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MARKOV PARTITIONS REFLECTING THE GEOMETRY OF $\times 2, \times 3$

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Abstract. We give an explicit geometric description of the $\times 2, \times 3$ system, and use this to study a uniform family of Markov partitions related to those of Wilson and Abramov. The behaviour of these partitions is stable across expansive cones and transitions in this behaviour detect the non-expansive lines.

1. **Introduction.** Markov partitions are a powerful tool in the study of hyperbolic diffeomorphisms of manifolds. Explicit constructions of Markov partitions for hyperbolic toral automorphisms of the 2-torus $\mathbb{T}^2$ in the work of Adler and Weiss [3] are an important paradigmatic example, and in special situations the tight connection between the geometry of the map and the partition found in [3] is extended to automorphisms of $\mathbb{T}^d$ with $d > 2$ by Manning [13]. On the other hand, maps of objects that are not quite manifolds arise naturally in dynamics, notably as attractors of smooth maps in work of Bowen [4] and Williams [20]. Thus a natural extension of the classical theory of smooth maps of compact manifolds concerns maps of solenoids; a useful overview and the history may be found in a paper of Takens [16]. The simplest solenoids are algebraic: compact groups that are locally isometric to products of local fields.

The structure of a tangent space comprising a product of local fields including non-Archimedean ones may be used to study various dynamical properties of automorphisms of solenoids: exotic orbit-growth properties by Chothi, Everest, Miles, Stevens and the first author [6], [9]; entropy and structure of $\mathbb{Z}^d$-actions of entropy rank one by Einsiedler and Lind [7]; topological entropy by Lind and the first author [12], [18].

Our purpose here is to study geometrically natural Markov partitions like those used by Abramov [1] and Wilson [21] for one of the simplest examples in which non-Archimedean directions arise in the tangent space, and to study how the structure of those partitions changes in expansive cones. This gives a simple geometrical instance of the 'subdynamics philosophy' of Boyle and Lind [5]. A combinatorial instance of

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the same kind of structure appears in work of Miles and the first author [15], where it is shown that directional zeta functions detect the non-expansive set for systems of entropy rank one.

In order to do this, we describe the structure of the space obtained by taking the invertible extension of the $\mathbb{N}^2$-action generated by the maps $x \mapsto 2x \pmod 1$ and $x \mapsto 3x \pmod 1$ on the additive circle in a geometric way. To simplify matters we concentrate on this one example: the same kind of construction works in those systems of entropy rank one with an adelic covering space, but is significantly more involved. In principle the Markov and generating properties of the partitions can be shown from our geometric description, but for brevity we deduce some of these properties from Wilson's results.

2. **The geometry of $\times 2, \times 3$.** We make use of a simple version of the adelic machinery; an elegant account may be found in Weil [19, Ch. 4]. We wish to describe the group $X = \widehat{\mathbb{Z}[\frac{1}{6}]}$ of characters on $\mathbb{Z}[\frac{1}{6}]$ and its metric structure: this group carries the automorphisms $\alpha^{(1,0)}$ and $\alpha^{(0,1)}$ dual to the automorphisms $x \mapsto 2x$ and $x \mapsto 3x$ on $\mathbb{Z}[\frac{1}{6}]$, and is a presentation of the invertible extension of the $\mathbb{N}^2$ action generated by $x \mapsto 2x \pmod 1$ and $x \mapsto 3x \pmod 1$ on $\mathbb{T}$. For a prime p, define the local field $\mathbb{Q}_p$ to be the set of formal power series $\sum_{n\geqslant k} a_n p^n$ with digits $a_n \in \{0, 1, \dots, p-1\}$ and $k \in \mathbb{Z}$, and with the non-Archimedean metric $|\cdot|_p$ induced by the p-adic norm $|\sum_{n\geqslant k} a_n p^n|_p = p^{-k}$ if $a_k \neq 0$. Notice that $\mathbb{Q}$ is a subfield of each $\mathbb{Q}_p$ and each $\mathbb{Q}_p$ has a maximal compact subring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_P \leqslant 1\}$.

The homomorphism

$$\begin{array}{rcl} \Delta : \mathbb{Z}[\frac{1}{6}] & \longrightarrow & \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \\ r & \longmapsto & (r, r, r) \end{array}$$

embeds $\mathbb{Z}[\frac{1}{6}]$ as a discrete (and hence closed) subgroup of $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$ with respect to the metric $\mathsf{d}(x, y) = \max\{|x_\infty - y_\infty|, |x_2 - y_2|_2, |x_3 - y_3|_3\}$, where

$$x = (x_\infty, x_2, x_3) \in \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3.$$

Write $G = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$ and $\Gamma = \Delta(\mathbb{Z}[\frac{1}{6}])$. The group X is the quotient G/Γ (this may be seen from Weil [19, Ch. 4]), and we wish to describe this quotient space in a concrete way. In order to motivate this, notice that a toral automorphism may be constructed as follows. The identity map embeds $\mathbb{Z}^d$ as a discrete subgroup of $\mathbb{R}^d$, and a choice of coset representatives for $\mathbb{R}^d/\mathbb{Z}^d$ gives an explicit geometric description of the map induced on the torus by any automorphism of $\mathbb{R}^d$ preserving $\mathbb{Z}^d$. In order to make this note self-contained and to rehearse the kind of calculation needed later, we include the proof of the following two lemmas, which are simple instance of a well-known principle (see Weil [19, Ch. 4] or Hewitt and Ross [11, § II.10, Th. 10.15]).

Lemma 2.1. *The set $F = [0, 1) \times \mathbb{Z}_2 \times \mathbb{Z}_3$ is a fundamental domain for Γ in G.*

Proof. The first step is to check that F is *big* enough: given $x \in G$, can we find some $\gamma = (r, r, r) \in \Gamma$ with $x - \gamma \in F$? To do this, write $\{\sum_{n\geqslant k} a_n p^n\} = \sum_{n=k}^{-1} a_n p^n$ for the fractional part of $x \in \mathbb{Q}_p$; $\{t\}$ for the fractional part and $\lfloor t \rfloor$ for the integer part of $t \in \mathbb{R}$. A calculation shows that if

$$r = \{x_2\} + \{x_3\} + \lfloor (x_\infty - \{x_2\} - \{x_3\} \rfloor$$

then $r \in \mathbb{Z}[\frac{1}{6}]$ and $(x_\infty - r, x_2 - r, x_3 - r) \in F$ as required.

The second step is to check that F is *small* enough: if $x, y \in F$ define the same coset of Γ then they are equal. Assume therefore that $x, y \in F$ are given with the property that $x - y = (r, r, r) \in \Gamma$. Then $x_2 - y_2 \in \mathbb{Z}_2 \cap \mathbb{Z}[\frac{1}{6}] = \mathbb{Z}[\frac{1}{3}]$ and $x_3 - y_3 \in \mathbb{Z}_3 \cap \mathbb{Z}[\frac{1}{6}] = \mathbb{Z}[\frac{1}{2}]$, so $r \in \mathbb{Z}[\frac{1}{3}] \cap \mathbb{Z}[\frac{1}{2}] = \mathbb{Z}$, and therefore $\{x_\infty\} = \{y_\infty\}$, so $x_\infty = y_\infty$ and $r = 0$ as required. □

This means that there is a bijection $G/\Gamma \longleftrightarrow F$; to go further we need to describe the image of the group operation on G/Γ under this bijection.

Lemma 2.2. *For $s, t \in G$,*

$$(t + \Gamma) + (s + \Gamma) = (\{t_\infty + s_\infty\}, t_2 + s_2 - \lfloor t_\infty + s_\infty \rfloor, t_3 + s_3 - \lfloor t_\infty + s_\infty \rfloor) + \Gamma$$

is the unique coset representative for $t + s$ in F.

Proof. We wish to find the unique $u \in F$ with the property that there is some (r, r, r) in Γ with $u = t + s - r$. We must have $u_\infty = \{t_\infty + s_\infty\}$, which forces r to be $\lfloor t_\infty + s_\infty \rfloor$; notice that we also then have

$$u_2 = t_2 + s_2 - \lfloor t_\infty + s_\infty \rfloor \in \mathbb{Z}_2$$

and

$$u_3 = t_3 + s_3 - \lfloor t_\infty + s_\infty \rfloor \in \mathbb{Z}_3$$

since $\mathbb{Z}_2, \mathbb{Z}_3$ are rings. □

Lemma 2.2 may be written as follows: the operation

$$t \rtimes s = (\{t_\infty + s_\infty\}, t_2 + s_2 - \lfloor t_\infty + s_\infty \rfloor, t_3 + s_3 - \lfloor t_\infty + s_\infty \rfloor) \tag{1}$$

makes F into a group $X = (F, \rtimes)$ isomorphic to G/Γ. An explicit metric on X is given by

$$\overline{\mathsf{d}}(x + \Gamma, y + \Gamma) = \min_{r \in \mathbb{Z}[\frac{1}{6}]} \max\{|x_\infty - y_\infty + r|_\infty, |x_2 - y_2 + r|_2, |x_3 - y_3 + r|_3\}.$$

Wilson [21] describes the same solenoid in a different way, as a projective limit of circles

$$X \cong \{z \in \mathbb{T}^{\mathbb{N}_0} \mid 6z_{k+1} = z_k \pmod 1 \text{ for all } k \geqslant 1\}; \tag{2}$$

points z, z' in this description are close if their coordinates z_k, z'_k are close in $\mathbb{T}$ for $1 \leqslant k \leqslant K$ for large K. The isomorphism in (2) may be thought of as follows. A given point $z = (z_k)_{k \geqslant 0}$ in the right-hand side of (2) defines an element $z_0 \in \mathbb{T}$; each choice of z_{k+1} given z_k defines a unique pair $x_2^{(k)} \in \{0, 1\}$ and $x_3^{(k)} \in \{0, 1, 2\}$ satisfying $z_{k+1} = \frac{1}{6} z_k + \frac{x_2^{(k)}}{2} + \frac{x_3^{(k)}}{6}$ (thinking of z_{k+1} as a real number in $[0, 1)$). The isomorphism is then defined by sending z to the point $\left(z_0, \sum_{k \geqslant 0} x_2^{(k)} 2^k, \sum_{k \geqslant 0} x_3^{(k)} 3^k\right) \in X$. This isomorphism respects the metric structures (nearby points in X correspond to nearby points in the projective limit) and is equivariant with respect to the automorphisms we study. The automorphisms $\alpha^{(1,0)} : x \mapsto 2x$ and $\alpha^{(0,1)} : x \mapsto 3x$ on G preserve Γ and therefore define automorphisms of $X = (F, \rtimes)$.

To see how the group X works, we compute the automorphisms $\alpha^{(0,1)}$ (multiplication by 3), $\alpha^{(-1,0)}$ (multiplication by $\frac{1}{2}$), and $\alpha^{(-1,1)}$ (multiplication by $\frac{3}{2}$) explicitly. By (1),

$$\alpha^{(0,1)}(x) = x \rtimes x \rtimes x = (\{3x_\infty\}, 3x_2 - \lfloor 3x_\infty \rfloor, 3x_3 - \lfloor 3x_\infty \rfloor).$$

TABLE 1. Stable and unstable directions.

region	$\mathbb{R}$	$\mathbb{Q}_2$	$\mathbb{Q}_3$
$a>0, b>0$	u	s	s
$a<0, b>0, 2^a3^b>1$	u	u	s
$a>0, b<0, 2^a3^b>1$	u	s	u
$a<0, b<0$	s	u	u
$a>0, b<0, 2^a3^b<1$	s	s	u
$a<0, b>0, 2^a3^b<1$	s	u	s

Notice that the map $\alpha^{(0,1)}$ locally *expands* the real component by a factor of 3, locally *contracts* the 3-adic component by a factor of 3, and is a local isometry on the 2-adic component.

Write $x_p = \sum_{n\geqslant k} x_p^{(n)} p^n$ with digits $x_p^{(n)} \in \{0,1,\dots,p-1\}$ for $n \geqslant k$. Then

$$\alpha^{(-1,0)}(x) = \left(\tfrac{1}{2} + \tfrac{1}{2}x_2^{(0)}, \tfrac{1}{2}x_2 + \tfrac{1}{2}x_2^{(0)}, \tfrac{1}{2}x_3 + \tfrac{1}{2}x_2^{(0)}\right)$$

(this is most easily verified by doubling the right-hand side).

Finally, by combining the two calculations we see that $\alpha^{(-1,1)}(x)$ is

$$\left(\left\{\tfrac{3}{2}x_\infty + \tfrac{3}{2}x_2^{(0)}\right\}, \tfrac{3}{2}x_2 + \tfrac{3}{2}x_2^{(0)} - \left\lfloor \tfrac{3}{2}x_\infty + \tfrac{3}{2}x_2^{(0)} \right\rfloor, \tfrac{3}{2}x_3 + \tfrac{3}{2}x_2^{(0)} - \left\lfloor \tfrac{3}{2}x_\infty + \tfrac{3}{2}x_2^{(0)} \right\rfloor\right).$$

Locally the action of $\alpha^{(a,b)}$ multiplies by 2^a3^b, and therefore acts on each of the three directions in the tangent space as shown in Table 1 (u, s denote unstable and stable directions).

The first three regions shown in Table 1 are the expansive regions in the sense of [5] and [8] (expansive regions are defined in the Grassmannian space of lines in $\mathbb{R}^2$, of which the circle is a two-fold cover; the table shows the six regions in the cover). There are three non-expansive lines $a = 0$ (containing maps like $\alpha^{(0,1)}$, which behaves like an isometry on the 2-adic direction), $b = 0$ (containing maps like $\alpha^{(1,0)}$, which behaves like an isometry on the 3-adic direction) and $2^a3^b = 1$ (which does not contain any lattice points, but has a sequence of lattice points (a_k, b_k) converging to it with the property that the real Lyapunov exponent $\log|2^{a_k}3^{b_k}|$ of the map $\alpha^{(a_k,b_k)}$ converges to zero as $k \to \infty$).

3. **Stable Markov partitions.** It is clear that there cannot be a single finite partition that is generating for all the maps $\alpha^{(a,b)}$ as (a,b) varies inside an expansive cone because the set of topological entropies of the maps in a cone is unbounded. Thus, what we mean by "stable" is that the Markov partition for $\alpha^{(a,b)}$ is constructed in a uniform manner across all $(a,b) \in \mathbb{Z}^2$. We will see later that the geometry of how the map acts on an atom of the partition is uniform across each expansive cone but changes at each non-expansive direction.

Recall that the naïve height (in the sense of Diophantine geometry) of a non-zero rational r/s is defined to be $H(r/s) = \max\{|r|,|s|\}$. Thus Abramov's formula [1] for the entropy of an automorphism of a one-dimensional solenoid may be written $h(T) = \log H(r/s)$ if T is the map dual to multiplication by r/s.

Main Theorem. *For each $(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ let $\xi^{(a,b)}$ denote the partition*

$$\left\{A_j = \left[\tfrac{j}{H(2^a3^b)}, \tfrac{j+1}{H(2^a3^b)}\right) \times \mathbb{Z}_2 \times \mathbb{Z}_3 \mid 0 \leqslant j < H(2^a3^b)\right\}.$$

Then $\{\xi^{(a,b)}\}$ is a stable family of Markov partitions whose geometry detects the non-expansive directions of α. The partition $\xi^{(a,b)}$ is generating for $\alpha^{(a,b)}$ if and only if $\alpha^{(a,b)}$ is expansive.

The theory of Markov partitions in the topological (rather than smooth) setting is developed by Adler [2]; by 'Markov' we mean that the partition obtained from $\xi^{(a,b)}$ by using open intervals in the real coordinate instead of half-open intervals satisfies [2, Def. 6.1]. Much of the proof of Theorem 3 will use results from Wilson [21] that conceal the geometry of the actions. In order to see how the maps act geometrically, we illustrate the result by describing the partition and the action of the map on the partition in some representative directions. In each figure the image of the atom A_0 of the partition is shaded.

Example 3.1. Consider the direction $(1,0)$, with corresponding map

$$\alpha^{(1,0)}(x) = (\{2x_\infty\}, 2x_2 - \lfloor 2x_\infty \rfloor, 2x_2 - \lfloor 2x_\infty \rfloor).$$

The partition $\xi^{(1,0)}$ simply divides the real coordinate into $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$. We compute

$$\alpha^{(1,0)}(\xi^{(1,0)}) = \{[0,1) \times 2\mathbb{Z}_2 \times \mathbb{Z}_3, [0,1) \times (1 + 2\mathbb{Z}_2) \times \mathbb{Z}_3\}$$

and

$$\alpha^{(2,0)}(\xi^{(1,0)}) = \{[0,1) \times (4\mathbb{Z}_2 \cup 1 + 4\mathbb{Z}_2) \times \mathbb{Z}_3, [0,1) \times (2 + 4\mathbb{Z}_2 \cup 3 + 4\mathbb{Z}_2) \times \mathbb{Z}_3\}.$$

Similarly,

$$\alpha^{(-1,0)}(\xi^{(1,0)}) = \{([0, \tfrac{1}{4}) \cup [\tfrac{1}{2}, \tfrac{3}{4})) \times \mathbb{Z}_2 \times \mathbb{Z}_3, ([\tfrac{1}{4}, \tfrac{1}{2}) \cup [\tfrac{3}{4}, 1)) \times \mathbb{Z}_2 \times \mathbb{Z}_3\}$$

and $\alpha^{(-2,0)}(\xi^{(1,0)})$ is the partition into the sets

$$\left([0, \tfrac{1}{8}) \cup [\tfrac{1}{4}, \tfrac{3}{8}) \cup [\tfrac{1}{2}, \tfrac{5}{8}) \cup [\tfrac{3}{4}, \tfrac{7}{8})\right) \times \mathbb{Z}_2 \times \mathbb{Z}_3$$

and

$$\left([\tfrac{1}{8}, \tfrac{1}{4}) \cup [\tfrac{3}{8}, \tfrac{1}{2}) \cup [\tfrac{5}{8}, \tfrac{3}{4}) \cup [\tfrac{7}{8}, 1)\right) \times \mathbb{Z}_2 \times \mathbb{Z}_3.$$

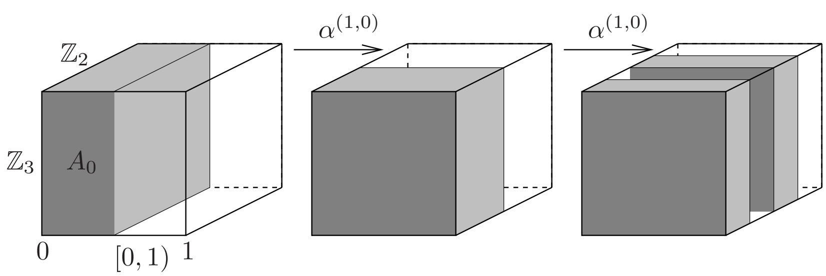

FIGURE 1. $\xi^{(1,0)}$, $\alpha^{(1,0)}(\xi^{(1,0)})$ and $\alpha^{(2,0)}(\xi^{(1,0)})$.

These partitions are illustrated in Figure 1 for the forward direction and Figure 2 for the reverse direction. Notice that $\bigvee_{n=-\infty}^{\infty} \alpha^{(n,0)}(\xi^{(1,0)})$ does not separate the $\mathbb{Z}_3$ coordinate, so the partition is not generating for $\alpha^{(1,0)}$. However, this does show that the system $(X, \alpha^{(1,0)})$ may be realized as an isometric extension of a base system (which is an almost 1-1 image of a full Bernoulli 2-shift) by $\mathbb{Z}_3$.

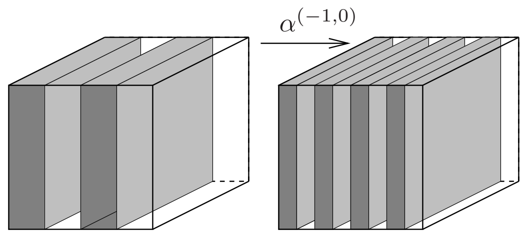

FIGURE 2. $\alpha^{(-1,0)}(\xi^{(1,0)})$ and $\alpha^{(-2,0)})(\xi^{(1,0)})$.

Example 3.2. The expansive region $ab > 0$ is particularly simple because the system $(X, \alpha^{(a,b)})$ is (at each point with $a > 0, b > 0$) simply the invertible extension of the map $x \mapsto 2^a 3^b x \pmod 1$ on the circle, and $\xi^{(a,b)}$ is the usual partition into intervals of width $\frac{1}{2^a 3^b}$ on $[0, 1)$ lifted to X. The action of $\alpha^{(1,1)}$ (multiplication by 6) is illustrated in Figure 3 for the forward direction and Figure 4 for the reverse direction.

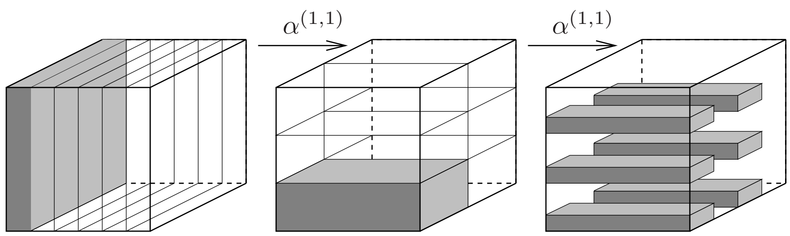

FIGURE 3. $\xi^{(1,1)}$, $\alpha^{(1,1)}(\xi^{(1,1)})$ and $\alpha^{(2,2)}(\xi^{(1,1)})$.

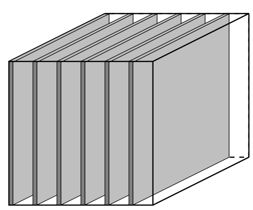

FIGURE 4. $\alpha^{(-1,-1)}(\xi^{(1,1)})$.

Example 3.3. Now consider the map $\alpha^{(-1,1)}$ (multiplication by $\frac{3}{2}$). For this map the real and the 2-adic directions are unstable and the 3-adic direction is stable. The partition $\xi^{(-1,1)}$ divides the real coordinate into three pieces. A calculation shows that $\alpha^{(-1,1)}(\xi^{(-1,1)})$ consists of the sets

$$[0, \tfrac{1}{2}) \times \mathbb{Z}_2 \times 3\mathbb{Z}_3 \cup [\tfrac{1}{2}, 1) \times \mathbb{Z}_2 \times (3\mathbb{Z}_3 + 2),$$

$$[\tfrac{1}{2}, 1) \times \mathbb{Z}_2 \times 3\mathbb{Z}_3 \cup [0, \tfrac{1}{2}) \times \mathbb{Z}_2 \times (3\mathbb{Z}_3 + 1),$$

and

$$[0, \tfrac{1}{2}) \times \mathbb{Z}_2 \times (3\mathbb{Z}_3 + 2) \cup [\tfrac{1}{2}, 1) \times \mathbb{Z}_2 \times (3\mathbb{Z}_3 + 1).$$

The image of A_0 under the maps $\alpha^{(-1,1)}$ and $\alpha^{(1,-1)}$ are shown in Figure 5.

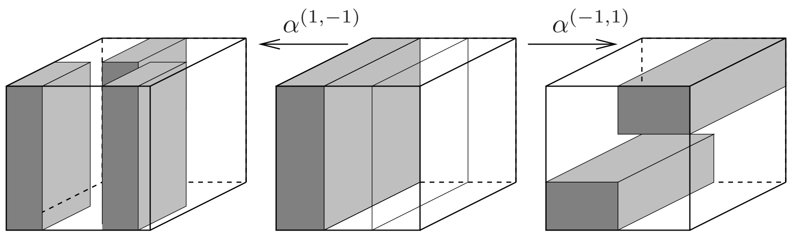

FIGURE 5. $\alpha^{(1,-1)}(\xi^{(-1,1)})$, $\xi^{(-1,1)}$ and $\alpha^{(-1,1)}(\xi^{(-1,1)})$.

A similar calculation shows that $\alpha^{(1,-1)}(\xi^{(-1,1)})$ consists of the sets

$$[0,\tfrac{2}{9})\times 2\mathbb{Z}_2\times\mathbb{Z}_3\cup[\tfrac{1}{3},\tfrac{5}{9})\times(1+2\mathbb{Z}_2)\times\mathbb{Z}_3\cup[\tfrac{2}{3},\tfrac{8}{9})\times 2\mathbb{Z}_2\times\mathbb{Z}_3,$$

$$[\tfrac{2}{9},\tfrac{4}{9})\times 2\mathbb{Z}_2\times\mathbb{Z}_3\cup[\tfrac{5}{9},\tfrac{7}{9})\times(1+2\mathbb{Z}_2)\times\mathbb{Z}_3\cup\big([\tfrac{8}{9},1)\times 2\mathbb{Z}_2\times\mathbb{Z}_3\cup[0,\tfrac{1}{9})\times(1+2\mathbb{Z}_2)\times\mathbb{Z}_3\big),$$

and the complement of their union. Notice that (for example) $\alpha^{-1}A_0\cap A_1\cap\alpha A_0$ does not consist of a single rectangle.

Proof of Theorem 3 in the region $ab>0$. Assume first that a and b are both positive, so that $2^a3^b\in\mathbb{N}$, and write $\alpha=\alpha^{(a,b)}$, $\xi=\xi^{(a,b)}$ throughout; the partitions $\alpha^{-1}(\xi),\xi,\alpha(\xi)$ are illustrated in Figures 3 and 4 with the image and pre-image of A_0 shaded for the case $(a,b)=(1,1)$. We claim that the combinatorics of a full shift on 6 symbols suggested by Figures 3 and 4 is indeed the case. This (and other steps flagged below) may in principle be extracted from Wilson's paper [21] but we prove it here to show how the map works. We first need to check that an atom of the form

$$\alpha(A_{i_1})\cap\cdots\cap\alpha^n(A_{i_n}),$$

for any choice of $i_1,\dots,i_n\in\{0,1,\dots,2^a3^b-1\}$, is a rectangle of the shape

$$[0,1)\times(t_n+2^{an}\mathbb{Z}_2)\times(s_n+3^{bn}\mathbb{Z}_3)$$

with an explicit description of $t_n\in\{0,1,\dots 2^{an}-1\}$ and $s_n\in\{0,1,\dots,3^{bn}-1\}$. In order to do this, we need some notation for the sets arising as the map is iterated. The first iteration is straightforward, and we can write

$$\alpha(A_k)=[0,1)\times(2^a\mathbb{Z}_2-k)\times(3^b\mathbb{Z}_3-k)$$

for $0\leqslant k\leqslant 2^a3^b-1$. The next iteration is more complicated, because the image involves reduction modulo Γ. We compute

$$\alpha^2(A_k)=\bigsqcup_{\ell_1=0}^{2^a3^b-1}A_{k,\ell_1},\tag{3}$$

where

$$A_{k,\ell_1}=[0,1)\times\big(2^a3^b(2^a\mathbb{Z}_2-k)-\ell_1\big)\times\big(2^a3^b(3^b\mathbb{Z}_3-k)-\ell_1\big)$$

($\sqcup$ denoting a disjoint union). Continue, arriving at the notation

$$\alpha^n(A_k)=\bigsqcup_{\ell_1=0}^{2^a3^b-1}\cdots\bigsqcup_{\ell_{n-1}=0}^{2^a3^b-1}A_{k,\ell_1,\dots,\ell_{n-1}}\tag{4}$$

for $n\geqslant 2$, in which each $A_{k,\ell_1,\dots,\ell_{n-1}}$ is a set of the form

$$[0,1)\times(2^{an}\mathbb{Z}_2-C(k,\ell_1,\dots,\ell_{n-2})-\ell_{n-1})\times\big(3^{bn}\mathbb{Z}_3-C(k,\ell_1,\dots,\ell_{n-2})-\ell_{n-1}\big)$$

where

$$C(k,\ell_1,\dots,\ell_{n-2}) = k(2^a3^b)^{n-1} + \ell_1(2^a3^b)^{n-2} + \ell_2(2^a3^b)^{n-3} + \cdots + \ell_{n-2}2^a3^b.$$

Using this description, we claim that an atom in $\bigvee_{j=1}^{n}\alpha^j(\xi)$ can be written in the form

$$\alpha(A_{i_1}) \cap \alpha^2(A_{i_2}) \cap \cdots \cap \alpha^n(A_{i_n}) = A_{i_n,i_{n-1},\dots,i_1} \tag{5}$$

for $n \geqslant 2$ and some $0 \leqslant i_j < 2^a3^b$, $1 \leqslant j \leqslant n$ where the right-hand side is defined as above.

We prove the claim in (5) by induction on the length n starting with $n = 2$. Clearly $\alpha^2(A_{i_2}) \supseteq A_{i_2,i_1}$ by definition. Now

$$\begin{aligned} A_{i_2,i_1} &= [0,1) \times \left(2^{2a}\mathbb{Z}_2 - i_2 2^a 3^b - i_1\right) \times \left(3^{2b}\mathbb{Z}_3 - i_2 2^a 3^b - i_1\right) \\ &\subseteq [0,1) \times (2^a\mathbb{Z}_2 - i_1) \times \left(3^b\mathbb{Z}_3 - i_1\right) = \alpha(A_{i_1}) \end{aligned}$$

since $i_2 2^a 3^b\mathbb{Z}_2 \subseteq 2^a\mathbb{Z}_2$, and similarly for the other terms, so $\alpha(A_{i_1}) \supseteq A_{i_2,i_1}$. Thus

$$\alpha(A_{i_1}) \cap \alpha^2(A_{i_2}) \supseteq A_{i_2,i_1}.$$

We now claim that $\alpha(A_{i_1}) \cap \alpha^2(A_{i_2}) = A_{i_2,i_1}$ by using (3) and showing that

$$A_{i_2,\ell} \cap \alpha(A_{i_1}) \neq \emptyset$$

for $0 \leqslant \ell < 2^a3^b$ implies that $\ell = i_1$. To see this, note first that if $A_{i_2,\ell} \cap \alpha(A_{i_1}) \neq \emptyset$ then $A_{i_2,\ell} \subset \alpha(A_{i_1})$. Suppose that there is some $i_1' \neq i_1$, both in $\{0,\dots,2^a3^b-1\}$, with $A_{i_2,i_1'} \cap \alpha(A_{i_1}) \neq \emptyset$. Then $i_1 - i_1' = 2^a k_1$ and $i_1 - i_1' = 3^b k_2$ for some $k_1, k_2 \in \mathbb{Z}$, so (since 2 and 3 are coprime), $i_1 \equiv i_1' \pmod{2^a3^b}$ and therefore $i_1 = i_1'$.

Now assume that (5) holds for $n \leqslant k$. First notice that

$$\alpha^{k+1}(A_{i_{k+1}}) \supset A_{i_{k+1},i_k,\dots,i_1},$$

and we claim that

$$A_{i_k,\dots,i_1} \supset A_{i_{k+1},i_k,\dots,i_1}. \tag{6}$$

Since

$$\begin{aligned} A_{i_k,\dots,i_1} &= [0,1) \times \left(2^{ak}\mathbb{Z}_2 - i_k(2^a3^b)^{k-1} - i_{k-1}(2^a3^b)^{k-2} - \cdots - i_1\right) \\ &\qquad \times \left(3^{bk}\mathbb{Z}_3 - i_k(2^a3^b)^{k-1} - i_{k-1}(2^a3^b)^{k-2} - \cdots - i_1\right), \end{aligned}$$

$$\begin{aligned} A_{i_{k+1},i_k,\dots,i_1} &= [0,1) \times \Big(2^{a(k+1)}\mathbb{Z}_2 - i_{k+1}(2^a3^b)^{k} - i_k(2^a3^b)^{k-1} - \cdots - i_1\Big) \\ &\qquad \times \Big(3^{b(k+1)}\mathbb{Z}_3 - i_{k+1}(2^a3^b)^{k} - i_k(2^a3^b)^{k-1} - \cdots - i_1\Big), \end{aligned}$$

and

$$\begin{aligned} 2^{a(k+1)}\mathbb{Z}_2 &\subseteq 2^{ak}\mathbb{Z}_2 \\ 3^{b(k+1)}\mathbb{Z}_3 &\subseteq 3^{bk}\mathbb{Z}_3 \\ i_{k+1}(2^a3^b)^k\mathbb{Z}_2 &\subseteq 2^{ak}\mathbb{Z}_2, \\ i_{k+1}(2^a3^b)^k\mathbb{Z}_3 &\subseteq 3^{bk}\mathbb{Z}_3, \end{aligned}$$

we have (6), and therefore

$$A_{i_k,\dots,i_1} \cap \alpha^{k+1}(A_{i_{k+1}}) \supseteq A_{i_{k+1},\dots,i_1}. \tag{7}$$

We now claim that there is equality in (7). To see this, assume that there is a choice of $\ell_1,\dots,\ell_k \in \{0,\dots,2^a3^b-1\}$ with $A_{i_{k+1},\ell_k,\dots,\ell_1} \cap A_{i_k,i_{k-1},\dots,i_1} \neq \emptyset$. By (6), and noting that $A_{i_{k+1},\ell_k,\dots,\ell_1} \cap A_{i_k,i_{k-1},\dots,i_1} \neq \emptyset$ implies that $A_{i_{k+1},\ell_k,\dots,\ell_1}$ is a subset of $A_{i_k,i_{k-1},\dots,i_1}$, it follows that

$$2^{a(k+1)}\mathbb{Z}_2 - i_{k+1}(2^a 3^b)^k - i_k(2^a 3^b)^{k-1} - \cdots - i_1$$

and

$$2^{a(k+1)}\mathbb{Z}_2 - i_{k+1}(2^a 3^b)^k - \ell_k(2^a 3^b)^{k-1} - \cdots - \ell_1$$

are both subsets of

$$2^{ak}\mathbb{Z}_2 - i_k(2^a 3^b)^{k-1} - \cdots - i_1,$$

and similarly for the $\mathbb{Z}_3$ component. Thus

$$(i_k - \ell_k)(2^a 3^b)^{k-1} + (i_{k-1} - \ell_{k-1})(2^a 3^b)^{k-2} + \cdots + (i_1 - \ell_1) \equiv 0 \pmod{(2^a 3^b)^k}.$$

Reducing this identity modulo $2^a 3^b$ shows that $i_1 = \ell_1$, reducing modulo $(2^a 3^b)^2$ shows that $i_2 = \ell_2$, and so on. Using (4), it follows that there is equality in (7) as required, proving (5).

Now we consider an atom of the form

$$A_{i_0} \cap \alpha^{-1}(A_{i_1}) \cap \cdots \cap \alpha^{-n}(A_{i_n});$$

we wish to prove a statement like (5) for these atoms, by showing that each such atom is a rectangle of the form $J \times \mathbb{Z}_2 \times \mathbb{Z}_3$ for an explicitly described interval J of width $\frac{1}{(2^a 3^b)^{n+1}}$. A calculation shows that

$$\alpha^{-1}(A_k) = \bigsqcup_{\ell=0}^{2^a 3^b - 1} \left[\tfrac{k}{(2^a 3^b)^2} + \tfrac{\ell}{2^a 3^b}, \tfrac{k+1}{(2^a 3^b)^2} + \tfrac{\ell}{2^a 3^b}\right) \times \mathbb{Z}_2 \times \mathbb{Z}_3 = \bigsqcup_{\ell=0}^{2^a 3^b - 1} A^{k,\ell},$$

and in general we have

$$\alpha^{-n}(A_k) = \bigsqcup_{\ell_1=0}^{2^a 3^b - 1} \cdots \bigsqcup_{\ell_n=0}^{2^a 3^b - 1} A^{k,\ell_1,\dots,\ell_n} \tag{8}$$

for $n \geqslant 1$, with

$$A^{k,\ell_1,\dots,\ell_n} = \left[\tfrac{k}{(2^a 3^b)^{n+1}} + D(\ell_1,\dots,\ell_n), \tfrac{k+1}{(2^a 3^b)^{n+1}} + D(\ell_1,\dots,\ell_n)\right) \times \mathbb{Z}_2 \times \mathbb{Z}_3$$

where

$$D(\ell_1,\dots,\ell_n) = \frac{\ell_1}{(2^a 3^b)^n} + \frac{\ell_2}{(2^a 3^b)^{n-1}} + \cdots + \frac{\ell_n}{2^a 3^b}.$$

We claim that

$$A_{i_0} \cap \alpha^{-1}(A_{i_1}) \cap \cdots \cap \alpha^{-n}(A_{i_n}) = A^{i_n,i_{n-1},\dots,i_0}. \tag{9}$$

for $n \geqslant 1$. For $n = 1$,

$$A_{i_0} \cap \alpha^{-1}(A_{i_1}) \supseteq \left[\tfrac{i_0}{2^a 3^b}, \tfrac{i_0+1}{2^a 3^b}\right) \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cap A^{i_1,i_0} = A^{i_1,i_0}$$

since $[\frac{i_0}{2^a 3^b}, \frac{i_0+1}{2^a 3^b}) \supseteq [\frac{i_1}{(2^a 3^b)^2} + \frac{i_0}{2^a 3^b}, \frac{i_1+1}{(2^a 3^b)^2} + \frac{i_0}{2^a 3^b})$. Thus $A_{i_0} \cap \alpha^{-1}(A_{i_1}) = A^{i_1,i_0}$ since the width of the real interval defining A_{i_0} is $\frac{1}{2^a 3^b}$ and by (8) the real coordinates of the sets in $\alpha^{-1}(A_{i_1})$ are intervals, each of width $\frac{1}{(2^a 3^b)^2}$ and with the property that the left end-points of distinct intervals are at least $\frac{1}{2^a 3^b}$ apart.

Now assume that (9) holds for $n \leqslant k$, so that $\bigcap_{j=0}^{k+1} \alpha^{-j}(A_{i_j})$ can be written as the intersection of

$$\left[D(i_k,\dots,i_0), \tfrac{1}{(2^a 3^b)^{k+1}} + D(i_k,\dots,i_0)\right) \times \mathbb{Z}_2 \times \mathbb{Z}_3 = A^{i_k,\dots,i_0}$$

with

$$\bigsqcup_{0\leqslant j_1,\dots,j_{k+1}<2^a3^b} A^{i_{k+1},j_1,\dots,j_{k+1}}.$$

It follows that

$$\bigcap_{j=0}^{k+1}\alpha^{-j}(A_{i_j})\supseteq\Big[D(i_{k+1},\dots,i_0),\tfrac{1}{(2^a3^b)^{k+2}}+D(i_{k+1},\dots,i_0)\Big)\times\mathbb{Z}_2\times\mathbb{Z}_3{=}A^{i_{k+1},\dots,i_0}.$$

Notice that the width of the real interval defining the set $A^{i_k,\dots,i_0}$ is $\frac{1}{(2^a3^b)^{k+1}}$. Now by (8) each member of the real projection of $\alpha^{-(k+1)}(A_{i_{k+1}})$ has length $\frac{1}{(2^a3^b)^{k+2}}$ and each of these intervals has the property that the left end-points of distinct intervals are at least distance $\frac{1}{(2^a3^b)^{k+1}}$ apart, showing (9) for $n=k+1$ and hence for all n by induction.

By (5) and (9), the atom

$$\bigcap_{j=-n}^{n}\alpha^{j}(A_{i_j})=A^{i_n,\dots,i_0}\cap A_{i_n,\dots,i_1}$$

is a rectangle with real width $\frac{1}{(2^a3^b)^{n+1}}$, 2-adic width $\frac{1}{(2^a)^n}$ and 3-adic width $\frac{1}{(3^b)^n}$. It follows that ξ satisfies a strong form of the condition [2, Exercise 6.1]. Moreover,

$$\operatorname{diam}\Big(\bigvee_{j=-n}^{n}\alpha^j(\xi)\Big)\to 0$$

as $n\to\infty$, so ξ is a generating Markov partition in the sense of [2]. □

Proof of Theorem 3 in other regions. Away from the positive and negative quadrants $ab>0$ the behaviour of $\xi=\xi^{(a,b)}$ under the map $\alpha=\alpha^{(a,b)}$ is more complicated. In particular, as seen in Figure 5, an atom in $\xi\vee\alpha\xi$ need not be a rectangle even in expansive directions. However, in an expansive direction the partition ξ corresponds under the map described after (2) to the partition $\pi_0^{-1}S(H(2^a3^b))$ used by Wilson [21, Th. 2.4]. Notice that for any (a,b) in an expansive region, the group Σ_{mn} in the notation of [21], where $\frac{m}{n}=2^a3^b$, is X. Wilson shows that this partition is a Bernoulli generator, so

$$\bigcap_{j=0}^{n}\alpha^j(A_{i_j})\neq\emptyset,\ \bigcap_{j=-n}^{0}\alpha^j(A_{i_j})\neq\emptyset\implies\bigcap_{j=-n}^{n}\alpha^j(A_{i_j})\neq\emptyset$$

(as in [2, Exercise 6.1]); he also shows that an atom in $\bigvee_{j=-n}^{n}\alpha^j(\xi)$ lies inside a cylinder defined by small intervals in many coordinates in the description (2), so

$$\operatorname{diam}\left(\bigvee_{j=-n}^{n}\alpha^j(\xi)\right)\to 0.$$

It follows that ξ is a generating Markov partition for $\alpha^{(a,b)}$.

There are three non-expansive directions, but only two of them contain non-trivial lattice points: Example 3.1 shows that $\xi^{(1,0)}$ is not generating under $\alpha^{(1,0)}$; the other direction $(0,1)$ is similar. □

An impression of the complexity of a generating Markov partition may be gained by comparing the dynamical zeta function of the resulting symbolic cover shift map $\sigma^{(a,b)}$ to the zeta function of the original map $\alpha^{(a,b)}$. In the positive quadrant $a > 0, b > 0$, where we have seen that the partition ξ behaves very simply, we have $\zeta_{\sigma^{(a,b)}}(z) = \frac{1}{1-H(a,b)z}$ while $\zeta_{\alpha^{(a,b)}}(z) = \frac{1-z}{1-H(a,b)z}$ since only one pair of points of each period are identified by the factor map defined by the partition. In contrast, in the region $a < 0, b > 0, 2^a 3^b > 1$ (for example) we have $\zeta_{\sigma^{(a,b)}}(z) = \frac{1}{1-3^b z}$ while $\zeta_{\alpha^{(a,b)}}(z) = \frac{1-2^a z}{1-3^b z}$, reflecting the fact that more periodic points in the full 3^b-shift are identified under the factor map. Finally, in a non-expansive direction (like $a = 1, b = 0$) the zeta function of $\alpha^{(a,b)}$ is not even a rational function (it is shown in [10] that the zeta function has a natural boundary on the circle $|z| = \frac{1}{2}$ in this case; the influence on the zeta function of further directions in which an automorphism of a solenoid acts like an isometry is studied by Miles [14] and the first author [17]).

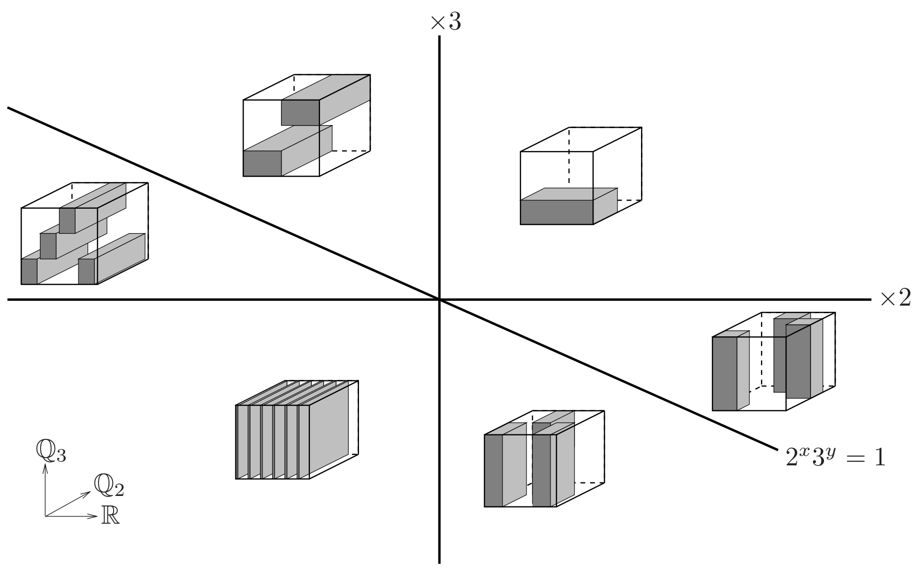

FIGURE 6. Geometry of $\alpha^{(a,b)}(A_0)$ in expansive cones.

The variation in geometrical properties of the partition $\xi^{(a,b)}$ across each expansive cone is illustrated in Figure 6: a representative shape of $\alpha^{(a,b)}(A_0)$ is shown shaded in each expansive cone. The transitions across the axes are clear; at the line $2^x 3^y = 1$ all that changes is the sign of the real Lyapunov exponent.

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