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Periodic point data detects subdynamics in entropy rank one

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Abstract. A framework for understanding the geometry of continuous actions of $\mathbb{Z}^d$ was developed by Boyle and Lind using the notion of expansive behaviour along lower-dimensional subspaces. For algebraic $\mathbb{Z}^d$-actions of entropy rank one, the expansive subdynamics are readily described in terms of Lyapunov exponents. Here we show that periodic point counts for elements of an entropy rank-one action determine the expansive subdynamics. Moreover, the finer structure of the non-expansive set is visible in the topological and smooth structure of a set of functions associated to the periodic point data.

1. *Introduction*

Let β be an action of $\mathbb{Z}^d$ by homeomorphisms of a compact metric space (X, ρ); thus, for each $\mathbf{n} \in \mathbb{Z}^d$ there is an associated homeomorphism $\beta^{\mathbf{n}}$, and $\beta^{\mathbf{m}} \circ \beta^{\mathbf{n}} = \beta^{\mathbf{m}+\mathbf{n}}$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. Such an action is called *expansive* if there is some $\delta > 0$ with the property that if x, y are distinct points in X, then there is some $\mathbf{n}$ for which $\rho(\beta^{\mathbf{n}}x, \beta^{\mathbf{n}}y) > \delta$. Any such δ is called an expansive constant for the action. Boyle and Lind [**1**] introduced the following notion, which reveals a rich geometrical structure inside an expansive action. A subset $A \subset \mathbb{R}^d$ is called *expansive for* β, or β is *expansive along* A, if there exist constants $\delta > 0$ and $t > 0$ with the property that

$$\sup_{\mathbf{n}, d(\mathbf{n}, A) < t} \rho(\beta^{\mathbf{n}}x, \beta^{\mathbf{n}}y) \leq \delta \quad \Longrightarrow \quad x = y \quad \text{for all } x, y \in X$$

where $d(\mathbf{n}, A)$ denotes the distance from the point $\mathbf{n}$ to the set A in the Euclidean metric on $\mathbb{R}^d$. Of particular importance is the behaviour along subspaces. Write G_k for the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$; this is a compact $k(d-k)$-dimensional manifold in the usual topology (subspaces are close if their intersections with the unit $(d-1)$-sphere S_{d-1} are close in the Hausdorff topology). Following Boyle and Lind, write

$$\mathsf{N}_k(\beta) = \{V \in \mathsf{G}_k \mid V \text{ is not expansive for } \beta\}.$$

The main structural result from [**1**] is that if X is infinite, then $\mathsf{N}_{d-1}(\beta)$ is a non-empty compact set, and the set $\mathsf{N}_{d-1}(\beta)$ governs all of the non-expansive behaviour in the sense that any element of $\mathsf{N}_k(\beta)$ must be a subspace of some element of $\mathsf{N}_{d-1}(\beta)$. For algebraic systems, in which X is a compact metric group and each map $\beta^{\mathbf{n}}$ is a continuous group automorphism, the subdynamical structure was determined by Einsiedler *et al* [**6**], where a finer structure was found inside the set $\mathsf{N}_{d-1}(\beta)$ reflecting the two different ways in which an algebraic dynamical system can fail to be expansive.

A different insight into a topological $\mathbb{Z}^d$ action is a combinatorial one coming from periodic points. Write $F_{\mathbf{n}}(\beta) = \{x \in X \mid \beta^{\mathbf{n}}x = x\}$ for the set of points fixed by the homeomorphism $\beta^{\mathbf{n}}$. The combinatorial data of all of these numbers may be thought of as a map

$$\mathbf{n} \mapsto |F_{\mathbf{n}}(\beta)| \in \mathbb{N} \cup \{\infty\},$$

where ∞ denotes the cardinality of an infinite compact group.

Our purpose here is to show that the combinatorial data contained in this map determine the expansive subdynamics for a certain class of systems (Theorem 4.8). These systems are the expansive algebraic systems of entropy rank one. In particular, for these systems the set $F_{\mathbf{n}}(\beta)$ is finite for $\mathbf{n} \neq 0$ except in degenerate situations.

2. *Ranks and subdynamics*

The following notions come from [**6**, §7]. Let β be an action of $\mathbb{Z}^d$ by homeomorphisms of a compact metric space (X, ρ) as before. The *expansive rank* of β is

$$\text{exprk}(\beta) = \min\{k \mid \mathsf{N}_k(\beta) \neq \mathsf{G}_k\},$$

that is the smallest dimension in which some expansive subspaces are seen. The *entropy rank* of β is

$$\text{entrk}(\beta) = \max\{k \mid \text{there is a rational } k\text{-plane } V \text{ with } h(\beta, V) > 0\},$$

where $h(\beta, V)$ denotes the topological entropy of the $\mathbb{Z}^{\dim(V)}$-action given by restricting β to $V \cap \mathbb{Z}^d$. By [**6**, Proposition 7.2],

$$\text{entrk}(\beta) \leq \text{exprk}(\beta).$$

Algebraic $\mathbb{Z}^d$-actions have a convenient description in terms of commutative algebra due to Kitchens and Schmidt [**10**] which we will need. Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in commuting variables $u_1, \dots, u_d$ with integer coefficients. If X is a compact metrizable abelian group and α is a $\mathbb{Z}^d$-action by continuous automorphisms $\alpha^{\mathbf{n}}$ of X, then the Pontryagin dual group $M = \widehat{X}$ has the structure of a discrete countable R_d-module, obtained by first identifying the dual automorphism $\widehat{\alpha}^{\mathbf{n}}$ with multiplication by the monomial $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$, and then extending additively to multiplication by polynomials. Conversely, for any countable R_d-module M, there is an associated $\mathbb{Z}^d$-action on a compact group obtained by dualizing the action induced by multiplying by monomials on M. A full account of this correspondence and the resulting theory is given in Schmidt's monograph [**19**]. An important aspect of this approach is the interpretation of dynamical properties as algebraic properties of M, particularly in terms of the set of associated prime ideals of M, written $\mathrm{Asc}(M)$. We will describe systems as Noetherian if they correspond to Noetherian modules, and in the reverse direction will describe modules as having various dynamical properties if the corresponding system has those properties.

The simplest algebraic systems are those corresponding to cyclic modules $R_d/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \subset R_d$, and these will be called *prime actions*. This gives a third natural notion of 'rank' to an algebraic $\mathbb{Z}^d$-action. Recall that the *Krull dimension* $\mathrm{kdim}(S)$ of a commutative ring S is the maximum of the lengths r taken over all strictly decreasing chains $\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r$ of prime ideals in S (see Matsumura [**14**, Ch. 1, §5]). Boyle and Lind [**1**, Theorem 7.5] show that if $\mathfrak{p}$ is a prime ideal generated by g elements, then

$$\mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}) \geq \mathrm{kdim}(R_d/\mathfrak{p}) \geq d - g$$

and

$$\mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}) \geq d - g + 1.$$

Moreover, [**6**, Proposition 7.3] shows that

$$\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}}) = \mathrm{kdim}(R_d/\mathfrak{p}) \leq \mathrm{exprk}(\alpha_{R_d/\mathfrak{p}})$$

if $\mathfrak{p}$ is non-principal. The *height* $\mathrm{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \subset R_d$ is equal to the Krull dimension of R_d localized at $\mathfrak{p}$, equivalently the maximal length r of a strictly decreasing chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r = (0).$$

The *co-height* $\mathrm{coht}(\mathfrak{p})$ of $\mathfrak{p}$ is equal to the Krull dimension of the domain $R_d/\mathfrak{p}$, equivalently it is the maximal length r of a strictly increasing chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r.$$

The domain R_d is universally catenary [**7**, Proposition 18.9 and Corollary 18.10] and, hence, [**7**, Theorem 13.8] shows that for each $\mathfrak{p} \in R_d$,

$$\mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) = \mathrm{kdim}(R_d) = d + 1.$$

Using associated primes, Einsiedler and Lind [**5**] provide the following classification of entropy rank-one actions for which the associated module M is Noetherian (see Proposition 2.1). When M is not Noetherian, problems arise in relation to finding the set of possible entropy values for general algebraic $\mathbb{Z}^d$-actions; this is closely related to Lehmer's problem and is discussed more fully in [**5**].

PROPOSITION 2.1. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action. Then:*

(1) *α_M has entropy rank one if and only if each of the associated prime actions $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one; equivalently, for each prime $\mathfrak{p} \in \mathrm{Asc}(M)$, coht$(\mathfrak{p}) \leq 1$;*

(2) *α_M has expansive rank one if and only if each of the associated prime actions $\alpha_{R_d/\mathfrak{p}}$ has expansive rank one; and*

(3) *if α_M is expansive, then α_M has expansive rank one if and only if α_M has entropy rank one.*

Proof. See [**5**, Propositions 4.4 and 6.1 and Theorems 7.1 and 7.2]. □

In particular, an expansive rank-one action may also be thought of as an expansive entropy rank-one action. Further properties of entropy rank-one actions are discussed in [**5**] and [**15**]. Of particular importance is the observation that if $\mathrm{coht}(\mathfrak{p}) = 1$, then the field of fractions K of the domain $R_d/\mathfrak{p}$ is a *global field* by [**5**, Proposition 6.1]. Moreover, the places of K, denoted by $\mathcal{P}(K)$, are determined by the ideal $\mathfrak{p}$. From this infinite set of places, we isolate

$$\mathcal{S}_{\mathfrak{p}} = \{w \in \mathcal{P}(K) \mid w \text{ is unbounded on } R_d/\mathfrak{p}\}.$$

Here w being *unbounded* means that $|R_d/\mathfrak{p}|_w$ is an unbounded subset of $\mathbb{R}$. Note that $\mathcal{S}_{\mathfrak{p}}$ contains all of the infinite places of K. Furthermore, $\mathcal{S}_{\mathfrak{p}}$ is finite because $R_d/\mathfrak{p}$ is finitely generated.

The description of expansive subdynamics for algebraic $\mathbb{Z}^d$-actions is further refined in [**6**, §8] to reflect the two ways in which an algebraic dynamical system can fail to be expansive. It can fail in a way which relates to the Noetherian condition for modules, and this failure will result in a set of directions denoted by N^{n}. It can also fail to be expansive in the way a quasihyperbolic toral automorphism fails to be expansive, by having the higher-rank analogue of an eigenvalue with unit modulus; this failure arising from the varieties of the associated prime ideals results in a set of directions denoted by N^{v}.

The Noetherian condition is described as follows. Each $\mathbf{n} \in \mathbb{Z}^d$ defines a half-space

$$H = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{n} \leq 0\} \subset \mathbb{R}^d,$$

which has an associated ring $R_H = \mathbb{Z}[u^{\mathbf{m}} \mid \mathbf{m} \in H \cap \mathbb{Z}^d] \subset R_d$. A module over R_d is also a module over R_H.

Definition 2.2. Let M be a Noetherian R_d-module and let $V \subset \mathbb{R}^d$ be a k-dimensional subspace. Then M is said to be *Noetherian along* V if M is a Noetherian R_H-module for every half-space H containing V. The collection of all k-dimensional subspaces along which M is not Noetherian is denoted by $\mathsf{N}_k^{\mathrm{n}}(\alpha_M)$.

For the variety condition, let $\mathfrak{a} \subset R_d$ be any ideal. Write

$$\mathsf{V}(\mathfrak{a}) = \{\mathbf{z} \in (\mathbb{C} \setminus \{0\})^d \mid f(\mathbf{z}) = 0 \text{ for all } f \in \mathfrak{a}\}$$

and define the *amoeba* associated to $\mathfrak{a}$ to be

$$\log|\mathsf{V}(\mathfrak{a})| = \{(\log|z_1|, \ldots, \log|z_d|) \mid \mathbf{z} \in \mathsf{V}(\mathfrak{a})\}.$$

Now let M be a Noetherian R_d module. Then define

$$\mathsf{N}_k^{\mathrm{v}}(\alpha_M) = \bigcup_{\mathfrak{p} \in \mathrm{Asc}(M)} \{V \in \mathsf{G}_k \mid V^{\perp} \cap \log|V(\mathfrak{p})| \neq \varnothing\}$$

where $V^{\perp}$ denotes the orthogonal complement of V in $\mathbb{R}^d$. The main result in [**6**, Theorem 8.4] says that

$$\mathsf{N}_k(\alpha_M) = \mathsf{N}_k^{\mathrm{n}}(\alpha_M) \cup \mathsf{N}_k^{\mathrm{v}}(\alpha_M)$$

for any Noetherian R_d-module M.

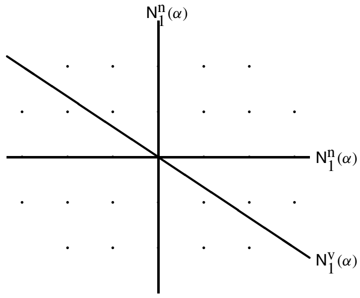

FIGURE 1. The three non-expansive lines for $\times 2$, $\times 3$.

3. *Periodic points*

Recall that the dynamical zeta function of a map T is defined formally as

$$\zeta_T(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} |F_n(T)|. \tag{3.1}$$

If $|F_n(T)|$ is finite for all $n \geq 1$ and grows at most exponentially, then (3.1) defines a complex function in some disc. In our setting there is a fixed $\mathbb{Z}^d$-action α, so write $\zeta_{\mathbf{n}}$ for the zeta function of the map $\alpha^{\mathbf{n}}$. Define $\mathsf{Q}(\alpha)$ to be the set of $\mathbf{n} \in \mathbb{Z}^d$ for which $\zeta_{\mathbf{n}}$ is a rational function. Note that any $\mathbf{n} \in \mathbb{Z}^d$ with the property that $F_j(\alpha^{\mathbf{n}})$ is infinite for some $j \geq 1$ is not a member of $\mathsf{Q}(\alpha)$.

The simplest non-trivial $\mathbb{Z}^2$-action is the '$\times 2$, $\times 3$' system, and the idea behind what follows is already visible in this example. In non-expansive directions, the periodic orbits for this system exhibit very complex growth properties (see [**8**] and [**9**]).

Example 3.1. Consider the $\mathbb{Z}^2$-action α dual to the $\mathbb{Z}^2$-action generated by the commuting maps $\times 2$ and $\times 3$ on $\mathbb{Z}[\frac{1}{6}]$. This is the dynamical system corresponding to the cyclic R_2-module $M = R_2/(u_1 - 2, u_2 - 3)$. The set $\mathsf{N}_1(\alpha)$ for this example is shown in Figure 1; it consists of three lines with $\mathsf{N}_1^{\mathsf{n}}(\alpha)$ comprising $2^{n_1} = 1$ and $3^{n_2} = 1$ and $\mathsf{N}_1^{\mathsf{v}}(\alpha)$ being the single irrational line $2^{n_1}3^{n_2} = 1$.

The map $\mathbf{n} \mapsto |F_{\mathbf{n}}(\alpha)| \in \mathbb{N}$ for the same system is given by

$$|F_{\mathbf{n}}(\alpha)| = |2^{n_1}3^{n_2} - 1||2^{n_1}3^{n_2} - 1|_2|2^{n_1}3^{n_2} - 1|_3. \tag{3.2}$$

Thus, for example, in an expansive direction such as $(1, 1)$ the formula reduces to $|F_{j(1,1)}| = 6^j - 1$. In a non-expansive direction such as $(1, 0)$ the ultrametric terms cause more exotic behaviour.

It may be shown (see [**9**] and [**15**, Theorem 4.7]) that

$$\mathsf{Q}(\alpha) = \{\mathbf{n} \in \mathbb{Z}^d \mid n_1 n_2 \neq 0\}$$

(the issue here is to show that the zeta functions $\zeta_{(1,0)}$ and $\zeta_{(0,1)}$ in the two rational non-expansive lines are not rational). The question addressed in this paper is the following:

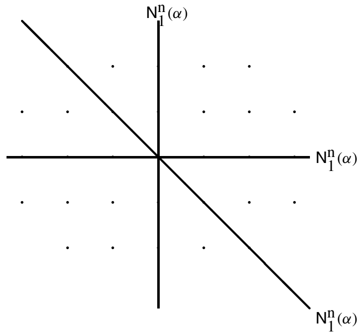

FIGURE 2. The three non-expansive lines for Ledrappier's example.

does the formula (3.2) determine the subdynamical portrait in Figure 1? As this example shows, the rationality set $\mathsf{Q}(\alpha)$ certainly does not determine $\mathsf{N}_1(\alpha)$, so, in particular, we are asking whether the periodic point data seen along the rational directions can detect the presence of an irrational non-expansive direction.

Example 3.2. The zero-dimensional analogue of the $\times 2, \times 3$ system is Ledrappier's example [**11**], which is the action α corresponding to the module $M = R_2/(2, 1+u_1+u_2)$. Using the local structure of X_M in terms of completions of the function field $\mathbb{F}_2(t)$ and the periodic point formula from [**2**] we have

$$|F_{(n_1,n_2)}(\alpha)| = |t^{n_1}(1+t)^{n_2} - 1|_\infty |t^{n_1}(1+t)^{n_2} - 1|_t |t^{n_1}(1+t)^{n_2} - 1|_{1+t},$$

the three absolute values being given by

$$|r(t)|_t = 2^{-\operatorname{ord}_t(r(t))}, \quad |r(t)|_\infty = |r(t^{-1})|_t \quad \text{and} \quad |r(t)|_{1+t} = 2^{-\operatorname{ord}_{1+t}(r(t))}$$

where $r(t) \in \mathbb{F}_2(t)$ (see [**2**], [**5**] or [**20**] for the details). The set $\mathsf{N}_1(\alpha)$ for this example is shown in Figure 2; $\mathsf{N}_1^{\mathsf{n}}(\alpha)$ comprises the lines

$$n_1 = 0, \quad n_2 = 0 \quad \text{and} \quad n_1 + n_2 = 0,$$

while $\mathsf{N}_1^{\mathsf{v}}(\alpha)$ is automatically empty since the associated prime ideal has an empty variety.

Once again the zeta function is known to be irrational in the non-expansive directions, and in this example $\mathsf{Q}(\alpha)$ does indeed detect all of the non-expansive behaviour. In each of the expansive regions, the periodic point formula simplifies significantly. For example, in the expansive region $n_1 < 0, n_2 > 0, n_1 + n_2 > 0$ we have

$$|t^{n_1}(1+t)^{n_2} - 1|_t = 2^{-n_1}, \quad |t^{n_1}(1+t)^{n_2} - 1|_\infty = 2^{n_1+n_2}$$

and

$$|t^{n_1}(1+t)^{n_2} - 1|_t = 1,$$

giving $|F_{(n_1,n_2)}(\alpha)| = 2^{n_2}$.

The formula in non-expansive directions may be found similarly, although the resulting expression is a little more involved. For example, in [**2**, Example 8.5] it is shown that $|F_{(n,0)}(\alpha)| = 2^{n-2^{\operatorname{ord}_2(n)}}$.

An alternative way to compute the number of periodic points, better adapted to more complicated situations, may be found in [**15**, Lemma 4.8].

To convert the periodic point data into a form which exposes the expansive subdynamics, we introduce a normalized encoding of the rational zeta functions arising from elements of the action.

Definition 3.3. Given a rational function $h \in \mathbb{C}(z)$, denote the set of poles and zeros of h by $\Psi(h) \subset \mathbb{C}$. Let α be an algebraic $\mathbb{Z}^d$-action of entropy rank one, and define

$$\Omega_\alpha = \{(\hat{\mathbf{n}}, |z|^{1/\|\mathbf{n}\|}) \mid z \in \Psi(\zeta_{\mathbf{n}}), \mathbf{n} \in \mathsf{Q}(\alpha)\} \subset \mathsf{S}_{d-1} \times \mathbb{R}$$

where $\hat{\mathbf{n}}$ denotes the unit vector in the direction of $\mathbf{n}$.

In order to exhibit the relationship between Ω_α and $\mathsf{N}(\alpha)$ we need a 'formula' for $|F_{\mathbf{n}}(\alpha)|$, and this has been found by Miles [**15**] using the structure of entropy rank-one systems from [**5**].

A *character* is a continuous homomorphism from an abelian group into $\mathbb{C}^\times$. We will be particularly interested in characters of the form $\chi : \mathbb{Z}^d \to \mathbb{C}^\times$. A *real character* is one with a real image. By a *list* we mean a finite sequence of the form $L = \langle \chi_1, \ldots, \chi_n \rangle$ which allows for multiplicities. The notation

$$\chi_L = \chi_1 \chi_2 \cdots \chi_n$$

is used to denote the product over all elements of L, with the understanding that $\chi_\varnothing \equiv 1$.

Let $\mathfrak{p} \subset R_d$ be a prime ideal with $\mathrm{coht}(\mathfrak{p}) = 1$ and let K be the field of fractions of $R_d/\mathfrak{p}$. Assume that $\mathrm{char}(R_d/\mathfrak{p}) = 0$, so all of the infinite places are archimedean. These infinite places are uniquely determined by the embeddings of $R_d/\mathfrak{p}$ into $\mathbb{C}$. A point $z \in V_{\mathbb{C}}(\mathfrak{p})$ determines a ring homomorphism into $\mathbb{C}$ via the substitution map $f + \mathfrak{p} \mapsto f(z)$. The map is injective because $R_d/\mathfrak{p}$ has Krull dimension one. Each $z \in V_{\mathbb{C}}(\mathfrak{p})$ induces a character on $\mathbb{Z}^d$ in an obvious way; there are finitely many such characters and the coordinates of these are all algebraic numbers. More generally, any place w of a domain of the form $R_d/\mathfrak{p}$ induces a real character on $\mathbb{Z}^d$ via the map

$$(n_1, \ldots, n_d) \mapsto (|\overline{u}_i|_w^{n_1}, \ldots, |\overline{u}_d|_w^{n_d}),$$

where $\overline{u}_i$ denotes the image of u_i in $R_d/\mathfrak{p}$, $i = 1 \ldots d$. This will always be our method of constructing characters using non-archimedean places.

Using the construction of characters given above, for a prime $\mathbb{Z}^d$-action $\alpha_{R_d/\mathfrak{p}}$ with $\mathrm{coht}(\mathfrak{p}) = 1$, let $\mathcal{W}(R_d/\mathfrak{p})$ be the list of characters induced by the non-archimedean $v \in \mathcal{S}_{\mathfrak{p}}$ and let $\mathcal{V}(R_d/\mathfrak{p})$ be the list of characters induced by the distinct complex embeddings of $R_d/\mathfrak{p}$. Note that $\mathcal{V}(R_d/\mathfrak{p}) = \varnothing$ when $\mathrm{char}(R_d/\mathfrak{p}) > 0$.

Now suppose that α_M is a Noetherian entropy rank-one action. The module M admits a *prime filtration*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M \tag{3.3}$$

where for each k, $1 \leq k \leq n$ we have $M_k/M_{k-1} \cong R_d/\mathfrak{q}_k$ for a prime ideal $\mathfrak{q}_k \subset R_d$ which is either an associated prime of M or which contains an associated prime of M. Lemma 8.2 of [**5**] shows that each minimal element of $\mathrm{Asc}(M)$ always appears in such a filtration with a fixed multiplicity $m(\mathfrak{p})$, so $m(\mathfrak{p})$ is well defined for all $\mathfrak{p} \in \mathrm{Asc}(M)$ with $\mathrm{coht}(\mathfrak{p}) = 1$.

Set

$$\mathcal{W}(M) = \bigsqcup \mathcal{W}(R_d/\mathfrak{p}),$$
$$\mathcal{V}(M) = \bigsqcup \mathcal{V}(R_d/\mathfrak{p}),$$

where the union of lists is taken over all $\mathfrak{p} \in \operatorname{Asc}(M)$ with $\operatorname{coht}(\mathfrak{p}) = 1$, ensuring that each prime $\mathfrak{p}$ appears with the appropriate multiplicity $m(\mathfrak{p})$.

If M has torsion-free rank one, then $\mathcal{V}(M)$ has a particularly simple form.

LEMMA 3.4. *Let M be a Noetherian R_d-module of torsion-free rank one, and suppose that α_M has entropy rank one. Then $\mathcal{V}(M)$ contains one element.*

Proof. Consider a prime filtration of M of the form (3.3). Since M has torsion-free rank one, there is at least one associated prime $\mathfrak{p}$ such that $\operatorname{char}(R_d/\mathfrak{p}) = 0$. Let $k \leq n$ be the lowest integer such that $\operatorname{char}(R_d/\mathfrak{q}_k) = 0$. Then $\operatorname{coht}(\mathfrak{q}_k) = 1$ and $\mathfrak{q}_k \in \operatorname{Asc}(M)$. Suppose $k < n$. Since $M_{k+1}/M_k \cong R_d/\mathfrak{q}_{k+1}$, there exists $a \in M_{k+1} \setminus M_k$ such that any element of M_{k+1} can be written in the form $x + fa$ for some $x \in M_k$ and $f \in R_d$ with $fa \in M_k$ if and only if $f \in \mathfrak{q}_{k+1}$. However, both M_k and M have torsion-free rank one so there exists $c \in \mathbb{Z}$ such that $ca \in M_k$. Therefore, $c \in \mathfrak{q}_{k+1}$ and $\operatorname{char}(R_d/\mathfrak{q}_{k+1}) > 0$. In a similar way, it follows that $\operatorname{char}(R_d/\mathfrak{q}_j) > 0$ for all $j > k$. Hence, $\mathfrak{q}_k$ is the only prime with $\operatorname{char}(R_d/\mathfrak{q}_k) = 0$; moreover, $m(\mathfrak{q}_k) = 1$. So

$$\mathcal{V}(M) = \mathcal{V}(R_d/\mathfrak{q}_k).$$

If $k = n$, then again $\mathcal{V}(M) = \mathcal{V}(R_d/\mathfrak{q}_k)$. Finally, $R_d/\mathfrak{q}_k$ is isomorphic to a subring of $\mathbb{Q}$, so $\mathcal{V}(M)$ contains one character induced by the single infinite place of $\mathbb{Q}$. □

Any character $\chi : \mathbb{Z}^d \to \mathbb{C}^\times$ induces a real character χ^* on $\mathbb{R}^d$, by setting

$$\chi^*(\kappa \mathbf{e}_i) = |\chi(\mathbf{e}_i)|^\kappa$$

where $\kappa \in \mathbb{R}$ and $\mathbf{e}_i$ is the standard ith basis vector in $\mathbb{Z}^d$, $i = 1 \ldots d$. Applying this construction to an element of $\mathcal{W}(M)$ yields a genuine extension, but the same is not necessarily true for elements of $\mathcal{V}(M)$.

PROPOSITION 3.5. *Suppose that α_M is an algebraic $\mathbb{Z}^d$-action of expansive rank one. Then $\mathsf{N}_{d-1}(\alpha_M)$ consists precisely of the finite set of hyperplanes defined by the equations*

$$\chi^*(\mathbf{n}) = 1$$

where $\chi \in \mathcal{V}(M) \cup \mathcal{W}(M)$. Furthermore, $\mathsf{N}^{\mathrm{v}}_{d-1}(\alpha_M)$ is determined by those characters in $\mathcal{V}(M)$ and $\mathsf{N}^{\mathrm{n}}_{d-1}(\alpha_M)$ by those characters in $\mathcal{W}(M)$.

Proof. This is a combination of [**16**, Theorem 4.3.10] and [**6**, Theorem 8.4]. □

By expressing $\mathsf{N}_1(\alpha_M)$ in terms of the intersection of non-expansive lines with S_{d-1} and referring to [**1**, Theorem 3.6], we also find the following description of the expansive subdynamics.

COROLLARY 3.6. *If α_M is an algebraic $\mathbb{Z}^d$-action of expansive rank one, then*

$$\mathsf{N}_1^{\mathrm{v}}(\alpha) = \bigcup_{\chi \in \mathcal{V}(M)} \{\mathbf{v} \in \mathsf{S}_{d-1} \mid \chi^*(\mathbf{v}) = 1\},$$

$$\mathsf{N}_1^{\mathrm{n}}(\alpha) = \bigcup_{\chi \in \mathcal{W}(M)} \{\mathbf{v} \in \mathsf{S}_{d-1} \mid \chi^*(\mathbf{v}) = 1\}$$

and the set of expansive directions is dense in S_{d-1}.

4. *Main results*

To begin this section, we return to the examples in §3.

Example 4.1. Let α be the $\mathbb{Z}^2$-action corresponding to the R_2-module

$$M = R_2/(u_1 - 2, u_2 - 3)$$

discussed in Example 3.1. Recall that

$$\mathsf{Q}(\alpha) = \{\mathbf{n} \in \mathbb{Z}^d \mid n_1 n_2 \neq 0\}.$$

Here, $\mathsf{Q}(\alpha)$ consists precisely of those $\mathbf{n} \in \mathbb{Z}^2$ for which $\alpha_M^{\mathbf{n}}$ is expansive, but this need not always be the case (see [**15**, Example 4.3] for an example). Note that expansiveness of the elements $\alpha_M^{\mathbf{n}}$ of the action can only ever detect non-expansiveness in rational directions, so the irrational line in $\mathsf{N}_1(\alpha)$ will be missed. For $\mathbf{n} = (n_1, n_2) \in \mathsf{Q}(\alpha)$, using the periodic point formula from [**15**],

$$|F_j(\alpha_M^{\mathbf{n}})| = |2^{jn_1}3^{jn_2} - 1|_\infty |2^{jn_1}3^{jn_2} - 1|_2 |2^{jn_1}3^{jn_2} - 1|_3.$$

It follows that

$$\zeta_{\mathbf{n}}(z) = (1 - g(\mathbf{n})z)^{\lambda_1}(1 - g(\mathbf{n})2^{n_1}3^{n_2}z)^{\lambda_1},$$

where $\lambda_1, \lambda_2 \in \{-1, 1\}$ and

$$g(\mathbf{n}) = |2^{n_1} - 1|_2 |3^{n_2} - 1|_3.$$

The resulting directional pole and zero data $\overline{\Omega}_\alpha$, realized as a subset of $[0, 2\pi) \times \mathbb{R}$, are shown in Figure 3. Non-expansive directions are marked with a dashed line.

Example 4.2. Let α be the $\mathbb{Z}^2$-action corresponding to the R_2-module

$$M = R_2/(2, 1 + u_1 + u_2)$$

discussed in Example 3.2. Recall that

$$\mathsf{Q}(\alpha) = \{\mathbf{n} \in \mathbb{Z}^d \mid n_1 n_2 \neq 0 \text{ and } n_1 + n_2 \neq 0\}.$$

For $\mathbf{n} = (n_1, n_2) \in \mathsf{Q}(\alpha)$, using the periodic point formula from [**15**],

$$\zeta_{\mathbf{n}}(z) = (1 - g(\mathbf{n})z)^{-1}$$

where

$$g(\mathbf{n}) = |t^{n_1}(1+t)^{n_2} - 1|_{1+t} |t^{n_1}(1+t)^{n_2} - 1|_t |t^{n_1}(1+t)^{n_2} - 1|_\infty.$$

The resulting directional pole and zero data $\overline{\Omega}_\alpha$, realized as a subset of $[0, 2\pi) \times \mathbb{R}$, are shown in Figure 4. Non-expansive directions are marked with a dashed line.

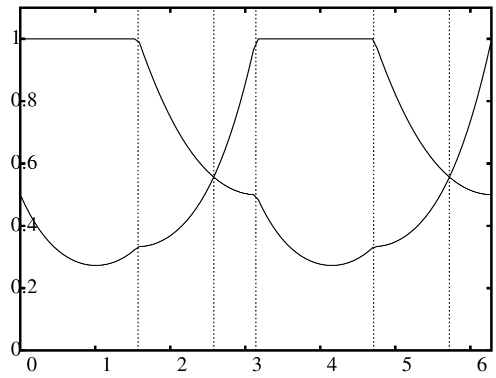

FIGURE 3. $\overline{\Omega}_\alpha$ for the $\times 2$, $\times 3$ system in Example 4.1.

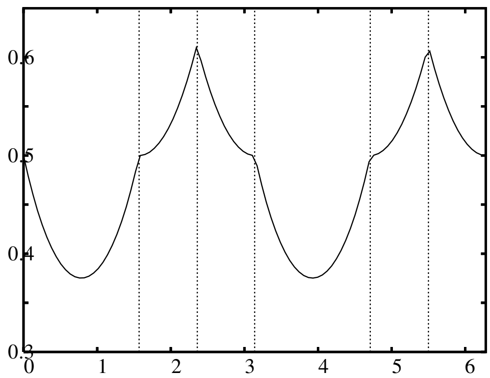

FIGURE 4. $\overline{\Omega}_\alpha$ for Ledrappier's example.

In Examples 4.1 and 4.2 the periodic point 'formula' is particularly simple, mainly because the module M is cyclic. In order to deal with more general modules, some machinery is needed.

PROPOSITION 4.3. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action, $j \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{Z}^d$. The following conditions are equivalent.*

(1) *$\alpha_M^{\mathbf{n}}$ has finitely many points of period j.*

(2) *$\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}$ has finitely many points of period j for every $\mathfrak{p} \in \mathrm{Asc}(M)$.*

For the proof, the following lemma is needed.

LEMMA 4.4. *Let $j \in \mathbb{N}$, $\mathbf{n} \in \mathbb{Z}^d$, $\mathfrak{b} = (u^{j\mathbf{n}} - 1) \subset R_d$ and let N be an R_d-module.*

(1) *Whenever $N/\mathfrak{b}N$ or $|F_j(\alpha_N^{\mathbf{n}})|$ is finite,*

$$|F_j(\alpha_N^{\mathbf{n}})| = |N/\mathfrak{b}N|.$$

(2) *If $L \subset N$ is a submodule satisfying $N/L \cong R_d/\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset R_d$, then*

$$\mathfrak{b}N \cap L = \begin{cases} \mathfrak{b}N & \text{when } u^{j\mathbf{n}} - 1 \in \mathfrak{q}, \\ \mathfrak{b}L & \text{otherwise}. \end{cases}$$

Proof. Part (1) is a simpler version of (for example) [**12**, Lemma 7.2]; the hypothesis means that $F_j(\alpha_N^{\mathbf{n}})$ is a finite abelian group and the statement just says that such a group has the same number of elements as its character group, which is well known. Part (2) is a modification of [**15**, Lemma 3.4] which uses an ergodicity assumption to get a slightly stronger result, but this is not needed here. □

Proof of Proposition 4.3. Set $\mathfrak{b} = (u^{j\mathbf{n}} - 1)$ and let $L \subset N$ be submodules of M satisfying $N/L \cong R_d/\mathfrak{q}$ for a prime ideal $\mathfrak{q} \subset R_d$. Then by Lemma 4.4(2),

$$\begin{aligned} |N/\mathfrak{b}N| &= \left|\frac{N/L}{(\mathfrak{b}N+L)/L}\right|\left|\frac{\mathfrak{b}N+L}{\mathfrak{b}N}\right| \\ &= \left|\frac{N/L}{\mathfrak{b}(N/L)}\right|\left|\frac{L}{\mathfrak{b}N\cap L}\right| \\ &= \begin{cases} |N/L|\left|\dfrac{L/\mathfrak{b}L}{\mathfrak{b}N/\mathfrak{b}L}\right| & \text{when } \mathfrak{b}\subset\mathfrak{q}, \\ \left|\dfrac{N/L}{\mathfrak{b}(N/L)}\right|\left|\dfrac{L}{\mathfrak{b}L}\right| & \text{otherwise.} \end{cases} \end{aligned} \tag{4.1}$$

To prove the result, we will first show

$$|N/\mathfrak{b}N| < \infty \quad \Leftrightarrow \quad |L/\mathfrak{b}L| < \infty \quad \text{and} \quad \left|\frac{N/L}{\mathfrak{b}(N/L)}\right| < \infty, \tag{4.2}$$

and then proceed by induction.

($\Rightarrow$) If $\mathfrak{b} \not\subset \mathfrak{q}$, then the required implication follows immediately from (4.1). Assume now that $\mathfrak{b} \subset \mathfrak{q}$. The finiteness of (4.1) forces N/L to be finite and, hence, $|(N/L)/\mathfrak{b}(N/L)| < \infty$. To see that $|L/\mathfrak{b}L| < \infty$, let $\phi : N \to \mathfrak{b}N/\mathfrak{b}L$ be defined by multiplication by $u^{j\mathbf{n}} - 1$, followed by the natural quotient map. Clearly ϕ is surjective and $L \subset \ker(\phi)$ which means there is a surjective homomorphism from N/L to $\mathfrak{b}N/\mathfrak{b}L$. Hence, $\mathfrak{b}N/\mathfrak{b}L$ is finite and $L/\mathfrak{b}L$ must also be finite to ensure that (4.1) holds.

($\Leftarrow$) If $\mathfrak{b} \not\subset \mathfrak{q}$, then (4.1) immediately gives $|N/\mathfrak{b}L| < \infty$. For the case $\mathfrak{b} \subset \mathfrak{q}$, again apply (4.1), noting that

$$|N/L| = \left|\frac{N/L}{\mathfrak{b}(N/L)}\right| \quad \text{and} \quad \left|\frac{L/\mathfrak{b}L}{\mathfrak{b}N/\mathfrak{b}L}\right| \leq |L/\mathfrak{b}L|.$$

(1) $\Rightarrow$ (2) A prime filtration of M of the form (3.3) may be taken so that any chosen $\mathfrak{p} \in \mathrm{Asc}(M)$ appears as $\mathfrak{q}_1$ (see, for example, the proof of [**7**, Proposition 3.7]). Starting with $M_n = M$ and descending through the submodules appearing in (3.3), successively applying (4.2) gives $|M_1/\mathfrak{b}M_1| < \infty$. The required result follows by Lemma 4.4(1).

(2) $\Rightarrow$ (1) Again take a prime filtration of M of the form (3.3). For each $1 \leq k \leq n$, $\mathfrak{q}_k$ is an associated prime of M or contains such a prime. Let $\mathfrak{p} \in \mathrm{Asc}(M)$ and suppose that $\mathfrak{q}_k \supset \mathfrak{p}$. There is a natural surjective homomorphism

$$\frac{R_d/\mathfrak{p}}{\mathfrak{b}(R_d/\mathfrak{p})} \cong \frac{R_d}{\mathfrak{b}+\mathfrak{p}} \longrightarrow \frac{R_d}{\mathfrak{b}+\mathfrak{q}_k} \cong \frac{R_d/\mathfrak{q}_k}{\mathfrak{b}(R_d/\mathfrak{q}_k)}.$$

Since $|F_j(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}})| < \infty$ for all $\mathfrak{p} \in \mathrm{Asc}(M)$, by the above and of Lemma 4.4(1),

$$\left|\frac{M_k/M_{k-1}}{\mathfrak{b}(M_k/M_{k-1})}\right| < \infty$$

for all $1 \leq k \leq n$. Starting with $k = 1$ and ascending through the modules appearing in (3.3), successively applying (4.2), gives

$$|M_n/\mathfrak{b}M_n| < \infty$$

and, hence, $|F_j(\alpha_M^{\mathbf{n}})| < \infty$. □

The following consequence of Proposition 4.3 shows that for Noetherian actions with entropy rank greater than one, periodic point data for individual elements are much less useful than in the rank-one case.

COROLLARY 4.5. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action with entropy rank greater than one. For each $\mathbf{n} \in \mathbb{Z}^d$, the automorphism $\alpha_M^{\mathbf{n}}$ has infinitely many points of any given period.*

Proof. First note that, by assumption, $d > 1$. Since $\operatorname{entrk}(\alpha_M) > 1$, by Proposition 2.1, there is at least one $\mathfrak{p} \in \operatorname{Asc}(M)$ with $\operatorname{coht}(\mathfrak{p}) > 1$. Let $j \in \mathbb{N}$, $D = R_d/\mathfrak{p}$ and consider $|F_j(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})|$. Suppose that this is finite. Then by Lemma 4.4(1), there is a proper principal ideal $\mathfrak{a} \subset D$ such that

$$|F_j(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})| = |D/\mathfrak{a}|.$$

By the principal ideal theorem [**7**, Theorem 10.1] there is a prime ideal $\mathfrak{q} \subset D$ containing $\mathfrak{a}$ with $\operatorname{ht}(\mathfrak{q}) = 1$. Since D is a quotient of R_d by a prime ideal of coheight at least 2 and since all maximal ideals of R_d have the same height, every maximal ideal of D has height at least 2. Hence, $\operatorname{kdim}(D/\mathfrak{q}) = \operatorname{coht}(\mathfrak{q}) \geq 1$. Therefore, $D/\mathfrak{q}$ cannot be finite as it either contains $\mathbb{Z}$ or has non-zero transcendence degree over a finite field. As there is a surjection from $D/\mathfrak{a}$ to $D/\mathfrak{q}$, the supposition that $|F_j(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})| < \infty$ must therefore be false. The result now follows by Proposition 4.3. □

Recall that an algebraic $\mathbb{Z}^d$-action α is ergodic (always with respect to Haar measure) if there are no non-trivial invariant sets for the whole action; Schmidt [**19**, Theorem 6.5] shows that α is ergodic if and only if there is some $\mathbf{n} \in \mathbb{Z}^d$ for which the single automorphism $\alpha^{\mathbf{n}}$ is ergodic. The next result gives a criterion to identify certain elements of the action that must be ergodic. Note that an automorphism of a finite group cannot be ergodic since it must fix the identity, which has positive Haar measure.

COROLLARY 4.6. *Suppose that α_M is an ergodic Noetherian algebraic $\mathbb{Z}^d$-action and $\mathbf{n} \in \mathbb{Z}^d$. If the automorphism $\alpha_M^{\mathbf{n}}$ has finitely many points of every period then $\alpha_M^{\mathbf{n}}$ is ergodic.*

Proof. Since α_M is ergodic, [**19**, Theorem 6.5] shows that there exists $\mathbf{m} \in \mathbb{Z}^d$ such that $\alpha_{R_d/\mathfrak{p}}^{\mathbf{m}}$ is ergodic for every $\mathfrak{p} \in \operatorname{Asc}(M)$. If $\mathfrak{p}$ is maximal, then $\alpha_{R_d/\mathfrak{p}}^{\mathbf{m}}$ is an automorphism of a finite field which cannot be ergodic. Hence, $\operatorname{Asc}(M)$ contains no maximal ideals.

For a contradiction suppose that $\alpha_M^{\mathbf{n}}$ is not ergodic. Then, again using [**19**, Theorem 6.5], there exists $j \geq 1$ and a prime ideal $\mathfrak{p} \in \operatorname{Asc}(M)$ such that $u^{j\mathbf{n}} - 1 \in \mathfrak{p}$. Therefore, the ideal $(u^{j\mathbf{n}} - 1) \subset R_d$ annihilates $R_d/\mathfrak{p}$ and, by Lemma 4.4(1),

$$|F_j(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})| = |R_d/\mathfrak{p}|.$$

Since $\mathfrak{p}$ is not maximal this is not finite, which contradicts Proposition 4.3. □

This allows us to describe the set $\Psi(\zeta_{\mathbf{n}})$ in terms of places and characters.

PROPOSITION 4.7. *Suppose that α_M is an ergodic Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one. If $\mathbf{n} \in \mathsf{Q}(\alpha_M)$, then*

$$\Psi(\zeta_{\mathbf{n}}) = \{\mu_{\mathbf{n}}\chi_T\chi_L(-\mathbf{n}) \mid L \subset \mathcal{V}(M)\} \tag{4.3}$$

where $T = \langle \chi \in \mathcal{W}(M) \mid \chi(\mathbf{n}) > 1\rangle$ and $\mu_{\mathbf{n}} \in \{-1, 1\}$.

Proof. Since $\mathbf{n} \in \mathsf{Q}(\alpha_M)$, $|F_j(\alpha_M^{\mathbf{n}})| < \infty$ for all $j \in \mathbb{N}$. Therefore, by Corollary 4.6, we may assume that $\alpha_M^{\mathbf{n}}$ is ergodic. Hence, the explicit formula for $\zeta_{\mathbf{n}}$ given in [**15**] may be used. Let

$$R = \langle \chi \in \mathcal{V}(M) \mid |\chi(\mathbf{n})| > 1\rangle$$

and denote the complement of $J \subset \mathcal{V}(M)$ in $\mathcal{V}(M)$ by J'. Applying [**15**, Theorem 4.7 and Lemma 4.2], if $\zeta_{\mathbf{n}}$ is rational, then

$$\zeta_{\mathbf{n}}(z) = \prod_{J\subset\mathcal{V}(M)} (1 - c_J z)^{\lambda_J},$$

where

$$c_J = \mu_{\mathbf{n}}\chi_T\chi_{R\cap J'}\chi_{R'\cap J}(\mathbf{n}) \tag{4.4}$$

and $\mu_{\mathbf{n}}, \lambda_J \in \{-1, 1\}$. Hence, setting $L = (R\cap J')\cup(R'\cap J)$ shows that the corresponding pole or zero of $\zeta_{\mathbf{n}}$ arising from the coefficient (4.4) is accounted for in (4.3). Conversely, to see that for a given $L \subset \mathcal{V}(M)$, $\mu_{\mathbf{n}}\chi_T\chi_L(-\mathbf{n}) \in \Psi(\zeta_{\mathbf{n}})$, simply set $J = (R \cap L') \cup (R' \cap L)$ in (4.4). □

We can now prove the main result, which roughly says that $\overline{\Omega}_\alpha$ (the closure of the set from Definition 3.3, which is determined completely by the periodic point data for α) is a union of graphs of continuous functions, and that these graphs pick out two sets of non-expansive directions. The first is the *crossing set*, where graphs of functions cross, and this turns out to contain $\mathsf{N}_1^{\mathsf{v}}(\alpha)$. The second is the set of points where one of the functions is not differentiable, and this turns out to be exactly $\mathsf{N}_1^{\mathsf{n}}(\alpha)$. In the simplest situations the crossing set is exactly $\mathsf{N}_1^{\mathsf{v}}(\alpha)$, and the extra directions identified in the general case reflect vanishing of other combinations of Lyapunov exponents (see Einsiedler and Lind [**5**, §8] for a discussion of how to interpret these as Lyapunov exponents).

THEOREM 4.8. *Let α_M be an ergodic algebraic $\mathbb{Z}^d$-action of expansive rank one corresponding to the R_d-module M. Then a collection of continuous functions $f_1, \ldots, f_r \in \mathcal{C}(\mathsf{S}_{d-1}, \mathbb{R})$ may be constructed from the periodic point data of α in such a way that*

$$\overline{\Omega}_\alpha = \bigcup_{k=1}^{r}\{(\mathbf{v}, f_k(\mathbf{v})) \mid \mathbf{v} \in \mathsf{S}_{d-1}\}$$

and

(1) $\mathsf{N}_1^{\mathsf{v}}(\alpha) \subset \{\mathbf{v} \in \mathsf{S}_{d-1} \mid f_j(\mathbf{v}) = f_k(\mathbf{v})$ *for some* $j \neq k\}$,

(2) $\mathsf{N}_1^{\mathsf{n}}(\alpha) = \{\mathbf{v} \in \mathsf{S}_{d-1} \mid f_k$ *is not smooth at* $\mathbf{v}$ *for some* $k\}$.

If $\dim(\widehat{M}) \leq 1$ *then equality holds in (1).*

Proof. First note that given $\chi \in \mathcal{V}(M) \cup \mathcal{W}(M)$, the induced character χ^* is continuous on $\mathbb{R}^d$. For each $\chi \in \mathcal{W}(M)$, define $g_\chi : \mathbb{R}^d \to \mathbb{R}^\times$ by

$$g_\chi(\mathbf{n}) = \max\{\chi^*(\mathbf{n}), 1\}.$$

Then $g = \prod_{\chi \in \mathcal{W}(M)} g_\chi$ is continuous since $\mathcal{W}(M)$ is finite. Now, for each $L \subset \mathcal{V}(M)$ define $f_L : \mathbb{R}^d \to \mathbb{R}^\times$ by

$$f_L(\mathbf{n}) = g(-\mathbf{n})\chi_L^*(-\mathbf{n}).$$

Let $\mathbf{n} \in \mathbb{R}^d$ and define

$$T(\mathbf{n}) = \{\chi \in \mathcal{W}(M) \mid \chi^*(\mathbf{n}) > 1\}.$$

Then $T(\mathbf{n}) = T(\hat{\mathbf{n}})$ and

$$\chi_{T(\hat{\mathbf{n}})}^* \chi_L^*(-\hat{\mathbf{n}}) = f_L(\hat{\mathbf{n}}).$$

Thus, if $\mathbf{n} \in \mathsf{Q}(\alpha)$, then Proposition 4.7 shows that

$$\{(\hat{\mathbf{n}}, |z|^{1/\|\mathbf{n}\|}) \mid z \in \Psi(\zeta_{\mathbf{n}})\} = \{(\hat{\mathbf{n}}, f_L(\hat{\mathbf{n}})) \mid L \subset \mathcal{V}(M)\}. \tag{4.5}$$

Since α has expansive rank one, by Corollary 3.6, the intersection of the set of expansive lines with S_{d-1} is dense. Since α is ergodic, [**15**, Theorem 4.4] shows that each expansive automorphism has a rational zeta function. Hence, the projection of $\mathsf{Q}(\alpha)$ to S_{d-1} is also dense. Using (4.5), it follows that

$$\overline{\Omega}_\alpha = \bigcup_{L \subset \mathcal{V}(M)} \{(\mathbf{v}, f_L(\mathbf{v})) \mid \mathbf{v} \in \mathsf{S}_{d-1}\}.$$

After making the obvious restriction of f_L to unit vectors, for $L \subset \mathcal{V}(M)$, the first part of the theorem follows.

To prove (1), first note by Corollary 3.6

$$\mathsf{N}_1^{\mathrm{v}}(\alpha_M) = \bigcup_{\chi \in \mathcal{V}(M)} \{\mathbf{v} \in \mathsf{S}_{d-1} \mid \chi^*(\mathbf{v}) = 1\}. \tag{4.6}$$

Furthermore, since $g(\mathbf{v}) \neq 0$ for all $\mathbf{v} \in \mathsf{S}_{d-1}$, for any $\chi \in \mathcal{V}(M)$,

$$\chi^*(\mathbf{v}) = 1 \quad \Leftrightarrow \quad f_{\{\chi\}}(\mathbf{v}) = f_\varnothing(\mathbf{v}). \tag{4.7}$$

To establish (2), again by Corollary 3.6

$$\mathsf{N}_1^{\mathrm{n}}(\alpha_M) = \bigcup_{\chi \in \mathcal{W}(M)} \{\mathbf{v} \in \mathsf{S}_{d-1} \mid \chi^*(\mathbf{v}) = 1\}.$$

Since for each $\chi \in \mathcal{V}(M)$, χ^* is smooth, f_L fails to be smooth only at points where g is not smooth. It follows from the definition of g that this happens precisely when $\chi^*(\mathbf{v}) = 1$ for some $\chi \in \mathcal{W}(M)$ and $\mathbf{v} \in \mathsf{S}_{d-1}$.

Finally, we show that equality holds in (1) if $\dim(\widehat{M}) \leq 1$.

If $\dim(\widehat{M}) = 0$, then $\operatorname{char}(R_d/\mathfrak{p}) > 0$ for all $\mathfrak{p} \in \operatorname{Asc}(M)$. Hence, both $\mathcal{V}(M)$ and $\mathsf{N}_1^{\mathrm{v}}(\alpha)$ are empty and equality in (1) holds trivially.

If $\dim(\widehat{M}) = 1$, then Lemma 3.4 shows that $\mathcal{V}(M)$ consists of a single character χ. Hence, for any $\mathbf{v} \in \mathsf{S}_{d-1}$, the only possible relation of the form $f_J(\mathbf{v}) = f_L(\mathbf{v})$ for $J, L \subset \mathcal{V}(M)$, $J \neq L$, is given by (4.7). Therefore, equality in (1) follows from (4.6). □

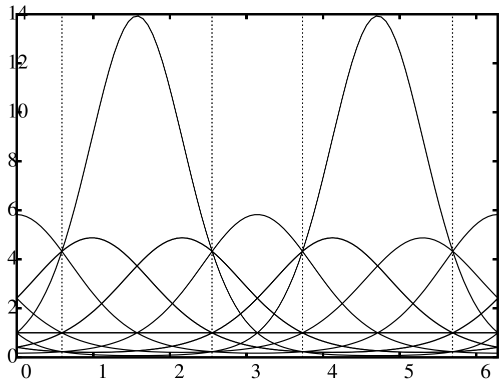

FIGURE 5. $\overline{\Omega}_\alpha$ for Example 5.1.

Remark 4.9. The set $\overline{\Omega}_\alpha$ carries information about the growth rate of periodic points in all of the expansive directions, and therefore knowledge of the directional entropies in the sense of Einsiedler and Lind [**5**, Proposition 8.5]. In the notation of Theorem 4.8,

$$h(\alpha^{\mathbf{n}}) = \|\mathbf{n}\| \log \min_{1\le i\le r} \{f_i(\hat{\mathbf{n}})\}.$$

5. *Further examples*

The following example illustrates that equality cannot hold in Theorem 4.8(1) in general.

Example 5.1. The following example is algebraically conjugate to the $\mathbb{Z}^2$-action of the 4-torus in [**5**, Example 6.6]. Let $\alpha = \alpha_M$ be the $\mathbb{Z}^2$-action corresponding to the prime R_2-module $R_2/\mathfrak{p}$, where

$$\mathfrak{p} = (u_1^2 - 2u_1 - 1, u_2^2 - 4u_2 + 1) \subset R_2.$$

The zeros of the first polynomial are $1 \pm \sqrt{2}$ and of the second are $2 \pm \sqrt{3}$. Hence, $M = R_2/\mathfrak{p} \cong \mathbb{Z}[\sqrt{2}, \sqrt{3}]$ and $\mathcal{S}_\mathfrak{p}$ consists of four infinite places induced by the elements of $G = \mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})|\mathbb{Q})$. Therefore,

$$\mathcal{V}(M) = \{(\sigma(1+\sqrt{2}), \sigma(2+\sqrt{3})) \mid \sigma \in G\}$$

and $\mathcal{V}(M)$ may be used to establish the poles and zeros of $\zeta_\mathbf{n}$ for all $\mathbf{n}$ in $\mathsf{Q}(\alpha)$. Note that in this example $\mathcal{W}(M) = \varnothing$. Consider

$$L = \{(1+\sqrt{2}, 2+\sqrt{3}), (1+\sqrt{2}, 2-\sqrt{3})\} \subset \mathcal{V}(M).$$

The equation

$$\chi_L^*(\mathbf{v}) = \chi_\varnothing^*(\mathbf{v}) = 1$$

has a solution $\mathbf{v} = (0, 1)$. However, for all $\chi \in \mathcal{V}(M)$, $\chi((0,1)) \neq 1$ and clearly $(0,1) \notin \mathsf{N}_1(\alpha)$. The complete set of directional pole and zero data $\overline{\Omega}_\alpha$ are shown in Figure 5. The non-expansive directions are marked with vertical dashed lines.

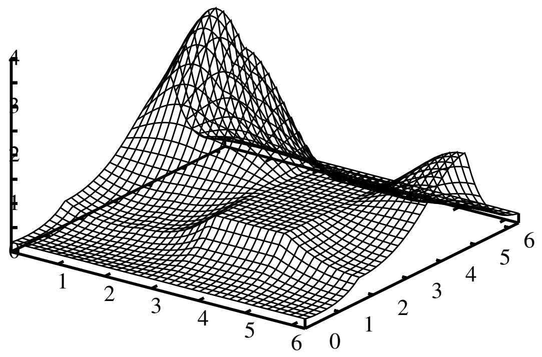

FIGURE 6. A layer of $\overline{\Omega}_\alpha$ for Example 5.2.

Example 5.2. Consider the $\mathbb{Z}^3$-action α which is dual to the three commuting automorphisms $\times 2$, $\times 3$ and $\times 5$ on $\mathbb{Z}[\frac{1}{30}]$. This is the dynamical system corresponding to the R_3-module

$$M = R_3/(u_1 - 2, u_2 - 3, u_3 - 5).$$

The set $\mathcal{V}(M) \cup \mathcal{W}(M)$ consists of four characters induced by the 2-adic, 3-adic, 5-adic and archimedean valuations on $\mathbb{Q}$. Thus, $\mathsf{N}_1(\alpha)$ may be obtained from the solutions to the equation

$$\chi^*(\mathbf{n}) = 1$$

where $\mathbf{n} \in \mathbb{R}^3$, and $\chi \in \mathcal{V}(M) \cup \mathcal{W}(M)$. Projecting points in the four resulting planes to S_2 presents $\mathsf{N}_1(\alpha)$ as a union of four great circles.

The set of directional pole and zero data Ω_α also contains this information. Proceeding in a similar way to Example 4.1, $\overline{\Omega}_\alpha$ comprises two continuous surfaces, with curves of non-differentiability detecting points in $\mathsf{N}_1^{\mathrm{n}}(\alpha)$. In terms of spherical coordinates θ, $\phi \in [0, 2\pi)$, these are given by

$$\theta, \phi \in \{k\pi/2 \mid k = 0, 1, 2, 3\}.$$

Figure 6 shows one of the two surfaces as a subset of $[0, 2\pi)^2 \times \mathbb{R}$. The other surface intersects this surface at points detecting $\mathsf{N}_1^{\mathrm{v}}(\alpha)$, shown as a subset of $[0, 2\pi)^2$ in Figure 7. This is the locus

$$2^{\cos\theta \sin\phi} 3^{\sin\theta \sin\phi} 5^{\cos\phi} = 1.$$

Example 5.3. A very important feature of algebraic dynamical systems in higher rank is the possibility of *genuinely partially hyperbolic systems.* Damjanović and Katok [**4**] show that for $r \geq 6$ there are such actions on the r-torus. The following example is taken from [**3**] and its directional zeta functions are discussed in [**15**, Example 4.4], where the following description may be found. Let

$$f(x) = x^6 - 2x^5 - 5x^4 - 3x^3 - 5x^2 - 2x + 1$$

and write $k = \mathbb{Q}(a)$, where a is a complex root of f. Then a and

$$b = 2a^5 - 6a^4 - 3a^3 - 6a^2 - 6a$$

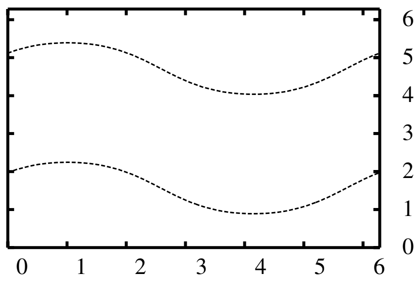

FIGURE 7. $\mathsf{N}_1^{\mathrm{v}}(\alpha)$ for Example 5.2.

are fundamental units for the ring of integers in k, and $M = \mathbb{Z}[a]$ is an R_2-module under the substitutions $u_1 = a, u_2 = b$. Since

$$\{1, a, a^2, a^3, a^4, a^5\}$$

is an integral basis for M, α_M is a $\mathbb{Z}^2$-action on $X_M = \mathbb{T}^6$, the 6-torus. Moreover, M is a domain and so it has one associated prime ideal $\mathfrak{p}$ in R_2 (this ideal is the kernel of the substitution map), and $\mathrm{coht}(\mathfrak{p}) = 1$. It follows that α_M has entropy rank one. The only places unbounded on $M = R_2/\mathfrak{p}$ are archimedean; one of these is complex and four are real. The complex place w has $|a|_w = |b|_w = 1$ (this is the part of the spectrum of the action that is not hyperbolic) and so $\alpha_M^{\mathbf{n}}$ is non-expansive for all $\mathbf{n} \in \mathbb{Z}^2$. The multiplicative independence of a and b implies that $\alpha_M^{\mathbf{n}}$ is ergodic for every $\mathbf{n} \in \mathbb{Z}^2$. Thus, [**15**, Lemma 4.1] shows that $\zeta_{\mathbf{n}}$ is rational for every $\mathbf{n} \in \mathbb{Z}^2$. This action is not expansive, so the results above do not apply; in particular, $\mathsf{N}_1(\alpha)$ consists of all directions. The crossing set and the non-smooth set for $\overline{\Omega}_\alpha$ may be found (in principle, there are a great many functions involved) and these points correspond to directions $\mathbf{n} \in \mathbb{R}^2$ with the property that the product of some subset of the set of eigenvalues of the integer matrix defining the automorphism $\widehat{\alpha_M^{\mathbf{n}}}$ is 1. Thus, the portrait obtained from $\overline{\Omega}_\alpha$ contains those directions in which the action is non-expansive transverse to the two-dimensional stable foliation arising from the common two-dimensional eigenspace for the two commuting matrices defining the automorphisms $\widehat{\alpha_M^{(1,0)}}$ and $\widehat{\alpha_M^{(0,1)}}$ on which they act like a pair of irrational rotations.

Example 5.4. Our emphasis has been on $\mathbb{Z}^d$-actions, but the notion of non-expansive subdynamics makes sense for actions of countable abelian groups. Miles [**17**] has extended Schmidt's characterization [**18**] of expansive $\mathbb{Z}^d$-actions to this setting. When the acting group has infinite torsion-free rank its integral group ring is no longer Noetherian as a ring, and new phenomena are possible. The simplest example in this setting shows that Theorem 4.8 does not extend without modification.

Consider the natural action α of $\mathbb{Q}_{>0}^\times$ on $X = \widehat{\mathbb{Q}}$ dual to the action $x \overset{\times r}{\mapsto} rx$ of $r \in \mathbb{Q}_{>0}^\times$ on $\mathbb{Q}$. Algebraically, this is associated to a cyclic (hence, Noetherian) module over the integral group ring $\mathbb{Z}[\mathbb{Q}_{>0}^\times]$, but that is deceptive since the ring is not Noetherian. A neighbourhood of the identity in X is isometric to an open set in the adele ring $\mathbb{Q}_{\mathbb{A}}$ (see [**13**] for the details); in this picture any $x \in X \setminus \{0\}$ must have some real or p-adic coordinate not equal to zero. Multiplication by a large real rational, or a large negative power of that prime p, will move such a point far from the identity, showing the whole action to be expansive. Just as in Boyle and Lind [**1**], any non-expansive subgroup lies in a non-expansive subgroup Γ with $\mathbb{Q}^{\times}_{>0}/\,\Gamma \cong \mathbb{Z}$ (that is, lies in some subgroup in what one is tempted to call $\mathsf{N}_{\infty-1}(\alpha)$). Any such subgroup acts non-expansively, since it must omit some prime p, and will therefore act isometrically on that direction in X.

On the other hand, it is clear that the set $\overline{\Omega}_\alpha$ does not detect this rich non-expansive structure. Since $\widehat{X}$ is a field, every map α^r for $r \in \mathbb{Q}^{\times}_{>0}$ has $\zeta_{\alpha^r}(z) = 1/(1-z)$, so the set $\overline{\Omega}_\alpha$ comprises the graph of the constant function 1.

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