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# A dichotomy in orbit-growth for commuting automorphisms

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Abstract

We consider asymptotic orbit-counting problems for certain expansive actions by commuting automorphisms of compact groups. A dichotomy is found between systems with asymptotically more periodic orbits than the topological entropy predicts, and those for which there is no excess of periodic orbits.

## 1. *Introduction*

Let $G$ be a countable group acting on some set $X$, with the action of $g \in G$ written $x \mapsto g{\scriptstyle\bullet}x$. Let $\mathcal{L} = \mathcal{L}(G)$ denote the poset of finite index subgroups of $G$, and write

$$a_n(G) = |\{L \in \mathcal{L} \mid [G : L] = n\}|.$$

We assume that $\mathcal{L}$ is locally finite (a finiteness assumption on $G$, guaranteed if $G$ is finitely-generated). For $L \in \mathcal{L}$, the set of $L$-periodic points in $X$ under the action is

$$\mathsf{F}(L) = \{x \in X \mid g{\scriptstyle\bullet}x = x \text{ for all } g \in L\}.$$

An $L$-periodic orbit $\tau$ is the orbit of a point with stabilizer $L$, and the length of the orbit is denoted $[L] = [G : L]$, the index of $L$ in $G$. We always assume that there are only finitely many orbits of length $n$ for each $n \geqslant 1$ (a finiteness assumption on the action, guaranteed if the action is expansive). The number of $L$-periodic orbits is

$$\mathsf{O}(L) = \frac{1}{[L]} \left|\{x \in X \mid g{\scriptstyle\bullet}x = x \iff g \in L\}\right|.$$

Orbit growth may be studied via the asymptotic behaviour of the orbit-counting function

$$\pi(N) = \sum_{[L]\leqslant N} \mathsf{O}(L).$$

Our focus is on actions with an exponential rate of orbit growth $\mathsf{g} > 0$, and for these it is also natural to consider the weighted sum

$$\mathcal{M}(N) = \sum_{[L]\leqslant N} \frac{\mathsf{O}(L)}{\mathrm{e}^{\mathsf{g}[L]}}.$$

The topological entropy $\mathsf{h}$ is a global measure of orbit complexity, and the dichotomy we explore here concerns the relationship between $\mathsf{g}$ and $\mathsf{h}$. In the case $\mathsf{g} > \mathsf{h}$ for a $\mathbb{Z}^2$-action, there is a preferred direction in which long thin rectangular orbit shapes have an abundance of periodic orbits, and these dominate the count to such an extent that the orbit-counting asymptotics resemble the case of a single transformation. In the case $\mathsf{g} = \mathsf{h}$ there are no preferred directions, and distinctly higher-dimensional asymptotics can arise.

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Any $L$-periodic point lives on a unique $L'$-periodic orbit for some subgroup $L' \geqslant L$, so

$$\mathsf{F}(L) = \sum_{L' \geqslant L} [L']\mathsf{O}(L'), \tag{1.1}$$

and therefore

$$\mathsf{O}(L) = \frac{1}{[L]} \sum_{L' \geqslant L} \mu(L', L)\mathsf{F}(L'), \tag{1.2}$$

where $\mu$ is the Möbius function on the incidence algebra of $\mathcal{L}$ (the equivalence of (1.1) and (1.2) for all functions $\mathsf{F} : \mathcal{L} \to \mathbb{N}_0$ defines the function $\mu$ by induction).

Example 1. The familiar setting for dynamical systems has $G = \mathbb{Z}$, where the action is generated by the transformation $x \mapsto 1{\bullet}x$. If there are parameters $\mathsf{h} > \mathsf{h}' > 0$ with

$$\mathsf{F}(n\mathbb{Z}) = \mathrm{e}^{\mathsf{h}n} + \mathrm{O}(\mathrm{e}^{\mathsf{h}'n}),$$

then it is easy to check that

$$\pi(N) \sim \frac{\mathrm{e}^{\mathsf{h}(N+1)}}{N}$$

and

$$\mathcal{M}(N) = \log N + C_1 + \mathrm{O}(1/N),$$

with $\mathsf{g} = \mathsf{h}$. Asymptotics of this shape arise in hyperbolic dynamical systems (see Parry and Pollicott [**14**] and Sharp [**17**]), and in combinatorics (see Pakapongpun and the second author [**13**]). Natural examples with slower growth rates are studied in [**1**], [**4**], [**5**]. For example, in [**4**] it is shown that for certain algebraic dynamical systems of finite combinatorial rank the asymptotic growth rate takes the form

$$\pi(N) \sim N^{\sigma}(\log N)^{\kappa}$$

for some $\sigma, \kappa \geqslant 0$. In all these cases the growth comes entirely from the action, because there is no growth in the group: $a_n(\mathbb{Z}) = 1$ for all $n \geqslant 1$, and $|\mu(L, L')| \leqslant 1$ for all $L, L' \in \mathcal{L}(\mathbb{Z})$.

Example 2. Let $G$ be a finitely-generated nilpotent group and $B$ a finite alphabet. The *full $G$-shift* on $b = |B|$ symbols is the $G$-action on $B^G$ given by $(g{\bullet}x)_h = x_{gh}$, where $x$ denotes a point $(x_h) \in B^G$. For this action

$$\mathsf{F}(L) = b^{[L]}$$

for all $L \in \mathcal{L}(G)$ and there is a characteristic exponential growth rate of $\log b$. We showed in [**12**] that there are constants $C_2 > 0$, $\alpha \in \mathbb{Q}_{\geqslant 0}$ and $\beta \in \mathbb{N}_0$ for which

$$\mathcal{M}(N) \sim C_2 N^{\alpha} (\log N)^{\beta}.$$

For $G = \mathbb{Z}^d$, $d \geqslant 2$, there are constants $C_3, C_4, C_5 > 0$ such that

$$C_3 \leqslant \frac{\pi(N)}{N^{d-2}b^N} \leqslant C_4(\log N)^{d-1}$$

and

$$\mathcal{M}(N) \sim C_5 N^{d-1}.$$

In these examples there is exponential growth due to the action and some growth from the group: in this setting both $a_n$ and $\mu$ are unbounded functions unless $G = \mathbb{Z}$.

In this paper we start to bridge the gap between these two examples, by considering some actions of $\mathbb{Z}^2$ less trivial than the full shift. It is hoped that, for example, asymptotics for any expansive $\mathbb{Z}^d$-action by automorphisms of a compact group may be found, but the simple case considered here already throws up new phenomena and suggests that more complex dominant orbit-counting phenomena may occur in intermediate dimensions.

## 2. *Actions defined by polynomials*

Fix a polynomial $f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$, written $f(x,y) = \sum c_{(a,b)} x^a y^b$ for some finitely-supported function $c : \mathbb{Z}^2 \to \mathbb{Z}$. Associate to $f$ a compact abelian group

$$X_f = \{x \in \mathbb{T}^{\mathbb{Z}^2} \mid \sum c_{(a,b)} x_{(a+m,b+n)} = 0 \pmod 1 \text{ for all } m,n \in \mathbb{Z}\},$$

with the $\mathbb{Z}^2$-action defined by the shift,

$$((m,n)\cdot x)_{(k,\ell)} = x_{(m+k,n+\ell)}.$$

Assume that $f(\mathrm{e}^{2\pi \mathrm{i}s}, \mathrm{e}^{2\pi \mathrm{i}t}) \neq 0$ for all $(s,t) \in \mathbb{T}^2$; by Schmidt [**15**] this is equivalent to the action being *expansive* with respect to the natural topology on $X_f$ inherited from that of $\mathbb{T}^{\mathbb{Z}^2} = X_0$ (that is, there is a neighbourhood $U$ of $0 \in X_f$ such that $\bigcap_{(m,n)\in\mathbb{Z}^2} (m,n)\cdot U = \{0\}$).

If $f(x,y) = b \in \mathbb{N}$ is a constant, then $X_f$ is the full $\mathbb{Z}^2$-shift on $b$ symbols as in Example 2. In a wider context, the connection between algebraic $G$-actions and polynomials (or ideals) in the integral group ring $\mathbb{Z}[G]$ plays a central role in algebraic dynamics. An overview of this theory may be found in Schmidt's monograph [**16**], and some recent developments for groups other than $\mathbb{Z}^d$ include work of the first author [**11**], of Einsiedler and Rindler [**3**], and of Deninger and Schmidt [**2**].

For brevity we write

$$\mathrm{L}_f(s,t) = \log |f(\mathrm{e}^{2\pi \mathrm{i}s}, \mathrm{e}^{2\pi \mathrm{i}t})|,$$

which is a continuous function on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ under the standing assumption of expansiveness. Following Lind [**10**], let $\mathcal{C}$ denote the set of compact subgroups of $\mathbb{T}^2$, and define $\mathsf{m} : \mathcal{C} \to \mathbb{R}$, continuous in the Hausdorff metric on $\mathcal{C}$, by

$$\mathsf{m}(K) = \int_K \mathrm{L}_f(s,t)\, \mathrm{d}m_K(s,t)$$

where the integration is with respect to Haar measure $m_K$. In particular, if $K$ is a finite subgroup, then $\mathsf{m}(K) = \frac{1}{|K|} \sum_{(s,t)\in K} \mathrm{L}_f(s,t)$.

By Lind, Schmidt and the second author [**9**], the topological entropy of the action is given by

$$\mathsf{h} = \mathsf{m}(\mathbb{T}^2),$$

the Mahler measure of $f$. The growth in periodic points is also studied in [**9**], and in particular it is shown that

$$\lim_{\mathrm{girth}(L)\to\infty} \frac{1}{[L]} \log \mathsf{F}(L) = \mathsf{h},$$

where $\mathrm{girth}(L) = \min\{\|(a,b)\| \mid (a,b) \in L \setminus \{(0,0)\}\}$. The upper growth rate is found by Lind [**10**],

$$\mathsf{g} = \lim_{N\to\infty} \sup_{[L]\geqslant N} \frac{1}{[L]} \log \mathsf{F}(L) = \sup_{C\in\mathcal{C}_\infty} \mathsf{m}(K), \tag{2.1}$$

where $\mathcal{C}_\infty \subset \mathcal{C}$ is the set of infinite compact subgroups of $\mathbb{T}^2$.

If $\mathsf{g} > \mathsf{h}$, then the action has favoured directions corresponding to sequences of infinite subgroups along which there is convergence to $\mathsf{g}$ in (2.1) (and along which an abundance of periodic points are found, in excess of the amount predicted by the topological entropy, which is a global invariant for the whole action). If $\mathsf{g} = \mathsf{h}$ then there are no preferred directions. Systems with $\mathsf{g} > \mathsf{h}$ as a result behave more like the familiar case $G = \mathbb{Z}$, while systems with $\mathsf{g} = \mathsf{h}$ have the potential for orbit growth asymptotics peculiar to higher rank actions (see Table 1 for explicit asymptotics for the simplest $d$-dimensional systems for small values of $d$).

Any $L \in \mathcal{L} = \mathcal{L}(\mathbb{Z}^2)$ may be written in the form

$$L = L(a, b, c) = \langle (a, 0), (b, c) \rangle,$$

where $a, c \geqslant 1$, $0 \leqslant b \leqslant a - 1$, and $[L] = ac$ (this is the canonical form for lattices due originally to Hermite [**8**]). Write

$$L^{\perp} = \left\{ \left( \tfrac{j}{a}, \tfrac{k}{c} - \tfrac{jb}{ac} \right) \mid 0 \leqslant j \leqslant a-1, 0 \leqslant k \leqslant c-1 \right\}$$

for the annihilator of $L$ under the Pontryagin duality between $\mathbb{T}^2$ and $\mathbb{Z}^2$. By [**9**] we have

$$\mathsf{F}(L) = \prod_{(s,t) \in L^{\perp}} |f(\mathrm{e}^{2\pi \mathrm{i} s}, \mathrm{e}^{2\pi \mathrm{i} t})| = \mathrm{e}^{[L] \mathsf{m}(L^{\perp})}.$$

THEOREM 1. *If $\mathsf{g} > \mathsf{h}$, then there are constants $C_6, C_7 > 0$ such that*

$$C_6 \log N \leqslant \mathcal{M}(N) \leqslant C_7 \log N \tag{2.2}$$

*and*

$$C_6 \leqslant \frac{\pi(N)}{\mathrm{e}^{\mathsf{g} N}} \leqslant C_7. \tag{2.3}$$

*Proof.* Just as in [**4**] and [**12**], part of the proof involves isolating a main term. However, the more complex geometry of the acting group and the action requires additional steps to take account of the preferred directions with an abundance of periodic orbits.

Associate to $L(a, b, c) \in \mathcal{L}$ subgroups

$$J(a) = \{ (\tfrac{j}{a}, t) \mid t \in \mathbb{T}, j = 0, \dots, a-1 \},$$

$$J(b, c) = \{ (t, \tfrac{k}{c} - \tfrac{bt}{c}) \mid t \in \mathbb{T}, k = 0, \dots, c-1 \} \subset \mathcal{C}_{\infty},$$

and set

$$K(L) = \begin{cases} J(a) & \text{if } a < c; \\ J(b, c) & \text{if } a \geqslant c. \end{cases}$$

The subgroup $K(L)$ approximates $L$-periodic points in the following sense.

LEMMA 2. *There is a constant $C_8$, depending only on $f$, with*

$$\left| \mathsf{m}(L^{\perp}) - \mathsf{m}(K(L)) \right| \leqslant \frac{C_8}{\max\{a, c\}} \tag{2.4}$$

*for any $L = L(a, b, c) \in \mathcal{L}$.*

*Proof.* For a point $w = (w_1, w_2, w_3, w_4) \in \mathbb{T}^4$, let $\Lambda(w)$ denote the line segment from $(w_1, w_2)$ to $(w_3, w_4)$, and let $V(w)$ be the total variation of the curve $(s, t) \mapsto \mathrm{L}_f(s, t)$ for $(s, t) \in \Lambda(w)$. By the hypothesis of expansiveness, $V : \mathbb{T}^4 \to \mathbb{R}$ is continuous and hence bounded by some constant $\alpha$. Thus

$$\left| \frac{1}{c} \sum_{k=0}^{c-1} \mathrm{L}_f(\tfrac{j}{a}, \tfrac{k}{c} - \tfrac{bj}{ac}) - \int_0^1 \mathrm{L}_f(\tfrac{j}{a}, t) \, \mathrm{d}t \right| \leqslant \frac{\alpha}{c} \tag{2.5}$$

and

$$\left|\frac{1}{a}\sum_{j=0}^{a-1}\mathrm{L}_f(\tfrac{j}{a},\tfrac{k}{c}-\tfrac{bj}{ac})-\int_0^1\mathrm{L}_f(t,\tfrac{k}{c}-\tfrac{b}{c}t)\,\mathrm{d}t\right|\leqslant\frac{\alpha}{a}. \tag{2.6}$$

If $a<c$, then

$$|\mathsf{m}(L^{\perp})-\mathsf{m}(K(L))|\leqslant\frac{1}{a}\sum_{j=0}^{a-1}\left|\frac{1}{c}\sum_{k=0}^{c-1}\mathrm{L}_f(\tfrac{j}{a},\tfrac{k}{c}-\tfrac{bj}{ca})-\int_0^1\mathrm{L}_f(\tfrac{j}{a},t)\,\mathrm{d}t\right|,$$

and so (2.5) gives (2.4). If $a\geqslant c$, then

$$|\mathsf{m}(L^{\perp})-\mathsf{m}(K(L))|\leqslant\frac{1}{c}\sum_{k=0}^{c-1}\left|\frac{1}{a}\sum_{j=0}^{a-1}\mathrm{L}_f(\tfrac{j}{a},\tfrac{k}{c}-\tfrac{bj}{ca})-\int_0^1\mathrm{L}_f(t,\tfrac{k}{c}-\tfrac{b}{c}t)\,\mathrm{d}t\right|,$$

and in this case (2.6) implies (2.4). □

Write $\mathcal{L}(n)$ for the set of subgroups of index $n$, and isolate the term corresponding to the largest subgroups arising in $\mathcal{M}(N)$ by writing

$$\mathcal{M}_1(N)=\sum_{n\leqslant N}\frac{1}{n}\sum_{L\in\mathcal{L}(n)}\frac{\mathsf{F}(L)}{\mathrm{e}^{\mathsf{g}n}}$$

and

$$\mathcal{M}_2(N)=\sum_{n\leqslant N}\frac{1}{n}\sum_{L\in\mathcal{L}(n)}\sum_{L'>L}\frac{\mu(L',L)\mathsf{F}(L')}{\mathrm{e}^{\mathsf{g}n}},$$

so that $\mathcal{M}(N)=\mathcal{M}_1(N)+\mathcal{M}_2(N)$.

Now fix a subgroup $L\in\mathcal{L}(n)$ and assume that $L'=L(a,b,c)>L$. Then $[L']=ac\leqslant\frac{n}{2}$, so either $a\leqslant\sqrt{n/2}$ or $c\leqslant\sqrt{n/2}$. By (2.4),

$$[L']\left(\mathsf{m}(L'^{\perp})-\mathsf{m}(K(L'))\right)\leqslant\frac{C_8ac}{\max\{a,c\}}=C_8\min\{a,c\}\leqslant C_8\sqrt{n}.$$

It follows that

$$\begin{aligned}\log\mathsf{F}(L')-\mathsf{g}n&=[L']\mathsf{m}(L'^{\perp})-\mathsf{g}n\\&=[L']\mathsf{m}(K(L'))-\mathsf{g}n+[L']\left(\mathsf{m}(L'^{\perp})-\mathsf{m}(K(L'))\right)\\&\leqslant\frac{n}{2}\left(\mathsf{m}(K(L'))-\mathsf{g}\right)+C_8\sqrt{n}-\frac{\mathsf{g}n}{2}\\&\leqslant C_8\sqrt{n}-\frac{\mathsf{g}n}{2},\end{aligned}$$

since $\mathsf{m}(K(L'))\leqslant\mathsf{g}$ by (2.1). By [**12**, Lem. 2] there is a constant $C_9$ with

$$|\mu(L',L)|\leqslant\mathrm{e}^{C_9(n/2)^2}$$

(since $[L']<n/2$); moreover

$$a_n(\mathbb{Z}^2)\leqslant 9n\log n$$

by [**12**, Lem. 3]. Thus

$$\begin{aligned}|\mathcal{M}_2(N)|&\leqslant\sum_{n\leqslant N}\frac{1}{n}\exp\left(C_8\sqrt{n}-\mathsf{g}n/2\sum_{L\in\mathcal{L}(n)}\right)\sum_{L'>L}|\mu(L',L)|\\&\leqslant\sum_{n\leqslant N}\frac{9}{n}\exp\left(C_8\sqrt{n}+C_9(\log(n/2))^2-\mathsf{g}n/2\right)n\log n\\&=\mathrm{O}(1).\end{aligned}$$

It follows that the asymptotic growth is controlled by $\mathcal{M}_1(N)$. In order to isolate the subgroups responsible for the excess of periodic orbits above the level predicted by the topological entropy, let

$$A = \{a \geqslant 1 \mid \mathsf{m}(J(a)) = \mathsf{g}\}$$

and

$$B = \{(b,c) \mid c \geqslant 1, 0 \leqslant b \leqslant a-1, \mathsf{m}(J(b,c)) = \mathsf{g}\}.$$

Partition the subgroups $\mathcal{L}(n)$ into

$$\begin{aligned}
\mathcal{L}_1(n) &= \{L(a,b,c) \in \mathcal{L}(n) \mid a \in A, (b,c) \in B\},\\
\mathcal{L}_2(n) &= \{L(a,b,c) \in \mathcal{L}(n) \mid a \in A, (b,c) \notin B\},\\
\mathcal{L}_3(n) &= \{L(a,b,c) \in \mathcal{L}(n) \mid a \notin A, (b,c) \in B\}, \text{ and}\\
\mathcal{L}_4(n) &= \{L(a,b,c) \in \mathcal{L}(n) \mid a \notin A, (b,c) \notin B\}.
\end{aligned}$$

The supremum in (2.1) is attained, so $\mathcal{L}_2(n) \cup \mathcal{L}_3(n) \neq \emptyset$ for any $n \geqslant 1$. This main term decomposes as

$$\mathcal{M}_1(N) = \sum_{j=1}^{4} \underbrace{\sum_{n \leqslant N} \frac{1}{n} \sum_{L \in \mathcal{L}_j(n)} \exp(n(\mathsf{m}(L^\perp) - \mathsf{g}))}_{\mathcal{N}_j(N)}. \tag{2.7}$$

Let $\mathcal{K} = \{K(L) \mid L \in \mathcal{L}\} \subset \mathcal{C}$, and enumerate $\mathcal{K} = \{K_1, K_2, \dots\}$. In the Hausdorff metric, $K_j \to \mathbb{T}^2$ as $j \to \infty$, so $\mathsf{m}(K_j) \to \mathsf{h}$ as $j \to \infty$. Since we have $\mathsf{g} > \mathsf{h}$, it follows that

$$\lambda = \inf\{|\mathsf{g} - \mathsf{m}(K)| \mid K \in \mathcal{K}, \mathsf{m}(K) \neq \mathsf{g}\} > 0 \tag{2.8}$$

and

$$\{J(a) \mid a \in A\} \cup \{J(b,c) \mid (b,c) \in B\} = \{K \in \mathcal{K} \mid \mathsf{m}(K) = \mathsf{g}\}$$

must be finite. In particular, both $A$ and $B$ are finite, so $\mathcal{L}_1(n) \neq \emptyset$ for only finitely many values of $n \geqslant 1$, and therefore $\mathcal{N}_1(N) = \mathrm{O}(1)$.

If $L = L(a,b,c) \in \mathcal{L}_4(N)$, then $a \leqslant \sqrt{n}$ or $c \leqslant \sqrt{n}$ since $[L] = ac = n$. By (2.4) and (2.8), it follows that

$$\begin{aligned}
n\left(\mathsf{m}(L^\perp) - \mathsf{g}\right) &= n\left(\mathsf{m}(K(L)) - \mathsf{g}\right) + n\left(\mathsf{m}(L^\perp) - \mathsf{m}(K(L))\right)\\
&\leqslant -\lambda n + C_8\sqrt{n}
\end{aligned}$$

and so

$$\mathcal{N}_4(N) \leqslant \sum_{n \leqslant N} \frac{1}{n} |\mathcal{L}_4(N)| \exp(-\lambda n + C_8\sqrt{n}).$$

By [**12**, Lem. 3],

$$|\mathcal{L}_4(n)| \leqslant |\mathcal{L}(n)| \leqslant 9n\log n,$$

so $\mathcal{N}_4(N) = \mathrm{O}(1)$.

We are left with $\mathcal{N}_2$ and $\mathcal{N}_3$. Let

$$B_a(x) = \left\{(b,c) \in \mathbb{Z}^2 \mid \max_{a \in A}\{a\} < c \leqslant \lfloor x \rfloor, 0 \leqslant b \leqslant a-1, (b,c) \notin B\right\},$$

so that $\mathcal{N}_2(N) = \Theta(N) + \mathrm{O}(1)$, where

$$\Theta(N) = \sum_{a \in A} \sum_{(b,c) \in B_a(N/a)} \frac{1}{ac} \exp\left(ac(\mathsf{m}(L(a,b,c)^\perp) - \mathsf{g})\right).$$

If $a \in A$, $(b,c) \in B_a(N/a)$, and $L = L(a,b,c)$, then $K(L) = J(a)$, so

$$
\begin{aligned}
|ac(\mathsf{m}(L^\perp) - \mathsf{g})| &= |ac(\mathsf{m}(K(L)) - \mathsf{g}) + ac(\mathsf{m}(L^\perp) - \mathsf{m}(K(L)))| \\
&= |ac(\mathsf{m}(L^\perp) - \mathsf{m}(K(L)))| \leqslant C_8 a
\end{aligned}
$$

by (2.4). Thus

$$\sum_{a \in A} \frac{1}{a} \exp(-C_8 a) \sum_{(b,c) \in B_a(N/a)} \frac{1}{c} \leqslant \Theta(N) \leqslant \sum_{a \in A} \frac{1}{a} \exp(C_8 a) \sum_{(b,c) \in B_a(N/a)} \frac{1}{c}.$$

Now

$$\sum_{(b,c) \in B_a(N/a)} \frac{1}{c} = a \log \lfloor N/a \rfloor + \mathrm{O}(1) = a \log N + \mathrm{O}(1),$$

which when summed over the finitely many possible $a$ gives the contribution from $\mathcal{N}_2(N)$.

Now let

$$A(x) = \left\{ a \in \mathbb{Z} \mid \max_{(b,c) \in B} \{c\} < a \leqslant \lfloor x \rfloor, a \notin A \right\},$$

so that $\mathcal{N}_3(N) = \Phi(N) + \mathrm{O}(1)$, where

$$\Phi(N) = \sum_{(b,c) \in B} \sum_{a \in A(N/c)} \frac{1}{ac} \exp(ac(\mathsf{m}(L(a,b,c)^\perp) - \mathsf{g})).$$

If $(b,c) \in B$, $a \in A(N/c)$, and $L = L(a,b,c)$, then $K(L) = J(b,c)$, so (2.4) says that

$$|ac(\mathsf{m}(L^\perp) - \mathsf{g})| \leqslant C_8 c,$$

and hence

$$\sum_{(b,c) \in B} \frac{1}{c} \exp(-C_8 c) \sum_{a \in A(N/c)} \frac{1}{a} \leqslant \Phi(N) \leqslant \sum_{(b,c) \in B} \frac{1}{c} \exp(C_8 c) \sum_{a \in A(N/c)} \frac{1}{a}.$$

Once again the Euler formula for $\sum_{a \in A(N/c)} \frac{1}{a}$ gives the contribution from $\mathcal{N}_3(N)$.

Finally, we need to check that the constants associated with $\mathcal{N}_2(N)$ and $\mathcal{N}_3(N)$ cannot both vanish. This follows from the fact that

$$\mathcal{L}_2(n) \cup \mathcal{L}_3(n) \neq \emptyset,$$

which in turn is a consequence of the fact that the supremum in (2.1) is attained by [**10**], completing the proof of (2.2).

Turning to (2.3), we isolate a dominant term as before,

$$
\begin{aligned}
\pi(N) = \sum_{n \leqslant N} \sum_{L \in \mathcal{L}(n)} \mathsf{O}(L) &= \underbrace{\sum_{n \leqslant N} \frac{1}{n} \sum_{L \in \mathcal{L}(n)} \mathsf{F}(L)}_{\pi_1(N)} \\
&\quad + \underbrace{\sum_{n \leqslant N} \frac{1}{n} \sum_{L \in \mathcal{L}(n)} \sum_{L' > L} \mu(L', L) \mathsf{F}(L')}_{\pi_2(N)}.
\end{aligned}
$$

Then (using estimates from [**12**, Lem. 2,3] and Lemma 2 as before)

$$\begin{aligned}\frac{\pi_2(N)}{e^{\mathsf{g}N}} &\leqslant \sum_{n\leqslant N}\tfrac{1}{n}\exp(-\mathsf{g}n)\sum_{L\in\mathcal{L}(n)}\sum_{L'>L}\left(\exp(C_9(\log[L])^2)\right)\mathsf{F}(L')\\ &\leqslant \sum_{n\leqslant N}\tfrac{1}{n}\sum_{L\in\mathcal{L}(n)}\sum_{L'>L}\exp\Big(C_9(\log n)^2\\ &\qquad\qquad + \tfrac{n}{2}\mathsf{m}(K(L')) + C_8\tfrac{\sqrt{n}}{\sqrt{2}} - \mathsf{g}N\Big)\\ &\leqslant \sum_{n\leqslant N}\tfrac{1}{n}\sum_{L\in\mathcal{L}(n)}\sum_{L'>L}\exp\Big(\ C_9(\log n)^2 + \tfrac{n}{2}\underbrace{(\mathsf{m}(K(L'))-\mathsf{g})}_{\leqslant 0}\\ &\qquad\qquad + C_8\tfrac{\sqrt{n}}{\sqrt{2}} - \mathsf{g}\underbrace{(N-\tfrac{n}{2})}_{\geqslant N/2}\ \Big)\\ &\leqslant \sum_{n\leqslant N}\tfrac{9n^4\log n}{n}\exp\Big(C_9(\log n)^2 + C_8\tfrac{\sqrt{n}}{\sqrt{2}} - \mathsf{g}\tfrac{N}{2}\Big) = \mathrm{O}(1).\end{aligned}$$

We decompose the main term $\pi_1(N)$ as $\sum_{j=1}^{4}\rho_j(N)$, corresponding to the decomposition $\mathcal{L}(n) = \mathcal{L}_1(n)\sqcup\mathcal{L}_2(n)\sqcup\mathcal{L}_3(n)\sqcup\mathcal{L}_4(n)$ as before.

Since $A$ and $B$ are finite, it is easy to check that $\exp(-\mathsf{g}N)\rho_1(N)\to 0$ as $N\to\infty$.

If $L\in\mathcal{L}_4(n)$ then $a\leqslant\sqrt{n}$ or $c\leqslant\sqrt{n}$, so

$$\begin{aligned}n\mathsf{m}(L^{\perp}) - N\mathsf{g} = n\underbrace{(\mathsf{m}(K(L))-\mathsf{g})}_{\leqslant -\lambda} + n\underbrace{(\mathsf{m}(L^{\perp})-\mathsf{m}(K(L)))}_{\leqslant C_8\sqrt{n}}\\ -\mathsf{g}(N-n),\end{aligned}$$

and therefore

$$\frac{\rho_4(N)}{e^{\mathsf{g}N}} = \sum_{n\leqslant N}\frac{1}{n}\sum_{L\in\mathcal{L}_4(N)}\exp\left(n\mathsf{m}(L^{\perp}) - N\mathsf{g}\right)\to 0$$

as $N\to\infty$.

Now

$$\begin{aligned}\frac{\rho_2(N)}{e^{\mathsf{g}N}} &= \exp(-\mathsf{g}N)\sum\tfrac{1}{ac}\exp(ac(\mathsf{m}(L(a,b,c)^{\perp})-\mathsf{g}))\exp(ac\mathsf{g})\\ &= \overbrace{\exp(-\mathsf{g}N)\sum 1}^{\Upsilon(N)} + \mathrm{o}(1),\end{aligned}$$

where the sum runs over all positive integers $a,b,c$ such that $ac\leqslant N$, $b\leqslant a-1$, $a\in A$ and $(b,c)\notin B$. If $a\in A$ and $(b,c)\in B_a(N/a)$, then $K(L(a,b,c)) = J(a)$, so $\mathsf{m}(K(L)) = \mathsf{g}$ and

$$|ac\left(\mathsf{m}(L(a,b,c)^{\perp})-\mathsf{g}\right)|\leqslant C_8 a.$$

Thus $\Upsilon(N)$ lies between

$$\exp(-\mathsf{g}N)\sum_{a\in A}\frac{\exp(-C_8 a)}{a}\sum_{(b,c)\in B_a(N/a)}\frac{1}{c}\exp(ac\mathsf{g})$$

and

$$\exp(\mathsf{g}N)\sum_{a\in A}\frac{\exp(-C_8 a)}{a}\sum_{(b,c)\in B_a(N/a)}\frac{1}{c}\exp(ac\mathsf{g}).$$

A similar argument applies to $\rho_3(N)$. This gives the lower bound in (2.3) by considering a single value of $a$, and the upper bound by easy estimates. □

## 3. *Examples*

Theorem 1 is a weak result – in that it does not give a single asymptotic – and it only applies when $\mathsf{g}$ exceeds $\mathsf{h}$. This section provides exact asymptotics for examples in both the cases $\mathsf{g} > \mathsf{h}$ and $\mathsf{g} = \mathsf{h}$, and shows that there are actions defined by non-constant polynomials that behave more like the $\mathbb{Z}^2$-actions in [**12**]. That is, there are examples beyond full-shifts for which $\mathcal{M}(N)$ behaves like $N$ rather than $\log N$.

EXAMPLE 3. Let $f(x,y) = 2 + xy^2$, so that $\mathsf{h} = \log 2$ (see [**6**]; this and all subsequent integrations may be carried out using Jensen's formula). Moreover,

$$\begin{aligned}\mathsf{m}(J(a)) &= \frac{1}{a}\sum_{j=0}^{a-1}\int_0^1 \log|2 + \mathrm{e}^{2\pi \mathrm{i} j/a}\mathrm{e}^{4\pi \mathrm{i} t}|\,\mathrm{d}t\\ &= \log 2.\end{aligned}$$

We calculate $\mathsf{m}(J(b,c))$ by exploiting the periodicity of $(s,t) \mapsto \mathrm{L}_f(s,t)$:

$$\mathsf{m}(J(b,c)) = \frac{1}{\gcd(b,c)}\sum_{\ell=0}^{\gcd(b,c)-1}\int_0^1 \log|2 + \xi_{\ell,c}\mathrm{e}^{2\pi \mathrm{i}(c-2b)t}|\,\mathrm{d}t,$$

where $\xi_{\ell,c} = \mathrm{e}^{4\pi \mathrm{i}\ell/c}$.

If $c \neq 2b$ then $\mathsf{m}(J(b,c)) = \log 2$. If $c = 2b$ then the integrand is $\log|2 + \xi_{\ell,c}|$, so

$$\begin{aligned}\mathsf{m}(J(b,c)) &= \frac{1}{b}\sum_{\ell=0}^{b-1}\log|2 + \xi_{\ell,2b}|\\ &= \frac{1}{b}\log\prod_{\ell=0}^{b-1}|2 + \xi_{\ell,2b}|\\ &= \frac{1}{b}\log(2^b - (-1)^b),\end{aligned}$$

which is $\log 3$ when $c = 2b = 2$ and is strictly smaller than $\log 3$ otherwise. It follows that $\mathsf{g} = \log 3$. Following the proof of Theorem 1, the significant contribution to $\mathcal{M}_1(N)$ comes from $\mathcal{N}_3(N)$ in (2.7). Now

$$\begin{aligned}\mathsf{F}(L(a,1,2)) &= \prod_{(s,t)\in L(a,1,2)^{\perp}} |f(\mathrm{e}^{2\pi \mathrm{i}s}, \mathrm{e}^{2\pi \mathrm{i}t})|\\ &= \prod_{j=0}^{a-1}\prod_{k=0}^{1}|2 + \mathrm{e}^{2\pi \mathrm{i}j/a}\mathrm{e}^{4\pi \mathrm{i}(k/2 - j/2a)}| = 3^{2a}.\end{aligned}$$

Thus

$$\mathcal{M}(N) = \mathcal{N}_3(N) + \mathrm{O}(1) = \tfrac{1}{2}\log N + \mathrm{O}(1).$$

EXAMPLE 4. Let $f(x,y) = 3 + x + y$, so that $\mathsf{h} = \log 3$. From Lind [**10**] we have

$$\begin{aligned}\mathsf{m}(J(0,1)) &= \log 4,\\ \mathsf{m}(J(1)) &= \log 4,\\ \mathsf{m}(K(L(a,b,c))) &< \log 4 \quad (\text{for } (b,c) \neq (0,1)), \text{ and}\\ \mathsf{m}(K(L(a,b,c))) &< \log 4 \quad (\text{for } a \neq 1).\end{aligned}$$

Thus we must take both $\mathcal{N}_2(N)$ and $\mathcal{N}_3(N)$ into account. A calculation using circulants shows that

$$\mathsf{F}(L(a,0,1)) = 4^a - (-1)^a$$

and

$$\mathsf{F}(L(1,b,c)) = 4^c - (-1)^c,$$

so

$$\mathcal{N}_2(N) = \sum_{c=2}^{N} \tfrac{1}{c}\left(1 - (-4)^{-c}\right) = \log N + \mathrm{O}(1)$$

and

$$\mathcal{N}_3(N) = \sum_{a=2}^{N} \tfrac{1}{a}\left(1 - (-4)^{-a}\right) = \log N + \mathrm{O}(1).$$

As in the proof of Theorem 1, all other contributions are bounded, so

$$\mathcal{M}(N) = 2\log N + \mathrm{O}(1).$$

EXAMPLE 5. Consider the $d$-dimensional full shift on $b$ symbols, which has $\mathsf{h} = \mathsf{g} = \log b$ (for $d = 2$ this is the case corresponding to the polynomial $f = b$). Then the estimates from [**12**] show that the growth in $\mathcal{M}(N)$ is determined by the main term

$$\sum_{n \leqslant N} \frac{1}{b^n}\frac{1}{n} \sum_{L \in \mathcal{L}(n)} \mathsf{F}(L),$$

and $\mathsf{F}(L) = b^{[L]}$. Then

$$\begin{aligned}\sum_{n \leqslant N} \frac{b^{-n}\frac{1}{n}a_n(\mathbb{Z}^d)\mathrm{e}^{\mathsf{g}n}}{n^z} &= \sum_{n \geqslant 1} \frac{a_n(\mathbb{Z}^d)}{n^{z+1}} \\ &= \zeta(z+1)\zeta(z)\cdots\zeta(z-d+2),\end{aligned}$$

so by Perron's theorem [**7**] we have

$$\begin{aligned}\mathcal{M}(N) &\sim \mathrm{Res}_{z=d-1}\left(\tfrac{\zeta(z+1)\cdots\zeta(z-d+2)N^z}{z}\right) \\ &= N^{d-1}\prod_{j=2}^{d}\zeta(j)/(d-1) \\ &= N^{d-1}\frac{\pi^{\lfloor\frac{d}{2}\rfloor(\lfloor\frac{d}{2}\rfloor+1)}}{r_d}\prod_{j=1}^{\lfloor(d-1)/2\rfloor}\zeta(2j+1)\end{aligned}$$

for some $r_d \in \mathbb{Q}$ ($r_d \in \mathbb{N}$ for $d \leqslant 11$; the numerator and denominator of $r_d$ as $d$ varies are sequences A159283 and A159282 in the On-line Encyclopedia of Integer Sequences). This gives the main term in the dynamical Mertens' theorem for the full $\mathbb{Z}^d$-shift considered in [**12**] in closed form; the first few expressions are shown in Table 1. The authors admit that this closed form was overlooked in [**12**].

TABLE 1. *Orbit growth for the full $\mathbb{Z}^d$-shift.*

| $d$ | $\mathcal{M}(N)$ |
|---|---|
| 1 | $\log N + \gamma$ |
| 2 | $\frac{1}{6}\pi^2 N$ |
| 3 | $\frac{1}{12}\zeta(3)\pi^2 N^2$ |
| 4 | $\frac{1}{1620}\zeta(3)\pi^6 N^3$ |
| 5 | $\frac{1}{2160}\zeta(3)\zeta(5)\pi^6 N^4$ |
| 6 | $\frac{1}{2551500}\zeta(3)\zeta(5)\pi^{12} N^5$ |
| 7 | $\frac{1}{3061800}\zeta(3)\zeta(5)\zeta(7)\pi^{12} N^6$ |
| 8 | $\frac{1}{33756345000}\zeta(3)\zeta(5)\zeta(7)\pi^{20} N^7$ |

EXAMPLE 6. A simple example beyond the full shift but still with $\mathsf{g} = \mathsf{h}$ is given by $f(x, y) = x - 2$. Here $\mathsf{h} = \log 2$,

$$\begin{aligned}\mathsf{m}(J(a)) &= \frac{1}{a}\sum_{j=0}^{a-1}\int_0^1 \log|\mathrm{e}^{2\pi\mathrm{i}j/a} - 2|\,\mathrm{d}t \\ &= \frac{1}{a}\sum_{j=0}^{a-1}\log|\mathrm{e}^{2\pi\mathrm{i}j/a} - 2| \\ &= \frac{1}{a}\log(2^a - 1),\end{aligned}$$

and

$$\mathsf{m}(J(b,c)) = \frac{1}{c}\sum_{k=0}^{c-1}\int_0^1 \log|\mathrm{e}^{2\pi\mathrm{i}t} - 2|\,\mathrm{d}t = \log 2,$$

so $\mathsf{g} = \log 2$. Now

$$\mathsf{F}(L(a,b,c)) = \prod_{j=0}^{a-1}\prod_{k=0}^{c-1}|\mathrm{e}^{2\pi\mathrm{i}j/a} - 2| = (2^a - 1)^c,$$

so $\mathrm{e}^{-\mathsf{g}ac}\mathsf{F}(L(a,b,c)) \leqslant 1$ and

$$\begin{aligned}\mathcal{M}_1(N) &\leqslant \sum_{\substack{a,b,c\geqslant 1,\\ 0\leqslant b\leqslant a-1; ac\leqslant N}} \frac{1}{ac} \\ &= \sum_{c=1}^{N}\frac{1}{c}\sum_{a=1}^{\lfloor N/c\rfloor} 1 \\ &= \sum_{c=1}^{N}\frac{1}{c}\left(N/c + \mathrm{O}(1)\right) \\ &\leqslant C_{10}N\end{aligned}$$

for some constant $C_{10} > 0$. On the other hand, if $2^a \geqslant N$ then

$$1 - 2^{-a} \geqslant 1 - 1/N$$

so

$$\underbrace{\exp(-\mathsf{g}ac)}_{1/2^{ac}}\mathsf{F}(L(a,b,c)) = (1 - 2^{-a})^c \geqslant (1 - 1/N)^N \geqslant \tfrac{1}{4}$$

for $N \geqslant 2$. It follows that

$$\mathcal{M}_1(N) \geqslant \frac{1}{4}\sum_{c=1}^{N}\frac{1}{c}\sum_{a=\lceil \log_2 N\rceil}^{\lfloor N/c\rfloor} 1$$
$$\geqslant \frac{1}{4}\sum_{c=1}^{\lfloor N/2\log_2 N\rfloor}\frac{1}{c}\left(\lfloor N/c\rfloor - \lceil \log_2 n\rceil\right) \geqslant C_{11}N$$

for some constant $C_{11} > 0$ and all sufficiently large $N$. Thus

$$0 < C_{11}N \leqslant \mathcal{M}(N) \leqslant C_{12}N$$

for all large $N$.

## *References*


**1.** V. Chothi, G. Everest, and T. Ward, '$S$-integer dynamical systems: periodic points', *J. Reine Angew. Math.* **489** (1997), 99–132.
**2.** C. Deninger and K. Schmidt, 'Expansive algebraic actions of discrete residually finite amenable groups and their entropy', *Ergodic Theory Dynam. Systems* **27** (2007), no. 3, 769–786.
**3.** M. Einsiedler and H. Rindler, 'Algebraic actions of the discrete Heisenberg group and other non-abelian groups', *Aequationes Math.* **62** (2001), no. 1-2, 117–135.
**4.** G. Everest, R. Miles, S. Stevens, and T. Ward, 'Orbit-counting in non-hyperbolic dynamical systems', *J. Reine Angew. Math.* **608** (2007), 155–182.
**5.** G. Everest, V. Stangoe, and T. Ward, 'Orbit counting with an isometric direction', in *Algebraic and topological dynamics*, in *Contemp. Math.* **385**, pp. 293–302 (Amer. Math. Soc., Providence, RI, 2005).
**6.** G. Everest and T. Ward, *Heights of polynomials and entropy in algebraic dynamics*, in *Universitext* (Springer-Verlag London Ltd., London, 1999).
**7.** G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series*, in *Cambridge Tracts in Mathematics and Mathematical Physics, No. 18* (Stechert-Hafner, Inc., New York, 1964).
**8.** C. Hermite, 'Sur l'introduction des variables continues dans la théorie des nombres.', *J. Reine Angew. Math.* **41** (1851), 191–216.
**9.** D. Lind, K. Schmidt, and T. Ward, 'Mahler measure and entropy for commuting automorphisms of compact groups', *Invent. Math.* **101** (1990), no. 3, 593–629.
**10.** D. A. Lind, 'A zeta function for $\mathbb{Z}^d$-actions', in *Ergodic theory of $\mathbb{Z}^d$ actions (Warwick, 1993–1994)*, in *London Math. Soc. Lecture Note Ser.* **228**, pp. 433–450 (Cambridge Univ. Press, Cambridge, 1996).
**11.** R. Miles, 'Expansive algebraic actions of countable abelian groups', *Monatsh. Math.* **147** (2006), no. 2, 155–164.
**12.** R. Miles and T. Ward, 'Orbit-counting for nilpotent group shifts', *Proc. Amer. Math. Soc.* **137** (2009), no. 4, 1499–1507.
**13.** A. Pakapongpun and T. Ward, 'Functorial orbit counting', *J. Integer Sequences* **12** (2009), Article 09.2.4.
**14.** W. Parry and M. Pollicott, 'An analogue of the prime number theorem for closed orbits of Axiom A flows', *Ann. of Math. (2)* **118** (1983), no. 3, 573–591.
**15.** K. Schmidt, 'Automorphisms of compact abelian groups and affine varieties', *Proc. London Math. Soc. (3)* **61** (1990), no. 3, 480–496.
**16.** K. Schmidt, *Dynamical systems of algebraic origin*, in *Progress in Mathematics* **128** (Birkhäuser Verlag, Basel, 1995).
**17.** R. Sharp, 'An analogue of Mertens' theorem for closed orbits of Axiom A flows', *Bol. Soc. Brasil. Mat. (N.S.)* **21** (1991), no. 2, 205–229.