\mathbb{E}[\sigma_z^p] < \infty$ .*

The next result is contained in Theorem 3.2 of [29].

**Lemma 4.2.** *Let  $(Z_t)_{t \in \mathbb{Z}^+}$  be an  $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$ -adapted process on  $[0, \infty)$ . Suppose that for some  $B < \infty$ ,  $\mathbb{E}[Z_{t+1} - Z_t \mid \mathcal{F}_t] \leq B$ , a.s. Then for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t \in \mathbb{Z}^+$ ,*

$$\max_{0 \leq s \leq t} Z_s \leq t(\log t)^{1+\varepsilon}.$$

Finally, we give a maximal inequality that generalizes Lemma 3.1 of [29], which covered the case where  $\nu$  is a fixed, deterministic time.

**Lemma 4.3.** *Let  $(Z_t)_{t \in \mathbb{Z}^+}$  be an  $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$ -adapted process on  $[0, \infty)$ , and let  $\nu$  be an  $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$  stopping time. Suppose that for some  $B < \infty$ , on  $\{t < \nu\}$ , a.s.,  $\mathbb{E}[Z_{t+1} - Z_t \mid \mathcal{F}_t] \leq B$ . Then for any  $x > 0$ ,*

$$\mathbb{P} \left[ \max_{0 \leq s \leq \nu} Z_s \geq x \right] \leq \frac{B\mathbb{E}[\nu] + \mathbb{E}[Z_0]}{x}. \tag{4.1}$$

*Proof.* It suffices to suppose that  $\mathbb{E}[\nu] < \infty$ , in which case  $\nu < \infty$  a.s. Write  $A_t = \mathbb{E}[Z_{t+1} - Z_t \mid \mathcal{F}_t]$ , and let  $Y_t = Z_t + \sum_{s=0}^{t-1} A_s^-$ ; so  $Y_0 = Z_0$  and  $Y_t \geq Z_t$  for all  $t$ . Then

$$\mathbb{E}[Y_{t+1} - Y_t \mid \mathcal{F}_t] = \mathbb{E}[Z_{t+1} - Z_t \mid \mathcal{F}_t] + A_t^- = A_t^+ \in [0, B], \text{ a.s.,}$$

on  $\{t < \nu\}$ . Hence  $Y_{t \wedge \nu}$  is a nonnegative  $(\mathcal{F}_t)_{t \in \mathbb{Z}^+}$ -adapted submartingale with

$$\mathbb{E}[Y_{(s+1) \wedge \nu} - Y_{s \wedge \nu} \mid \mathcal{F}_s] \leq B \mathbb{1}\{s < \nu\}, \text{ a.s.}$$

Taking expectations in the last display and summing from  $s = 0$  to  $t - 1$  we have

$$\mathbb{E}[Y_{t \wedge \nu}] - \mathbb{E}[Y_0] \leq B \sum_{s=0}^{t-1} \mathbb{P}[\nu > s] \leq B \mathbb{E}[\nu].$$

Doob's submartingale inequality gives, for any  $x > 0$ ,

$$\mathbb{P} \left[ \max_{0 \leq s \leq t} Y_{s \wedge \nu} \geq x \right] \leq \frac{\mathbb{E}[Y_{t \wedge \nu}]}{x} \leq \frac{B \mathbb{E}[\nu] + \mathbb{E}[Z_0]}{x},$$

where the final inequality follows from the preceding display and the fact that  $Y_0 = Z_0$ . Since  $Z_t \leq Y_t$  for all  $t$ , the same bound holds with  $Z_{s \wedge \nu}$  replacing  $Y_{s \wedge \nu}$ ; since  $\nu < \infty$  a.s., letting  $t \rightarrow \infty$  we see  $\max_{0 \leq s \leq t} Z_{s \wedge \nu} \rightarrow \max_{0 \leq s \leq \nu} Z_s$  a.s., completing the proof.  $\square$

### 4.3 Technical results

In this section we prepare the ground for the proofs of our theorems from Section 2; we complete the proofs in Section 4.4. In the first two results, we study our first Lyapunov function, and obtain conditions under which a local submartingale/supermartingale condition holds. Our first Lyapunov function  $f_{z,\delta} : \mathbb{R} \rightarrow [0, 1]$  satisfies  $f_{z,\delta}(y) \rightarrow 0$  as  $y \rightarrow \infty$ ; it will enable us to estimate, among other things, hitting probabilities for  $X_t$ .

**Lemma 4.4.** *Let  $\alpha \in (0, 1)$  and  $\beta > \alpha$ . Suppose that there exist  $c > 0$ ,  $C < \infty$ , and  $x_0 < \infty$  for which (2.2) holds and, for all  $x \geq x_0$ , (2.3) holds. For  $z \in \mathbb{R}$  and  $\delta > 0$ , define the non-increasing function  $f_{z,\delta} : \mathbb{R} \rightarrow [0, 1]$  by*

$$f_{z,\delta}(y) := \begin{cases} 1 & \text{if } y \leq z \\ (1 + y - z)^{-\delta} & \text{if } y > z \end{cases}. \tag{4.2}$$

Then for any  $\delta \in (0, \beta - \alpha)$  and some  $A > 0$  sufficiently large, for any  $z \in \mathbb{R}$ , a.s.,

$$\mathbb{E}[f_{z,\delta}(X_{t+1}) - f_{z,\delta}(X_t) \mid \mathcal{F}_t] \leq 0, \text{ on } \{X_t > z + A\}.$$

*Proof.* It suffices to suppose that  $z = 1$ . Let  $\delta > 0$ , and let  $f_\delta := f_{1,\delta}$  be as defined at (4.2). Let  $\gamma \in (0, 1)$ ; we will specify  $\delta$  and  $\gamma$  later. Since  $f_\delta$  is non-increasing and  $[0, 1]$ -valued, we have for  $y > 1$  that

$$\begin{aligned} f_\delta(y + \Delta_t) - f_\delta(y) &\leq [(y + \Delta_t^+)^{-\delta} - y^{-\delta}] \mathbb{1}\{\Delta_t^+ \leq y^\gamma\} \\ &\quad + [(y - \Delta_t^-)^{-\delta} - y^{-\delta}] \mathbb{1}\{\Delta_t^- \leq y^\gamma\} + \mathbb{1}\{\Delta_t^- > y^\gamma\}. \end{aligned} \tag{4.3}$$

We will take expectations on both sides of (4.3), conditioning on  $\mathcal{F}_t$  and setting  $y = X_t$ . The final term in (4.3) then becomes, by Markov's inequality and (2.2),

$$\mathbb{P}[\Delta_t^- > X_t^\gamma \mid \mathcal{F}_t] = \mathbb{P}[(\Delta_t^-)^\beta > X_t^{\gamma\beta} \mid \mathcal{F}_t] \leq C X_t^{-\gamma\beta}, \text{ a.s.} \tag{4.4}$$

For the second term on the right-hand side of (4.3), since  $\gamma < 1$ , Taylor's formula implies that, as  $y \rightarrow \infty$ ,

$$[(y - \Delta_t^-)^{-\delta} - y^{-\delta}] \mathbb{1}\{\Delta_t^- \leq y^\gamma\} = \delta(1 + o(1))y^{-1-\delta} \Delta_t^- \mathbb{1}\{\Delta_t^- \leq y^\gamma\}, \tag{4.5}$$

where the  $o(1)$  term is uniform in  $t$  and  $\omega$ . Here we have for the product of the final two terms in (4.5) that

$$\Delta_t^- \mathbb{1}\{\Delta_t^- \leq y^\gamma\} = (\Delta_t^-)^{\beta \wedge 1} (\Delta_t^-)^{(1-\beta)^+} \mathbb{1}\{\Delta_t^- \leq y^\gamma\} \leq (\Delta_t^-)^{\beta \wedge 1} y^{\gamma(1-\beta)^+}. \quad (4.6)$$

Combining (4.5) and (4.6), taking  $y = X_t$  and using (2.2), we obtain that, a.s.,

$$\mathbb{E} \left[ [(X_t - \Delta_t^-)^{-\delta} - X_t^{-\delta}] \mathbb{1}\{\Delta_t^- \leq X_t^\gamma\} \mid \mathcal{F}_t \right] = O(X_t^{-1-\delta+\gamma(1-\beta)^+}), \quad (4.7)$$

on  $\{X_t > 1\}$ , uniformly in  $t$  and  $\omega$ . For the first term on the right-hand side of (4.3), another application of Taylor's formula implies that, as  $y \rightarrow \infty$ ,

$$[(y + \Delta_t^+)^{-\delta} - y^{-\delta}] \mathbb{1}\{\Delta_t^+ \leq y^\gamma\} = -\delta(1 + o(1))y^{-1-\delta} \Delta_t^+ \mathbb{1}\{\Delta_t^+ \leq y^\gamma\}.$$

Setting  $y = X_t$ , taking expectations, and using (2.3) we obtain, for  $X_t$  sufficiently large,

$$\mathbb{E} \left[ [(X_t + \Delta_t^+)^{-\delta} - X_t^{-\delta}] \mathbb{1}\{\Delta_t^+ \leq X_t^\gamma\} \mid \mathcal{F}_t \right] \leq -(c\delta/2)X_t^{-1-\delta+\gamma(1-\alpha)}, \text{ a.s.} \quad (4.8)$$

Thus from (4.3), using the estimates (4.4), (4.7) and (4.8), we verify that  $\mathbb{E}[f_\delta(X_{t+1}) - f_\delta(X_t) \mid \mathcal{F}_t] \leq 0$ , on  $\{X_t > A\}$  for some  $A$  sufficiently large, provided that the negative term arising from (4.8) dominates, i.e.,

$$-1 - \delta + \gamma(1 - \alpha) > -\gamma\beta, \quad \text{and} \quad -1 - \delta + \gamma(1 - \alpha) > -1 - \delta + \gamma(1 - \beta)^+.$$

The second inequality holds since  $\alpha < \beta \wedge 1$ . The first inequality holds provided we choose  $\delta \in (0, \beta - \alpha)$ , which we may do since  $\alpha < \beta$ , and then choose  $\gamma \in (\frac{1+\delta}{1+\beta-\alpha}, 1)$ .  $\square$

**Lemma 4.5.** *Let  $\alpha \in (0, 1)$  and  $\beta > \alpha$ . Suppose that there exist  $C < \infty$ ,  $c > 0$ , and  $x_0 < \infty$  for which  $\mathbb{E}[(\Delta_t^+)^{\alpha} \mid \mathcal{F}_t] \leq C$  a.s. and, for all  $x \geq x_0$ ,  $\mathbb{P}[\Delta_t^- \geq x \mid \mathcal{F}_t] \geq cx^{-\beta}$  a.s. For  $z \in \mathbb{R}$  and  $\delta > 0$ , define  $f_{z,\delta}$  as at (4.2). Then for any  $\delta > \beta - \alpha$  and some  $A > 0$  sufficiently large, for any  $z \in \mathbb{R}$ , a.s.,*

$$\mathbb{E}[f_{z,\delta}(X_{t+1}) - f_{z,\delta}(X_t) \mid \mathcal{F}_t] \geq 0, \text{ on } \{X_t > z + A\}.$$

*Proof.* As in the proof of Lemma 4.4, it suffices to take  $z = 1$ . Let  $\delta > 0$ , and let  $f_\delta := f_{1,\delta}$  be as defined at (4.2). Let  $\gamma \in (0, 1)$ ; we will specify  $\delta$  and  $\gamma$  later. For  $y > 1$  we have

$$\begin{aligned} f_\delta(y + \Delta_t) - f_\delta(y) &\geq [(y + \Delta_t^+)^{-\delta} - y^{-\delta}] \mathbb{1}\{\Delta_t^+ \leq y^\gamma\} \\ &\quad + (1 - y^{-\delta}) \mathbb{1}\{\Delta_t^- \geq y\} - y^{-\delta} \mathbb{1}\{\Delta_t^+ > y^\gamma\}. \end{aligned} \quad (4.9)$$

In (4.9), we will set  $y = X_t$ . We bound the three terms on the right-hand side of (4.9). For the first term, we have that by Taylor's formula, as  $y \rightarrow \infty$ , since  $\gamma < 1$ ,

$$[(y + \Delta_t^+)^{-\delta} - y^{-\delta}] \mathbb{1}\{\Delta_t^+ \leq y^\gamma\} = -\delta(1 + o(1))y^{-1-\delta} \Delta_t^+ \mathbb{1}\{\Delta_t^+ \leq y^\gamma\}, \text{ a.s.,}$$

where, as usual, the  $o(1)$  term is uniform in  $t$  and  $\omega$ . Similarly to (4.6), we have that  $\Delta_t^+ \mathbb{1}\{\Delta_t^+ \leq y^\gamma\} \leq (\Delta_t^+)^{\alpha} y^{(1-\alpha)\gamma}$ , so that

$$|(y + \Delta_t^+)^{-\delta} - y^{-\delta}| \mathbb{1}\{\Delta_t^+ \leq y^\gamma\} = O((\Delta_t^+)^{\alpha} y^{(1-\alpha)\gamma-1-\delta}), \text{ a.s.,}$$

uniformly in  $t$  and  $\omega$ . It follows that, on  $\{X_t > 1\}$ , a.s.,

$$\begin{aligned} \mathbb{E} \left[ [(X_t + \Delta_t^+)^{-\delta} - X_t^{-\delta}] \mathbb{1}\{\Delta_t^+ \leq X_t^\gamma\} \mid \mathcal{F}_t \right] &= O(X_t^{(1-\alpha)\gamma-1-\delta} \mathbb{E}[(\Delta_t^+)^{\alpha} \mid \mathcal{F}_t]) \\ &= O(X_t^{(1-\alpha)\gamma-1-\delta}), \end{aligned} \quad (4.10)$$

uniformly in  $t$  and  $\omega$ . For the second term on the right-hand side of (4.9), we have that for some  $A > 1$  sufficiently large, on  $\{X_t > A\}$ , a.s.,

$$\mathbb{E}[(1 - X_t^{-\delta})\mathbb{1}\{\Delta_t^- \geq X_t\} \mid \mathcal{F}_t] \geq (1/2)\mathbb{P}[\Delta_t^- \geq X_t \mid \mathcal{F}_t] \geq (c/2)X_t^{-\beta}. \tag{4.11}$$

For the third term on the right-hand side of (4.9), we have that, by Markov's inequality,

$$\mathbb{E}[X_t^{-\delta}\mathbb{1}\{\Delta_t^+ > X_t^\gamma\} \mid \mathcal{F}_t] \leq X_t^{-\delta}X_t^{-\alpha\gamma}\mathbb{E}[(\Delta_t^+)^\alpha \mid \mathcal{F}_t] = O(X_t^{-\delta-\alpha\gamma}). \tag{4.12}$$

Combining (4.9) with (4.10), (4.11) and (4.12) we have that on  $\{X_t > A\}$ , a.s.,

$$\mathbb{E}[f_\delta(X_{t+1}) - f_\delta(X_t) \mid \mathcal{F}_t] \geq (c/2)X_t^{-\beta} + O(X_t^{-\delta-\alpha\gamma}) + O(X_t^{(1-\alpha)\gamma-1-\delta}).$$

The positive  $X_t^{-\beta}$  term here dominates for  $A$  large enough provided that

$$-\beta > -\delta - \alpha\gamma \quad \text{and} \quad -\beta > (1 - \alpha)\gamma - 1 - \delta.$$

For any  $\delta > \beta - \alpha$ , the second inequality holds since  $\alpha \in (0, 1)$  and  $\gamma < 1$ . Given any such  $\delta$ , the first inequality holds provided we choose  $\gamma \in (\frac{\beta-\delta}{\alpha}, 1)$ .  $\square$

Our next result deals with a Lyapunov function of a different kind:  $W_t \rightarrow \infty$  as  $X_t \rightarrow -\infty$ . This function will allow us to study, amongst other things, passage-times for  $X_t$ . In particular, Lemma 4.6 will be central to the proofs of Theorems 2.9 and 2.11.

**Lemma 4.6.** *Let  $\alpha \in (0, 1)$  and  $\beta > \alpha$ . Suppose that there exist  $c > 0$ ,  $C < \infty$ , and  $x_0 < \infty$  for which (2.2) holds and (2.3) holds for all  $x \geq x_0$ . For  $\gamma \in (\alpha, \beta)$  and  $y \in \mathbb{R}$  let  $W_t := (y - X_t)^\gamma \mathbb{1}\{X_t < y\}$ . Then the following hold.*

(i) *Take  $\gamma = \beta - \varepsilon$ . Then for any  $\varepsilon \in (0, \frac{\beta(\beta-\alpha)}{1+\beta-\alpha})$  there exists a finite constant  $K$  such that, for all  $t$ ,*

$$\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] \leq K, \text{ a.s.}$$

(ii) *For any  $\eta \in (0, 1 - (\alpha/\beta))$ , we can choose  $x < y$  and  $\gamma \in (\alpha, \beta)$  such that, for some  $\varepsilon > 0$ , for all  $t$ , on  $\{X_t < x\}$ ,*

$$\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] \leq -\varepsilon W_t^\eta, \text{ a.s.}$$

*Proof.* Fix  $y \in \mathbb{R}$  and let  $x < y - 1$ . Also take  $\gamma \in (\alpha, \beta)$  and  $\theta \in (0, 1)$ ; we will make more restrictive specifications for these parameters later. On  $\{X_t < y - 1\}$ , we have  $(y - X_t)^\theta < y - X_t$  and so

$$\begin{aligned} W_{t+1} - W_t &= (y - X_t - \Delta_t)^\gamma \mathbb{1}\{X_{t+1} < y\} - (y - X_t)^\gamma \\ &\leq [(y - X_t - \Delta_t^+)^\gamma - (y - X_t)^\gamma] \mathbb{1}\{\Delta_t^+ \leq (y - X_t)^\theta\} \\ &\quad + [(y - X_t + \Delta_t^-)^\gamma - (y - X_t)^\gamma] \mathbb{1}\{\Delta_t^- \leq (y - X_t)^\theta\} \\ &\quad + (y - X_t + \Delta_t^-)^\gamma \mathbb{1}\{\Delta_t^- \geq (y - X_t)^\theta\}. \end{aligned} \tag{4.13}$$

We bound the three terms on the right-hand side of (4.13) in turn. For the first term, we have from Taylor's formula that, on  $\{X_t < x\}$ ,

$$\begin{aligned} &[(y - X_t - \Delta_t^+)^\gamma - (y - X_t)^\gamma] \mathbb{1}\{\Delta_t^+ \leq (y - X_t)^\theta\} \\ &= -\gamma \Delta_t^+ (y - X_t)^{\gamma-1} (1 + o(1)) \mathbb{1}\{\Delta_t^+ \leq (y - X_t)^\theta\}, \end{aligned}$$

where the  $o(1)$  is uniform in  $t$  and  $\omega$  as  $y - x \rightarrow \infty$ . Hence, taking expectations and using (2.3), it follows that for a fixed  $y$  and any  $x$  for which  $y - x$  is large enough, a.s.,

$$\mathbb{E} \left[ [(y - X_t - \Delta_t^+)^\gamma - (y - X_t)^\gamma] \mathbb{1}\{\Delta_t^+ \leq (y - X_t)^\theta\} \mid \mathcal{F}_t \right] \leq -(c\gamma/2)(y - X_t)^{\gamma-1+\theta(1-\alpha)},$$

on  $\{X_t < x\}$ . For the second term on the right-hand side of (4.13), a similar application of Taylor's formula yields, on  $\{X_t < x\}$ , for  $y - x$  sufficiently large,

$$\begin{aligned} & [(y - X_t + \Delta_t^-)^\gamma - (y - X_t)^\gamma] \mathbb{1}\{\Delta_t^- \leq (y - X_t)^\theta\} \\ & \leq 2\gamma(y - X_t)^{\gamma-1}(\Delta_t^-)^{\beta \wedge 1}(\Delta_t^-)^{(1-\beta)^+} \mathbb{1}\{\Delta_t^- \leq (y - X_t)^\theta\} \\ & \leq 2\gamma(y - X_t)^{\gamma-1+\theta(1-\beta)^+}(\Delta_t^-)^{\beta \wedge 1}. \end{aligned}$$

Taking expectations and using (2.2), we obtain, on  $\{X_t < x\}$ ,

$$\mathbb{E} [ [(y - X_t + \Delta_t^-)^\gamma - (y - X_t)^\gamma] \mathbb{1}\{\Delta_t^- \leq (y - X_t)^\theta\} \mid \mathcal{F}_t ] \leq K(y - X_t)^{\gamma-1+\theta(1-\beta)^+}, \text{ a.s.},$$

for a constant  $K$  not depending on  $t$  or  $\omega$ . For the final term in (4.13), on  $\{X_t < y - 1\}$ ,

$$(y - X_t + \Delta_t^-)^\gamma \mathbb{1}\{\Delta_t^- \geq (y - X_t)^\theta\} \leq ((\Delta_t^-)^{1/\theta} + \Delta_t^-)^\gamma \leq 2^\gamma (\Delta_t^-)^{\gamma/\theta}.$$

Taking  $\gamma = \theta\beta$ , which requires  $\theta \in (\alpha/\beta, 1)$ , and using (2.2), we see that, on  $\{X_t < y - 1\}$ ,

$$\mathbb{E} [(y - X_t + \Delta_t^-)^\gamma \mathbb{1}\{\Delta_t^- \geq (y - X_t)^\theta\} \mid \mathcal{F}_t ] \leq 2^\gamma C, \text{ a.s.}$$

Combining these estimates and taking expectations in (4.13) we see that the negative term dominates asymptotically provided

$$\gamma - 1 + \theta(1 - \alpha) > 0 \text{ and } \gamma - 1 + \theta(1 - \alpha) > \gamma - 1 + \theta(1 - \beta)^+.$$

The first inequality requires  $\theta > 1/(1 + \beta - \alpha)$ , which is a stronger condition than  $\theta > \alpha/\beta$  that we had already imposed, but which can be achieved with  $\theta \in (\alpha/\beta, 1)$  since  $\alpha < \beta$ . The second inequality reduces to  $1 - \alpha > (1 - \beta)^+$  which is satisfied since  $\alpha < \beta \wedge 1$ . Part (i) follows. Moreover, for  $\gamma = \theta\beta$ ,  $1/(1 + \beta - \alpha) < \theta < 1$ , we can take  $y - x$  large enough so that, for some  $\varepsilon > 0$ , on  $\{X_t < x\}$ ,

$$\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] \leq -\varepsilon(y - X_t)^{\theta\beta-1+\theta(1-\alpha)} = -\varepsilon W_t^\eta, \text{ a.s.},$$

where  $\eta = (\theta\beta - 1 + \theta(1 - \alpha))/(\theta\beta)$  can be anywhere in  $(0, 1 - (\alpha/\beta))$ , by appropriate choice of  $\theta$ , which proves part (ii).  $\square$

Lemma 4.6 has as a consequence the following tail bound, which is essentially a large deviations result of the same kind as (but much more general than) those obtained in [15] for the case  $X_t = S_t$ , a sum of i.i.d. *nonnegative* random variables; indeed, the results in [15] show that Lemma 4.7 is close to best possible.

**Lemma 4.7.** *Let  $\alpha \in (0, 1)$  and  $\beta > \alpha$ . Suppose that there exist  $c > 0$ ,  $C < \infty$ , and  $x_0 < \infty$  for which (2.2) holds and (2.3) holds for all  $x \geq x_0$ . Then for any  $\phi > 0$  and any  $\varepsilon > 0$ , as  $t \rightarrow \infty$ ,*

$$\mathbb{P} \left[ \min_{0 \leq s \leq t} X_s \leq -t^\phi \right] = O(t^{1-\beta\phi+\varepsilon}).$$

*Proof.* As in Lemma 4.6, choosing  $y = 0$  there, let  $W_t = (-X_t)^\gamma \mathbb{1}\{X_t < 0\}$ . For  $t > 0$ ,

$$\mathbb{P} \left[ \min_{0 \leq s \leq t} X_s \leq -t^\phi \right] \leq \mathbb{P} \left[ \max_{0 \leq s \leq t} W_s \geq t^{\phi\gamma} \right].$$

Take  $\gamma = \beta - \varepsilon$  for  $\varepsilon \in (0, \frac{\beta(\beta-\alpha)}{1+\beta-\alpha})$ . Then by Lemma 4.6(i) and Lemma 4.3 with  $\nu = t$  (or [29, Lemma 3.1]),  $\mathbb{P} [\max_{0 \leq s \leq t} W_s \geq t^{\phi\gamma}] = O(t^{1-\phi\gamma})$ , which implies the result.  $\square$

The next result gives a general condition for obtaining almost-sure upper bounds.

**Lemma 4.8.** *Let  $h : [0, \infty) \rightarrow [0, \infty)$  be increasing and concave. Suppose that there exists  $C < \infty$  such that  $\mathbb{E}[h(\Delta_t^+) | \mathcal{F}_t] \leq C$ , a.s. Then for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t \in \mathbb{Z}^+$ ,*

$$X_t \leq \sum_{s=0}^{t-1} \Delta_s^+ \leq h^{-1}(t(\log t)^{1+\varepsilon}).$$

*Proof.* Set  $Y_0 := 0$  and for  $t \in \mathbb{N}$  let  $Y_t := \sum_{s=0}^{t-1} \Delta_s^+$ . Then  $Y_t \geq 0$  is non-decreasing and  $X_t \leq X_0 + Y_t = Y_t$ , since  $X_0 = 0$ . Since  $h$  is nonnegative and concave, it is subadditive, i.e.,  $h(a + b) \leq h(a) + h(b)$  for  $a, b \in [0, \infty)$ . Hence

$$\mathbb{E}[h(Y_t + \Delta_t^+) - h(Y_t) | \mathcal{F}_t] \leq \mathbb{E}[h(\Delta_t^+) | \mathcal{F}_t] \leq C, \text{ a.s.}, \tag{4.14}$$

by hypothesis. The almost-sure upper bound in Lemma 4.2 to  $Z_t = h(Y_t)$  implies that, for any  $\varepsilon > 0$ , a.s.,  $h(Y_t) \leq t(\log t)^{1+\varepsilon}$ , for all but finitely many  $t$ . Since  $h$  is increasing and  $X_t \leq Y_t$ , it follows that for any  $\varepsilon > 0$ , a.s.,  $h(X_t) \leq t(\log t)^{1+\varepsilon}$ .  $\square$

Finally, we need a result on the maxima of the increments of  $X_t$ .

**Lemma 4.9.** *Suppose that for some  $\alpha \in (0, \infty)$ ,  $c > 0$ , and  $x_0 < \infty$ , for all  $x \geq x_0$ , (2.6) holds. Then for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t \in \mathbb{Z}^+$ ,*

$$\max_{0 \leq s \leq t} \Delta_s^+ \geq t^{1/\alpha}(\log t)^{-(1/\alpha)-\varepsilon}.$$

*Proof.* By a telescoping conditioning argument, for  $x > 0$ ,

$$\mathbb{P} \left[ \max_{0 \leq s \leq t} \Delta_s^+ < x \right] = \mathbb{E} \left[ \mathbb{1}\{\Delta_0^+ < x\} \cdots \mathbb{E} \left[ \mathbb{1}\{\Delta_{t-1}^+ < x\} \mathbb{E} \left[ \mathbb{1}\{\Delta_t^+ < x\} | \mathcal{F}_t \right] | \mathcal{F}_{t-1} \right] \cdots | \mathcal{F}_0 \right].$$

Hence for any  $x \geq x_0$ , by repeated applications of (2.6),

$$\mathbb{P} \left[ \max_{0 \leq s \leq t} \Delta_s^+ < x \right] \leq \prod_{s=0}^t (1 - cx^{-\alpha}) \leq (1 - cx^{-\alpha})^t. \tag{4.15}$$

Taking  $x = t^{1/\alpha}(\log t)^q$  in (4.15) we obtain, for  $t$  sufficiently large,

$$\mathbb{P} \left[ \max_{0 \leq s \leq t} \Delta_s^+ < t^{1/\alpha}(\log t)^q \right] \leq (1 - ct^{-1}(\log t)^{-\alpha q})^t = O(\exp(-c(\log t)^{-\alpha q})),$$

which is summable over  $t \geq 2$  provided  $q < -1/\alpha$ . Hence the Borel–Cantelli lemma completes the proof.  $\square$

#### 4.4 Proofs of results in Section 2

First we give the proof of Proposition 2.1.

*Proof of Proposition 2.1.* We claim that under any of the conditions in the proposition, it is the case that for any  $y \geq 0$  there exists  $\delta(y) > 0$  for which, for all  $t$ ,

$$\mathbb{P}[|\Delta_t| > y | \mathcal{F}_t] \geq \delta(y), \text{ a.s.} \tag{4.16}$$

Given (4.16), for any  $B < \infty$ ,  $\mathbb{P}[|X_{t+1}| > 2B | \mathcal{F}_t] \geq \delta(3B)$ , a.s., on  $\{|X_t| \leq B\}$ . Suppose that  $\limsup_{t \rightarrow \infty} |X_t| = B < \infty$ . But then,  $\sum_t \mathbb{P}[|X_{t+1}| > 2B | \mathcal{F}_t] = \infty$  a.s., which leads to a contradiction by Lévy’s extension of the Borel–Cantelli lemma (see e.g. [17, Corollary 7.20]), and (2.1) is proved.

It remains to verify (4.16). Since  $|\Delta_t| = \Delta_t^+ + \Delta_t^-$ , it suffices to verify (4.16) with one of  $\Delta_t^+$  or  $\Delta_t^-$  in place of  $|\Delta_t|$ . In the case where, say,  $\mathbb{P}[\Delta_t^+ > x | \mathcal{F}_t] \geq cx^{-\gamma}$ , a.s.,



for  $x \geq x_0$  (condition (i) in the statement of the proposition), the claim is immediate. So suppose that  $\mathbb{E}[\Delta_t^+ \mathbf{1}\{\Delta_t^+ \leq x\} \mid \mathcal{F}_t] \geq cx^{1-\gamma}$ , a.s., for  $x \geq x_0$  (condition (ii)). Then, for any  $y \geq 0$ , for  $x > y$ ,

$$\mathbb{E}[\Delta_t^+ \mathbf{1}\{y \leq \Delta_t^+ \leq x\} \mid \mathcal{F}_t] \geq \mathbb{E}[\Delta_t^+ \mathbf{1}\{\Delta_t^+ \leq x\} \mid \mathcal{F}_t] - y > 1, \text{ a.s.,}$$

provided  $x > x_0 + ((1 + y)/c)^{1/(1-\gamma)}$ , say. Then, a.s.,

$$1 < \mathbb{E}[\Delta_t^+ \mathbf{1}\{y \leq \Delta_t^+ \leq x\} \mid \mathcal{F}_t] \leq x\mathbb{P}[y \leq \Delta_t^+ \leq x \mid \mathcal{F}_t] \leq x\mathbb{P}[\Delta_t^+ \geq y \mid \mathcal{F}_t],$$

which implies (4.16) in this case also. □

Next, in the proof of Theorem 2.2, we use the Lyapunov function  $f_{z,\delta}$  defined at (4.2) to estimate hitting probabilities for  $X_t$ .

*Proof of Theorem 2.2.* First we show that, under the conditions of the theorem,

$$\mathbb{P} \left[ \liminf_{t \rightarrow \infty} X_t = -\infty \right] = 0. \tag{4.17}$$

Let  $a > 0$ , to be chosen later. For  $x \in \mathbb{R}$ , set

$$\nu_x := \min\{t \in \mathbb{Z}^+ : X_t > x + a\}; \quad \eta_x := \min\{t \geq \nu_x : X_t \leq x\}.$$

In particular, since  $X_0 = 0$ , we have that  $\nu_x = 0$  for all  $x < -a$ .

Let  $\delta \in (0, \beta - \alpha)$ . Then Lemma 4.4 shows that, on  $\{\nu_x < \infty\}$ ,  $(f_{x-A,\delta}(X_{t \wedge \eta_x}))_{t \geq \nu_x}$  is a nonnegative supermartingale adapted to  $(\mathcal{F}_t)_{t \geq \nu_x}$ , and so converges a.s. as  $t \rightarrow \infty$  to a finite limit,  $L_x$ , say. On  $\{\nu_x < \infty\}$ , we have by the supermartingale property that

$$\mathbb{E}[f_{x-A,\delta}(X_{t \wedge \nu_x}) \mid \mathcal{F}_{\nu_x}] \leq f_{x-A,\delta}(X_{\nu_x}) \leq (1 + A + a)^{-\delta}, \text{ a.s.,}$$

while by Fatou's lemma, also on  $\{\nu_x < \infty\}$ ,

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}[f_{x-A,\delta}(X_{t \wedge \eta_x}) \mid \mathcal{F}_{\nu_x}] &\geq \mathbb{E}[L_x \mid \mathcal{F}_{\nu_x}] \\ &\geq \mathbb{E}[L_x \mathbf{1}\{\eta_x < \infty\} \mid \mathcal{F}_{\nu_x}] \\ &\geq (1 + A)^{-\delta} \mathbb{P}[\eta_x < \infty \mid \mathcal{F}_{\nu_x}], \end{aligned}$$

since, on  $\{\eta_x < \infty\}$ ,  $X_{t \wedge \eta_x} \leq x$  for all  $t$  sufficiently large. So on  $\{\nu_x < \infty\}$  we have, a.s.,

$$\mathbb{P}[\eta_x < \infty \mid \mathcal{F}_{\nu_x}] \leq \left( \frac{1 + A + a}{1 + A} \right)^{-\delta}.$$

Let  $\varepsilon > 0$ . Then we can take  $a$  sufficiently large so that  $\mathbb{P}[\eta_x = \infty \mid \mathcal{F}_{\nu_x}] \geq 1 - \varepsilon$ , a.s., on  $\{\nu_x < \infty\}$ . For such a choice of  $a$ , suppose that  $x < -a$ ; then  $\nu_x < \infty$  a.s. (indeed, since  $X_0 = 0$ ,  $\nu_x = 0$  a.s.). Hence for such an  $x$ ,

$$\mathbb{P} \left[ \liminf_{t \rightarrow \infty} X_t > x \right] = \mathbb{P}[\eta_x = \infty] \geq \mathbb{E}[\mathbb{P}[\eta_x = \infty \mid \mathcal{F}_{\nu_x}] \mathbf{1}\{\nu_x < \infty\}] \geq 1 - \varepsilon. \tag{4.18}$$

It follows from (4.18) that

$$\mathbb{P} \left[ \liminf_{t \rightarrow \infty} X_t = -\infty \right] \leq \mathbb{P} \left[ \liminf_{t \rightarrow \infty} X_t \leq -a - 1 \right] \leq \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, (4.17) follows.

Proposition 2.1 applies under condition (2.3). Hence, together with (2.1), (4.17) implies that, a.s.,  $\limsup_{t \rightarrow \infty} X_t = \infty$ ; in other words, for any  $a > 0$  and any  $x \in \mathbb{R}$ ,  $\nu_x < \infty$  a.s. Hence the argument for (4.18) extends to any  $x \in \mathbb{R}$ , which implies that for any  $x \in \mathbb{R}$ , a.s.,  $\liminf_{t \rightarrow \infty} X_t > x$ , so  $X_t \rightarrow \infty$  a.s. □

Next we give the proofs of Theorems 2.4 and 2.6, based on the almost-sure bounds given in Lemmas 4.8 and 4.9.

*Proof of Theorem 2.4.* First we prove part (i), so let  $\theta \in (0, 1)$ . For  $\varepsilon > 0$ , take  $h(x) = (K + x)^\theta (\log(K + x))^{-\phi-1-\varepsilon}$ . For a large enough choice of  $K \geq 1$ ,  $h$  is nonnegative, increasing, and concave. Moreover,  $\mathbb{E}[h(\Delta_t^+) \mid \mathcal{F}_t]$  is uniformly bounded provided  $\sum_{k=1}^\infty h'(k) \mathbb{P}[\Delta_t^+ > k \mid \mathcal{F}_t]$  is uniformly bounded; see e.g. [14, p. 76]. This is indeed the case under the hypothesis of the theorem, by (2.4), since  $h'(x) = O(x^{\theta-1} (\log x)^{-\phi-1-\varepsilon})$ . Now (i) follows from Lemma 4.8, noting that  $h^{-1}(x) = O(x^{1/\theta} (\log x)^{\frac{\phi+1}{\theta}+\varepsilon})$ . The proof of (ii) is similar, this time taking  $h(x) = (K + x)(\log(K + x))^{-(\phi+1)^+-\varepsilon}$ .  $\square$

*Proof of Theorem 2.6.* Let  $\varepsilon > 0$ . Lemma 4.8 applied to  $-X_t$  with  $h(x) = x^{\beta \wedge 1}$ , using (2.2), shows that, a.s., for all but finitely many  $t$ ,

$$\sum_{s=0}^{t-1} \Delta_s^- \leq t^{1/(\beta \wedge 1)} (\log t)^{(1/(\beta \wedge 1))+\varepsilon}.$$

On the other hand, Lemma 4.9 implies that, a.s., for all but finitely many  $t$ ,

$$\sum_{s=0}^{t-1} \Delta_s^+ \geq \max_{0 \leq s \leq t-1} \Delta_s^+ \geq t^{1/\alpha} (\log t)^{-(1/\alpha)-\varepsilon},$$

Combining these bounds and using the fact that  $\alpha < \beta \wedge 1$  we complete the proof.  $\square$

Now we turn to the proofs of our results on first passage times. First we prove Theorem 2.9, which uses the Lyapunov function  $W_t$  given in Lemma 4.6, together with the general criterion Lemma 4.1.

*Proof of Theorem 2.9.* Define  $W_t = (y - X_t)^\gamma \mathbb{1}\{X_t < y\}$  as in Lemma 4.6. For  $z > 0$ , let  $\sigma_z = \min\{t \in \mathbb{Z}^+ : W_t \leq z\}$ . Since  $\{W_t \leq z\} = \{X_t \geq y - z^{1/\gamma}\}$ , we have with  $\tau_x$  as defined by (2.7) that  $\tau_x = \sigma_{(y-x)^\gamma}$  for  $x \leq y$ . Now fix  $x \in \mathbb{R}$ . Under the conditions of the theorem, Lemma 4.6(ii) implies that, for any  $\eta \in (0, 1 - (\alpha/\beta))$ , for  $y > x$  sufficiently large, on  $\{t < \sigma_{(y-x)^\gamma}\}$ , a.s.,

$$\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] \leq -\varepsilon W_t^\eta.$$

Then Lemma 4.1 shows that for any  $x \in \mathbb{R}$ ,  $\mathbb{E}[\tau_x^p] = \mathbb{E}[\sigma_{(y-x)^\gamma}^p] < \infty$ , for any  $p < \beta/\alpha$ .  $\square$

Next we prove our non-existence of moments result for  $\tau_x$ . General semimartingale analogues of Lemma 4.1 are available for non-existence results (see e.g. [2]) but typically require strong control (such as uniform boundedness) of the increments of the process. Thus we use a different idea, based on Lemma 4.3: roughly speaking, we show that with good probability  $X_t$  travels a long way in the negative direction with a single heavy-tailed jump, and then must take a long time to come back.

*Proof of Theorem 2.10.* Fix  $x > 0$  and let  $y < x$ . Let  $W'_t = (X_t - y)^\alpha \mathbb{1}\{X_t > y\}$ . Then on  $\{X_t \leq y\}$ ,  $W'_{t+1} - W'_t \leq (\Delta_t^+)^\alpha$ . On the other hand, on  $\{X_t > y\}$ ,

$$W'_{t+1} - W'_t \leq (X_t + \Delta_t^+ - y)^\alpha - (X_t - y)^\alpha \leq (\Delta_t^+)^\alpha,$$

by concavity since  $\alpha \in (0, 1]$ . Hence for  $C < \infty$  (not depending on  $y$ ),  $\mathbb{E}[W'_{t+1} - W'_t \mid \mathcal{F}_t] \leq C$ , a.s., so the maximal inequality (4.1) implies that, for any  $y < x$ ,

$$\mathbb{P} \left[ \max_{0 \leq r \leq s} W'_{t+r} \geq (x - y)^\alpha \mid \mathcal{F}_t \right] \leq \frac{Cs + W'_t}{(x - y)^\alpha}, \text{ a.s.}$$

In particular, on  $\{X_t \leq y\}$ ,  $W'_t = 0$  and so

$$\mathbb{P} \left[ \max_{0 \leq r \leq s} X_{t+r} \geq x \mid \mathcal{F}_t \right] \leq \mathbb{P} \left[ \max_{0 \leq r \leq s} W'_{t+r} \geq (x - y)^\alpha \mid \mathcal{F}_t \right] \leq \frac{Cs}{(x - y)^\alpha}, \text{ a.s.}$$

Setting  $s = (x - y)^\alpha / (2C)$  in the last display, we obtain that for some  $\varepsilon > 0$  (not depending on  $x$  or  $y$ ), on  $\{t < \tau_x\} \cap \{X_t \leq y\}$ , for any  $y < x$ ,

$$\mathbb{P}[\tau_x \geq \varepsilon(x - y)^\alpha \mid \mathcal{F}_t] \geq 1/2, \text{ a.s.} \tag{4.19}$$

Since  $X_0 = 0$  and  $x > 0$ , we have that  $\{\Delta_0^- > y^-\}$  implies  $\{\tau_x > 1\}$  and  $\{X_1 \leq y\}$ . So applying (4.19) at  $t = 1$  we have that

$$\mathbb{P}[\tau_x \geq \varepsilon(x - y)^\alpha] \geq \mathbb{E} \left[ \mathbb{1}\{\Delta_0^- > y^-\} \mathbb{P}[\tau_x \geq \varepsilon(x - y)^\alpha \mid \mathcal{F}_1] \right] \geq \frac{1}{2} \mathbb{P}[\Delta_0^- > y^-].$$

Taking  $y = -\varepsilon^{-1/\alpha} z^{1/\alpha} < 0$ , we have that for any  $z > 0$ ,

$$\mathbb{P}[\tau_x \geq z] \geq \mathbb{P}[\tau_x \geq \varepsilon(x - y)^\alpha] \geq \frac{1}{2} \mathbb{P}[\Delta_0^- > \varepsilon^{-1/\alpha} z^{1/\alpha}].$$

Hence for any  $\gamma > 0$ ,

$$\mathbb{E}[\tau_x^\gamma] = \int_0^\infty \mathbb{P}[\tau_x > z^{1/\gamma}] dz \geq \frac{1}{2} \int_0^\infty \mathbb{P}[\Delta_0^- > \varepsilon^{-1/\alpha} z^{1/(\alpha\gamma)}] dz.$$

Using the substitution  $w = \varepsilon^{-\gamma} z$  we obtain

$$\mathbb{E}[\tau_x^\gamma] \geq \frac{1}{2} \varepsilon^\gamma \int_0^\infty \mathbb{P}[\Delta_0^- > w^{1/(\alpha\gamma)}] dw = \frac{1}{2} \varepsilon^\gamma \mathbb{E}[(\Delta_0^-)^{\alpha\gamma}],$$

which is infinite provided  $\alpha\gamma \geq \beta$ , i.e.,  $\gamma \geq \beta/\alpha$ . □

The final two proofs for this section concern our results on last exit times.

*Proof of Theorem 2.11.* Recall the definition of  $\tau_x$  and  $\lambda_x$  from (2.7) and (2.8) respectively. Fix  $x \in \mathbb{R}$  and let  $y > x$ , to be specified later. For this proof, define the stopping time  $\eta_{y,x} := \min\{t \geq \tau_y : X_t \leq x\}$ , the time of reaching  $(-\infty, x]$  after having first reached  $[y, \infty)$ . To prove our result on finiteness of moments for  $\lambda_x$ , we prove an upper tail bound for  $\lambda_x$ . For  $y > x$ ,  $\{\tau_y \leq t\} \cap \{\eta_{y,x} = \infty\}$  implies  $\{\lambda_x \leq t\}$ , so

$$\mathbb{P}[\lambda_x > t] \leq \mathbb{P}[\eta_{y,x} < \infty] + \mathbb{P}[\tau_y > t]. \tag{4.20}$$

We obtain an upper bound for  $\mathbb{P}[\eta_{y,x} < \infty]$ . Under the conditions of the theorem, Lemma 4.4 applies. It follows that for  $\delta \in (0, \beta - \alpha)$ , on  $\{\tau_y < \infty\}$ ,  $(f_{x-A,\delta}(X_{t \wedge \eta_{y,x}}))_{t \geq \tau_y}$  is a nonnegative supermartingale adapted to  $(\mathcal{F}_t)_{t \geq \tau_y}$ , and hence converges a.s. as  $t \rightarrow \infty$  to a limit,  $L_{y,x}$ , say. Then, on  $\{\tau_y < \infty\}$ , by Fatou's lemma,

$$\begin{aligned} f_{x-A,\delta}(X_{\tau_y}) &\geq \mathbb{E}[L_{y,x} \mid \mathcal{F}_{\tau_y}] \geq \mathbb{E}[f_{x-A,\delta}(X_{\eta_{y,x}}) \mathbb{1}\{\eta_{y,x} < \infty\} \mid \mathcal{F}_{\tau_y}] \\ &\geq (1 + A)^{-\delta} \mathbb{P}[\eta_{y,x} < \infty \mid \mathcal{F}_{\tau_y}]. \end{aligned}$$

By definition, on  $\{\tau_y < \infty\}$ ,  $X_{\tau_y} \geq y$ , so  $f_{x-A,\delta}(X_{\tau_y}) \leq (1 + A + y - x)^{-\delta}$ . Hence,

$$\mathbb{P}[\eta_{y,x} < \infty] = \mathbb{E}[\mathbb{P}[\eta_{y,x} < \infty \mid \mathcal{F}_{\tau_y}] \mathbb{1}\{\tau_y < \infty\}] = O(y^{-\delta}). \tag{4.21}$$

For the final term in (4.20), for  $y > 0$ ,  $\mathbb{P}[\tau_y > t] = \mathbb{P}[\max_{0 \leq s \leq t} X_s < y]$ , where

$$\begin{aligned} \mathbb{P} \left[ \max_{0 \leq s \leq t} X_s < y \right] &\leq \mathbb{P} \left[ \max_{0 \leq s \leq t} X_s \leq y, \min_{0 \leq s \leq t} X_s \geq -y \right] + \mathbb{P} \left[ \min_{0 \leq s \leq t} X_s \leq -y \right] \\ &\leq \mathbb{P} \left[ \max_{0 \leq s \leq t-1} \Delta_s^+ \leq 2y \right] + \mathbb{P} \left[ \min_{0 \leq s \leq t} X_s \leq -y \right]. \end{aligned} \tag{4.22}$$

We choose  $y = t^{(1/\alpha)-\varepsilon}$ , for  $\varepsilon \in (0, 1/\alpha)$ . Then we have from (4.15) that for  $c' > 0$ ,

$$\mathbb{P} \left[ \max_{0 \leq s \leq t-1} \Delta_s^+ \leq 2t^{(1/\alpha)-\varepsilon} \right] = O(\exp\{-c't^{\alpha\varepsilon}\}). \tag{4.23}$$

On the other hand, the  $\phi = (1/\alpha) - \varepsilon$  case of Lemma 4.7 implies that

$$\mathbb{P} \left[ \min_{0 \leq s \leq t} X_s \leq -t^{(1/\alpha)-\varepsilon} \right] = O(t^{1-(\beta/\alpha)+\varepsilon}). \tag{4.24}$$

Using the bounds (4.23) and (4.24) in the  $y = t^{(1/\alpha)-\varepsilon}$  case of (4.22), we obtain

$$\mathbb{P} \left[ \max_{0 \leq s \leq t} X_s < t^{(1/\alpha)-\varepsilon} \right] = O(t^{1-(\beta/\alpha)+(\beta+1)\varepsilon}). \tag{4.25}$$

Thus taking  $y = t^{(1/\alpha)-\varepsilon}$  in (4.20) and  $\delta$  as close as we wish to  $\beta - \alpha$ , and combining (4.21) with (4.25), we conclude that, for any  $\varepsilon > 0$ ,  $\mathbb{P}[\lambda_x > t] = O(t^{1-(\beta/\alpha)+\varepsilon})$ , which yields the claimed moment bounds.  $\square$

*Proof of Theorem 2.12.* Fix  $x \in \mathbb{R}$  and let  $y > x$ . For this proof, define  $\nu_{t,x} := \min\{s \geq t : X_s \leq x\}$ , the first time of reaching  $(-\infty, x]$  after time  $t$ . Similarly, set  $\tau_{t,y} := \min\{s \geq t : X_s \geq y\}$ . We have that, for  $r > 0$ ,

$$\mathbb{P}[\lambda_x > t] \geq \mathbb{E} [\mathbf{1}\{X_t \leq r\} \mathbb{P}[\nu_{t,x} < \infty \mid \mathcal{F}_t]]. \tag{4.26}$$

Under the conditions of the theorem, Lemma 4.5 applies. It follows that for  $\delta > \beta - \alpha$ ,  $(f_{x-A,\delta}(X_{s \wedge \nu_{t,x} \wedge \tau_{t,y}}))_{s \geq t}$  is a nonnegative submartingale adapted to  $(\mathcal{F}_s)_{s \geq t}$ ; moreover, it is uniformly bounded and so converges a.s. and in  $L^1$ , as  $s \rightarrow \infty$ , to the limit  $f_{x-A,\delta}(X_{\nu_{t,x} \wedge \tau_{t,y}})$ , since  $\nu_{t,x} \wedge \tau_{t,y} < \infty$  a.s., by (2.1), which is available since Proposition 2.1 applies under the conditions of the theorem. Hence, a.s.,

$$f_{x-A,\delta}(X_t) \leq \mathbb{E}[f_{x-A,\delta}(X_{\nu_{t,x} \wedge \tau_{t,y}}) \mid \mathcal{F}_t] \leq \mathbb{P}[\nu_{t,x} < \infty \mid \mathcal{F}_t] + f_{x-A,\delta}(y).$$

Since  $y$  was arbitrary, and  $f_{x-A,\delta}(y) \rightarrow 0$  as  $y \rightarrow \infty$ , it follows that, a.s.,

$$\mathbb{P}[\nu_{t,x} < \infty \mid \mathcal{F}_t] \geq f_{x-A,\delta}(X_t) \geq f_{x-A,\delta}(r),$$

on  $\{X_t \leq r\}$ . Hence from (4.26) we obtain for  $r \geq x$ ,

$$\mathbb{P}[\lambda_x > t] \geq f_{x-A,\delta}(r) \mathbb{P}[X_t \leq r] \geq (1 + A + r - x)^{-\delta} \mathbb{P}[X_t \leq r]. \tag{4.27}$$

It remains to obtain a lower bound for  $\mathbb{P}[X_t \leq r]$ , for a suitable choice of  $r$ . Let  $Y_t = \sum_{s=0}^{t-1} \Delta_s^+$ . Following the argument for (4.14), with  $h(y) = y^\alpha$ ,  $\alpha \in (0, 1]$ , we may apply Lemma 4.3 with  $\nu = t$  (or [29, Lemma 3.1]) to  $Z_t = Y_t^\alpha$  to obtain

$$\mathbb{P} \left[ \max_{0 \leq s \leq t} Y_s^\alpha \geq x \right] = \mathbb{P}[Y_t \geq x^{1/\alpha}] \leq Ctx^{-1},$$

for some  $C < \infty$  and all  $t \in \mathbb{Z}^+$ ,  $x > 0$ , which implies that

$$\mathbb{P} \left[ X_t \leq (2Ct)^{1/\alpha} \right] \geq \mathbb{P} \left[ Y_t \leq (2Ct)^{1/\alpha} \right] \geq 1/2,$$

since  $X_t \leq X_0 + Y_t = Y_t$ . Thus taking  $r = (2Ct)^{1/\alpha}$ , we have  $\mathbb{P}[X_t \leq r] \geq 1/2$ , and with this choice of  $r$  in (4.27) we obtain  $\mathbb{P}[\lambda_x > t] \geq \varepsilon t^{-\delta/\alpha}$ , for some  $\varepsilon > 0$  and all  $t$  sufficiently large. Since  $\delta > \beta - \alpha$  was arbitrary, the result follows.  $\square$

## 5 Proofs for Section 3

### 5.1 Overview

In this section we first prove our results from Section 3.2, from which the results on the strip model given in Section 3.1 will follow. For our results from Section 3.2 on the random walk  $Y_t$  with a distinguished subset  $\mathcal{C}$  of the state-space, we use two related but different proof ideas. We prove Theorem 3.7 in Section 5.3 by an explicit use of the embedded process  $X_t = Y_{\sigma_t}$ , which observes the process at successive visits to  $\mathcal{C}$ . We give estimates on the tails of the increments of  $X_t$  given our assumptions on the tails of the increments of  $Y_t$ , and then apply the one-dimensional results of Section 2 to  $X_t$ ; a small additional amount of work is then needed to recover the result for  $Y_t$  itself. In contrast, in Section 5.2 we give the proofs of Theorems 3.8 and 3.9, which work directly with the process  $Y_t$ , but again make repeated use of the results from Section 2, not only for analysing the random walk but for estimating the almost-sure growth rate of  $\sigma_n$  as well. Finally, in Section 5.4, we derive the results on the strip model of Section 3.1.

### 5.2 Proofs of Theorems 3.8 and 3.9

We recall some notation introduced in Section 3.2. The stochastic process  $Y_t$  has state space  $\mathcal{S}$  and increments  $D_t = Y_{t+1} - Y_t$ . The successive hitting times of  $\mathcal{C} \subset \mathcal{S}$  are  $\sigma_0 = 0, \sigma_1, \sigma_2, \dots$ , and  $\nu_n = \sigma_{n+1} - \sigma_n$ . We write  $\mathcal{G}_n = \sigma(Y_0, \dots, Y_n)$ . To start this section we give some preparatory results on the hitting times  $\sigma_n = \sum_{i=1}^{n-1} \nu_i$ .

**Lemma 5.1.** *Suppose that (C1) holds.*

- (i) *Suppose that for some  $\gamma > 0$  and  $C < \infty$ ,  $\mathbb{E}[\nu_n^\gamma \mid \mathcal{G}_{\sigma_n}] \leq C$  a.s. for all  $n$ . Then for any  $\varepsilon > 0$ , a.s., for all but finitely many  $n$ ,  $\sigma_n \leq n^{(1/(\gamma \wedge 1)) + \varepsilon}$ .*
- (ii) *Suppose that for some  $\gamma \in (0, 1]$ ,  $y_0 < \infty$ , and  $c > 0$ , for all  $y \geq y_0$ ,  $\mathbb{P}[\nu_n \geq y \mid \mathcal{G}_{\sigma_n}] \geq cy^{-\gamma}$  a.s. for all  $n$ . Then for any  $\varepsilon > 0$ , a.s., for all but finitely many  $n$ ,  $\sigma_n \geq n^{(1/\gamma) - \varepsilon}$ .*

*Proof.* For part (i), Markov's inequality yields  $\mathbb{P}[\nu_n \geq y \mid \mathcal{G}_{\sigma_n}] = O(y^{-\gamma})$ , uniformly in  $n$  and  $\omega$ . Now apply Theorem 2.4 with  $X_t = \sigma_t$ ,  $\Delta_t = \Delta_t^+ = \nu_t$ ,  $\mathcal{F}_t = \mathcal{G}_{\sigma_t}$ ,  $\theta = \gamma \wedge 1$ , and  $\phi = 0$ . For part (ii), apply Theorem 2.6, noting that  $\Delta_t^- = 0$  a.s. since  $\nu_t \geq 0$  a.s.  $\square$

Denote the number of visits to  $\mathcal{C}$  by time  $t$  by

$$N(t) := \max\{n \in \mathbb{Z}^+ : \sigma_n \leq t\}. \tag{5.1}$$

An inversion of Lemma 5.1 yields the following result.

**Lemma 5.2.** *Suppose that (C1) holds.*

- (i) *Suppose that for some  $\gamma > 0$  and  $C < \infty$ ,  $\mathbb{E}[\nu_n^\gamma \mid \mathcal{G}_{\sigma_n}] \leq C$  a.s. for all  $n$ . Then for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $N(t) \geq t^{\gamma \wedge 1 - \varepsilon}$ .*
- (ii) *Suppose that for some  $\gamma \in (0, 1]$ ,  $y_0 < \infty$ , and  $c > 0$ , for all  $y \geq y_0$ ,  $\mathbb{P}[\nu_n \geq y \mid \mathcal{G}_{\sigma_n}] \geq cy^{-\gamma}$  a.s. for all  $n$ . Then for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $N(t) \leq t^{\gamma + \varepsilon}$ .*

*Proof.* Since  $\sigma_n < \infty$  a.s., we have  $N(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , a.s. Also note that, by definition of  $N(t)$ ,  $\sigma_{N(t)} \leq t$  but  $\sigma_{N(t)+1} > t$ . Thus under the conditions of part (i) we have that for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,

$$t < \sigma_{N(t)+1} \leq (N(t) + 1)^{\frac{1}{\gamma \wedge 1} + \varepsilon},$$

by Lemma 5.1(i), which yields part (i). On the other hand, under the conditions of part (ii), for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,

$$t \geq \sigma_{N(t)} \geq N(t)^{(1/\gamma) - \varepsilon},$$

by Lemma 5.1(ii), which yields part (ii).  $\square$

Now we can prove Theorems 3.8 and 3.9, starting with the former.

*Proof of Theorem 3.8.* By (C3), for any  $\varepsilon > 0$ , there exists  $y_0 < \infty$  such that, a.s., for all  $y \geq y_0$ ,

$$y^{-\gamma-\varepsilon} \leq \mathbb{P}[\nu_n \geq y \mid \mathcal{G}_{\sigma_n}] \leq y^{-\gamma+\varepsilon}, \tag{5.2}$$

uniformly in  $n$ . The upper bound in (5.2) in turn implies that, for  $p > 0$ , for any  $\varepsilon > 0$ ,

$$\mathbb{E}[\nu_n^p \mid \mathcal{G}_{\sigma_n}] = \int_0^\infty \mathbb{P}[\nu_n > y^{1/p} \mid \mathcal{G}_{\sigma_n}] dy \leq y_0 + \int_{y_0}^\infty y^{-(\gamma-\varepsilon)/p} dy, \text{ a.s.}, \tag{5.3}$$

which is bounded uniformly in  $n$  and  $\omega$  provided  $p < \gamma - \varepsilon$ . First we prove the lower bound for  $Y_t$ . Recall the definition of  $N(t)$  from (5.1), and that  $\sigma_m \rightarrow \infty$  as  $m \rightarrow \infty$ . Since  $Y_t = Y_0 + \sum_{s=0}^{t-1} D_s$ , we observe that

$$Y_t \geq Y_0 + \sum_{m=0}^{N(t-1)} D_{\sigma_m}^+ - \sum_{s=0}^t D_s^-. \tag{5.4}$$

We have from (3.4) that, for any  $\varepsilon > 0$ , there exists  $x_0 < \infty$  such that, a.s., for all  $x \geq x_0$ ,

$$x^{-\alpha-\varepsilon} \leq \mathbb{P}[D_{\sigma_n}^+ \geq x \mid \mathcal{G}_{\sigma_n}] \leq x^{-\alpha+\varepsilon}, \tag{5.5}$$

uniformly in  $n$ . An application of Theorem 2.6 with  $X_t = \sum_{m=0}^{t-1} D_{\sigma_m}^+$  and  $\mathcal{F}_t = \mathcal{G}_{\sigma_t}$  (noting that, since  $\sigma_{t-1} + 1 \leq \sigma_t$ ,  $X_t$  is then  $\mathcal{F}_t$ -measurable, and  $\Delta_t = X_{t+1} - X_t = D_{\sigma_t}^+$ ), using the lower bound in (5.5), then implies that for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $\sum_{m=1}^t D_{\sigma_m}^+ \geq t^{(1/\alpha)-\varepsilon}$ . Together with Lemma 5.2(i) and (5.3), this implies that for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,

$$\sum_{m=0}^{N(t)} D_{\sigma_m}^+ \geq t^{(\gamma/\alpha)-\varepsilon}. \tag{5.6}$$

On the other hand, condition (i) in Theorem 3.8 with Markov's inequality implies that  $\mathbb{P}[D_t^- \geq x \mid \mathcal{G}_t] \leq Cx^{-\beta}$ , uniformly in  $t$  and  $\omega$ . Then an application of Theorem 2.4 with  $X_t = \sum_{s=0}^{t-1} D_s^-$  (so that  $\Delta_t = D_t^-$ ),  $\mathcal{F}_t = \mathcal{G}_t$ ,  $\theta = \beta \wedge 1$  and  $\phi = 0$  implies that for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,

$$\sum_{s=0}^t D_s^- \leq t^{\frac{1}{\beta \wedge 1} + \varepsilon}. \tag{5.7}$$

Thus from (5.4) with (5.6) and (5.7), and the fact that  $\alpha < \gamma(\beta \wedge 1)$ , we obtain, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $Y_t \geq t^{(\gamma/\alpha)-\varepsilon}$ . Since  $\varepsilon > 0$  was arbitrary,

$$\liminf_{t \rightarrow \infty} \frac{\log Y_t}{\log t} \geq \frac{\gamma}{\alpha}, \text{ a.s.}$$

Now we prove the upper bound for  $Y_t$ . Observe that

$$Y_t \leq Y_0 + \sum_{m=0}^{N(t)} D_{\sigma_m}^+ + \sum_{s=0}^t D_s^+ \mathbf{1}\{Y_s \notin \mathcal{C}\}. \tag{5.8}$$

Here we have from Lemma 5.2(ii) and the lower bound in (5.2) that, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $N(t) \leq t^{\gamma+\varepsilon}$ . Moreover, an application of Theorem 2.4(i) with  $X_t = \sum_{m=0}^{t-1} D_{\sigma_m}^+$  (so  $\Delta_t = D_{\sigma_t}^+$ ),  $\mathcal{F}_t = \mathcal{G}_{\sigma_t}$ ,  $\theta = \alpha - \varepsilon$  and  $\phi = 0$ , using the upper bound

in (5.5), implies that for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $\sum_{m=0}^t D_{\sigma_m}^+ \leq t^{(1/\alpha)+\varepsilon}$ . Hence for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,

$$\sum_{m=0}^{N(t)} D_{\sigma_m}^+ \leq t^{(\gamma/\alpha)+\varepsilon}. \tag{5.9}$$

Another application of Theorem 2.4, this time with  $X_t = \sum_{s=0}^{t-1} D_s^+ \mathbb{1}\{Y_s \notin \mathcal{C}\}$  (so  $\Delta_t = D_t^+ \mathbb{1}\{Y_t \notin \mathcal{C}\}$ ),  $\mathcal{F}_t = \mathcal{G}_t$ ,  $\theta = \beta \wedge 1$  and  $\phi = 0$ , using condition (iii) in Theorem 3.8, implies that for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,

$$\sum_{s=0}^t D_s^+ \mathbb{1}\{Y_s \notin \mathcal{C}\} \leq t^{(1/(\beta \wedge 1))+\varepsilon}. \tag{5.10}$$

Then from (5.8) with (5.9) and (5.10), using the fact that  $\alpha < \gamma(\beta \wedge 1)$ , we obtain, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $Y_t \leq t^{(\gamma/\alpha)+\varepsilon}$ . Since  $\varepsilon > 0$  was arbitrary,

$$\limsup_{t \rightarrow \infty} \frac{\log Y_t}{\log t} \leq \frac{\gamma}{\alpha}, \text{ a.s.}$$

Combining this with the lim inf result obtained above completes the proof. □

We finish this section with the proof of Theorem 3.9.

*Proof of Theorem 3.9.* Parts of this proof are similar to the proof of Theorem 3.8 above, so we omit some details this time around. Again, (5.2) holds. Observe that

$$Y_t \geq Y_0 - \sum_{m=0}^{N(t)} D_{\sigma_m}^- - \sum_{s=0}^t D_s^- \mathbb{1}\{Y_s \notin \mathcal{C}\}. \tag{5.11}$$

Similarly to the argument for (5.9) above, from Lemma 5.2(i) and Theorem 2.4, using condition (i) in Theorem 3.9, we have that, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $\sum_{m=0}^{N(t)} D_{\sigma_m}^- \leq t^{(\gamma/\alpha)+\varepsilon}$ . Also, similarly to the argument for (5.10) above, we have from Theorem 2.4 with condition (ii) in Theorem 3.9 that, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $\sum_{s=0}^t D_s^- \mathbb{1}\{Y_s \notin \mathcal{C}\} \leq t^{(1/\beta)+\varepsilon}$ . Since  $\alpha > \gamma\beta$  it follows from (5.11) that for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $Y_t \geq -t^{(1/\beta)+\varepsilon}$ .

Next we prove the upper bound for  $Y_t$ . Observe that

$$Y_t \leq Y_0 + \sum_{m=0}^{N(t)} D_{\sigma_m}^+ + \sum_{s=0}^t D_s^+ \mathbb{1}\{Y_s \notin \mathcal{C}\} - \sum_{s=0}^{t-1} D_s^- \mathbb{1}\{Y_s \notin \mathcal{C}\}. \tag{5.12}$$

Similarly to the analogous term in (5.11), for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $\sum_{m=0}^{N(t)} D_{\sigma_m}^+ \leq t^{(\gamma/\alpha)+\varepsilon}$ . Yet another application of Theorem 2.4, using condition (iii) in Theorem 3.9, yields, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $\sum_{s=0}^t D_s^+ \mathbb{1}\{Y_s \notin \mathcal{C}\} \leq t^{\frac{1}{(\beta+\delta)\wedge 1}+\varepsilon}$ . Since  $\alpha > \beta\gamma$ ,  $\beta < 1$ , and  $\delta > 0$ , we may choose  $\varepsilon > 0$  small enough so that both of these upper bounds are  $o_\omega(t^{(1/\beta)-\varepsilon})$ . So, by (5.12), to complete the proof, it remains to show that, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,

$$\sum_{s=0}^{t-1} D_s^- \mathbb{1}\{Y_s \notin \mathcal{C}\} \geq t^{(1/\beta)-\varepsilon}. \tag{5.13}$$

Let  $\kappa_1, \kappa_2, \dots$  be the successive (stopping) times at which  $Y_t \notin \mathcal{C}$ , and let  $M(t) = \max\{m : \kappa_m \leq t\}$ . Since  $\gamma \in (0, 1)$ , we have from Lemma 5.2(ii) that  $N(t) = o_\omega(t)$ , a.s., so  $M(t) > t/2$  a.s., for all  $t$  sufficiently large. Then  $\sum_{s=0}^{t-1} D_s^- \mathbb{1}\{Y_s \notin \mathcal{C}\} \geq \sum_{m=1}^{M(t-1)} D_{\kappa_m}^-$ . For this latter sum, Theorem 2.6 with condition (ii) in Theorem 3.9 shows that, for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $X_t = \sum_{m=1}^{t-1} D_{\kappa_m}^- \geq t^{(1/\beta)-\varepsilon}$ . Then the claim (5.13) follows, using the a.s. lower bound on  $M(t)$ . □

**5.3 Proof of Theorem 3.7**

For this section we take  $X_t = Y_{\sigma_t}$  and  $\mathcal{F}_t = \mathcal{G}_{\sigma_t}$ . Thus  $X_t$  is the *embedded process* obtained by observing  $Y_t$  at those instants at which it is in the distinguished class  $\mathcal{C}$ ;  $X_t$  is an  $(\mathcal{F}_t)$ -adapted process on the state space  $\mathcal{C}$ . As before, we write  $D_t := Y_{t+1} - Y_t$  and  $\Delta_t := X_{t+1} - X_t$  for the increments of  $Y_t$  and  $X_t$ , respectively. The next two results derive properties of the increments  $\Delta_t$  of the embedded process  $X_t$  from conditions on the increments  $D_t$  of the original process  $Y_t$ . First we have an upper tail bound.

**Lemma 5.3.** *Suppose that (C1) and (C2) hold. Suppose that for some  $C < \infty$  and some  $\beta > 0$ ,  $\mathbb{E}[(D_t^+)^{\beta} | \mathcal{G}_t] \leq C$  a.s. for all  $t$ . Then there exists  $C' < \infty$  such that for all  $x > 0$  and all  $t$ ,  $\mathbb{P}[\Delta_t^+ \geq x | \mathcal{F}_t] \leq C'x^{-(\beta \wedge 1)}$  a.s.*

*Proof.* For the duration of this proof, let  $Z_s := \sum_{r=\sigma_t}^{\sigma_t+s-1} D_r^+$  for  $s \geq 0$ , so that  $Z_0 = 0$ ,  $Z_s \geq 0$  and  $Y_{\sigma_t+s} \leq Z_s + Y_{\sigma_t}$ . Then for any  $s \geq 0$ , by concavity,

$$\mathbb{E}[Z_{s+1}^{\beta \wedge 1} - Z_s^{\beta \wedge 1} | \mathcal{G}_{\sigma_t+s}] \leq \mathbb{E}[(D_{\sigma_t+s}^+)^{\beta \wedge 1} | \mathcal{G}_{\sigma_t+s}] \leq C, \text{ a.s.}$$

Hence, by Lemma 4.3, for  $x > 0$ ,

$$\mathbb{P}\left[\max_{0 \leq s \leq \nu_t} Z_s^{\beta \wedge 1} \geq x | \mathcal{F}_t\right] \leq Cx^{-1}\mathbb{E}[\nu_t | \mathcal{F}_t] \leq BCx^{-1}, \text{ a.s.}, \tag{5.14}$$

for all  $t$ , since, by (C2),  $\mathbb{E}[\nu_t | \mathcal{F}_t] \leq B$ . In particular, since  $\Delta_t = Y_{\sigma_t+\nu_t} - Y_{\sigma_t} \leq Z_{\nu_t}$ , (5.14) implies that  $\mathbb{P}[(\Delta_t^+)^{\beta \wedge 1} \geq x | \mathcal{F}_t] = O(x^{-1})$ , uniformly in  $t$  and  $\omega$ .  $\square$

Next we prove the following lower tail bound.

**Lemma 5.4.** *Suppose that (C1) and (C2) hold. Suppose that for some  $c > 0$ ,  $\alpha > 0$ , and  $x_0 < \infty$ , for all  $x \geq x_0$  and all  $t$ ,  $\mathbb{P}[D_t^+ \geq x | \mathcal{G}_t] \geq cx^{-\alpha}$  a.s. on  $\{Y_t \in \mathcal{C}\}$ . Suppose also that there exist  $C < \infty$  and  $\beta > 0$  with  $\alpha < \beta \wedge 1$  such that  $\mathbb{E}[(D_t^-)^{\beta} | \mathcal{G}_t] \leq C$  a.s. for all  $t$ . Then there exist  $c' > 0$  and  $x_1 < \infty$  for which  $\mathbb{P}[\Delta_t^+ \geq x | \mathcal{F}_t] \geq c'x^{-\alpha}$  a.s. for all  $x \geq x_1$  and all  $t$ .*

*Proof.* Recall that  $\Delta_t = Y_{\sigma_{t+1}} - Y_{\sigma_t} = \sum_{s=0}^{\nu_t-1} D_{\sigma_t+s}$ . Then  $\Delta_t^+ \geq D_{\sigma_t}^+ - \sum_{r=\sigma_t}^{\sigma_t+\nu_t-1} D_r^-$ , so

$$\begin{aligned} \mathbb{P}[\Delta_t^+ \geq x | \mathcal{F}_t] &\geq \mathbb{P}[D_{\sigma_t}^+ \geq 2x | \mathcal{G}_{\sigma_t}] - \mathbb{P}\left[\sum_{r=\sigma_t}^{\sigma_t+\nu_t-1} D_r^- \geq x | \mathcal{G}_{\sigma_t}\right] \\ &= \mathbb{P}[D_{\sigma_t}^+ \geq 2x | \mathcal{G}_{\sigma_t}] - O(x^{-(\beta \wedge 1)}), \end{aligned}$$

by the argument for (5.14) but with a change of sign.  $\square$

Recall the definition of  $N(t)$  from (5.1).

*Proof of Theorem 3.7.* Condition (ii) of the theorem implies that for any  $\varepsilon > 0$ , there exists  $x_0 < \infty$  for which,

$$x^{-\alpha-\varepsilon} \leq \mathbb{P}[D_t^+ > x | \mathcal{G}_t] \leq x^{-\alpha+\varepsilon}, \text{ a.s.}, \tag{5.15}$$

for all  $x \geq x_0$  and all  $t$ . Using the lower bound in (5.15) and (C2), Lemma 5.4 implies that, for any  $\varepsilon > 0$ , there exists  $x_1 < \infty$  such that  $\mathbb{P}[\Delta_t^+ \geq x | \mathcal{F}_t] \geq x^{-\alpha-\varepsilon}$ , a.s., for all  $x \geq x_1$  and all  $t$ . On the other hand, Lemma 5.3 with (C2) and the upper bound in (5.15) (which shows that, for  $\varepsilon > 0$ ,  $\mathbb{E}[(D_t^+)^{\alpha-\varepsilon} | \mathcal{G}_t]$  is bounded uniformly in  $t$  and  $\omega$ ) implies that, for any  $\varepsilon > 0$ ,  $\mathbb{P}[\Delta_t^+ \geq x | \mathcal{F}_t] = O(x^{-\alpha+\varepsilon})$  a.s., uniformly in  $t$  and  $\omega$ . Moreover, another application of Lemma 5.3, now using condition (i) of the theorem as well as (C2), yields  $\mathbb{P}[\Delta_t^- \geq x | \mathcal{F}_t] = O(x^{-(\beta \wedge 1)})$  a.s., uniformly in  $t$  and  $\omega$ , so that, for  $\varepsilon > 0$ ,



$\mathbb{E}[(\Delta_t^-)^{(\beta \wedge 1) - \varepsilon} | \mathcal{F}_t]$  is bounded uniformly in  $t$  and  $\omega$ . With these tail and moment bounds, since  $\alpha < \beta \wedge 1$ , we obtain from Theorem 2.4 that for any  $\varepsilon > 0$ ,  $X_t = O_\omega(t^{(1/\alpha) + \varepsilon})$ , a.s., and we obtain from Theorem 2.6 that for any  $\varepsilon > 0$ , a.s., for all but finitely many  $t$ ,  $X_t \geq t^{(1/\alpha) - \varepsilon}$ . Thus, since  $\varepsilon > 0$  was arbitrary,

$$\lim_{t \rightarrow \infty} \frac{\log X_t}{\log t} = \frac{1}{\alpha}, \text{ a.s.} \tag{5.16}$$

We need to show that the same limit holds for  $Y_t$  instead of  $X_t$ . Note that  $\sigma_{N(t)} \leq t < \sigma_{N(t)+1}$ . If  $t = \sigma_{N(t)}$ , we have

$$Y_t \mathbb{1}\{Y_t \in \mathcal{C}\} = X_{N(t)} = (N(t))^{(1/\alpha) + o_\omega(1)}, \text{ a.s.,} \tag{5.17}$$

by (5.16). On the other hand, for  $\sigma_{N(t)} < t < \sigma_{N(t)+1}$  we have the estimate

$$|Y_t - Y_{\sigma_{N(t)+1}}| \mathbb{1}\{Y_t \notin \mathcal{C}\} \leq \sum_{s=\sigma_{N(t)+1}^{\sigma_{N(t)+1}-1} |D_s| = \sum_{s=\sigma_{N(t)}^{\sigma_{N(t)+1}-1} |D_s| \mathbb{1}\{Y_s \notin \mathcal{C}\}.$$

It follows that

$$\begin{aligned} \max_{0 \leq s \leq t} (|Y_s - X_{N(s)+1}| \mathbb{1}\{Y_s \notin \mathcal{C}\}) &\leq \max_{0 \leq n \leq N(t)} \max_{\sigma_n \leq s < \sigma_{n+1}} |Y_s - Y_{\sigma_{N(s)+1}}| \mathbb{1}\{Y_s \notin \mathcal{C}\} \\ &\leq \sum_{s=0}^{\sigma_{N(t)+1}} |D_s| \mathbb{1}\{Y_s \notin \mathcal{C}\} \leq \sum_{s=0}^{\sigma_{t+1}} |D_s| \mathbb{1}\{Y_s \notin \mathcal{C}\}, \end{aligned}$$

using the trivial bound  $N(t) \leq t$  for the final inequality. By conditions (i) and (iii) of the theorem,  $\mathbb{E}[|D_t|^\beta | \mathcal{G}_t] \leq C$  on  $\{Y_t \notin \mathcal{C}\}$ , a.s., so Theorem 2.4 yields, for any  $\varepsilon > 0$ ,

$$\sum_{s=0}^{\sigma_t} |D_s| \mathbb{1}\{Y_s \notin \mathcal{C}\} \leq (\sigma_t)^{\frac{1}{\beta \wedge 1} + \varepsilon},$$

where, for any  $\varepsilon > 0$ ,  $\sigma_t = O_\omega(t^{1+\varepsilon})$ , by Lemma 5.1(i). Thus we obtain, for any  $\varepsilon > 0$ ,

$$\max_{0 \leq s \leq t} (|Y_s - X_{N(s)+1}| \mathbb{1}\{Y_s \notin \mathcal{C}\}) = O_\omega(t^{\frac{1}{\beta \wedge 1} + \varepsilon}),$$

which is  $o_\omega(t^{(1/\alpha) - \varepsilon})$  for small enough  $\varepsilon$ , since  $\alpha < \beta \wedge 1$ . Hence

$$Y_t \mathbb{1}\{Y_t \notin \mathcal{C}\} = X_{N(t)+1} + o_\omega(t^{(1/\alpha) - \varepsilon}) = (N(t) + 1)^{(1/\alpha) + o_\omega(1)} + o_\omega(t^{(1/\alpha) - \varepsilon}), \text{ a.s.,} \tag{5.18}$$

by (5.16). The result of the theorem now follows from (5.17) and (5.18) provided we can show that  $N(t) = t^{1+o_\omega(1)}$ , a.s. The upper bound here is trivial since  $N(t) \leq t$ , and the lower bound follows from Lemma 5.2(i) with (C2). This completes the proof.  $\square$

### 5.4 Proofs for heavy-tailed random walks on strips

The model of Section 3.2 generalizes the strip model as follows. Set  $Y_t = V_t + \frac{1}{2+U_t}$ . Then  $(U_t, V_t)$  can be recovered from  $Y_t$  via  $V_t = \lfloor Y_t \rfloor$  and  $U_t = (Y_t - \lfloor Y_t \rfloor)^{-1} - 2$ . In this case, the state-space  $\mathcal{S}$  of  $Y_t$  is a subset of the rationals  $\mathbb{Q}$ ; the distinguished subset  $\mathcal{C}$  corresponds to  $U_t = 0$ , i.e.,  $\mathcal{C} = \frac{1}{2} + \mathbb{Z}$ , a translate of  $\mathbb{Z}$ . The increments of  $Y_t$  have the same tail behaviour as the increments of  $V_t$ .

Thus Theorems 3.2, 3.3, and 3.4 follow immediately from Theorems 3.7, 3.8, and 3.9, respectively. It remains to prove Proposition 3.6.

*Proof of Proposition 3.6.* Proposition 1 of [2, p. 957] implies that  $\mathbb{E}[\nu^\gamma] < \infty$ , which implies the upper bound in (3.3) by Markov’s inequality. On the other hand, for the lower tail bound we appeal to a result of [1]. For  $p > 0$ , Taylor’s formula implies that

$$\begin{aligned} & \mathbb{E}[U_{t+1}^p - U_t^p \mid U_t = x] \\ &= px^{p-1} \left( \mathbb{E}[U_{t+1} - U_t \mid U_t = x] + \frac{p-1}{2x} \mathbb{E}[(U_{t+1} - U_t)^2 \mid U_t = x] + O(x^{-2}) \right), \end{aligned}$$

using the uniform bound on  $U_{t+1} - U_t$  for the error term. By our assumptions on the moments of  $U_{t+1} - U_t$ , we have

$$\mathbb{E}[U_{t+1}^p - U_t^p \mid U_t = x] = px^{p-2}\sigma^2 \left( \left( \frac{1}{2} - \gamma \right) + \frac{p-1}{2} + o(1) \right) \geq 0,$$

for all  $x$  sufficiently large, provided  $p > 2\gamma$ . So Corollary 1 of [1, p. 119] implies that for any  $\varepsilon > 0$ ,  $\mathbb{P}[\nu \geq t] \geq t^{-\gamma-\varepsilon}$ , for all  $t$  sufficiently large.  $\square$

## 6 Appendix

In this appendix we make some additional remarks concerning the nature of our conditions (2.3), (2.2), and (2.6), and how they relate to the formulation of the results of Erickson [8] and Kesten and Maller [19] on sums of i.i.d. random variables.

For any nonnegative random variable  $Z$  with distribution function  $F(z) := \mathbb{P}[Z \leq z]$ ,

$$\begin{aligned} \mathbb{E}[Z\mathbb{1}\{Z \leq z\}] &= \int_0^z y dF(y) \\ &= \int_0^\infty \mathbb{P}[Z\mathbb{1}\{Z \leq z\} > y] dy = \int_0^z \mathbb{P}[y < Z \leq z] dy \\ &= \int_0^z \mathbb{P}[Z > y] dy - z\mathbb{P}[Z > z]. \end{aligned} \tag{6.1}$$

Our condition (2.3) concerns  $\mathbb{E}[\Delta_t^+ \mathbb{1}\{\Delta_t^+ \leq x\} \mid \mathcal{F}_t]$ ; conditions in [8, 19] are stated in terms of the analogue in the i.i.d. case of  $\int_0^x \mathbb{P}[\Delta_t^+ > y \mid \mathcal{F}_t] dy$ , which is denoted  $m_+(x)$  by Erickson [8, p. 372] and  $A_+(y)$  by Kesten and Maller [19, p. 3]. It follows from (6.1) that, for  $x > 0$ ,  $\int_0^x \mathbb{P}[\Delta_t^+ > y \mid \mathcal{F}_t] dy \geq \mathbb{E}[\Delta_t^+ \mathbb{1}\{\Delta_t^+ \leq x\} \mid \mathcal{F}_t]$ , so (2.3) implies that  $\int_0^x \mathbb{P}[\Delta_t^+ > y \mid \mathcal{F}_t] dy \geq cx^{1-\alpha}$  a.s. for  $x$  sufficiently large.

On the other hand, (2.2) together with Markov’s inequality implies that  $\int_0^x \mathbb{P}[\Delta_t^- \geq y \mid \mathcal{F}_t] dy = O(x^{1-\beta})$ ; here  $\int_0^x \mathbb{P}[\Delta_t^- \geq y \mid \mathcal{F}_t] dy$  is the analogue in our more general setting of Erickson’s  $m_-(x)$  [8, p. 372] and Kesten and Maller’s  $A_-(x)$  [19, p. 3].

We state one result on the relationship between conditions (2.3) and (2.6), using the concept of slow variation (see e.g. [27, pp. 354–356]).

**Lemma 6.1.** *Suppose that for some  $\alpha \in (0, 1)$  the nonnegative random variable  $Z$  satisfies  $\mathbb{P}[Z > z] = z^{-\alpha}L(z)$  for some slowly varying function  $L$ . Then  $\mathbb{E}[Z\mathbb{1}\{Z \leq z\}] \sim \frac{\alpha}{1-\alpha}z^{1-\alpha}L(z)$  as  $z \rightarrow \infty$ .*

*Proof.* Karamata’s theorem (see e.g. [27, p. 356]) implies that

$$\int_0^z \mathbb{P}[Z > y] dy = \int_0^z y^{-\alpha}L(y) dy \sim \frac{1}{1-\alpha}z^{1-\alpha}L(z),$$

as  $z \rightarrow \infty$ . The result follows from (6.1).  $\square$

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