

GROUP AUTOMORPHISMS WITH FEW AND WITH MANY PERIODIC POINTS

THOMAS WARD

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ABSTRACT. For each $C \in [0, \infty]$ a compact group automorphism $T : X \rightarrow X$ is constructed with the property that

$$\frac{1}{n} \log |\{x \in X \mid T^n(x) = x\}| \rightarrow C.$$

This may be interpreted as a combinatorial analogue of the (still open) problem of whether compact group automorphisms with any given topological entropy exist.

1. INTRODUCTION

One of the outstanding open problems in the dynamics of compact group automorphisms concerns the existence of group automorphisms with small entropy. Taking the infimum over all compact group automorphisms T , and writing $h_{\text{top}}(T)$ for the topological entropy of T , is

$$(1) \quad \inf\{h_{\text{top}}(T) \mid h_{\text{top}}(T) > 0\} > 0?$$

As pointed out in [9], the infimum being zero is equivalent to the statement that for any $H \in (0, \infty]$ there is an (ergodic) compact group automorphism with topological entropy H . The infimum being positive implies that the set of possible values of topological entropies of compact group automorphisms is countable.

The problem in (1) turns out to be exactly equivalent to Lehmer's Problem (see [9]). In [8], Lehmer associated to any monic polynomial

$$f(x) = \prod_{i=1}^d (x - \alpha_i) \in \mathbb{Z}[x]$$

the integer sequence defined by $\Delta_n(f) = \prod_{i=1}^d |\alpha_i^n - 1|$, and asked how fast this sequence grows. To avoid degeneracies, we assume that $\Delta_n(f) \neq 0$ for all $n \geq 1$ (equivalently, f does not vanish on any root of unity). It turns out that the measure of growth $\lim_{n \rightarrow \infty} \frac{\Delta_{n+1}(f)}{\Delta_n(f)}$ he used does not always exist, but a consequence of

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Baker's Theorem is that the logarithmic growth rate

$$(2) \quad m(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \Delta_n(f) = \sum_{i=1}^d \max\{\log |\alpha_i|, 0\}$$

exists (see [6, Chap. 1] for the details). Lehmer's problem is the following question: is

$$(3) \quad \inf\{m(f) \mid f \in \mathbb{Z}[x], m(f) > 0\} > 0?$$

The equivalence of (1) and (3) is proved in [9]; to see why this equivalence is plausible notice that if T_f is the endomorphism of the d -torus \mathbb{T}^d associated to the companion matrix of the polynomial f , then

$$h_{\text{top}}(T_f) = m(f).$$

More is true. The sequence $(\Delta_n(f))$ counts the periodic points under the automorphism T_f ,

$$(4) \quad F_n(T_f) = |\{x \in \mathbb{T}^d \mid T_f^n(x) = x\}| = \Delta_n(f)$$

whenever either side is finite (ergodicity of T_f is equivalent to $F_n(T_f)$ being finite for all $n \geq 1$). Thus (2) means that the logarithmic growth rate of the number of periodic points for an ergodic toral automorphism coincides with the topological entropy. Indeed, any continuous map on a compact metric space with sufficiently strong specification properties will have this property. For compact group automorphisms, expansiveness guarantees that the growth rate of periodic points is the topological entropy (this is shown in [10] in greater generality).

Without the assumption of expansiveness or ergodicity, we show that in contrast to the conjectured answer 'yes' to the questions (1) and (3) above, all possible logarithmic growth rates of periodic points arise for compact group automorphisms.

2. PERIODIC ORBITS

The examples will be constructed by controlling the growth in the number of orbits of length n , so the first step is to relate this to the number of points of period n . Given any bijection $T : X \rightarrow X$, define

$$(5) \quad F_n(T) = |\{x \in X \mid T^n(x) = x\}|$$

to be the number of points of period n and

$$(6) \quad L_n(T) = |\{x \in X \mid |\{T^j(x)\}_{j \in \mathbb{N}}| = n\}|$$

to be the number of points with least period n . The sequences $(F_n(T))$ and $(L_n(T))$ determine each other via the relation (7) and its Möbius inversion (see [5, Sect. 11.2] or [15]).

Lemma 1. *For $C \in (0, \infty)$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log F_n(T) = C$$

if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log L_n(T) = C.$$

Proof. This is shown in [15, Sect. 4]; a short proof is included here. First notice that

$$(7) \quad F_n(T) = \sum_{d|n} L_d(T).$$

If $\frac{1}{n} \log L_n(T) \rightarrow C > 0$ then, for n large enough to have $F_n(T) > 0$,

$$\begin{aligned} \frac{1}{n} \log L_n(T) &\leq \frac{1}{n} \log F_n(T) = \frac{1}{n} \log \left(\sum_{d|n} L_d(T) \right) \\ &\leq \frac{1}{n} \log n + \frac{1}{n} \log \max_{d|n} \{L_d(T)\}. \end{aligned}$$

For each such n , choose \tilde{n} so that $L_{\tilde{n}}(T) = \max_{d|n} \{L_d(T)\}$ and $\frac{\tilde{n}}{n} \leq 1$. Now $\tilde{n} \rightarrow \infty$ as $n \rightarrow \infty$ since $C > 0$. It follows that

$$\begin{aligned} \frac{1}{n} \log L_n(T) &\leq \frac{1}{n} \log F_n(T) \leq \frac{1}{n} \log n + \frac{\tilde{n}}{n} \cdot \frac{1}{\tilde{n}} \log L_{\tilde{n}}(T) \\ &\leq \frac{1}{n} \log n + \frac{1}{\tilde{n}} \log L_{\tilde{n}}(T) \rightarrow C, \end{aligned}$$

so $\frac{1}{n} \log F_n(T) \rightarrow C$.

Now assume that $\frac{1}{n} \log F_n(T) \rightarrow C$. For $r \geq 1$,

$$F_r(T) \geq L_r(T) = - \sum_{d|r, d \neq r} L_d(T) + F_r(T) \geq F_r(T) - \sum_{d|r, d \neq r} F_d(T).$$

Let R be an upper bound for $\{\frac{1}{n} \log F_n(T) \mid F_n(T) \neq 0\}$; pick $\epsilon \in (0, 3C)$. Choose N so that

$$r > N \implies e^{r(C-\epsilon)} \leq F_r(T) \leq e^{r(C+\epsilon)}.$$

Then for $r > 2N$,

$$\begin{aligned} F_r(T) \geq L_r(T) &\geq F_r(T) - \sum_{n=1}^N F_n(T) - \sum_{n=N+1}^{\lfloor r/2 \rfloor} F_n(T) \\ &\geq F_r(T) - \left(N e^{NR} + (r/2 - N) e^{r(C+\epsilon)/2} \right) \\ &\geq F_r(T) \left(1 - N e^{NR-r(C-\epsilon)} - (r/2 - N) e^{-r(C-3\epsilon)/2} \right), \end{aligned}$$

and the bracketed expression converges to 1 as $r \rightarrow \infty$. Taking logs and dividing by r gives the result. \square

Write $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for the finite field with p elements. For any prime p , the field \mathbb{F}_p has a primitive root g that generates the cyclic multiplicative group \mathbb{F}_p^* .

Theorem 2. *For any $C \in [0, \infty]$, there is a compact group automorphism T with $F_n(T) < \infty$ for all n and with*

$$\frac{1}{n} \log F_n(T) \longrightarrow C.$$

Proof. If $C = 0$ then we may take X to be any finite group (more interesting ergodic examples with $X = \widehat{\mathbb{Q}}$ or $\widehat{\mathbb{F}_p}(t)$ may be found in [2] and [11]).

Assume now that $C = \infty$. Choose primes p_1, p_2, \dots with the following properties:

- (1) $\frac{1}{n} \log p_n \longrightarrow \infty$, and

(2) $n|p_n - 1$ for all $n \geq 1$.

This can be done since there are infinitely many primes in the arithmetic progression $(1 + nk)_{k \in \mathbb{N}}$ by Dirichlet's theorem. For each i , let g_i be a primitive root modulo p_i and define a map $T_i : \mathbb{F}_{p_i} \rightarrow \mathbb{F}_{p_i}$ by $T_i(x) = g_i^{(p_i-1)/i} x$ modulo p_i . Then $F_n(T_i) = p_i$ if and only if $i|n$; in all other cases $F_n(T_i) = 1$. Define $X = \prod_{i=1}^{\infty} \mathbb{F}_{p_i}$ and the automorphism $T = \prod_{i=1}^{\infty} T_i$. Then $F_n(T) \geq p_n$, so $\frac{1}{n} \log F_n(T) \rightarrow \infty$. Notice that $F_n(T)$ is finite for all n .

The argument above uses the fact that arithmetic progressions of the form $(1 + nk)_{k \in \mathbb{N}}$ contain infinitely many primes. For $C < \infty$, a related but deeper fact is used. Linnik [12] has shown that the least prime congruent to 1 modulo n is no larger than n^A for some absolute constant A . Later developments have culminated in Heath-Brown's proof [7] that the bound may be replaced by $Bn^{5.5}$ for an explicitly computable constant B .

The case $C \in (0, \infty)$ remains. By Lemma 1, it is enough to construct a group automorphism T with $L_n(T)$ close to e^{nC} . By Linnik's Theorem, choose primes p_1, p_2, \dots with the following properties:

- (1) $n|p_n - 1$ for all $n \geq 1$, and
- (2) $p_n \leq n^A$.

Notice that the first property ensures that $p_n \rightarrow \infty$ as $n \rightarrow \infty$ and the second property ensures that $\frac{n}{\log p_n} \rightarrow \infty$ as $n \rightarrow \infty$. Let g_n be a primitive root modulo p_n , and define

$$(8) \quad K_n = \left\lfloor \frac{nC}{\log p_n} \right\rfloor \text{ for } n \geq 1,$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x . For each n , let

$$T_n : (\mathbb{F}_{p_n})^{K_n} \rightarrow (\mathbb{F}_{p_n})^{K_n}$$

be defined by

$$T_n(x_1, \dots, x_{K_n}) = (g_n^{(p_n-1)/n} x_1, \dots, g_n^{(p_n-1)/n} x_{K_n}).$$

Define the compact group by

$$X = \prod_{i=1}^{\infty} (\mathbb{F}_{p_i})^{K_i},$$

and the automorphism by $T = \prod_{i=1}^{\infty} T_i$.

The map T_n^j is the identity if and only if $n|j$. So $T_n^j(x_1, \dots, x_{K_n}) = (x_1, \dots, x_{K_n})$ if and only if $x_1 = \dots = x_{K_n} = 0$ or $n|j$. It follows that the number of points of least period n under T is given by

$$(9) \quad L_n(T) = p_n^{K_n} - 1.$$

Thus

$$\begin{aligned} \frac{1}{n} \log L_n(T) &= \frac{1}{n} \log(p_n^{K_n} - 1) \\ &= \frac{K_n}{n} \log(p_n^{K_n} - 1)^{1/K_n} \\ &= \frac{1}{n} \left\lfloor \frac{nC}{\log p_n} \right\rfloor \log(p_n^{K_n} - 1)^{1/K_n} \rightarrow C \end{aligned}$$

as $n \rightarrow \infty$ since $\frac{n}{\log p_n} \rightarrow \infty$. It follows by Lemma 1 that

$$\frac{1}{n} \log F_n(T) \rightarrow C \text{ as } n \rightarrow \infty.$$

□

3. REMARKS

- (1) The only requirement in the proof is that $K_n \rightarrow \infty$, so the same argument allows the construction of group automorphisms with any super-polynomial growth rate, $\frac{\log F_n(T)}{\log n} \rightarrow \infty$.
- (2) A similar infinite product construction is used in [4] to exhibit a group automorphism whose periodic points count the Bernoulli denominators. Moss [13] has obtained results showing when a given realisable divisibility sequence can arise from a group automorphism.
- (3) The proof of Theorem 2 constructs a non-ergodic automorphism of a totally disconnected group. Does the same result hold for ergodic automorphisms of connected groups? On connected groups, the type of behaviour seen in automorphisms with few periodic points seems to be $\limsup_{n \rightarrow \infty} \frac{1}{n} \log F_n(T) = h_{\text{top}}(T)$ and $\liminf_{n \rightarrow \infty} \frac{1}{n} \log F_n(T) = 0$ (see [16], [17]).
- (4) The dynamical zeta function associated to a map T with $F_n(T) < \infty$ for all $n \geq 1$ is the formal power series $\exp \sum_{n=1}^{\infty} \frac{z^n}{n} F_n(T)$. Explicit examples of group automorphisms with irrational zeta functions are easy to find (see [2]; it should be noted that the examples of this kind in [3] do not seem to be correct). For values of C that are not reciprocals of algebraic integers, the examples constructed here cannot have rational zeta functions for a purely arithmetic reason (the poles and zeros of a rational zeta function can only occur at reciprocals of algebraic integers [1, Appendix] or [14, Part VIII, Chap. 4, No. 230]).

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SCHOOL OF MATHEMATICS, UNIVERSITY OF EAST ANGLIA, NORWICH NR4 7TJ, UNITED KINGDOM

E-mail address: `t.ward@uea.ac.uk`