Random walk with barycentric self-interaction

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13th October 2011

Abstract

We study the asymptotic behaviour of a d-dimensional self-interacting random walk $(X_n)_{n\in\mathbb{N}}$ ($\mathbb{N} := \{1,2,3,\ldots\}$) which is repelled or attracted by the centre of mass $G_n = n^{-1}\sum_{i=1}^n X_i$ of its previous trajectory. The walk's trajectory $(X_1,\ldots,X_n)$ models a random polymer chain in either poor or good solvent. In addition to some natural regularity conditions, we assume that the walk has one-step mean drift

$$\mathbb{E}[X_{n+1} - X_n \mid X_n - G_n = \mathbf{x}] \approx \rho \|\mathbf{x}\|^{-\beta}\hat{\mathbf{x}}$$

for $\rho \in \mathbb{R}$ and $\beta \geq 0$. When $\beta < 1$ and $\rho > 0$, we show that X_n is transient with a limiting (random) direction and satisfies a super-diffusive law of large numbers: $n^{-1/(1+\beta)}X_n$ converges almost surely to some random vector. When $\beta \in (0,1)$ there is sub-ballistic rate of escape. When $\beta \geq 0$ and $\rho \in \mathbb{R}$ we give almost-sure bounds on the norms $\|X_n\|$, which in the context of the polymer model reveal extended and collapsed phases.

Analysis of the random walk, and in particular of $X_n - G_n$, leads to the study of real-valued time-inhomogeneous non-Markov processes $(Z_n)_{n\in\mathbb{N}}$ on $[0,\infty)$ with mean drifts of the form

$$\mathbb{E}[Z_{n+1} - Z_n \mid Z_n = x] \approx \rho x^{-\beta} - \frac{x}{n}, \tag{0.1}$$

where $\beta \geq 0$ and $\rho \in \mathbb{R}$. The study of such processes is a time-dependent variation on a classical problem of Lamperti; moreover, they arise naturally in the context of the distance of simple random walk on $\mathbb{Z}^d$ from its centre of mass, for which we also give an apparently new result. We give a recurrence classification and asymptotic theory for processes Z_n satisfying (0.1), which enables us to deduce the complete recurrence classification (for any $\beta \geq 0$) of $X_n - G_n$ for our self-interacting walk.

Keywords: Self-interacting random walk; self-avoiding walk; random walk avoiding its convex hull; random polymer; centre of mass; simple random walk; random walk average; limiting direction; law of large numbers.
AMS 2010 Subject Classifications: 60J05 (Primary) 60K40; 60F15; 82C26 (Secondary)

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1 Introduction

We study a self-interacting random walk. Self-interacting random processes, in which the stochastic behaviour depends on the entire previous history of the process, present many challenges for mathematical analysis (see e.g. [4,34] and references therein) and are often motivated by real applications.

Although not a random process in the same sense, the *self-avoiding walk* is a prototypical example of a self-interacting random walk that gives rise to important and difficult problems. Random self-avoiding walks were introduced to model the configuration of polymer molecules in solution. The sites visited by the walk represent the locations of the polymer's constituent monomers; successive monomers are viewed as connected by chemical bonds. The classical self-avoiding walk (SAW) model takes uniform measure on n-step self-avoiding paths in $\mathbb{Z}^d$. In the important cases of $d \in \{2, 3\}$, there are still major open problems for such walks: see for example [26,28] and [17, Chapter 7], or [37, Chapter 7] for a mathematical physics perspective.

The *loop-erased random walk* (LERW), obtained by erasing chronologically the loops of a random walk, was introduced in [24] to study SAW, but it was soon realized that the two processes belong to different universality classes. For its independent interest, including applications to combinatorics and quantum field physics, LERW has received considerable attention and now there is a more precise picture of its behaviour, which shows fine dependence on the spatial dimension. In the planar case, the mean number of steps for LERW stopped at distance n is of order $n^{5/4}$ [20], and the scaling limit is conformally invariant, described by the radial Schramm–Loewner evolution with parameter 2 [25].

A different perspective on polymer models concerns directed polymers, where the self-interaction is reduced to a trivial form but interesting phenomena arise from the interaction with the medium: see [14, 16] for recent surveys for localization on interfaces (pinning, wetting) possibly with time-inhomogeneities (e.g. copolymers), and [9] for interactions with a time-space inhomogeneous medium leading to localization in the bulk.

In the standard framework, SAW cannot be interpreted as a dynamic (or progressive) stochastic process. There have been many attempts to formulate genuine stochastic processes with similar behaviour to that of, or at least conjectured for, SAW. A recent model is the random walk on $\mathbb{R}^2$ which at each step avoids the convex hull of its preceding values [2,39]. Unlike the conjectured behaviour of SAW, this model is ballistic (see [2,39]), i.e., it has a positive speed. The discrete version on $\mathbb{Z}^2$, the dynamic prudent walk, has been studied in [5]: it is ballistic with speed 3/7 (in the L^1 norm), but, in contrast to the (conjecture for the) continuous model, it does not have a fixed direction (see [5]). Ballisticity is known for other types of self-interacting random walks: see [7, 18].

In this paper we consider a self-interacting random walk model that is a tractable alternative to SAW, and is distinguished from the models of [2,5,7,18,39] by exhibiting a range of possible scaling behaviour, including sub-ballisticity (i.e., zero speed) and super-diffusivity. Our model is tunable, with parameters that in principle can be estimated from real data, and it can be used to represent polymers in the extended phase (for good solvent) or collapsed phase (poor solvent). The self-interaction in the model at time n is mediated through the *barycentre* or *centre of mass* of the past trajectory until time n. Specifically, our random walk will at each step have a mean drift (typically asymptotically zero in magnitude) pointing away from or towards the average of all previous positions. We now informally describe the probabilistic model; we give a brief description of the motivation and interpretation arising from polymer physics in Section 3.3.

Let $d \in \mathbb{N} := \{1, 2, 3, \ldots\}$. Our random walk will be a discrete-time stochastic process $X = (X_n)_{n \in \mathbb{N}}$ on $\mathbb{R}^d$. For $n \in \mathbb{N}$, set

$$G_n := \frac{1}{n} \sum_{i=1}^{n} X_i, \tag{1.1}$$

the centre of mass (average) of $\{X_1, \ldots, X_n\}$. In addition to some regularity conditions on X that we describe later, our main assumption will be that the one-step mean drift of the walk after n steps is of order $\|X_n - G_n\|^{-\beta}$ in the direction $\pm(X_n - G_n)$, where $\beta \geq 0$ is a fixed parameter; here and subsequently $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^d$. Loosely speaking for the moment, we will suppose that for some $\rho \in \mathbb{R}$ and $\beta \geq 0$,

$$\mathbb{E}[X_{n+1} - X_n \mid X_n - G_n = \mathbf{x}] \approx \rho \|\mathbf{x}\|^{-\beta} \hat{\mathbf{x}}, \tag{1.2}$$

for any $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, where $\hat{\mathbf{x}} := \mathbf{x}/\|\mathbf{x}\|$ denotes a unit vector in the $\mathbf{x}$-direction and $\mathbf{0}$ is the origin in $\mathbb{R}^d$. We attach no precise meaning to '$\approx$' in (1.2) (or elsewhere); it indicates that we are ignoring some terms and also that we have not yet formally defined all the terms present. We describe the model formally and in detail in Section 2 below.

The natural case of our model to compare to the walk that avoids its convex hull [2,39] has $\beta = 0$ and $\rho > 0$, when our walk has positive drift away from its current centre of mass. In our $\beta = 0$, $\rho > 0$ setting we show that the walk has an asymptotic speed and an asymptotic direction, properties which are conjectured but not yet proved for the walk avoiding its convex hull [2, 39]. Our results however cover much more than this special case. For example, the case of our model that we might expect to be in some sense comparable to SAW in $d = 2$ has $\beta = 1/3$, $\rho > 0$: see the discussion in Section 3.3 below.

To give a flavour of our more general results, described in more detail in Section 2.2 below, we now informally describe our results in the case where (1.2) holds with $\rho > 0$ and $\beta \in [0, 1)$. Under suitable regularity conditions, we show that X is transient, i.e. $\|X_n\| \to \infty$ a.s., and moreover we prove a strong law of large numbers that precisely quantifies this transience: $n^{-1/(1+\beta)} \|X_n\|$ is asymptotically constant, almost surely. In addition, we show that X_n has a limiting direction, that is, $X_n/\|X_n\|$ converges a.s. to some (random) unit vector. Thus we have, in this case, a rather complete picture of the asymptotic behaviour of X_n. For other regions of the (ρ, β) parameter space we have other results, although we also leave some interesting open problems.

The self-interaction in the model is introduced via the presence of G_n in (1.2). If the condition $\{X_n - G_n = \mathbf{x}\}$ in (1.2) is replaced by $\{X_n = \mathbf{x}\}$ then there is no self-interaction in the drift, which instead points away from a fixed origin. Such non-homogeneous 'centrally biased' walks were studied by Lamperti in [21, Section 4] and [23, Section 5]; for more recent work see e.g. [13, 27, 31]. Considering the process of norms $Z_n = \|X_n\|$ leads to a process on $[0, \infty)$ with mean drift

$$\mathbb{E}[Z_{n+1} - Z_n \mid Z_n = x] \approx \rho' x^{-\beta}, \tag{1.3}$$

ignoring higher-order terms. Such 'asymptotically zero-drift' processes are of independent interest; the asymptotic analysis of such (not necessarily Markov) processes is sometimes known as *Lamperti's problem* following pioneering work of Lamperti [21–23]. From the point of view of the recurrence classification of processes satisfying (1.3), the case $\beta = 1$ turns out to be critical, in which case the value of $\rho' \in \mathbb{R}$ is crucial: we give a brief summary of the relevant background in Section 3.1 below.

We shall see below that considering the process $Z_n = \|X_n - G_n\|$ with X_n satisfying (1.2) leads to a more complicated form of (1.3). Loosely speaking, we will obtain

$$\mathbb{E}[Z_{n+1} - Z_n \mid Z_n = x] \approx \rho' x^{-\beta} - \frac{x}{n}. \tag{1.4}$$

We note that the two terms on the right-hand side of (1.4) are typically of the same order, as can be predicted by solving the corresponding differential equation, and so both contribute to the asymptotic behaviour.

Comparing (1.4) with (1.3), we see that the drift is now *time*- as well as space-dependent. (A different variation on (1.3) with this property was studied in [30], where processes with drift $\rho x^\alpha n^{-\beta}$ were considered.) Thus (1.4) is an interesting starting point for analysis in its own right. Additional motivation for (1.4) arises naturally from simple random walk (SRW) and its centre of mass: if $Z_n = \|X_n - G_n\|$ where X_n is a symmetric SRW on $\mathbb{Z}^d$ and G_n its centre-of-mass as defined by (1.1), Z_n satisfies (1.4) with $\beta = 1$ and $\rho' = \rho'(d)$; see Section 3.2 below.

Let us step back from the general setting for a moment to state one consequence of our results, which is a (seemingly new) observation on SRW:

Theorem 1.1. *Let $d \in \mathbb{N}$. Suppose that $(X_n)_{n\in\mathbb{N}}$ is a symmetric SRW on $\mathbb{Z}^d$, and $(G_n)_{n\in\mathbb{N}}$ is its centre-of-mass process as defined by (1.1). Then*

(a) $\liminf_{n\to\infty} \|X_n - G_n\| < \infty$ *a.s. for $d \in \{1, 2\}$;*

(b) $\lim_{n\to\infty} \|X_n - G_n\| = \infty$ *a.s. for $d \geq 3$.*

Pólya's recurrence theorem says that X_n is recurrent in $d \leq 2$ and transient in $d \geq 3$, while results of Grill [15] say that the centre-of-mass process G_n is recurrent only in $d = 1$ and transient for $d \geq 2$. Thus the asymptotic behaviour of $X_n - G_n$ is not trivial; Theorem 1.1 says that it is recurrent if and only if $d \in \{1, 2\}$. In particular when $d = 2$, X_n and $X_n - G_n$ are both recurrent, but G_n is transient; see Figure 1 for a simulation.

Remark 1.1. *Theorem 1.1 exhibits an amusing feature. With the notation $\Delta_n := X_{n+1} - X_n$ it is not hard to see from (1.1) that we may write (with $X_0 := \mathbf{0}$)*

$$G_n = \sum_{i=0}^{n-1} \left(1 - \frac{i}{n}\right) \Delta_i; \quad X_n - G_n = \sum_{i=0}^{n-1} \left(\frac{i}{n}\right) \Delta_i.$$

It follows that (for fixed *n) $X_n - G_n$ and G_n are very nearly* time-reversals *of each other: writing $\Delta'_i := \Delta_{n-i}$ we see that*

$$X_n - G_n = \sum_{i=1}^{n} \left(1 - \frac{i}{n}\right) \Delta'_i.$$

Despite this, the two processes *behave very differently, as can be seen by contrasting Theorem 1.1 with Grill's result [15].*

It is natural to ask whether a continuous analogue of Theorem 1.1 holds. In the one-dimensional case, we would take B_t to be standard Brownian motion and $G_t = t^{-1} \int_0^t B_s \mathrm{d}s$, and ask about the joint behaviour of (B_t, G_t); in higher dimensions, writing the d-dimensional Brownian motion as $(B_t^{(1)}, \ldots, B_t^{(d)})$, the ith component $G_t^{(i)}$ of G_t is

$t^{-1} \int_0^t B_s^{(i)} \mathrm{d}s$, and different components are independent. We could not find a Brownian analogue of Grill's theorem [15] for (compact set) recurrence/transience of G_t explicitly stated in the literature. The process $(tG_t)_{t \geq 0}$ is *integrated Brownian motion*, or the *Langevin process*, see e.g. [3, 19] and references therein. The two-dimensional process $(B_t, tG_t)_{t \geq 0}$ is the *Kolmogorov diffusion* [19]. Theorem 1.1 gives basic information about the joint behaviour of a discrete version of this process, under a re-scaling of the second coordinate.

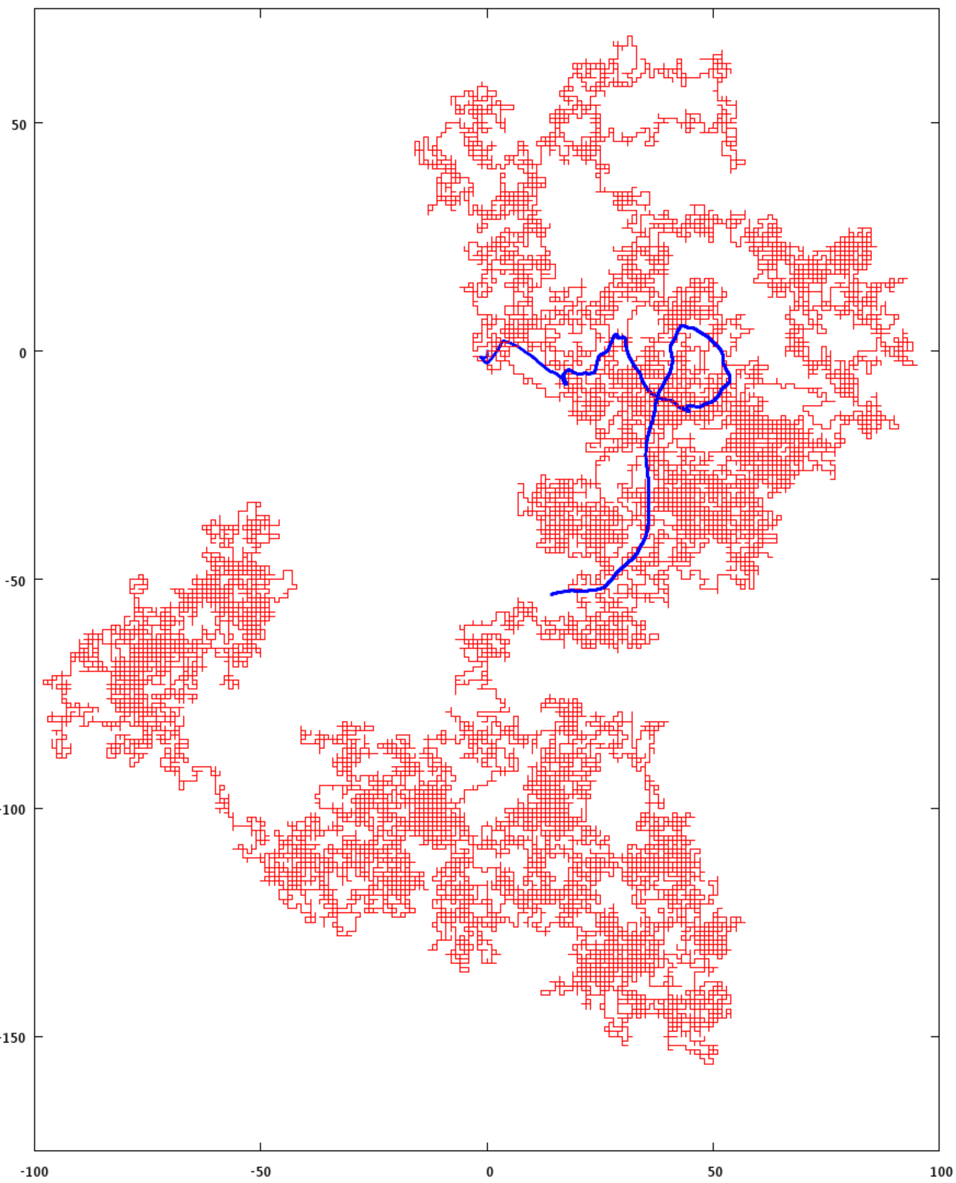

Figure 1: Simulation of 4×10^4 steps of symmetric SRW starting at the origin of $\mathbb{Z}^2$ and its centre of mass process (thick line). [color online]

In Section 2 we formally define our self-interacting random walk and state our main results. In Section 3 we discuss some more of the motivation behind our model (coming from the physics of polymers and also purely theoretical considerations) and also the one-dimensional problems associated with (1.3) and (1.4), and explain how SRW (and Theorem 1.1) fits into our picture. The subsequent sections are devoted to the proofs.

We finish this section with some comments on the relation of our model to the existing literature. We are not aware of any self-interacting random walk models similar to the one studied here (i.e., interacting with the previous history of the process, as summarized through the barycentre). In broad outline, our model is related to the vertex-reinforced random walk (see [34, Section 5.3]) in that the evolution of the walk depends on the sites previously visited. A significant difference is that in vertex-reinforced random walk this self-interaction is local, in that only the occupation of nearest-neighbours of the current site affects the law of the increment, whereas our interaction, mediated by the barycentre, is global. In the continuous setting, self-interacting diffusions (or 'Brownian polymers') with similar flavour and motivation to those of our model have also been studied over the last two decades or so, but are rather different in detail to the model considered here: see e.g. [4, 11, 32, 33] and references therein; some recent work on processes with self-attracting drift defined through a potential includes [6]. In the self-interacting diffusion setting, most of the results in the literature are concerned with the ergodic case; questions of recurrence/transience seem to have received little attention (particularly in dimensions greater than 1), and we do not know of any results on asymptotic directions. Also, it is typically assumed that the vector consisting of the process and its empirical average are Markovian, whereas our model is more general. See [32, Section 1] for a short survey.

2 The model and main results

2.1 Definitions and assumptions

We now define the stochastic process $X := (X_n)_{n\in\mathbb{N}}$ on $\mathbb{R}^d$ ($d \in \mathbb{N}$) that is our main object of study. (We start at time $n = 1$ only so that (1.1) has the neatest form.) The process X will not be Markovian, as the distribution of X_{n+1} will depend on the entire history $X_1, \ldots, X_n$, although to a large extent this dependence will be mediated through the current centre of mass G_n defined at (1.1). Formally, we suppose that $(X_n)_{n\in\mathbb{N}}$ is adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$; note that by (1.1) $G_1, \ldots, G_n$ are $\mathcal{F}_n$-measurable. We use the notation $\mathbb{P}_n[\,\cdot\,] := \mathbb{P}[\,\cdot \mid \mathcal{F}_n]$ and $\mathbb{E}_n[\,\cdot\,] := \mathbb{E}[\,\cdot \mid \mathcal{F}_n]$. Throughout the paper we understand $\log x$ to mean $\log x$ if $x \geq 1$ and 0 otherwise.

We impose some specific assumptions on the law of $\Delta_n := X_{n+1} - X_n$ given $\mathcal{F}_n$. We assume that for some $B \in (0, \infty)$ and all $n \in \mathbb{N}$,

$$\mathbb{P}_n[\|\Delta_n\| \leq B] = 1, \text{ a.s.}. \tag{2.1}$$

The assumption of uniformly bounded jumps can be replaced by an assumption on higher order moments at the expense of additional technical complications, but (2.1) is natural when the increments represent chemical bonds in a model for a polymer molecule.

Our next assumption will be a precise version of (1.2). We suppose that for some $\rho \in \mathbb{R}$ and $\beta \geq 0$, for any $n \in \mathbb{N}$, writing $\mathbf{x} = X_n - G_n$ for convenience,

$$\mathbb{E}_n[\Delta_n] = \rho\|\mathbf{x}\|^{-\beta}\hat{\mathbf{x}} + O(\|\mathbf{x}\|^{-\beta}(\log\|\mathbf{x}\|)^{-2}), \text{ a.s.}, \tag{2.2}$$

as $\|\mathbf{x}\| \to \infty$, where $\hat{\mathbf{x}} := \mathbf{x}/\|\mathbf{x}\|$. (In (2.2) the exponent -2 on the logarithm is chosen for simplicity; it could be replaced with any exponent strictly less than -1.) In equation (2.2) and similar vector equations in the sequel, terms such as $O(\,\cdot\,)$ indicate the presence of a vector whose norm satisfies the given $O(\,\cdot\,)$ asymptotics (similarly for $o(\,\cdot\,)$); error terms not involving n are understood to be uniform in n. To be clear, (2.2) is to be understood as, with $X_n - G_n = \mathbf{x}$, as $\|\mathbf{x}\| \to \infty$,

$$\sup_{n\in\mathbb{N}} \operatorname{ess\,sup} \|\mathbb{E}_n[\Delta_n] - \rho\|\mathbf{x}\|^{-\beta}\hat{\mathbf{x}}\| = O(\|\mathbf{x}\|^{-\beta}(\log\|\mathbf{x}\|)^{-2}).$$

We also need to assume a uniform ellipticity condition, to ensure that our random walk does not get 'trapped' in some subset of $\mathbb{R}^d$. Let $\mathbb{S}_d := \{\mathbf{e} \in \mathbb{R}^d : \|\mathbf{e}\| = 1\}$ denote the unit-radius sphere in $\mathbb{R}^d$. We suppose that there exists $\varepsilon_0 > 0$ such that

$$\operatorname*{ess\,inf}_{\mathbf{e}\in\mathbb{S}_d} \mathbb{P}_n[\Delta_n \cdot \mathbf{e} \geq \varepsilon_0] \geq \varepsilon_0. \tag{2.3}$$

Write $\Delta_n = (\Delta_n^{(1)}, \ldots, \Delta_n^{(d)})$ in Cartesian components. An immediate consequence of (2.3) is the following lower bound on second moments: a.s.,

$$\min_{i\in\{1,\ldots,d\}} \mathbb{E}_n[(\Delta_n^{(i)})^2] \geq 2\varepsilon_0^3 > 0. \tag{2.4}$$

Our primary standing assumption will be the following.

(A1) Let $d \in \mathbb{N}$. Let $X := (X_n)_{n\in\mathbb{N}}$ be a stochastic process on $\mathbb{R}^d$ and $G := (G_n)_{n\in\mathbb{N}}$ its associated centre-of-mass process defined by (1.1). For definiteness, take $X_1 \in \mathbb{R}^d$ to be fixed. Suppose that for some $B < \infty$, $\varepsilon_0 > 0$, $\rho \in \mathbb{R}$, and $\beta \geq 0$ the conditions (2.1), (2.2), and (2.3) hold.

In the examples discussed later (see Section 2.3), $(X_n, G_n)_{n\in\mathbb{N}}$ will be a Markov process, but we do not assume the Markov property in general.

When $\beta = 1$, as in the Lamperti case [21, 23] the value of ρ in (2.2) will turn out to be crucial. As in Lamperti's problem, the recurrence classification depends on the relationship between ρ and the covariance structure of Δ_n. To obtain an explicit criterion, we impose additional regularity conditions on that covariance structure. Specifically, we sometimes suppose that (a) there exists $\sigma^2 \in (0, \infty)$ such that, a.s.,

$$\mathbb{E}_n[(\Delta_n^{(i)})^2] = \sigma^2 + o((\log\|X_n - G_n\|)^{-1}), \quad (i \in \{1, \ldots, d\}); \tag{2.5}$$

and (b) for i, j distinct elements of $\{1, \ldots, d\}$, a.s.,

$$\mathbb{E}_n[\Delta_n^{(i)}\Delta_n^{(j)}] = o((\log\|X_n - G_n\|)^{-1}). \tag{2.6}$$

Thus for $\beta \geq 1$, when necessary we will impose the following additional assumption.

(A2) The conditions (2.5) and (2.6) hold for some $\sigma^2 \in (0, \infty)$.

2.2 Results on self-interacting walk

Our first result, Theorem 2.1, constitutes the first part of our complete recurrence classification for $X_n - G_n$. Since we are dealing with non-Markovian processes, we first formally define what we mean by recurrence and transience in this context.

Definition 2.1. *An $\mathbb{R}^d$-valued stochastic process $(\xi_n)_{n\in\mathbb{N}}$ is said to be* recurrent *if* $\liminf_{n\to\infty} \|\xi_n\| < \infty$ *a.s. and* transient *if* $\lim_{n\to\infty} \|\xi_n\| = \infty$ *a.s..*

Define

$$\rho_0 := \rho_0(d, \sigma^2) := \frac{1}{2}(2-d)\sigma^2. \tag{2.7}$$

Theorem 2.1. *Suppose that (A1) and (A2) hold with $d \in \mathbb{N}$, $\beta \geq 1$, and $\rho \in \mathbb{R}$.*

(i) Suppose that $\beta = 1$. Let $\rho_0 = \rho_0(d, \sigma^2)$ be as defined at (2.7). Then $X_n - G_n$ is recurrent if $\rho \leq \rho_0$ and transient if $\rho > \rho_0$.

(ii) Suppose that $\beta > 1$. Then $X_n - G_n$ is recurrent if $d \in \{1, 2\}$ and transient if $d \geq 3$.

For almost all our remaining results we do not need to assume (A2). Set

$$\ell(\rho, \beta) := \left(\frac{\rho(1+\beta)}{2+\beta} \right)^{1/(1+\beta)}. \tag{2.8}$$

In the case $\beta \in [0, 1)$, we have the following result, which completes the recurrence classification for $X_n - G_n$ and also gives a detailed account of the asymptotic behaviour of the random walk X_n. In particular, when $\rho > 0$, X_n and G_n are transient, and moreover have a limiting direction, and the escape is quantified by super-diffusive but, for $\beta > 0$, sub-ballistic strong laws of large numbers. The case $\beta = 0$ shows ballistic behaviour.

Theorem 2.2. *Suppose that (A1) holds with $d \in \mathbb{N}$, $\beta \in [0, 1)$, and $\rho \in \mathbb{R} \setminus \{0\}$. Then $X_n - G_n$ is transient if $\rho > 0$ and recurrent if $\rho < 0$. Moreover, if $\rho > 0$, there exists a random $\mathbf{u} \in \mathbb{S}_d$ such that, as $n \to \infty$, with $\ell(\rho, \beta)$ defined at (2.8),*

$$n^{-1/(1+\beta)} X_n \xrightarrow{\text{a.s.}} (2+\beta)\ell(\rho, \beta)\mathbf{u}, \text{ and } n^{-1/(1+\beta)} G_n \xrightarrow{\text{a.s.}} (1+\beta)\ell(\rho, \beta)\mathbf{u}.$$

At the level of detail displayed by Theorem 2.2, we can see a difference between the asymptotic behaviour of the $\beta \in [0, 1)$, $\rho > 0$ case of (2.2) compared to the 'supercritical Lamperti-type' case in which the drift is away from a fixed origin (i.e., the analogue of (2.2) holds but with $\mathbf{x} = X_n$ rather than $\mathbf{x} = X_n - G_n$). See Theorem 2.5 below and the remarks that precede it.

Our ultimate goal is a complete recurrence classification for the process X_n. Theorem 2.2 covers the case $\beta \in [0, 1)$, $\rho > 0$. Otherwise, we have at the moment only the following one-dimensional result (to be viewed in conjunction with Theorem 2.1).

Theorem 2.3. *Suppose that (A1) holds for $d = 1$. Then if $X_n - G_n$ is transient, X_n and G_n are also transient, i.e., $|X_n| \to \infty$ and $|G_n| \to \infty$ a.s. as $n \to \infty$.*

Our final result on our walk with barycentric interaction gives upper bounds on $\|X_n\|$ for general $d \in \mathbb{N}$. In view of the interpretation of $(X_1, \ldots, X_n)$ as a model for a polymer molecule in solution, we can describe the phases listed in Theorem 2.4 below as (i) *extended*, (ii) *transitional*, (iii) *partially collapsed*, and (iv) *fully collapsed*. See the discussion in Section 3.3 below. Theorem 2.4(i) is included for comparison only; Theorem 2.2 gives a much sharper result. Define

$$\gamma(d, \sigma^2, \rho) := \left(2 - d - \frac{2\rho}{\sigma^2} \right)^{-1}. \tag{2.9}$$

Theorem 2.4. *Suppose that (A1) holds with $d \in \mathbb{N}$, $\beta \geq 0$, and $\rho \in \mathbb{R}$. Then the following bounds apply.*

(i) *(Theorem 2.2.) If $\beta \in [0,1)$ and $\rho > 0$, there exists $C \in (0,\infty)$ such that, a.s., $\|X_n\| \le Cn^{1/(1+\beta)}$ for all but finitely many $n \in \mathbb{N}$.*

(ii) *If $\beta \ge 1$, then for any $\varepsilon > 0$, a.s., $\|X_n\| \le n^{1/2}(\log n)^{(1/2)+\varepsilon}$ for all but finitely many $n \in \mathbb{N}$.*

(iii) *Suppose that (A2) also holds. Suppose that $\beta = 1$ and $\rho < -d\sigma^2/2$, and let $\gamma(d,\sigma^2,\rho) \in (0,1/2)$ be as defined at (2.9). Then for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{N}$, $\|X_n\| \le n^{\gamma(d,\sigma^2,\rho)+\varepsilon}$.*

(iv) *If $\beta \in [0,1)$ and $\rho < 0$, then for any $\varepsilon > 0$, a.s., $\|X_n\| \le (\log n)^{1+\frac{1}{1-\beta}+\varepsilon}$ for all but finitely many $n \in \mathbb{N}$.*

We suspect that the bounds in Theorem 2.4 are close to sharp, in that corresponding lower bounds of almost the same order should be valid (only infinitely often, of course, in the recurrent cases). However, the lower bounds of [29, Section 4] do not apply directly.

Given (1.1) it is evident that the bounds for $\|X_n\|$ in Theorem 2.4 imply the same bounds (up to multiplication by a constant) for $\|G_n\|$, and hence $\|X_n - G_n\|$ too. In addition, the same upper bounds hold (again up to a constant factor) for the quantities of *diameter* D_n and root-mean-square *radius of gyration* R_n given by

$$D_n := \max_{1\le i<j\le n} \|X_i - X_j\|, \quad R_n^2 := \frac{1}{n}\sum_{i=1}^{n} \|X_i - G_n\|^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} \|X_i - X_j\|^2;$$

these are both physically significant in the interpretation of $(X_1,\ldots,X_n)$ as a polymer chain (see pp. 95–96 of [17] and Section 3.3 below).

Finally, we briefly describe how our results compare to the more classical model studied by Lamperti [21, 23]. That is, suppose that (A1) holds but that (2.2) holds with $\mathbf{x} = X_n$ *instead of* $\mathbf{x} = X_n - G_n$. In this case, there is no self-interaction in the drift term, and the drift is relative to a fixed origin. Lamperti studied examples of such processes (so-called centrally biased random walks) in [21, Section 4] and [23, Section 5]. We see that our recurrence classification for the self-interacting process $X_n - G_n$ in the case $\beta = 1$, Theorem 2.1, gives, surprisingly, essentially the same criteria as Lamperti's [21, Theorem 4.1]. In the case $\beta \in [0,1)$, the difference between the two settings is clearly manifest in the constant in the law of large numbers. The analogue of our Theorem 2.2 in the case of drift relative to the origin is an immediate consequence of Theorem 2.2 of [27] with Theorem 3.2 of [31] (see the discussion in [31, Section 3.2]):

Theorem 2.5. *[27, 31] Suppose that (A1) holds, with the modification that (2.2) holds with $\mathbf{x} = X_n$ instead of $\mathbf{x} = X_n - G_n$. Suppose that $d \in \mathbb{N}$, $\beta \in [0,1)$, and $\rho > 0$. Then there exists a random $\mathbf{u} \in \mathbb{S}_d$ such that, as $n \to \infty$,*

$$n^{-1/(1+\beta)}X_n \xrightarrow{\text{a.s.}} (2+\beta)^{1/(1+\beta)}\ell(\rho,\beta)\mathbf{u}, \text{ and } n^{-1/(1+\beta)}G_n \xrightarrow{\text{a.s.}} (1+\beta)(2+\beta)^{-\beta/(1+\beta)}\ell(\rho,\beta)\mathbf{u}.$$

The method of proof of Theorem 2.2 in the present paper (see Section 6.2) gives an alternative proof of Theorem 2.5, avoiding the rather involved argument for establishing a limiting direction used in [27]. Specifically, in the argument in Section 6.2, we can apply the relevant law of large numbers (Theorem 3.2 of [31]) in place of our Lemma 6.1. Note that under the assumption of bounded increments, the law of large numbers [31, Theorem 3.2] is available, unlike in the generality of Theorem 2.2 from [27]; thus in the more general setting, the proof of [27] is currently the only one that the authors are aware of.

2.3 Examples

To illustrate our assumptions and results, we give three examples of walks satisfying (A1) and (A2). In all of the following examples, the couple (X_n, G_n) is Markov.

Example 1. For $\mathbf{x} \in \mathbb{R}^d$, let $\mathbf{b}_1(\mathbf{x}), \ldots, \mathbf{b}_d(\mathbf{x})$ denote an orthonormal basis for $\mathbb{R}^d$ such that $\mathbf{b}_1(\mathbf{x}) = \hat{\mathbf{x}}$, the unit vector in the direction $\mathbf{x}$; we use the convention $\hat{\mathbf{0}} := \mathbf{e}_1 := (1, 0, \ldots, 0)$. Fix $\varepsilon_0 \in (0, 1/(2d))$, $\rho \in \mathbb{R}$, and $\beta > 0$. Take

$$\mathbb{P}_n[\Delta_n = \mathbf{b}_i(X_n - G_n)] = \mathbb{P}_n[\Delta_n = -\mathbf{b}_i(X_n - G_n)] = \frac{1}{2d}, \quad (i \in \{2, \ldots, d\}),$$

and

$$\mathbb{P}_n[\Delta_n = \mathbf{b}_1(X_n - G_n)] = \begin{cases} \frac{1}{2d} + \frac{\rho}{2}\|X_n - G_n\|^{-\beta} & \text{if } \frac{|\rho|}{2}\|X_n - G_n\|^{-\beta} \leq \frac{1}{2d} - \varepsilon_0 \\ \frac{1}{d} - \varepsilon_0 & \text{if } \frac{\rho}{2}\|X_n - G_n\|^{-\beta} > \frac{1}{2d} - \varepsilon_0 \\ \varepsilon_0 & \text{if } \frac{\rho}{2}\|X_n - G_n\|^{-\beta} < -\frac{1}{2d} + \varepsilon_0 \end{cases};$$

$$\mathbb{P}_n[\Delta_n = -\mathbf{b}_1(X_n - G_n)] = \frac{1}{d} - \mathbb{P}_n[\Delta_n = \mathbf{b}_1(X_n - G_n)].$$

In other words, for all $\|X_n - G_n\|$ sufficiently large,

$$\mathbb{P}_n[\Delta_n = \pm\mathbf{b}_1(X_n - G_n)] = \frac{1}{2d} \pm \frac{\rho}{2}\|X_n - G_n\|^{-\beta}.$$

Then writing $\mathbf{x} = X_n - G_n$, we have for $\mathbf{x} \in \mathbb{R}^d$ with $\|\mathbf{x}\|$ sufficiently large, a.s.,

$$\mathbb{E}_n[\Delta_n] = \rho\|\mathbf{x}\|^{-\beta}\hat{\mathbf{x}}; \quad \mathbb{E}_n[(\Delta_n^{(i)})^2] = \frac{1}{d}\sum_{j=1}^{d}(\mathbf{b}_j \cdot \mathbf{e}_i)^2 = \frac{1}{d}.$$

It is not hard to verify that (A1) and (A2) (with $\sigma^2 = 1/d$) hold in this case. In particular, if $\beta = 1$ Theorem 2.1 says that $X_n - G_n$ is transient if and only if $\rho > (2-d)/(2d)$. See Figure 2 for some simulations of this model.

Example 2. Here is another example satisfying (A1) and (A2), this time with jumps supported on a unit sphere rather than being restricted to a finite set of possibilities. Let $\beta > 0$ and $\rho \in \mathbb{R}$. Given $\mathcal{F}_n$ and $X_n - G_n = \mathbf{x}$, the jump Δ_n is obtained as follows.

(i) Choose $\mathbf{U}_n$ uniformly distributed on the unit sphere $\mathbb{S}_d$.

(ii) Take $\Delta_n = \mathbf{U}_n + \rho\|\mathbf{x}\|^{-\beta}\mathbf{1}_{\{\rho\|\mathbf{x}\|^{-\beta}<1/2\}}\hat{\mathbf{x}}$.

So the jumps of the walk are uniform on a sphere, but the centre of the sphere is (for $\|\mathbf{x}\|$ large enough) shifted slightly in the direction $\pm\hat{\mathbf{x}}$, depending on the sign of ρ. Conditions (A1) and (A2) (again with $\sigma^2 = 1/d$) are readily verified for this example.

Example 3. We sketch one more example with $d \geq 2$, $\beta > 0$ and $\rho > 0$, which is reminiscent of the walk avoiding its convex hull. Take the jump Δ_n uniform on $\mathbb{S}_d$ minus the circular cap of relative surface $\rho\|X_n - G_n\|^{-\beta}$ pointing towards the barycentre, i.e.,

$$\Delta_n \text{ is uniform on } \left\{\mathbf{y} \in \mathbb{S}_d : \mathbf{y} \cdot \hat{\mathbf{x}} > -1 + C(\rho)\|\mathbf{x}\|^{-\beta/(d-1)}\right\},$$

with $\mathbf{x} = X_n - G_n$, where $C(\rho)$ is a constant depending on ρ and d. Here we assume $\|\mathbf{x}\|$ is sufficiently large; if not we can take Δ_n uniform on $\mathbb{S}_d$.

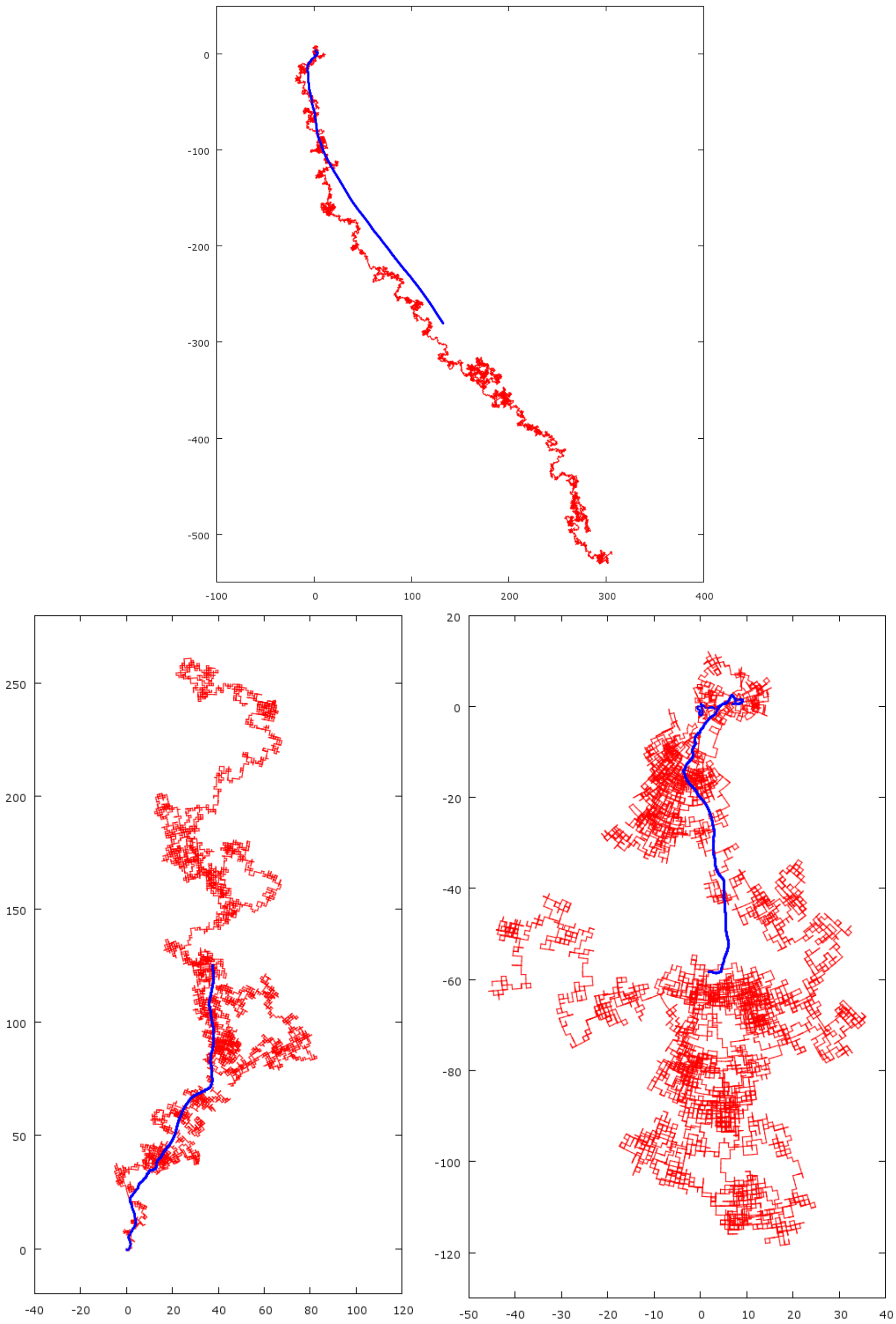

Figure 2: Simulation of 10^4 steps of the random walk and its centre of mass (thick line), as described in Example 1, with $d = 2$, $\rho = 0.1$, $\varepsilon_0 = 0.01$, and different values of $\beta \in (0, 1]$; the three pictures have $\beta = 0.1$ (top), $\beta = 0.5$ (bottom left), and $\beta = 1$ (bottom right). Theorem 2.2 shows that in the two $\beta < 1$ cases, the random walk X_n is transient with a limiting direction. In the $\beta = 1$ case, we know from Theorem 2.1 that $X_n - G_n$ is transient ($\rho_0 = 0$ here), but transience of X_n itself is an open problem. [color online]

2.4 Open problems and paper outline

Our results give a detailed recurrence classification (Theorem 2.1) for the process $X_n - G_n$. Of considerable interest is the asymptotic behaviour of X_n itself, for which we have a complete picture only in the case $\beta \in [0, 1)$, $\rho > 0$ (Theorem 2.2). We conjecture:

- $\|X_n\| \to \infty$ a.s. if and only if $\|X_n - G_n\| \to \infty$ a.s..

Theorems 2.2 and 2.3 verify the 'if' part of the conjecture when (i) $\beta \in [0, 1)$ and $\rho > 0$, and (ii) $d = 1$. Another open problem involves the angular behaviour of our model when $\beta \geq 1$. By analogy with [27] we suspect that there is *no* limiting direction in that case (in contrast to Theorem 2.2).

The remainder of the paper is arranged as follows. In Section 3 we describe in more detail how our model is related to Lamperti's problem (Section 3.1), and to the centre-of-mass of SRW (Section 3.2), and we prove Theorem 1.1. We also outline the motivation of our random walk as a model for a random polymer in solution (Section 3.3). Section 4 is devoted to preliminary computations for the processes X_n, G_n, and (especially) $X_n - G_n$. In Section 5 we take a somewhat more general view, and study the asymptotic properties of one-dimensional, not necessarily Markov, processes satisfying a precise version of (1.4). The recurrence classification is a time-varying, more complicated analogue of Lamperti's results [21, 23], and we use some martingale ideas related to those in [12, 31]. In the case $\beta \in [0, 1)$, $\rho > 0$ we prove a law of large numbers that is a cornerstone of our subsequent analysis for the random walk X_n. This law of large numbers is an analogue of that in [31] for the supercritical Lamperti problem. While the results of [31] supply an upper bound crucial to our approach, the law of large numbers in the present setting requires a new idea, and our key tool here is a stochastic approximation lemma (Lemma 5.1), which may be of independent interest. Section 6 is devoted to the proofs of our main theorems. The basic method is an application of the results of Section 5 to the process $\|X_n - G_n\|$, armed with our computations in Section 4. We carry out this approach to prove Theorems 2.1 and 2.3 in Section 6.1. A crucial ingredient is the proof, in Section 6.2, that $X_n - G_n$ has a limiting direction. This enables us to prove Theorem 2.2. Finally, in Section 6.3, we prove Theorem 2.4, building on some general results from [29].

3 Connections and further motivation

3.1 Lamperti's problem and simple random walk norms

Our problem is related to a time-dependent version of the so-called Lamperti problem. We briefly review the latter here. Let $Z = (Z_n)_{n \in \mathbb{N}}$ be a discrete-time stochastic process adapted to a filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and taking values in an unbounded subset $\mathcal{S}$ of $[0, \infty)$. The set $\mathcal{S}$ may be countable (as in the SRW example which follows in this section) or uncountable (as in the application to stochastic billiards described in [29]).

Lamperti [21–23] investigated the extent to which the asymptotic behaviour of Z is determined by the increment moments $\mathbb{E}_n[(Z_{n+1} - Z_n)^k]$ when viewed as (random) functions of Z_n. Formally, suppose that for some k, $\mathbb{E}_n[(Z_{n+1} - Z_n)^k]$ is well-defined for all n. Then by standard properties of conditional expectations (see e.g. [8, Section 9.1]), there exist a Borel-measurable function $\phi_k(n; \cdot\,)$ and an $\mathcal{F}_n$-measurable random variable $\psi_k(n)$ (orthogonal to Z_n) such that, a.s.,

$$\mathbb{E}_n[(Z_{n+1} - Z_n)^k] = \mathbb{E}[(Z_{n+1} - Z_n)^k \mid Z_n] + \psi_k(n) = \phi_k(n; Z_n) + \psi_k(n).$$

Define

$$\mu_k(n;x) := \phi_k(n;x) + \psi_k(n). \tag{3.1}$$

The $\mu_k(n;x)$ are, in general, $\mathcal{F}_n$-measurable random variables; if Z is a Markov process then $\mu_k(n;x) = \mathbb{E}[(Z_{n+1}-Z_n)^k \mid Z_n = x]$ is a deterministic function of x and n, and if in addition Z is time-homogeneous, $\mu_k(n;x) = \mu_k(x)$ is a function of x only. For many applications, including those described here, Z will not be time-homogeneous or Markovian, but nevertheless the $\mu_k(n;x)$ are well-behaved asymptotically.

In this section, $X = (X_n)_{n\in\mathbb{N}}$ will be the symmetric SRW on $\mathbb{Z}^d$ ($d \in \mathbb{N}$). That is, X has i.i.d. increments $\Delta_n := X_{n+1} - X_n$ such that if $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ is the standard orthonormal basis on $\mathbb{R}^d$, for $i \in \{1, \ldots, d\}$, $\mathbb{P}[\Delta_n = \mathbf{e}_i] = \mathbb{P}[\Delta_n = -\mathbf{e}_i] = (2d)^{-1}$.

Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and consider the $(\mathcal{F}_n)_{n\in\mathbb{N}}$-adapted process $Z = (Z_n)_{n\in\mathbb{N}}$ on $[0,\infty)$ defined by $Z_n = \|X_n\|$. Here Z takes values in the countable set $\mathcal{S} = \{\|\mathbf{x}\| : \mathbf{x} \in \mathbb{Z}^d\}$. Note that Z is not in general a Markov process: when $d = 2$, given one of the two $\mathcal{F}_n$-events $\{X_n = (5,0)\}$ and $\{X_n = (3,4)\}$ we have $Z_n = 5$ in each case but Z_{n+1} has two different distributions; for instance Z_{n+1} can take the value 6 (with probability 1/4) in the first case, but this is impossible in the second case.

We recall some simple facts about $\Delta_n = X_{n+1} - X_n$ in the case of SRW. We have

$$\mathbb{P}_n[\|\Delta_n\| \leq 1] = 1, \text{ a.s.,} \quad \text{and} \quad \mathbb{E}_n[\Delta_n] = \mathbf{0}, \text{ a.s.}. \tag{3.2}$$

Writing $\Delta_n = (\Delta_n^{(1)}, \ldots, \Delta_n^{(d)})$ in Cartesian components, we have that

$$\mathbb{E}_n[\Delta_n^{(i)}\Delta_n^{(j)}] = \frac{1}{d}\mathbf{1}\{i = j\}, \text{ a.s.}. \tag{3.3}$$

Elementary calculations based on Taylor's expansion and (3.2) and (3.3) show that

$$\begin{aligned}\mathbb{E}_n[Z_{n+1} - Z_n] &= \frac{1}{2d}\sum_{i=1}^{d}(\|X_n + \mathbf{e}_i\| + \|X_n - \mathbf{e}_i\| - 2\|X_n\|)\\ &= \frac{1}{2\|X_n\|}\left(1 - \frac{1}{d}\right) + O(\|X_n\|^{-2});\end{aligned}$$

in the above notation, $\mu_1(n;x) = \frac{1}{2x}(1-\frac{1}{d}) + O(x^{-2})$ as $x \to \infty$. As before, this asymptotic expression is the compact notation for

$$\sup_{n\in\mathbb{N}} \operatorname{ess\,sup} \mu_1(n;x) = \frac{1}{2x}\left(1 - \frac{1}{d}\right) + O(x^{-2}),$$

together with the same expression with 'inf' instead of each 'sup'. Similarly

$$\mathbb{E}_n[Z_{n+1}^2 - Z_n^2] = \frac{1}{2d}\sum_{i=1}^{d}\left(\|X_n + \mathbf{e}_i\|^2 + \|X_n - \mathbf{e}_i\|^2 - 2\|X_n\|^2\right) = 1.$$

Then since $(Z_{n+1} - Z_n)^2 = Z_{n+1}^2 - Z_n^2 - 2Z_n(Z_{n+1} - Z_n)$ we obtain

$$\mathbb{E}_n[(Z_{n+1} - Z_n)^2] = \frac{1}{d} + O(\|X_n\|^{-1}).$$

In particular, (1.3) holds (interpreted correctly) with $\beta = 1$ and $\rho' = (1 - (1/d))/2$.

3.2 Centre of mass for simple random walk

We saw in Section 3.1 how a mean drift described loosely by (1.3) arises from the process of norms of symmetric SRW. In this section we describe how a process with mean drift of the form (1.4) arises when considering the distance of a symmetric SRW to its centre of mass. The motion of the centre of mass of a random walk is of interest from a physical point of view, when, for example, the walk represents a growing polymer molecule: see e.g. [1] and [37], especially Chapter 6.

The centre-of-mass process (defined by (1.1)) corresponding to a symmetric SRW on $\mathbb{Z}^d$ was studied by Grill [15], who showed that the process $(G_n)_{n\in\mathbb{N}}$ returns to a fixed ball containing the origin with probability 1 if and only if $d = 1$. In particular the process is transient for $d \geq 2$ and Grill gives a sharp integral test for the rate of escape of the lower envelope. A consequence of his result is the following.

Theorem 3.1. *[15] Let $(X_n)_{n\in\mathbb{N}}$ be symmetric SRW on $\mathbb{Z}^d$ and $(G_n)_{n\in\mathbb{N}}$ the corresponding centre-of-mass process defined by (1.1). Let $d \in \{2, 3, 4, \ldots\}$. Then for any $\varepsilon > 0$,*

$$\|G_n\| \geq (\log n)^{-\frac{1}{d-1}-\varepsilon} n^{1/2}, \text{ a.s.},$$

for all but finitely many $n \in \mathbb{N}$. On the other hand, for infinitely many $n \in \mathbb{N}$,

$$\|G_n\| \leq (\log n)^{-\frac{1}{d-1}} n^{1/2}, \text{ a.s.}.$$

A crude upper bound for $\|G_n\|$, obtained by applying the triangle inequality $\|G_n\| \leq \frac{1}{n}\sum_{i=1}^{n} \|X_i\|$ and the law of the iterated logarithm for symmetric SRW on $\mathbb{Z}^d$ ($d \in \mathbb{N}$) to each $\|X_i\|$ (see e.g. Theorem 19.1 of [35]), is that for any $\varepsilon > 0$, a.s.,

$$\|G_n\| \leq \frac{2}{3d^{1/2}}(1+\varepsilon)(2n \log\log n)^{1/2},$$

for all but finitely many $n \in \mathbb{N}$; it seems likely that this is an overestimate. In $d = 1$, in the analogous continuous setting, a result of Watanabe [38, Corollary 1, p. 237] says that, for B_t standard Brownian motion, for any $\varepsilon > 0$, for all t large enough,

$$\frac{1}{t}\int_0^t B_s \mathrm{d}s \leq 3^{-1/2}(1+\varepsilon)(2t \log\log t)^{1/2}, \text{ a.s.},$$

and this bound is sharp in that the inequality fails infinitely often, a.s., when $\varepsilon = 0$. Standard strong approximation results show that this result can be transferred to $\|G_t\|$ in $d = 1$.

The next result shows how the drift equation (1.4) arises in this context. Lemma 3.1 is a consequence of the more general Lemma 4.2 below.

Lemma 3.1. *Let $d \in \mathbb{N}$. Suppose that $(X_n)_{n\in\mathbb{N}}$ is a symmetric SRW on $\mathbb{Z}^d$, and $(G_n)_{n\in\mathbb{N}}$ is its centre-of-mass process as defined by (1.1). Let $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$ and $Y_n := X_n - G_n$. Then, a.s.,*

$$\mathbb{E}_n[\|Y_{n+1}\| - \|Y_n\|] = \left(1 - \frac{1}{d}\right)\frac{1}{2\|Y_n\|} - \frac{\|Y_n\|}{n+1} + O(\|Y_n\|^{-2});$$

$$\mathbb{E}_n[(\|Y_{n+1}\| - \|Y_n\|)^2] = \frac{1}{d} + O(\|Y_n\| n^{-1}) + O(\|Y_n\|^{-1}).$$

Neglecting higher-order terms, the study of the process $\|X_n - G_n\|$ for SRW leads to the analysis of a process with drift given by (1.4). Lemma 3.1 can be generalized to zero-drift Markov chains $X = (X_n)_{n\in\mathbb{N}}$ satisfying appropriate versions of (3.2) and (3.3). We prove our results on SRW by applying our general results given in Sections 4 and 5.

Proof of Lemma 3.1. This follows from Lemma 4.2 stated and proved in Section 4. Taking expectations in (4.10) and using (3.2) and (3.3) we obtain the first equation in the statement of the lemma, using the fact that $\|Y_n\| = o(n)$ a.s. to simplify the error terms. Similarly, squaring both sides of (4.10) and taking expectations we obtain the second equation in the lemma. $\square$

Proof of Theorem 1.1. Let $Z_n = \|Y_n\| = \|X_n - G_n\|$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then by Lemma 3.1, a.s.,

$$\mathbb{E}_n[Z_{n+1} - Z_n] = \left(1 - \frac{1}{d}\right)\frac{1}{2Z_n} - \frac{Z_n}{n} + O(n^{-2}Z_n) + O(Z_n^{-2});$$
$$\mathbb{E}_n[(Z_{n+1} - Z_n)^2] = \frac{1}{d} + O(Z_n n^{-1}) + O(Z_n^{-1}).$$

Thus (5.4) and (5.5) hold with $\rho' = (d-1)/(2d)$ and $\sigma^2 = 1/d$. It follows from Theorems 5.1 and 5.2 (stated and proved in Section 5) that Z_n is transient if and only if $2\rho' > \sigma^2$, or equivalently $1 - (1/d) > (1/d)$, that is, $d > 2$. $\square$

3.3 The process viewed as a new random polymer model

In this section we briefly summarize motivation of self-interacting random walks arising from polymer physics, and give an interpretation of our model described by (A1) in that context. Much more background is provided by, for instance, [28, Section 2.2], [37, Chapter 7], [17, Chapter 7], and, for the underlying physics, [36]. Recent accounts of some of the relevant probability theory are given in [14, 16].

The sites visited by the walk X_n represent the monomers that make up a long polymer molecule in solution in $\mathbb{R}^d$ (of course, physically $d \in \{2, 3\}$ are most interesting). The line segments between successive sites X_n and X_{n+1} represent the chemical bonds holding the molecule together; in this regard our condition of uniformly bounded increments in (A1) is natural. We assume that the polymer solution is dilute, so that interaction between different polymer molecules can be neglected.

In real polymers, a phase transition is observed between polymers in *poor solvents* (or at low temperature) and *good solvents* (or high temperature) [36, Chapter 7]. In poor solvents, a polymer molecule collapses as the attraction between monomers overcomes the excluded volume effect caused by the fact that no two monomers can occupy the same physical space. In good solvents, a polymer molecule exists in an extended phase where the excluded volume effect dominates.

It is the extended phase that is believed to lie in the same universality class as SAW. Heuristic arguments dating back to P.J. Flory (see e.g. [28, Section 2.2]) suggest that in this phase $\|X_n\|$ should exist on 'macroscopic scale' of order n^ν for an exponent $\nu = \nu(d) \in [1/2, 1]$, with $\nu < 1$ for $d > 1$ and $\nu > 1/2$ for $d \leq 3$. So for $d \in \{2, 3\}$, the polymer is expected to be super-diffusive but sub-ballistic. According to Theorem 2.4(i), our model defined by (A1) has macroscopic scale exponent $\max\{1/2, 1/(1+\beta)\}$ when $\rho > 0$; for $\beta < 1$ this regime therefore corresponds to polymers in the extended phase, where the excluded volume effect, summarized by repulsion from the centre of mass, dominates. For instance, since $\nu(2) = 3/4$, in $d = 2$ the 'physical' choice of our model has $\beta = 1/3$ and $\rho > 0$; it is not clear to what extent that case of our model replicates the behaviour of SAW.

On the other hand, the collapsed phase corresponds to taking $\rho < 0$ in (A1), where the polymer's self-attraction, summarized through its centre of mass, is dominant. See Theorem 2.4(iii) and (iv). Between the poor and good solvent phases, there is a transitional phase at the so-called *θ-point* at which the temperature achieves a specific (critical) value $T = \theta$. Here the excluded volume effect and self-attraction are in balance, and the molecule behaves rather like a simple random walk path. Compare Theorem 2.4(ii).

4 Properties of the self-interacting random walk

Under the assumption (A1), we are going to study the process $X_n - G_n$ and in particular determine whether it is transient or recurrent. It suffices to study $\|X_n - G_n\|$. In this section we analyse the basic properties of the latter process; subsequently we will apply our general results of Section 5 on processes that satisfy, roughly speaking, (1.4).

We introduce some convenient notation that we use throughout. For $n \in \mathbb{N}$ set

$$Y_n := X_n - G_n, \quad \Delta_n := X_{n+1} - X_n.$$

We start with some elementary relations amongst X_n, G_n, and Y_n following from (1.1).

Lemma 4.1. *Suppose that $(X_n)_{n\in\mathbb{N}}$ is a stochastic process on $\mathbb{R}^d$, and $(G_n)_{n\in\mathbb{N}}$ is its centre-of-mass process as defined by (1.1). For $n \in \mathbb{N}$ we have*

$$G_{n+1} = \frac{n}{n+1} G_n + \frac{1}{n+1} X_{n+1}; \text{ and} \tag{4.1}$$

$$Y_{n+1} = \frac{n}{n+1}(Y_n + \Delta_n). \tag{4.2}$$

Moreover $G_1 = X_1$ and for $n \in \{2, 3, \ldots\}$,

$$G_n = X_1 + \sum_{j=2}^{n} \frac{1}{j-1} Y_j. \tag{4.3}$$

Proof. Equation (4.1) is immediate from (1.1). Then from (4.1) we have that for $n \in \mathbb{N}$,

$$Y_{n+1} = X_{n+1} - G_{n+1} = \frac{n}{n+1}\left(X_{n+1} - G_n\right), \tag{4.4}$$

from which (4.2) follows since $X_{n+1} - G_n = Y_n + \Delta_n$. For (4.3), we have from (4.1) again that for $n \in \mathbb{N}$,

$$G_{n+1} - G_n = \frac{1}{n+1}\left(X_{n+1} - G_n\right) = \frac{1}{n} Y_{n+1},$$

where the final equality is obtained from (4.4). Thus for $n \geq 2$,

$$G_n - G_1 = \sum_{j=1}^{n-1}(G_{j+1} - G_j) = \sum_{j=1}^{n-1} \frac{1}{j} Y_{j+1},$$

from which (4.3) follows. $\square$

The main result of this section concerns the increments of the process $\|Y_n\|$ under assumption (A1) and also possibly (A2). Part (i) of Proposition 4.1 gives basic regularity properties, including boundedness of jumps. Part (ii) gives an expression for the mean drift when $\beta \in [0, 1)$. Part (iii) deals with the case $\beta \geq 1$ when (A2) also holds.

Proposition 4.1. *Suppose that (A1) holds.*

(i) There exists $C \in (0, \infty)$ for which, for any $n \in \mathbb{N}$,

$$\mathbb{P}_n[|\|Y_{n+1}\| - \|Y_n\|| > C] = 0, \text{ a.s..} \tag{4.5}$$

In addition

$$\limsup_{n\to\infty} \|Y_n\| = \infty, \text{ a.s..} \tag{4.6}$$

(ii) If $\beta \in [0, 1)$ then, a.s.,

$$\mathbb{E}_n[\|Y_{n+1}\| - \|Y_n\|] = \rho\|Y_n\|^{-\beta} - \frac{\|Y_n\|}{n+1} + O(\|Y_n\|^{-\beta}(\log \|Y_n\|)^{-2}). \tag{4.7}$$

(iii) Suppose also that (A2) holds and $\beta \geq 1$. Then, a.s.,

$$\begin{aligned}\mathbb{E}_n[\|Y_{n+1}\| - \|Y_n\|] &= \left(\rho\mathbf{1}_{\{\beta=1\}} + \frac{1}{2}(d-1)\sigma^2\right)\|Y_n\|^{-1} - \frac{\|Y_n\|}{n+1} \\ &\quad + o(\|Y_n\|^{-1}(\log \|Y_n\|)^{-1}); \qquad (4.8)\\ \mathbb{E}_n[(\|Y_{n+1}\| - \|Y_n\|)^2] &= \sigma^2 + O(n^{-1}\|Y_n\|) + o((\log \|Y_n\|)^{-1}). \qquad (4.9)\end{aligned}$$

We prove Proposition 4.1 via a series of lemmas. The first gives information on the increments of the process given by the distance of a general stochastic process to its centre-of-mass. In particular, it shows that $\|Y_n\|$ inherits boundedness of jumps from X_n, and gives an expression for the increments of $\|Y_n\|$ in terms of Δ_n, the increments of X_n.

Lemma 4.2. *Suppose that $(X_n)_{n\in\mathbb{N}}$ is a stochastic process on $\mathbb{R}^d$, and $(G_n)_{n\in\mathbb{N}}$ is its centre-of-mass process as defined by (1.1). Suppose that $X_1 \in \mathbb{R}^d$ is fixed and that (2.1) holds for some $B \in (0, \infty)$. There exists $C \in (0, \infty)$ for which, for all $n \in \mathbb{N}$, (4.5) holds. Moreover, a.s.,*

$$\|Y_{n+1}\| - \|Y_n\| = \frac{n}{n+1}\left(\frac{Y_n \cdot \Delta_n}{\|Y_n\|} + \frac{\|\Delta_n\|^2}{2\|Y_n\|} - \frac{(Y_n \cdot \Delta_n)^2}{2\|Y_n\|^3}\right) + O(\|Y_n\|^{-2}) - \frac{\|Y_n\|}{n+1}. \tag{4.10}$$

Proof. We work with the process $(\|Y_n\|)_{n\in\mathbb{N}}$. From (2.1) and the triangle inequality, we have the simple bound $\|X_n\| \leq \|X_1\| + B(n-1)$ a.s., for all $n \in \mathbb{N}$. Applying the triangle inequality in (1.1) then yields the equally simple bound

$$\|G_n\| \leq \frac{1}{n}\sum_{i=1}^{n}(\|X_1\| + B(i-1)) \leq \|X_1\| + \frac{Bn}{2}.$$

Combining these two inequalities together with the fact that $\|Y_n\| \leq \|X_n\| + \|G_n\|$, it follows that $\|Y_n\| \leq 2\|X_1\| + (3Bn/2)$ a.s., for all $n \in \mathbb{N}$. Then from the triangle inequality and (4.2) we have that

$$|\|Y_{n+1}\| - \|Y_n\|| \leq \|Y_{n+1} - Y_n\| \leq \frac{1}{n}\|Y_n\| + \|\Delta_n\| \leq \frac{5B}{2} + \frac{2\|X_1\|}{n},$$

a.s., by (2.1), and this lattermost quantity is uniformly bounded. Thus we have (4.5).

For the final statement of the lemma, note that, from (4.2),

$$\|Y_{n+1}\| = \frac{n}{n+1}\left(\|Y_n\|^2 + \|\Delta_n\|^2 + 2Y_n \cdot \Delta_n\right)^{1/2}. \tag{4.11}$$

Now writing $\mathbf{y} = Y_n$ for convenience, we obtain from (4.11) that

$$\|Y_{n+1}\| - \|Y_n\| = \|\mathbf{y}\| \left[\frac{n}{n+1}\left(1 + \frac{\|\Delta_n\|^2 + 2\mathbf{y}\cdot\Delta_n}{\|\mathbf{y}\|^2}\right)^{1/2} - 1\right]. \tag{4.12}$$

Using Taylor's formula for $(1+x)^{1/2}$ with Lagrange remainder in (4.12) implies that

$$\|Y_{n+1}\| - \|Y_n\| = \frac{n\|\mathbf{y}\|}{n+1}\left(\frac{\|\Delta_n\|^2 + 2\mathbf{y}\cdot\Delta_n}{2\|\mathbf{y}\|^2} - \frac{(\|\Delta_n\|^2 + 2\mathbf{y}\cdot\Delta_n)^2}{8\|\mathbf{y}\|^4} + O(\|\mathbf{y}\|^{-3})\right) - \frac{\|\mathbf{y}\|}{n+1},$$

using (2.1) for the error bound. Simplifying and again using (2.1), this becomes

$$\|Y_{n+1}\| - \|Y_n\| = \frac{n\|\mathbf{y}\|}{n+1}\left(\frac{\|\Delta_n\|^2}{2\|\mathbf{y}\|^2} + \frac{\mathbf{y}\cdot\Delta_n}{\|\mathbf{y}\|^2} - \frac{(\mathbf{y}\cdot\Delta_n)^2}{2\|\mathbf{y}\|^4} + O(\|\mathbf{y}\|^{-3})\right) - \frac{\|\mathbf{y}\|}{n+1}.$$

Then equation (4.10) follows. □

Now we turn to the model defined by (A1), starting with the drift of $\|Y_n\|$. For $a, b \in \mathbb{R}$, we use the standard notation $a \wedge b := \min\{a, b\}$.

Lemma 4.3. *Suppose that (A1) holds. Then the drift of $\|Y_n\|$ satisfies, a.s.,*

$$\begin{aligned}\mathbb{E}_n[\|Y_{n+1}\| - \|Y_n\|] = &\frac{n}{n+1}\left(\rho\|Y_n\|^{-\beta} + \Theta_n\|Y_n\|^{-1}\right) - \frac{\|Y_n\|}{n+1} \\ &+ O(\|Y_n\|^{-(1\wedge\beta)}(\log\|Y_n\|)^{-2}),\end{aligned} \tag{4.13}$$

where Θ_n is the $\mathcal{F}_n$-measurable random variable given by

$$\Theta_n = \frac{1}{2}\|Y_n\|^{-2}\mathbb{E}_n[\|Y_n\|^2\|\Delta_n\|^2 - (Y_n\cdot\Delta_n)^2]. \tag{4.14}$$

Moreover, there exists $C < \infty$ such that $\Theta_n \in [0, C]$ a.s., and if $\beta \in [0, 1)$, (4.7) holds.

Proof. Taking expectations in (4.10), using the fact that

$$\|Y_n\|^{-1}\mathbb{E}_n[Y_n\cdot\Delta_n] = \rho\|Y_n\|^{-\beta} + O(\|Y_n\|^{-\beta}(\log\|Y_n\|)^{-2}),$$

by (2.2), we obtain

$$\begin{aligned}\mathbb{E}_n[\|Y_{n+1}\| - \|Y_n\|] = &\frac{n}{n+1}\left(\rho\|Y_n\|^{-\beta} + \frac{1}{2\|Y_n\|^3}\mathbb{E}_n[\|Y_n\|^2\|\Delta_n\|^2 - (Y_n\cdot\Delta_n)^2]\right) \\ &+ O(\|Y_n\|^{-(\beta\wedge 1)}(\log\|Y_n\|)^{-2}) - \frac{\|Y_n\|}{n+1}.\end{aligned}$$

By the fact that $|Y_n\cdot\Delta_n| \le \|Y_n\|\|\Delta_n\|$ and the jumps bound (2.1) we have that

$$0 \le \mathbb{E}_n[\|Y_n\|^2\|\Delta_n\|^2 - (Y_n\cdot\Delta_n)^2] \le C\|Y_n\|^2, \text{ a.s.},$$

for some $C \in (0, \infty)$. Thus defining Θ_n by (4.14) we obtain (4.13) and the fact that $\Theta_n \in [0, C]$ a.s.. Then (4.7) follows when $\beta \in [0, 1)$. □

The next result shows how the ellipticity condition (2.3) leads to (4.6).

Lemma 4.4. *Suppose that (A1) holds. Then* $\limsup_{n\to\infty} \|Y_n\| = \infty$ *a.s..*

Proof. We have from (4.11) that

$$\|Y_{n+1}\|^2 - \|Y_n\|^2 = \left(\frac{n}{n+1}\right)^2 \left(\|\Delta_n\|^2 + 2Y_n \cdot \Delta_n\right) - \frac{2n+1}{(n+1)^2}\|Y_n\|^2. \tag{4.15}$$

Fix $p \in \mathbb{N}$, and define $F_{n,1} := \cap_{i=np}^{np+(p-1)} \{\Delta_i \cdot \hat{Y}_i \geq \varepsilon_0\}$ and $F_{n,2} := \{\|Y_{np}\| \leq \frac{\varepsilon_0 np}{16}\}$. Fix also $n_p \in \mathbb{N}$ with $\varepsilon_0 n_p \geq 16C$, where C is as in (4.5), and consider $n \geq n_p$ only. By (2.3) we have that $\mathbb{P}_n[F_{n,1}] \geq \varepsilon_0^p$ a.s., and hence Lévy's extension of the second Borel–Cantelli lemma (see e.g. [10, Theorem 5.3.2]) implies that $\mathbb{P}[F_{n,1} \text{ i.o.}] = 1$.

Now, observe from (4.5) that $\|Y_{i+1}\| \leq \|Y_i\| + C$, a.s., which implies on $F_{n,2}$ that $\|Y_i\| \leq \frac{1}{8}\varepsilon_0 np$ for all $i \in \{np, \dots np+(p-1)\}$. Then, on $F_{n,1} \cap F_{n,2}$, we obtain from (4.15) that, a.s.,

$$\|Y_{i+1}\|^2 - \|Y_i\|^2 \geq -\frac{2}{i}\|Y_i\|^2 + \frac{1}{4}\left(\varepsilon_0^2 + 2\varepsilon_0\|Y_i\|\right) \geq \frac{1}{4}\varepsilon_0\|Y_i\| + \frac{1}{4}\varepsilon_0^2 \geq \frac{1}{4}\varepsilon_0^2,$$

for any i with $np \leq i \leq np + (p-1)$ and any $n \geq n_p$. Hence on $F_{n,1} \cap F_{n,2}$, a.s.,

$$\|Y_{(n+1)p}\|^2 = \|Y_{np}\|^2 + \sum_{i=np}^{np+(p-1)} \left(\|Y_{i+1}\|^2 - \|Y_i\|^2\right) \geq p\varepsilon_0^2/4.$$

Thus, up to sets of probability zero, $\{(F_{n,1} \cap F_{n,2}) \text{ i.o.}\} \subseteq \{\limsup_{n\to\infty} \|Y_n\| \geq p^{1/2}\varepsilon_0/2\}$. Moreover, by definition of $F_{n,2}$, $\{F_{n,2}^{\mathrm{c}} \text{ i.o.}\} \subseteq \{\limsup_{n\to\infty} \|Y_n\| = \infty\}$. Since $\{F_{n,1} \text{ i.o.}\} \subseteq \{(F_{n,1} \cap F_{n,2}) \text{ i.o.}\} \cup \{F_{n,2}^{\mathrm{c}} \text{ i.o}\}$, it follows that $\{F_{n,1} \text{ i.o.}\} \subseteq \{\limsup_{n\to\infty} \|Y_n\| \geq p^{1/2}\varepsilon_0/2\}$. Since p was arbitrary, the result follows from the fact that $\mathbb{P}[F_{n,1} \text{ i.o.}] = 1$, as shown in the first part of this proof. $\square$

When (A1) holds with $\beta \geq 1$, we need more regularity to obtain a well-behaved version of (4.13). Thus we impose (A2) and use the following result, which in addition gives an expression for the second moment of the increment $\|Y_{n+1}\| - \|Y_n\|$.

Lemma 4.5. *Suppose that (A1) and (A2) hold. Then* Θ_n *as defined by (4.14) satisfies*

$$\Theta_n = \frac{1}{2}(d-1)\sigma^2 + o((\log\|Y_n\|)^{-1}), \ \text{a.s.}. \tag{4.16}$$

Moreover, (4.9) holds.

Proof. First we prove (4.16). We have that

$$\mathbb{E}_n[\|\Delta_n\|^2] = \sum_{i=1}^{d} \mathbb{E}_n[(\Delta_n^{(i)})^2] = d\sigma^2 + o((\log\|Y_n\|)^{-1}),$$

by (2.5). Also if $Y_n = (y_1, \dots, y_d) \in \mathbb{R}^d$, with the convention that an empty sum is 0,

$$\mathbb{E}_n[(Y_n \cdot \Delta_n)^2] = \sum_{i=1}^{d} y_i^2 \mathbb{E}_n[(\Delta_n^{(i)})^2] + 2\sum_{i=2}^{d}\sum_{j=1}^{i-1} y_i y_j \mathbb{E}_n[\Delta_n^{(i)}\Delta_n^{(j)}]$$

$$= \|Y_n\|^2 \left[\sigma^2 + o((\log \|Y_n\|)^{-1})\right], \tag{4.17}$$

by (2.5) and (2.6). Then (4.16) follows from (4.14).

Next we prove (4.9). Squaring both sides of (4.10) and taking expectations we obtain

$$\mathbb{E}_n[(\|Y_{n+1}\| - \|Y_n\|)^2] = \|Y_n\|^{-2}\mathbb{E}_n[(Y_n \cdot \Delta_n)^2] + O(n^{-1}\|Y_n\|) + O(\|Y_n\|^{-1}).$$

Now using (4.17) yields (4.9). □

Proof of Proposition 4.1. We collect results from Lemmas 4.2, 4.3, 4.4, and 4.5. □

5 Recurrence classification for processes satisfying equation (1.4)

5.1 Introduction

In this section we state general results for processes with drift of the form (1.4). We will later apply these results to the process $\|X_n - G_n\|$ satisfying (A1) (and maybe also (A2)), but for this section we work in some generality.

Let $(Z_n)_{n\in\mathbb{N}}$ be a stochastic process taking values in an unbounded subset $\mathcal{S}$ of $[0,\infty)$, adapted to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Recall the definition of $\mu_k(n;x)$ from (3.1), so that $\mathbb{E}_n[(Z_{n+1} - Z_n)^k] = \mu_k(n; Z_n)$ a.s.. As discussed in Section 3.1, the case where $\mu_2(n;x)$ is $O(1)$ and $\mu_1(n;x) \to 0$ as $x \to \infty$ arises often in applications; the case where $\mu_1(n;x) \to 0$ uniformly in n is sometimes known as Lamperti's problem after Lamperti's work [21–23]. Roughly speaking, the Lamperti problem has $\mu_1(n;x) \approx \rho x^{-\beta}$, $\beta > 0$, $\rho \in \mathbb{R}$, ignoring higher-order terms. Results of Lamperti [21,23] imply that the case $\beta = 1$ is critical from the point of view of the recurrence classification. The supercritical case, when $\beta \in [0,1)$, $\rho > 0$, has also been studied (see [31] and references therein).

In this section we study the analogous problem for which $\mu_1(n;x) \approx \rho x^{-\beta} - (x/n)$. In keeping with the applications of the present paper, and to ease technical difficulties, we adopt some stronger regularity assumptions than imposed in [21,23] or [31]. Nevertheless, this version of the problem is more difficult than the classical case (without the extra $-x/n$ term in the drift). Thus although the ideas in this section are related to those in [21,23] and [31], we have to proceed somewhat differently. In particular, to obtain our $\beta < 1$ law of large numbers in this setting (an analogue of [31, Theorem 2.3] for the standard Lamperti case), we use a 'stochastic approximation' result (Lemma 5.1), the proof of which uses ideas somewhat similar to those in [30,31].

We impose some regularity conditions on $(Z_n)_{n\in\mathbb{N}}$. Suppose that there exists $C \in (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P}_n[|Z_{n+1} - Z_n| > C] = 0, \text{ a.s.}. \tag{5.1}$$

We also assume that

$$\limsup_{n\to\infty} Z_n = \infty, \text{ a.s.}, \tag{5.2}$$

without which the question of whether $(Z_n)_{n\in\mathbb{N}}$ is recurrent or transient (in the sense of Definition 2.1) is trivial. Note that (5.2) is implied by a suitable 'irreducibility' assumption, such as, for all $y > 0$, $\inf_{n\in\mathbb{N}} \mathbb{P}_n[Z_m - Z_n > y \text{ for some } m > n] > 0$, a.s.. In our case, as in the standard Lamperti problem, we will see a distinction between the 'critical' case where $\beta = 1$ and the 'supercritical' case where $\beta \in [0,1)$. Thus we deal with these two cases separately in the remainder of this section.

5.2 The critical case: $\beta = 1$

For $x > 0$ and $n \in \mathbb{N}$ define

$$r(n; x) := n^{-1}x^2 + (\log(1 + x))^{-1}. \tag{5.3}$$

For $p > 0$ we write $\log^p x$ for $(\log x)^p$. We impose the further assumptions that there exist $\rho' \in \mathbb{R}$ and $s^2 \in (0, \infty)$ such that

$$\mu_1(n; x) = \rho' x^{-1} - \frac{x}{n} + o(x^{-1}r(n; x)), \tag{5.4}$$

$$\mu_2(n; x) = s^2 + o(r(n; x)). \tag{5.5}$$

Theorem 5.1. *Suppose that the process $(Z_n)_{n\in\mathbb{N}}$ satisfies (5.1), (5.2), (5.4) and (5.5) for some $\rho' \in \mathbb{R}$ and $s^2 \in (0, \infty)$. Then if $2\rho' \le s^2$, Z_n is recurrent.*

Proof. Let $W_n := \log\log Z_n$. Write $D_n := Z_{n+1} - Z_n$. First note that Taylor's formula implies that for x, h with $x \to \infty$ and $h = o(x/\log x)$,

$$\log\log(x + h) = \log\log x + \frac{h}{x \log x} - \frac{(\log x + 1)h^2}{2x^2 \log^2 x} + O(h^3 x^{-3}(\log x)^{-1}).$$

Setting $x = Z_n$ and $h = D_n$ and then taking expectations, we obtain

$$\mathbb{E}_n[W_{n+1} - W_n] = \frac{\mu_1(n; Z_n)}{Z_n \log Z_n} - \frac{(\log Z_n + 1)\mu_2(n; Z_n)}{2Z_n^2 \log^2 Z_n} + O(Z_n^{-3}),$$

using (5.1) for the error term. By (5.4) and (5.5) this last expression is

$$\frac{2\rho' - s^2}{2Z_n^2 \log Z_n} - \frac{s^2}{2Z_n^2 \log^2 Z_n} - \frac{1}{n \log Z_n} + o(Z_n^{-2}(\log Z_n)^{-1}r(n; Z_n)) < 0,$$

for all n and Z_n large enough, provided $2\rho' - s^2 \le 0$, by (5.3). Thus there exist non-random constants $w_0 \in (0, \infty)$ and $n_1 \in \mathbb{N}$ for which, for all $n \ge n_1$, on $\{W_n > w_0\}$,

$$\mathbb{E}_n[W_{n+1} - W_n] < 0, \text{ a.s.}.$$

By Doob's decomposition, we may write $W_n = M_n + A_n$, $n \ge n_1$, where $W_{n_1} = M_{n_1}$, $(M_n)_{n\ge n_1}$ is a martingale, and the previsible sequence $(A_n)_{n\ge n_1}$ is defined by

$$A_n = \sum_{m=n_1}^{n-1} \mathbb{E}_m[W_{m+1} - W_m] \le \sum_{m=n_1}^{n-1} \mathbb{E}_m[W_{m+1} - W_m]\mathbf{1}\{W_m \le w_0\} \le C \sum_{m=n_1}^{n-1} \mathbf{1}\{W_m \le w_0\},$$

since the uniform jumps bound (5.1) for Z_n implies a uniform jumps bound for W_n, $n \ge n_1$. Hence $W_n \to \infty$ implies that $\limsup_{n\to\infty} A_n < \infty$ so $M_n \to \infty$ also. However, $(M_n)_{n\ge n_1}$ is a martingale with uniformly bounded increments (by (5.1)) so $\mathbb{P}[M_n \to \infty] = 0$ (see e.g. [10, Theorem 5.3.1, p. 204]). Hence $\mathbb{P}[\liminf_{n\to\infty} W_n < \infty] = 1$. □

Theorem 5.2. *Suppose that the process $(Z_n)_{n\in\mathbb{N}}$ satisfies (5.1), (5.2), (5.4) and (5.5) for some $\rho' \in \mathbb{R}$ and $s^2 \in (0, \infty)$. Then if $2\rho' > s^2$, Z_n is transient.*

Proof. This time set

$$W_n := \frac{1}{\log Z_n} + \frac{9}{\log n}.$$

Again write $D_n := Z_{n+1} - Z_n$. We want to compute

$$\begin{aligned}\mathbb{E}_n[W_{n+1} - W_n] = \mathbb{E}_n \left[(\log(Z_n + D_n))^{-1} - (\log Z_n)^{-1}\right] \\ + 9\left[(\log(n+1))^{-1} - (\log n)^{-1}\right].\end{aligned} \tag{5.6}$$

Observe that, for the final term on the right-hand side of (5.6),

$$(\log(n+1))^{-1} - (\log n)^{-1} = \frac{\log(1-(n+1)^{-1})}{\log n \log(n+1)} = -\frac{1}{n \log^2 n} + O(n^{-2}). \tag{5.7}$$

Also for the expectation on the right-hand side of (5.6) we have that

$$\begin{aligned}&\mathbb{E}_n \left[(\log(Z_n + D_n))^{-1} - (\log Z_n)^{-1}\right] \\ &= (\log Z_n)^{-1}\mathbb{E}_n \left[\left(1 + \frac{\log(1+(D_n/Z_n))}{\log Z_n}\right)^{-1} - 1\right].\end{aligned}$$

Taylor's formula implies that for $a = O(1)$ and $y = o(1)$,

$$(1 + a\log(1+y))^{-1} = 1 - ay + \frac{2a^2 + a}{2} y^2 + O(y^3).$$

Applying this formula with $a = 1/\log Z_n$ and $y = D_n/Z_n$ we obtain,

$$\begin{aligned}&\mathbb{E}_n \left[(\log(Z_n + D_n))^{-1} - (\log Z_n)^{-1}\right] \\ &= -\frac{\mu_1(n; Z_n)}{Z_n \log^2 Z_n} + \frac{\mu_2(n; Z_n)}{2Z_n^2 \log^2 Z_n} + \frac{\mu_2(n; Z_n)}{Z_n^2 \log^3 Z_n} + O(Z_n^{-3}),\end{aligned}$$

by (5.1). Now using (5.4) and (5.5) we obtain,

$$\begin{aligned}&\mathbb{E}_n \left[(\log(Z_n + D_n))^{-1} - (\log Z_n)^{-1}\right] \\ &= \frac{1}{2Z_n^2 \log^2 Z_n}\left(-(2\rho' - s^2) + o(r(n; Z_n)) + O((\log Z_n)^{-1})\right) + \frac{1}{n \log^2 Z_n}.\end{aligned} \tag{5.8}$$

Suppose that $2\rho' - s^2 \geq 2\varepsilon > 0$. Then by (5.3), (5.6), (5.7) and (5.8) we have that there exist non-random constants $n_0 \in \mathbb{N}$ and $x_0 \in (1, \infty)$ such that for all $n \geq n_0$, on $\{Z_n \geq x_0\}$, a.s.,

$$\mathbb{E}_n[W_{n+1} - W_n] \leq -\frac{\varepsilon}{2Z_n^2 \log^2 Z_n} - \frac{8}{n \log^2 n} + \frac{3}{2n \log^2 Z_n}. \tag{5.9}$$

We have that the right-hand side of (5.9) is bounded above by

$$\frac{1}{\log^2 Z_n}\left(\frac{-\varepsilon}{2Z_n^2} + \frac{3}{2n}\right) \leq \frac{1}{\log^2 Z_n}\left(\frac{-\varepsilon}{2Z_n^2} + \frac{3\varepsilon}{8Z_n^2}\right),$$

provided $n \geq 4Z_n^2\varepsilon^{-1}$, and this last upper bound is negative for $Z_n \geq x_0$. On the other hand, if $n \leq 4Z_n^2\varepsilon^{-1}$ the right-hand side of (5.9) is bounded above by

$$\frac{1}{n}\left(\frac{3/2}{\log^2 Z_n} - \frac{8}{\log^2 n}\right) \leq \frac{1}{n}\left(\frac{7}{\log^2 n} - \frac{8}{\log^2 n}\right) < 0,$$

for $Z_n \geq x_0$ and $n \geq n_0$. Thus in either case we have concluded that for all $n \geq n_0$, on $\{Z_n \geq x_0\}$,

$$\mathbb{E}_n[W_{n+1} - W_n] < 0, \text{ a.s..} \tag{5.10}$$

Now fix $K > 1$ and $x_1 \geq x_0$. Define the stopping times

$$\sigma_K := \min\{n \geq \max\{n_0, x_1^{18K}\} : Z_n \geq x_1^{4K}\}; \quad \tau_K := \min\{n \geq \sigma_K : Z_n \leq x_1\}.$$

By (5.2) we have that $\mathbb{P}[\sigma_K < \infty] = 1$. From (5.10) and the definition of τ_K we have that $(W_{n\wedge\tau_K})_{n\geq\sigma_K}$ is a non-negative supermartingale, and hence it converges almost surely to a $[0,\infty)$-valued random variable $W := W^{(K)}$. In particular, since $\sigma_K < \infty$ a.s., we have $\lim_{n\to\infty} W_{n\wedge\tau_K} = W$, a.s.. Moreover

$$\mathbb{E}[W] \geq \mathbb{E}[W\mathbf{1}_{\{\tau_K<\infty\}}] = \mathbb{E}[W_{\tau_K}\mathbf{1}_{\{\tau_K<\infty\}}] \geq \frac{\mathbb{P}[\tau_K < \infty]}{\log x_1}, \tag{5.11}$$

since $Z_{\tau_K} \leq x_1$. On the other hand, since $(W_{n\wedge\tau_K})_{n\geq\sigma_K}$ is a supermartingale,

$$\mathbb{E}[W] \leq \mathbb{E}[W_{\sigma_K}] \leq \frac{1}{4K\log x_1} + \frac{9}{18K\log x_1} = \frac{3}{4K\log x_1}, \tag{5.12}$$

using the facts that $Z_{\sigma_K} \geq x_1^{4K}$ and $\sigma_K \geq x_1^{18K}$. Combining (5.11) and (5.12) we see that

$$\frac{\mathbb{P}[\tau_K < \infty]}{\log x_1} \leq \frac{3}{4K\log x_1}.$$

On $\{\sigma_K < \infty\} \cap \{\tau_K = \infty\}$, we have that $\liminf_{n\to\infty} Z_n \geq x_1$, so the preceding argument shows that $\mathbb{P}[\liminf_{n\to\infty} Z_n \geq x_1] \geq 1 - \frac{3}{4K}$ for any K and any $x_1 \geq x_0$. It follows that $\mathbb{P}[Z_n \to \infty] = 1$. □

5.3 The supercritical case: $\beta \in [0, 1)$

Once again we will assume that (5.1) and (5.2) hold. We will also assume that there exist $\beta \in [0, 1)$ and $\rho \in \mathbb{R} \setminus \{0\}$ such that

$$\mu_1(n; x) = \rho x^{-\beta} - \frac{x}{n} + o(x^{-\beta}) + o(xn^{-1}). \tag{5.13}$$

Theorem 5.3. *Consider the process $(Z_n)_{n\in\mathbb{N}}$ satisfying (5.1), (5.2), and (5.13), where $\beta \in [0, 1)$. Then Z_n is transient if $\rho > 0$ and recurrent if $\rho < 0$.*

Proof. First suppose that $\rho > 0$. By (5.1) we can choose $\rho' \in (0, \infty)$ so that $2\rho' > C^2 > \mathbb{E}_n[(Z_{n+1} - Z_n)^2]$, a.s., and, by (5.13),

$$\mathbb{E}_n[Z_{n+1} - Z_n] \geq (\rho' + o(1))Z_n^{-1} - \frac{Z_n}{n} + o(Z_n^{-1}r(n; Z_n)), \text{ a.s..}$$

It is this inequality, rather than the equality (5.4), that is needed in the proof of Theorem 5.2. Hence following that proof implies transience. Similarly, if $\rho < 0$ we have, for any $\rho' \in (-\infty, 0)$, a.s.,

$$\mathbb{E}_n[Z_{n+1} - Z_n] \leq (\rho' + o(1))Z_n^{-1} - \frac{Z_n}{n} + o(Z_n^{-1}r(n; Z_n)).$$

Using this inequality in the proof of Theorem 5.1 implies recurrence. □

The rest of this section works towards a proof of the following law of large numbers.

Theorem 5.4. *Consider the process $(Z_n)_{n\in\mathbb{N}}$ satisfying (5.1), (5.2), and (5.13), where $\beta \in [0,1)$ and $\rho > 0$. Then, with $\ell(\rho,\beta)$ as defined at (2.8), as $n \to \infty$,*

$$\frac{Z_n}{n^{1/(1+\beta)}} \xrightarrow{\text{a.s.}} \ell(\rho,\beta). \tag{5.14}$$

The proof uses the following lemma, which is of some independent interest, and falls loosely into a family of "stochastic approximation" results; see e.g. [34, Section 2.4].

Lemma 5.1. *Suppose that $(V_n)_{n\in\mathbb{N}}$ is a non-negative process adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Suppose that there exists $r > 0$ such that the following hold.*

(a) There exists a non-negative sequence $(\gamma_n)_{n\in\mathbb{N}}$ adapted to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}} \gamma_n < \infty$ a.s. such that for any $b > 0$ and all $n \in \mathbb{N}$ we have that, a.s.,

$$\mathbb{E}_n\left[(V_{n+1} - V_n)^2\right] \le C(b)\gamma_n \text{ on } \{V_n \le b\},$$

where $C(b)$ is a constant depending only on b.

(b) There exists $\varepsilon > 0$, and for any $\delta \in (0,r)$ there is a sequence of events $A_n = A_n(\delta)$, $n \in \mathbb{N}$, such that $A_n \in \mathcal{F}_n$, $\mathbb{P}[A_n \text{ i.o.}] = 0$, and a.s. for all $n \in \mathbb{N}$,

$$\mathbb{E}_n[V_{n+1}] \le V_n \text{ on } \{V_n > r+\delta\} \cap A_n^{\mathrm{c}}, \quad \text{and} \quad \mathbb{E}_n[V_{n+1}] \ge V_n \text{ on } \{V_n < r-\delta\} \cap A_n^{\mathrm{c}},$$

and also $A_n^{\mathrm{c}} \subseteq \{V_{n+1} > (1-\varepsilon)V_n\}$.

Then a.s. $\lim_{n\to\infty} V_n = V_\infty$ exists in $[0,\infty)$. If, additionally,

(c) there exists a non-negative sequence $(\tilde{\gamma}_n)_{n\in\mathbb{N}}$ adapted to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}} \tilde{\gamma}_n = +\infty$ a.s. such that for any a, b with $0 < a < b$ and $r \notin [a,b]$, for all n large enough, on $\{V_n \in [a,b]\}$, a.s.,

$$\mathbb{E}_n[V_{n+1} - V_n] \le -\tilde{C}(a,b)\tilde{\gamma}_n \text{ if } r < a,$$
$$\mathbb{E}_n[V_{n+1} - V_n] \ge \tilde{C}(a,b)\tilde{\gamma}_n \text{ if } r > b,$$

where $\tilde{C}(a,b) > 0$ is a constant depending only on a and b,

then $V_\infty \in \{0, r\}$.

Proof. We first show that under conditions (a) and (b) of the lemma, V_n converges a.s. to some finite limit V_∞. We claim that

$$\mathbb{P}\left[\{\liminf_{n\to\infty} V_n \le r\} \cup \{\exists \lim_{n\to\infty} V_n > r\}\right] = 1. \tag{5.15}$$

Indeed, suppose that $\{\liminf_{n\to\infty} V_n \le r\}$ does not hold, so that $\liminf_{n\to\infty} V_n > r + \delta$ for some $\delta > 0$. For $M \in \mathbb{N}$ let

$$\tau^{(M)} := \inf\{n \ge M : \ V_n \le r + \delta \text{ or } A_n(\delta) \text{ occurs}\},$$

and define $V_n^{(M)} := V_{n\wedge\tau^{(M)}}$. Then, for each M, by (b), $(V_n^{(M)})_{n\ge M}$, is a non-negative supermartingale and hence converges a.s.. On the other hand, from (b) and our assumption that $\liminf_{n\to\infty} V_n > r + \delta$ it follows that a.s. $\tau^{(M)} = \infty$ for *some* M; in this case

$V_n^{(M)} \equiv V_n$ for all $n \geq M$ and hence V_n must also converge a.s., and the limit must be greater than r. This establishes (5.15).

By an analogous argument with a bounded submartingale we also establish

$$\mathbb{P}\left[\{\limsup_{n\to\infty} V_n \geq r\} \cup \{\exists \lim_{n\to\infty} V_n < r\}\right] = 1. \tag{5.16}$$

Given (5.15) and (5.16), to show that $\lim_{n\to\infty} V_n$ exists a.s. it suffices to demonstrate this convergence on the set

$$E := \{\limsup_{n\to\infty} V_n \geq r\} \cap \{\liminf_{n\to\infty} V_n \leq r\}.$$

Let us prove that on E in fact $\limsup_{n\to\infty} V_n = r$. For $\delta > 0$ define

$$E_\delta := E \cap \{\limsup V_n > y + \delta\} \text{ where } y = r + 2\delta.$$

We will show that $\mathbb{P}[E_\delta] = 0$ for any $\delta > 0$, which yields the desired conclusion.

Fix some ν_0 such that $V_{\nu_0} > y + \delta$. Iteratively for $i = 0, 1, 2, \dots$ define

$$\begin{aligned}
\tau_i &:= \min\{n > \nu_i : V_n \leq y - \delta\}, \\
\kappa_i &:= \min\{n > \tau_i : V_n > y - \delta\}, \\
\nu_{i+1} &:= \min\{n > \tau_i : V_n \geq y + \delta\}.
\end{aligned}$$

On E we have $V_n \leq r + \delta$ infinitely often, so that $V_n \leq y - \delta$ infinitely often. Thus our definitions imply that τ_i, κ_i, and ν_i are finite for all i on E_δ. Next, setting $B_n := \{V_{n-1} \leq y - \delta,\ V_n > y\}$, we have by Lévy's extension of the second Borel–Cantelli lemma (see e.g. [10, Theorem 5.3.2])

$$\{\{V_{\kappa_i} > y,\ \kappa_i < \infty\} \text{ i.o.}\} \subseteq \{B_n \text{ i.o.}\} = \left\{\sum_{n\in\mathbb{N}} \mathbb{P}_n[B_{n+1}] = \infty\right\},$$

up to events of probability 0. On the other hand,

$$\begin{aligned}
\mathbb{P}_n[B_{n+1}] &= \mathbb{P}_n[V_{n+1} > y]\mathbf{1}\{V_n \leq y - \delta\} \\
&\leq \mathbb{P}_n[|V_{n+1} - V_n| > \delta]\mathbf{1}\{V_n \leq y - \delta\} \\
&\leq \delta^{-2}\mathbb{E}_n[(V_{n+1} - V_n)^2]\mathbf{1}\{V_n \leq y - \delta\},
\end{aligned} \tag{5.17}$$

by Chebyshev's inequality, so that by (5.17) and (a),

$$\sum_{n\in\mathbb{N}} \mathbb{P}_n[B_{n+1}] \leq \sum_{n\in\mathbb{N}} \delta^{-2} C(y - \delta)\gamma_n < \infty, \text{ a.s.} \tag{5.18}$$

Thus on E_δ, by (5.18) and the Borel–Cantelli lemma, $\{V_{\kappa_i} > y\}$ occurs only finitely often a.s., so there is some $N_1 \in \mathbb{N}$ for which $V_{\kappa_i} \in (y - \delta, y]$ for all $i \geq N_1$. Now let

$$\eta_i := \min\{n > \kappa_i : V_n \leq y - \delta \text{ or } V_n \geq y + \delta\} \leq \nu_{i+1}.$$

On E_δ all the η_i are also finite (since the ν_i are finite). For $n \in \mathbb{N}$ define

$$I_n = \begin{cases} 1, & \text{if } \kappa_i \leq n < \eta_i \text{ for some } i \text{ and } A_n^{\mathrm{c}} \text{ occurs;} \\ 0, & \text{otherwise,} \end{cases}$$

$$D_n = \mathbb{E}_n[(V_{n+1} - V_n)I_n] \text{ and } M_n = \sum_{s=\kappa_0}^{n-1}[(V_{s+1} - V_s)I_s - D_s],$$

with an empty sum understood as zero so that $M_n = 0$ for $n \le \kappa_0$. Then $(M_n)_{n\in\mathbb{N}}$ is a zero-mean martingale adapted to $(\mathcal{F}_n)_{n\in\mathbb{N}}$, and it is not hard to see that

$$\mathbb{E}_n[M_{n+1}^2 - M_n^2] = \mathbb{E}_n[(M_{n+1} - M_n)^2] \le \mathbb{E}_n[(V_{n+1} - V_n)^2 I_n].$$

Moreover, since for $\kappa_i \le n < \eta_i$ we have $y - \delta < V_n \le y + \delta$, from (a) it follows that

$$\mathbb{E}_n[M_{n+1}^2 - M_n^2] \le C(y + \delta)\gamma_n.$$

This implies that the increasing process associated with M_n is bounded by a constant times $\sum_{n\in\mathbb{N}} \gamma_n$ and hence is a.s. finite by (a). Consequently, by [10, Theorem 5.4.9] the martingale M_n converges a.s. to some finite limit; in particular, there is some $N_2 \in \mathbb{N}$ for which $\sup_{n,m\ge N_2} |M_n - M_m| < \delta$ a.s.. Then for all $i \ge N_1$ such that $\kappa_i \ge N_2$ we have

$$V_{\eta_i} = V_{\kappa_i} + [M_{\eta_i} - M_{\kappa_i}] + \sum_{s=\kappa_i}^{\eta_i - 1} D_s < y + \delta,$$

since, by (b), $D_n \le 0$ for $n \in [\kappa_i, \eta_i)$, $n \ge N_2$, and $V_{\kappa_i} \le y$ for all $i \ge N_1$. Consequently, the process V_n eventually exits the interval $(y - \delta, y + \delta)$ only on the left (and it cannot jump over it, as we showed above), contradicting E_δ. So $\mathbb{P}[E_\delta] = 0$.

A similar argument shows that on E not only $\limsup_{n\to\infty} V_n = r$ but also $\liminf_{n\to\infty} V_n = r$; we sketch the changes needed to adapt the previous argument to this case. Analogously to E_δ above, we define $E'_\delta := E \cap \{\liminf V_n < y - \delta\}$ where $y = r - 2\delta$ and $\delta \in (0, r/3)$. Also fix some ν'_0 such that $V_{\nu'_0} < y - \delta$, and iteratively set

$$\begin{aligned} \tau'_i &:= \min\{n > \nu'_i : V_n \ge y + \delta\}, \\ \kappa'_i &:= \min\{n > \tau'_i : V_n < y + \delta\}, \\ \nu'_{i+1} &:= \min\{n > \tau'_i : V_n \le y - \delta\}. \end{aligned}$$

This time let $B'_n := \{V_{n-1} \ge y + \delta, V_n < y\}$. Now by definition of A_n^{c}, $\{V_n > (1-\varepsilon)^{-1}r\} \cap A_n^{\mathrm{c}} \subseteq \{V_{n+1} > r\} \subseteq (B'_{n+1})^{\mathrm{c}}$, so that

$$\begin{aligned} \mathbb{P}_n[B'_{n+1}] &\le \mathbb{P}_n[B'_{n+1}]\mathbf{1}(A_n^{\mathrm{c}}) + \mathbf{1}(A_n) \\ &\le \mathbb{P}_n[B'_{n+1}]\mathbf{1}\{V_n < (1-\varepsilon)^{-1}r\} + \mathbf{1}(A_n). \end{aligned}$$

A similar argument as that for (5.17) and (5.18), using Chebyshev's inequality and (a), with $C(y - \delta)$ in (5.18) now being replaced by $C((1 - \varepsilon)^{-1}r)$, shows that, a.s., $\mathbb{P}_n[B'_{n+1}]\mathbf{1}\{V_n < (1 - \varepsilon)^{-1}r\}$ is summable, while $\mathbf{1}(A_n)$ is a.s. summable by assumption in (b). As before, it follows that $\{V_{\kappa'_i} < y\}$ a.s. occurs only finitely often. Then a similar argument to the previous case, with the martingale M_n, shows that $\mathbb{P}[E'_\delta] = 0$ as well.

Consequently, on E, $\lim_{n\to\infty} V_n$ a.s. exists and equals r in this case. Thus the first claim of the lemma follows, and $V_\infty = \lim_{n\to\infty} V_n$ a.s. exists in $[0, \infty)$.

To prove the second claim of the lemma, under the additional condition (c), we show that $\mathbb{P}[V_\infty \in (0, r)\cup(r, \infty)] = 0$. To this end, suppose that $V_n \to y > r$ (the case $y \in (0, r)$ can be handled similarly). Choose a small $\delta > 0$ such that $y - \delta > r$. Then a.s. there exists an N_3 such that $|V_n - y| < \delta$ for all $n \ge N_3$. Now define $D'_n = \mathbb{E}_n[(V_{n+1} - V_n)]$ and the martingale $M'_n = \sum_{s=1}^{n-1}[(V_{s+1} - V_s) - D'_s]$. Then by (c) we have that for all $n \geq N_3$, $D'_n = \mathbb{E}_n[(V_{n+1} - V_n)] \leq -\tilde{C}(y-\delta, y+\delta)\tilde{\gamma}_n$. By a similar argument to that for M_n above, M'_n must converge a.s.. However this leads to a contradiction with the inequality

$$[V_n - V_{N_3}] - [M'_n - M'_{N_3}] = \sum_{s=N_3}^{n-1} D'_s \leq -\tilde{C}(y-\delta, y+\delta) \sum_{s=N_3}^{n-1} \tilde{\gamma}_s,$$

since, a.s., as $n \to \infty$ the right-hand side converges to $-\infty$ while the left-hand side converges to a finite limit. $\square$

Now we can give the proof of Theorem 5.4.

Proof of Theorem 5.4. It suffices to prove that

$$\lim_{n\to\infty} \frac{n}{Z_n^{1+\beta}} = \frac{2+\beta}{\rho(1+\beta)}, \text{ a.s.}. \tag{5.19}$$

Set $V_n = (n-1)/Z_n^{1+\beta}$ and $\tilde{V}_n = n/Z_n^{1+\beta} = \frac{n}{n-1}V_n$. Writing $D_n := Z_{n+1} - Z_n$, we have

$$V_{n+1} - V_n = \tilde{V}_n \left[\left(1 + \frac{D_n}{Z_n}\right)^{-(1+\beta)} - \left(1 - \frac{1}{n}\right) \right] = \tilde{V}_n \left[\frac{1}{n} - \frac{(1+\beta)D_n}{Z_n} + O(Z_n^{-2}) \right],$$

using Taylor's formula and (5.1) for the error term. Hence

$$V_{n+1} - V_n = \frac{\tilde{V}_n}{n} \left[1 - \frac{(1+\beta)nD_n}{Z_n} + O(nZ_n^{-2}) \right]. \tag{5.20}$$

Taking conditional expectations in (5.20) we obtain, on $\{Z_n \to \infty\}$, a.s.,

$$\begin{aligned} \mathbb{E}_n[V_{n+1} - V_n] &= \frac{\tilde{V}_n}{n} \left[1 - \frac{(1+\beta)n\mu_1(n; Z_n)}{Z_n} + O(nZ_n^{-2}) \right] \\ &= \frac{\tilde{V}_n}{n} \left[2 + \beta + o(1) - ((1+\beta)\rho + o(1))\, V_n \right], \end{aligned} \tag{5.21}$$

using (5.13), and then using the fact that $Z_n \to \infty$ to simplify the error terms. Similarly, squaring both sides of (5.20) and taking expectations, on $\{Z_n \to \infty\}$, a.s.,

$$\mathbb{E}_n[(V_{n+1} - V_n)^2] = \frac{\tilde{V}_n^2}{n^2} \left[1 - \frac{2(1+\beta)n\mu_1(n; Z_n)}{Z_n}(1 + o(1)) + \frac{(1+\beta)^2 n^2 \mu_2(n; Z_n)}{Z_n^2}(1 + o(1)) \right],$$

using (5.1) to obtain the error terms. Then from (5.5) and (5.13) we obtain

$$\mathbb{E}_n[(V_{n+1} - V_n)^2] = \frac{\tilde{V}_n^2}{n^2} \left[3 + 2\beta + o(1) - (2\rho(1+\beta) + o(1))\, V_n + \frac{(c + o(1))n^2}{Z_n^2} \right], \tag{5.22}$$

for some $c \in (0, \infty)$ (depending on s^2 and β) as $Z_n \to \infty$ and $n \to \infty$. For a fixed $b > 0$ and $A < \infty$, there exists a (non-random) n_0 for which $\{V_n \leq b\}$ implies that $\{Z_n \geq A\}$ for all $n \geq n_0$. In particular, from (5.22) we have that for some (non-random) $C(b) < \infty$, on $\{V_n \leq b\}$, for any $n \in \mathbb{N}$, a.s.,

$$\mathbb{E}_n[(V_{n+1} - V_n)^2] \leq \frac{\tilde{V}_n^2}{n^2} \left[O(1) + (c + o(1))n^2(\tilde{V}_n/n)^{\frac{2}{1+\beta}} \right] \leq C(b)n^{-\frac{2}{1+\beta}}.$$

Since $\beta < 1$, $\sum_{n\in\mathbb{N}} n^{-2/(1+\beta)} < \infty$ so that the conditions of part (a) of Lemma 5.1 are satisfied with the present choice of V_n and $\gamma_n = n^{-2/(1+\beta)}$. Let $A_n := \{Z_n < A\}$. By Theorem 5.3, $Z_n \to \infty$ a.s., so that A_n occurs only finitely often for any $A \in (0,\infty)$. Taking $r = \frac{2+\beta}{\rho(1+\beta)}$, the conditions on $\mathbb{E}_n[V_{n+1}]$ in part (b) of Lemma 5.1 are shown to hold for any $\delta \in (0,r)$, taking $A = A(\delta)$ sufficiently large, by (5.21). Indeed, from (5.21), on $\{V_n > r+\delta\}$ for some $\delta \in (0,r)$,

$$\mathbb{E}_n[V_{n+1} - V_n] \leq -\delta(1+\beta)\rho(1+o(1))n^{-1}\tilde{V}_n,$$

which is negative on A_n^{c} for our choice of $A = A(\delta)$. A similar argument holds for the other condition on $\mathbb{E}_n[V_{n+1}]$ in Lemma 5.1(b). The final condition in (b), that A_n^{c} implies that $V_{n+1} > (1-\varepsilon)V_n$ for some $\varepsilon \in (0,1)$, follows from (5.20) and the fact that D_n is uniformly bounded (by (5.1)), taking A and n sufficiently large in our choice of A_n.

The conditions in part (c) of Lemma 5.1 follow from (5.21) again, with $\tilde{\gamma}_n = n^{-1}$, noting that the $o(1)$ terms in (5.21) are uniformly small on $\{V_n \leq b\}$ for any $n \geq n_0$ (for some non-random $n_0 \in \mathbb{N}$).

Hence we conclude from Lemma 5.1 that $V_n \to V_\infty$ a.s. where $V_\infty \in \{0, r\}$. To complete the proof of the theorem we must show that $\mathbb{P}[V_n \to 0] = 0$. This, however, follows from the fact that $\limsup_{n\to\infty}(n^{-1/(1+\beta)}Z_n) < \infty$ a.s. due to [31, Theorem 2.3], noting the remark following that theorem. □

6 Proofs of main theorems on self-interacting walks

6.1 Recurrence classification: Proofs of Theorems 2.1 and 2.3

We apply the results of Section 5 to $Z_n = \|Y_n\| = \|X_n - G_n\|$.

Proof of Theorem 2.1. Suppose that (A1) and (A2) hold, and that $\beta \geq 1$. First note that with $Z_n = \|Y_n\|$, (4.5) and (4.6) imply (5.1) and (5.2). Now from (4.8) we obtain, with $r(n;x)$ defined by (5.3),

$$\mathbb{E}_n[Z_{n+1} - Z_n] = \left(\rho\mathbf{1}_{\{\beta=1\}} + \frac{1}{2}(d-1)\sigma^2\right)\frac{1}{Z_n} - \frac{Z_n}{n} + o(Z_n^{-1}r(n;Z_n)), \text{ a.s.}$$

Similarly, we have from (4.9) that

$$\mathbb{E}_n[(Z_{n+1} - Z_n)^2] = \sigma^2 + O(Z_n n^{-1}) + o((\log Z_n)^{-1}).$$

First suppose that $\beta = 1$. Thus (5.4) and (5.5) hold with $\rho' = \rho + (d-1)(\sigma^2/2)$ and $s^2 = \sigma^2$. It follows from Theorems 5.1 and 5.2 that Z_n is transient if and only if $2\rho' > s^2$, or equivalently $2\rho > \sigma^2(2-d)$, i.e., $\rho > \rho_0$. This proves part (i) of the theorem.

Finally suppose that $\beta > 1$. This time (5.4) and (5.5) hold with $\rho' = (d-1)(\sigma^2/2)$ and $s^2 = \sigma^2$. It follows from Theorems 5.1 and 5.2 that Z_n is transient if and only if $2\rho' > s^2$, or equivalently $\sigma^2(2-d) < 0$, i.e., $d > 2$. This proves part (ii). □

Proof of Theorem 2.3. Suppose that $d = 1$. If Y_n is transient, then by (4.5) we have that with probability 1 either: (i) $Y_n \to +\infty$; or (ii) $Y_n \to -\infty$. In case (i), there exists $N \in [2,\infty)$ for which $Y_n \geq 1$ for all $n \geq N$, so (4.3) with (4.5) implies that for $n \geq N$,

$$G_n \geq X_1 - CN + \sum_{j=N}^{n} \frac{1}{j-1} \to \infty, \text{ a.s.},$$

as $n \to \infty$; a similar argument applies in case (ii). Since $X_n = Y_n + G_n$, and Y_n, G_n are transient with the same sign, it follows that X_n is transient too. □

6.2 Limiting directions: Proof of Theorem 2.2

The key first step in the proof of Theorem 2.2 is the following application of the law of large numbers, Theorem 5.4.

Lemma 6.1. *Suppose that (A1) holds with $d \in \mathbb{N}$, $\beta \in [0,1)$, and $\rho > 0$. As $n \to \infty$,*

$$n^{-1/(1+\beta)} \|X_n - G_n\| \xrightarrow{\text{a.s.}} \ell(\rho, \beta).$$

Proof. We take $Z_n = \|Y_n\| = \|X_n - G_n\|$ and apply Theorem 5.4. The conditions of the latter are verified since (4.5), (4.6), and (4.7) imply (5.1), (5.2), and (5.13) respectively. □

The second step in the proof of Theorem 2.2 is to show that the process $Y_n = X_n - G_n$ has a limiting direction. Together with Lemma 6.1 and the simple but useful relation (4.3), we will then be able to deduce the asymptotic behaviour of X_n and G_n.

We use the notation $\hat{Y}_n := Y_n / \|Y_n\|$, with the convention that $\hat{\mathbf{0}} := \mathbf{0}$. Then $(\hat{Y}_n)_{n\in\mathbb{N}}$ is an $(\mathcal{F}_n)$-adapted process, and using the vector-valued version of Doob's decomposition we may write

$$\hat{Y}_n = A_n + M_n, \tag{6.1}$$

where $M_1 = \hat{Y}_1$, $(M_n)_{n\in\mathbb{N}}$ is an $(\mathcal{F}_n)$-adapted d-dimensional martingale and $(A_n)_{n\in\mathbb{N}}$ is the previsible sequence defined by $A_1 = \mathbf{0}$ and $A_n = \sum_{m=1}^{n-1} \mathbb{E}_m[\hat{Y}_{m+1} - \hat{Y}_m]$ for $n \geq 2$.

Lemma 6.2. *Suppose that (A1) holds with $d \in \mathbb{N}$, $\beta \in [0,1)$, and $\rho > 0$. Let the Doob decomposition of $\hat{Y}_n$ be as given at (6.1). There exists a d-dimensional random vector A_∞ such that $A_n \to A_\infty$ a.s., as $n \to \infty$.*

Proof. We have from (4.2) that, with $\Delta_n := X_{n+1} - X_n$ as usual,

$$\begin{aligned} A_{n+1} - A_n &= \mathbb{E}_n \left[\frac{Y_n + \Delta_n}{\|Y_n + \Delta_n\|} - \hat{Y}_n \right] = \mathbb{E}_n \left[\frac{\Delta_n}{\|Y_n + \Delta_n\|} \right] - \hat{Y}_n \mathbb{E}_n \left[\frac{\|Y_n + \Delta_n\| - \|Y_n\|}{\|Y_n + \Delta_n\|} \right] \\ &=: T_1 - \hat{Y}_n T_2. \end{aligned}$$

We deal with the expectations T_1 and T_2 separately. First,

$$T_1 = \|Y_n\|^{-1} \mathbb{E}_n[\Delta_n] - \mathbb{E}_n \left[\frac{(\|Y_n + \Delta_n\| - \|Y_n\|)\Delta_n}{\|Y_n\| \|Y_n + \Delta_n\|} \right].$$

The numerator in the last expectation is bounded in absolute value by $\|\Delta_n\|^2$, by the triangle inequality. Then using the fact that $\|\Delta_n\|$ is uniformly bounded, and that $\|Y_n\| \sim \ell(\rho, \beta) n^{1/(1+\beta)}$ by Lemma 6.1, it follows that

$$T_1 = \|Y_n\|^{-1} \mathbb{E}_n[\Delta_n] + O(n^{-2/(1+\beta)}), \text{ a.s.,}$$

as $n \to \infty$. Similarly, we have that

$$T_2 = \mathbb{E}_n \left[\frac{\|Y_n + \Delta_n\|^2 - \|Y_n\|^2}{\|Y_n + \Delta_n\| (\|Y_n + \Delta_n\| + \|Y_n\|)} \right].$$

Again using the boundedness of $\|\Delta_n\|$ and that $\|Y_n\| \sim \ell(\rho,\beta)n^{1/(1+\beta)}$, we obtain

$$T_2 = \mathbb{E}_n \left[\frac{2\Delta_n \cdot Y_n + \|\Delta_n\|^2}{\|Y_n + \Delta_n\|(\|Y_n + \Delta_n\| + \|Y_n\|)} \right] = \mathbb{E}_n \left[\frac{\hat{Y}_n \cdot \Delta_n}{\|Y_n\|} \right] + O(n^{-2/(1+\beta)}), \text{ a.s..}$$

On applying (2.2) to evaluate the terms $\mathbb{E}_n[\Delta_n]$ and $\mathbb{E}_n[\Delta_n \cdot \hat{Y}_n]$, the leading terms in T_1 and $\hat{Y}_n T_2$ cancel to give

$$A_{n+1} - A_n = O(\|Y_n\|^{-\beta-1}(\log \|Y_n\|)^{-2}) + O(n^{-2/(1+\beta)}).$$

Since $\|Y_n\| \sim \ell(\rho,\beta)n^{1/(1+\beta)}$, and $\beta < 1$, these two $O(\,\cdot\,)$ terms are summable, so that $\sum_{n=1}^{\infty} \|A_{n+1} - A_n\| < \infty$, a.s., implying that A_n converges a.s.. □

Lemma 6.3. *Suppose that (A1) holds with $d \in \mathbb{N}$, $\beta \in [0,1)$, and $\rho > 0$. Let the Doob decomposition of $\hat{Y}_n$ be as given at (6.1). There exists a d-dimensional random vector M_∞ such that $M_n \to M_\infty$ a.s., as $n \to \infty$.*

Proof. Taking expectations in the vector identity $\|M_{n+1} - M_n\|^2 = \|M_{n+1}\|^2 - \|M_n\|^2 - 2M_n \cdot (M_{n+1} - M_n)$ and using the martingale property, we have

$$\mathbb{E}_n[\|M_{n+1}\|^2 - \|M_n\|^2] = \mathbb{E}_n[\|M_{n+1} - M_n\|^2] = \mathbb{E}_n[\|\hat{Y}_{n+1} - \hat{Y}_n - \mathbb{E}_n[\hat{Y}_{n+1} - \hat{Y}_n]\|^2].$$

Expanding out the expression in the latter expectation, it follows that

$$\mathbb{E}_n[\|M_{n+1}\|^2 - \|M_n\|^2] = \mathbb{E}_n[\|\hat{Y}_{n+1} - \hat{Y}_n\|^2] - (\mathbb{E}_n[\hat{Y}_{n+1} - \hat{Y}_n])^2 \le \mathbb{E}_n[\|\hat{Y}_{n+1} - \hat{Y}_n\|^2].$$

Here we have from (4.2) that

$$\|\hat{Y}_{n+1} - \hat{Y}_n\| = \left\| \frac{Y_n(\|Y_n\| - \|Y_n + \Delta_n\|) + \Delta_n\|Y_n\|}{\|Y_n\|\|Y_n + \Delta_n\|} \right\| \le \frac{2\|\Delta_n\|}{\|Y_n + \Delta_n\|},$$

by the triangle inequality. Since $\|\Delta_n\|$ is uniformly bounded, and $\|Y_n\| \sim \ell(\rho,\beta)n^{1/(1+\beta)}$ by Lemma 6.1, it follows that $\|\hat{Y}_{n+1} - \hat{Y}_n\| = O(n^{-1/(1+\beta)})$, so that

$$\mathbb{E}_n[\|M_{n+1}\|^2 - \|M_n\|^2] = O(n^{-2/(1+\beta)}), \text{ a.s..}$$

Hence $\sum_{n=1}^{\infty} \mathbb{E}_n[\|M_{n+1}\|^2 - \|M_n\|^2] < \infty$, a.s., which implies that M_n has an almost-sure limit, by e.g. the d-dimensional version of [10, Theorem 5.4.9, p. 217]. □

Proof of Theorem 2.2. Combining Lemmas 6.2 and 6.3 with the decomposition (6.1), we conclude that $\hat{Y}_n \to A_\infty + M_\infty =: \mathbf{u}$, for some random unit vector $\mathbf{u}$, a.s., as $n \to \infty$. In other words, the process Y_n has a limiting direction. It follows from the representation (4.3) that the processes G_n and X_n have the same limiting direction. Specifically,

$$G_n = X_1 + \sum_{j=2}^{n} \frac{1}{j-1}\|Y_j\|\hat{Y}_j = X_1 + \sum_{j=2}^{n} \frac{1}{j-1}[\ell(\rho,\beta) + o(1)]j^{1/(1+\beta)}[\mathbf{u} + o(1)], \text{ a.s.,}$$

by Lemma 6.1. Hence

$$G_n = [(1+\beta)\ell(\rho,\beta)\mathbf{u} + o(1)]n^{1/(1+\beta)}, \text{ a.s.,}$$

and the result for X_n follows since $X_n = G_n + Y_n$. □

6.3 Upper bounds: Proof of Theorem 2.4

Theorem 2.4 will follow from the next result, which gives bounds for $\|Y_n\|$.

Proposition 6.1. *Suppose that (A1) holds with $d \in \mathbb{N}$, $\beta \geq 0$, and $\rho \in \mathbb{R}$. Then the following bounds apply.*

(i) If $\beta \geq 1$, then for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{N}$, $\|Y_n\| \leq n^{1/2}(\log n)^{(1/2)+\varepsilon}$.

(ii) If (A2) holds, $\beta = 1$, and $\rho < -(d\sigma^2/2)$, then for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{N}$, $\|Y_n\| \leq n^{\gamma(d,\sigma^2,\rho)+\varepsilon}$ where $\gamma(d, \sigma^2, \rho)$ is given by (2.9).

(iii) If $\beta \in [0, 1)$ and $\rho < 0$, then for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{N}$, $\|Y_n\| \leq (\log n)^{\frac{1}{1-\beta}+\varepsilon}$.

To prove this result we apply some general results from [29]. Section 4 of [29] dealt with stochastic processes that were time-homogeneous, but that condition was not used in the proofs of the results that we apply here, which relied on the very general results of Section 3 of [29]: the basic tool is Theorem 3.2 of [29].

It is most convenient to again work in some generality. Again let $(Z_n)_{n\in\mathbb{N}}$ denote a stochastic process on $[0, \infty)$. Recall the definition of $\mu_k(n; x)$ from (3.1). The next result gives the upper bounds that we need. Part (i) is contained in [29, Theorem 4.1(i)]. Part (ii) is a variation on [29, Theorem 4.3(i)] that is more suited to the present application. Part (iii) is also based on [29] but does not seem to have appeared before.

Lemma 6.4. *Suppose that $(Z_n)_{n\in\mathbb{N}}$ is such that (5.1) holds.*

(i) Suppose that for some $A < \infty$, $x\mu_1(n; x) \leq A$ for all n and x sufficiently large. Then for any $\varepsilon > 0$, a.s., $Z_n \leq n^{1/2}(\log n)^{(1/2)+\varepsilon}$ for all but finitely many $n \in \mathbb{N}$.

(ii) Suppose that for some $v > 0$ and $\kappa > 1$, $2x\mu_1(n; x) \leq -\kappa\mu_2(n; x) + o(1)$ and $\mu_2(n; x) \geq v$ for all n and x sufficiently large. Then, for any $\varepsilon > 0$, a.s. $Z_n \leq n^{\frac{1}{1+\kappa}+\varepsilon}$ for all but finitely many $n \in \mathbb{N}$.

(iii) Suppose that for some $\beta \in [0, 1)$ and $A > 0$, $x^\beta\mu_1(x; n) \leq -A$ for all n and x large enough. Then for any $\varepsilon > 0$, a.s., for all but finitely many $n \in \mathbb{N}$, $Z_n \leq (\log n)^{\frac{1}{1-\beta}+\varepsilon}$.

Proof. First we prove part (ii). Let $\kappa' = \kappa - \varepsilon$ for $\varepsilon \in (0, \kappa)$. Writing $D_n = Z_{n+1} - Z_n$,

$$\begin{aligned}
\mathbb{E}_n[Z_{n+1}^{1+\kappa'} - Z_n^{1+\kappa'}] &= Z_n^{1+\kappa'}\mathbb{E}_n[(1 + (D_n/Z_n))^{1+\kappa'} - 1] \\
&= (1 + \kappa')Z_n^{\kappa'}\left(\mu_1(n; Z_n) + \frac{\kappa'}{2Z_n}\mu_2(n; Z_n) + O(Z_n^{-2})\right),
\end{aligned}$$

using Taylor's formula and (5.1). Under the conditions of part (ii), we have

$$\mu_1(n; Z_n) + \frac{\kappa'}{2Z_n}\mu_2(n; Z_n) + O(Z_n^{-2}) \leq -\frac{\varepsilon}{2Z_n}\mu_2(n; Z_n) + o(Z_n^{-1}) < 0,$$

for all n and Z_n large enough. Hence $\mathbb{E}_n[Z_{n+1}^{1+\kappa'} - Z_n^{1+\kappa'}]$ is uniformly bounded above and the result follows from Theorem 3.2 of [29].

It remains to prove part (iii). For $\alpha > 0$, define $f_\alpha(x) := \exp\{x^\alpha\}$. First we show that, under the conditions of the lemma, for any $\alpha \in (0, 1-\beta)$, for some $C < \infty$,

$$\mathbb{E}_n[f_\alpha(Z_{n+1}) - f_\alpha(Z_n)] \leq C, \text{ a.s..} \tag{6.2}$$

Writing $D_n = Z_{n+1} - Z_n$, we have that

$$\mathbb{E}_n[f_\alpha(Z_{n+1}) - f_\alpha(Z_n)] = f_\alpha(Z_n)\mathbb{E}_n\left[\exp\{(Z_n + D_n)^\alpha - Z_n^\alpha\} - 1\right].$$

Since $D_n = O(1)$ a.s., by (5.1), Taylor's formula applied to the last expression yields

$$\mathbb{E}_n[f_\alpha(Z_{n+1}) - f_\alpha(Z_n)] = f_\alpha(Z_n)\mathbb{E}_n\left[\alpha D_n Z_n^{\alpha-1} + O(Z_n^{2\alpha-2})\right].$$

Here we have that $\mathbb{E}_n[D_n] \leq -AZ_n^{-\beta}$ for all Z_n, n large enough. Since $\alpha < 1-\beta$ we obtain (6.2). Now we can apply Theorem 3.2 of [29] to complete the proof. □

Finally we complete the proofs of Proposition 6.1 and Theorem 2.4.

Proof of Proposition 6.1. Under the conditions of part (i) of Proposition 6.1, we have from (4.5) and Lemma 4.3 that the conditions of Lemma 6.4(i) hold for $Z_n = \|Y_n\|$. Thus we obtain part (i) of the proposition. Similarly, under the conditions of part (ii), we have from (4.5), (4.8) and (4.9) that Lemma 6.4(ii) holds for $Z_n = \|Y_n\|$ and $\kappa = -\frac{2\rho}{\sigma^2} - (d-1)$, which is greater than 1 for $\rho < -d\sigma^2/2$. Finally, under the conditions of part (iii), we have from (4.5) and (4.7) that Lemma 6.4(iii) holds for $Z_n = \|Y_n\|$. □

Proof of Theorem 2.4. Part (i) of the theorem follows from Theorem 2.2. Parts (ii), (iii), and (iv) follow from Proposition 6.1 with (4.3) and the triangle inequality; note this introduces an extra logarithmic factor in the case of part (iv) of the theorem. □

Acknowledgements

FC was partially supported by CNRS UMR 7599. MM thanks the Fondation Sciences Mathématiques de Paris for support during part of this work. Some of this work was done while AW was at the University of Bristol, supported by the Heilbronn Institute for Mathematical Research.

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