Ergod. Th. & Dynam. Sys. (1988), **8**, 411–419
Printed in Great Britain

Automorphisms of solenoids and p-adic entropy*

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(*Received* 4 *June* 1987; *revised* 18 *August* 1987)

Abstract. We show that a full solenoid is locally the product of a euclidean component and p-adic components for each rational prime p. An automorphism of a solenoid preserves these components, and its topological entropy is shown to be the sum of the euclidean and p-adic contributions. The p-adic entropy of the corresponding rational matrix is computed using its p-adic eigenvalues, and this is used to recover Yuzvinskii's calculation of entropy for solenoidal automorphisms. The proofs apply Bowen's investigation of entropy for uniformly continuous transformations to linear maps over the adele ring of the rationals.

1. *Background and results*

A *solenoid* is a finite-dimensional, connected, compact abelian group. Equivalently, its dual group is a finite rank, torsion-free, discrete abelian group, i.e. a subgroup of $\mathbb{Q}^d$ for some $d \geq 1$. Solenoids generalize the familiar torus groups. Halmos [**H**] first observed that (continuous) automorphisms of compact groups must preserve Haar measure, providing an interesting class of examples for ergodic theory. Furthermore, Berg [**Be**] has shown that the entropy of such an automorphism with respect to Haar measure coincides with its topological entropy.

We are concerned here with the computation of the topological entropy of an automorphism of a solenoid. If A is such an automorphism, its dual automorphism extends to an element of $GL(d, \mathbb{Q})$, which we also call A (see § 3). When A is a toral automorphism, so $A \in GL(d, \mathbb{Z})$, then the topological entropy of A is given by the familiar formula

$$h(A) = \sum_{|\lambda_i|>1} \log |\lambda_i|, \tag{1}$$

where A has complex eigenvalues $\lambda_1, \ldots, \lambda_d$ counted with multiplicity. To state the generalization to solenoids, let $\chi_A(t)$ be the characteristic polynomial of $A \in GL(d, \mathbb{Q})$, and s denote the least common multiple of the denominators of the coefficients of $\chi_A(t)$. Yuzvinskii [**Y**] proved that

$$h(A) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|. \tag{2}$$

* The authors gratefully acknowledge support, respectively, by NSF Grant DMS-8320356 and SERC Award B85318868.

Our purpose here is to explain Yuzvinskii's calculation in terms of a combination of geometric and arithmetic hyperbolicity. We begin in § 3 by lifting A to an automorphism of the full solenoid $\Sigma^d \cong \hat{\mathbb{Q}}^d$ with the same entropy. In Lemma 4.1 we show that the full solenoid is locally a product of a euclidean component and p-adic components for each rational prime p. The entropy of A is computed in Theorem 1 to be the sum of a contribution from the euclidean component, generated by geometric expansion, and contributions from each of the p-adic components, generated by arithmetic expansions. If $\mathbb{Q}_p$ denotes the p-adic completion of the rationals, than a p-adic component contributes the Bowen entropy $h(A; \mathbb{Q}_p^d)$ of the uniformly continuous linear map A on the non-compact metric space $\mathbb{Q}_p^d$. Since the infinite place ∞ on $\mathbb{Q}$ gives the completion $\mathbb{Q}_\infty = \mathbb{R}$, and the Bowen entropy of a linear map on $\mathbb{R}^d$ is given by (1), we can summarize Theorem 1 by

$$h(A) = \sum_{p \le \infty} h(A; \mathbb{Q}_p^d), \tag{3}$$

i.e. the entropy of the solenoidal automorphism is the sum, over all inequivalent completions of $\mathbb{Q}$, of the entropies of the corresponding linear maps.

In Theorem 2 the p-adic entropy of A is explicitly computed as

$$h(A; \mathbb{Q}_p^d) = \sum_{|\lambda_j^{(p)}| > 1} \log |\lambda_j^{(p)}|_p, \tag{4}$$

where the $\lambda_j^{(p)}$ are the p-adic eigenvalues of A lying in a finite extension of $\mathbb{Q}_p$, and $|\cdot|_p$ is normalized so $|p|_p = p^{-1}$. As pointed out to us by the referee, this shows that $h(A; \Sigma^d)$ is the sum of the logarithmic heights of the eigenvalues of A in the sense of algebraic geometry (see [**Lan**], p. 52). Using (4), we show in Theorem 3 that

$$\sum_{p < \infty} h(A; \mathbb{Q}_p^d) = \log s,$$

so that the mysterious initial term in Yuzvinskii's formula (2) is just the sum of the p-adic entropies of A over $p < \infty$, while the second term is the euclidean term corresponding to $p = \infty$.

The calculation of entropy for group automorphisms has a history going back to the original papers defining entropy. Sinai [**S**, 1959] showed (1) for 2-dimensional toral automorphisms, and claimed the higher dimensional formula holds. The 2-dimensional case was reproved by Rohlin [**R**, 1961] as an application of his measurable partition machinery. Abromov [**Ab**, 1959] computed entropy for automorphisms of 1-dimensional solenoids. Here the map is specified by a rational number m/n in lowest terms and then

$$h([m/n]) = \max \{\log |m|, \log |n|\}. \tag{5}$$

In [**G**, 1961], Genis claimed without proof the formula (1) for general toral automorphisms. Next Arov [**A**, 1964] published a proof of (1), and generalized to solenoidal automorphisms whose characteristic polynomial has coefficients whose denominators are all powers of a fixed integer. Finally, Yuzvinskii [**Y**, 1967] obtained (2) in full generality, using rather complicated linear algebra.

In § 2 we give some examples to illustrate the interplay between geometric and arithmetic components, and describe a combinatorial formulation of entropy due to Peters. Reduction to full solenoids is carried out in § 3. Our approach in § 4 to the proof of the main formula (3) is to realize the full d-dimensional solenoid as the quotient of the adele ring $\mathbb{Q}_{\mathbb{A}}^d$ by the embedded lattice $\mathbb{Q}^d$. Then A lifts to a linear map of $\mathbb{Q}_{\mathbb{A}}^d$ preserving entropy, and a series of reductions shows $h(A; \mathbb{Q}_{\mathbb{A}}^d)$ equals the right side of (3). A technical problem is that Bowen's definition of entropy is in general not additive over products, so some care is needed. In § 5 we deduce the eigenvalue expression (4) for p-adic entropy, and this is used in § 6 to recover Yuzvinskii's formula (2). We remark that analogous results hold if $\mathbb{Q}$ is replaced by a finite algebraic extension k of $\mathbb{Q}$, and $A \in GL(d, k)$. The details of this generalization are contained in [W, § 4].

The possibility of computing entropy for solenoidal automorphisms using p-adic eigenvalues was described without proof in a lecture by the first author [L1], and ultimately goes back to a suggestion of H. Furstenberg. The authors discussed this topic during the 1986 Warwick Symposium, leading the second author to discover the appropriate framework and proofs [W]. The second author expresses his thanks to Klaus Schmidt for his help and advice.

2. *Examples and a combinatorial interpretation*

Before beginning the proof, we give some examples of (3) to illustrate the interplay between the geometric and arithmetic contributions. We also give an algebraic or combinatorial interpretation of entropy due to Peters.

First consider $A = [3/2]$ acting on the 1-dimensional solenoid $\Sigma = \hat{\mathbb{Q}}$. Abromov's result (5) shows $h(A; \Sigma) = \log 3$. Now (3) shows

$$h(A; \Sigma) = h(A; \mathbb{Q}_2) + h(A; \mathbb{Q}_3) + h(A; \mathbb{R}) \tag{6}$$

since by (4) the other components vanish. If $\log^+ x$ denotes $\max\{\log x, 0\}$, then by (4) and (1) we see $h(A; \mathbb{Q}_2) = \log^+ |3/2|_2 = \log 2$, $h(A; \mathbb{Q}_3) = \log^+ |3/2|_3 = 0$, and $h(A; \mathbb{R}) = \log 3/2$, combining to give $h(A; \Sigma) = \log 3$. Here there are positive contributions from the euclidean and 2-adic components. Next consider $A^{-1} = [2/3]$. Then (6) holds with A replaced by A^{-1}. Now (4) shows

$$h(A^{-1}; \mathbb{Q}_2) = \log^+ |2/3|_2 = 0, \qquad h(A^{-1}; \mathbb{Q}_3) = \log^+ |2/3|_3 = \log 3,$$

and $h(A^{-1}; \mathbb{R}) = 0$, again combining to give $h(A^{-1}, \Sigma) = \log 3$. Note, however, that here the euclidean and 2-adic components contribute nothing, and that all entropy comes from the 3-adic direction.

Next consider

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 6/5 \end{bmatrix}$$

acting on the 2-dimensional solenoid Σ^2. Here $\chi_A(t) = t^2 - \frac{6}{5}t + 1$. The complex eigenvalues of A have modulus 1, so $h(A; \mathbb{R}^2) = 0$. Since $p = 5$ is the only arithmetic component with a positive contribution, we see by (3) that $h(A; \Sigma^2) = h(A; \mathbb{Q}_5^2)$. Over $\mathbb{Q}_5$ we have $\chi_A(t) = (t - \lambda_1)(t - \lambda_2)$, where $|\lambda_1|_5 = 5$ and $|\lambda_2|_5 = 5^{-1}$. Thus by (4) we have $h(A; \Sigma^2) = \log 5$, and of course $h(A^{-1}; \Sigma^2) = \log 5$ by the same calculation. Here A is an isometry on the geometric component, while all hyperbolic behavior is concentrated on the 5-adic component. This example was given in [L2] to show that exponential recurrence for solenoidal automorphisms can be entirely due to arithmetic hyperbolicity.

There is an algebraic way to compute entropy using the growth of sums of images of a finite set in the dual group, due to Peters [P]. Let Γ be a discrete abelian group, and A be an automorphism of Γ. For a finite set $E \subset \Gamma$, let $|E|$ denote its cardinality. Put

$$h_{\mathrm{alg}}(A; E) = \limsup_{n\to\infty} \frac{1}{n} \log |E + A^{-1}E + \cdots + A^{-(n-1)}E|,$$

and

$$h_{\mathrm{alg}}(A; \Gamma) = \sup \{h(A; E) : E \subset \Gamma,\ E \text{ finite}\}. \tag{7}$$

Peters showed that $h_{\mathrm{alg}}(A; \Gamma)$ coincides with the topological entropy $h(A, \hat{\Gamma})$ of the dual automorphism.

In our case, $\Gamma = \mathbb{Q}^d$ and $A \in GL(d, \mathbb{Q})$. Then (3) and (4) can be used to compute $h_{\mathrm{alg}}(A; \mathbb{Q}^d)$. The reader may find it instructive to prove directly that $h_{\mathrm{alg}}([3/2]; \mathbb{Q}) = \log 3$.

3. *Full solenoids*

In this section we lift an automorphism of a solenoid to one of a full solenoid while preserving entropy. This allows us to assume from now on that G is a full d-dimensional solenoid Σ^d, i.e. the dual of G is $\mathbb{Q}^d$.

Let G be a solenoid, and Γ its dual group. Since Γ has finite rank, say d, and is torsion-free, it embeds in $\Gamma_{\mathbb{Q}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$. Hence an automorphism A of Γ extends to an automorphism $A_{\mathbb{Q}}$ of $\Gamma_{\mathbb{Q}}$. Let $G_{\mathbb{Q}} = \hat{\Gamma}_{\mathbb{Q}}$, and $A_{\mathbb{Q}}$ also denote the dual automorphism. By duality, A is a quotient of $A_{\mathbb{Q}}$, and we claim entropy is preserved. Note that $G_{\mathbb{Q}} \cong \Sigma^d$ is a full solenoid, and that $A_{\mathbb{Q}}$ can be considered a rational matrix in $GL(d, \mathbb{Q})$.

PROPOSITION 3.1. *With the above notations,* $h(A; G) = h(A_{\mathbb{Q}}; G_{\mathbb{Q}})$.

Proof. For $n \ge 1$ the subgroups $\Gamma_n = (n!)^{-1}\Gamma$ of $\Gamma_{\mathbb{Q}}$ are A-invariant, and increase to $\Gamma_{\mathbb{Q}}$. Hence $G_{\mathbb{Q}}$ is the inverse limit of the A-invariant $\hat{\Gamma}_n$, and

$$h(A_{\mathbb{Q}}; G_{\mathbb{Q}}) = \lim_{n\to\infty} h(A_{\mathbb{Q}}; \hat{\Gamma}_n).$$

Since Γ is torsion-free, the action of $A_{\mathbb{Q}}$ on Γ_n is isomorphic to its action on Γ, so $h(A_{\mathbb{Q}}; \hat{\Gamma}_n) = h(A; G)$ for $n \ge 1$. This proves the result. □

Although entropy is preserved when lifting A to $A_{\mathbb{Q}}$, other dynamical properties may be lost. For example, consider $\Gamma = \mathbb{Z}[1/6]$ and $A = [3/2]$, and put $G = \hat{\Gamma}$. The closure of the subgroup of A-periodic points in G has annihilator

$$\bigcap_{n=1}^{\infty} \left[\left(\frac{3}{2}\right)^n - 1\right]\Gamma = \{0\},$$

i.e. the periodic points of A in G are dense. However, passing to $\Gamma_{\mathbb{Q}} \cong \mathbb{Q}$, and noting that

$$\bigcap_{n=1}^{\infty} \left[\left(\frac{3}{2}\right)^n - 1\right]\Gamma_{\mathbb{Q}} = \Gamma_{\mathbb{Q}},$$

we see that $A_{\mathbb{Q}}$ has only 0 as a periodic point. Here A may be thought of as hyperbolic with eigendirections being the p-adic components with $p=2, 3$, and ∞. Passing to $A_{\mathbb{Q}}$ introduces isometric directions for the other primes that destroy periodic behavior.

For a general solenoidal automorphism A it is not difficult to show that its periodic points are dense precisely when it is expansive, and this occurs if and only if the dual group is finitely generated as a $\mathbb{Z}[A, A^{-1}]$-module [**La**].

4. *Proof of the main formula*

In this section we prove the main formula (3). Let $A \in GL(d, \mathbb{Q})$. By duality, A acts as an automorphism of the d-dimensional solenoid Σ^d, and also as a uniformly continuous linear map on $\mathbb{Q}_p^d$ for $p \leq \infty$. The relation between the entropies of these actions is as follows.

THEOREM 1. $h(A; \Sigma^d) = \sum_{p \leq \infty} h(A; \mathbb{Q}_p^d)$.

A convenient approach is the use of the adele ring $\mathbb{Q}_{\mathbb{A}}$ of $\mathbb{Q}$. We will use notations and results from Weil's elegant book [**We**].

For each rational prime p, let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to the p-adic valuation $|\cdot|_p$, normalized so $|p|_p = p^{-1}$. As is standard, $p = \infty$ corresponds to the usual absolute value $|\cdot|_\infty$ on $\mathbb{Q}$, so $\mathbb{Q}_\infty \cong \mathbb{R}$. The valuations $|\cdot|_p$ for $p \leq \infty$ form a complete list of mutually inequivalent valuations on $\mathbb{Q}$. The phrase "almost every p" will mean "all but a finite number of p." Define the *adele ring* $\mathbb{Q}_{\mathbb{A}}$ of $\mathbb{Q}$ by

$$\mathbb{Q}_{\mathbb{A}} = \left\{ x \in \prod_{p \leq \infty} \mathbb{Q}_p : |x_p|_p \leq 1 \quad \text{for almost every } p \right\}.$$

For a finite set $P \subset \{2, 3, 5, \ldots\} \cup \{\infty\}$ with $\infty \in P$, put

$$\mathbb{Q}_{\mathbb{A}}(P) = \{x \in \mathbb{Q}_{\mathbb{A}} : |x_p|_p \leq 1 \quad \text{if } p \notin P\}.$$

Each $\mathbb{Q}_{\mathbb{A}}(P)$ is locally compact under the product topology, and the topology on $\mathbb{Q}_{\mathbb{A}}$ is the coarsest making each of the $\mathbb{Q}_{\mathbb{A}}(P)$ an open subring. Under this topology $\mathbb{Q}_{\mathbb{A}}$ itself is locally compact. For $x \in \mathbb{Q}$, let $\delta(x) \in \mathbb{Q}_{\mathbb{A}}$ be the diagonal embedding given by $\delta(x)_p = x$ for $p \leq \infty$.

LEMMA 4.1. *The subgroup $\delta(\mathbb{Q})$ is discrete in $\mathbb{Q}_{\mathbb{A}}$, and $\mathbb{Q}_{\mathbb{A}}/\delta(\mathbb{Q}) \cong \Sigma$.*

Proof. See Theorems 2 and 3 of [**We**, § IV.2]. □

Identifying $\mathbb{Q}$ with $\delta(\mathbb{Q}) \subset \mathbb{Q}_{\mathbb{A}}$, we may consider $\mathbb{Q}_{\mathbb{A}}$ as a rational vector space. Therefore the action of A on $\mathbb{Q}^d$ extends to $\mathbb{Q}_{\mathbb{A}}^d$ by definining $(Ax)_p = A(x_p)$ for $x \in \mathbb{Q}_{\mathbb{A}}^d$. Using the identifications of Theorem 3 of [**We**, § IV.2], the quotient action of A on $\mathbb{Q}_{\mathbb{A}}^d/\delta(\mathbb{Q})^d$ is isomorphic to that of A on Σ^d. Since $\mathbb{Q}_{\mathbb{A}}^d$ is locally compact metric, and A is uniformly continuous, the definition of topological entropy $h(A; \mathbb{Q}_{\mathbb{A}}^d)$ of Bowen [**B**] applies.

LEMMA 4.2. $h(A; \Sigma^d) = h(A; \mathbb{Q}_{\mathbb{A}}^d)$.

Proof. By the above, $h(A; \Sigma^d) = h(A; \mathbb{Q}_{\mathbb{A}}^d/\delta(\mathbb{Q})^d)$. Since $\delta(\mathbb{Q})^d$ is a discrete subgroup of $\mathbb{Q}_{\mathbb{A}}^d$ with compact quotient, the result follows from [**B**, Thm. 20]. □

Our method to compute topological entropy uses Haar measure to count orbits, following Bowen [**B**]. Suppose (X, ρ) is a locally compact metric space, that $T: X \to X$ is uniformly continuous, and put

$$D_n(x, \varepsilon, T) = \bigcap_{k=0}^{n-1} T^{-k}(B_\varepsilon(T^k x)),$$

where $B_\varepsilon(y) = \{z \in X: \rho(y, z) < \varepsilon\}$. A Borel measure μ on X is called *T-homogeneous* [**B**, Def. 6] if

(1) $\mu(K) < \infty$ for all compact K,

(2) $\mu(K) > 0$ for some compact K,

(3) for each $\varepsilon > 0$ there are $\delta > 0$ and $c > 0$ so that

$$\mu(D_n(y, \delta, T)) \leq c\mu(D_n(x, \varepsilon, T))$$

for all $n \geq 0$ and $x, y \in X$.

If μ is T-homogeneous, put

$$k(\mu, T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(D_n(x, \varepsilon, T)).$$

By condition (3), this is independent of x. The key result from Bowen [**B**, Prop. 7] is that $k(\mu, T) = h(T)$ for any T-homogeneous measure μ. In particular, if A is an automorphism of a locally compact group G, then Haar measure μ_G is A-homogeneous, so $h(A) = k(\mu_G, A)$.

To apply this method, we find a finite set of primes for A which are those contributing to its entropy. If $\mathbb{Z}_p$ denotes the p-adic integers, then both A and A^{-1} have entries in $\mathbb{Z}_p$ for almost every p. Let P be the set of primes p for which some entry of A or of A^{-1} is not in $\mathbb{Z}_p$, together with ∞. Thus $A \in GL(d, \mathbb{Z}_p)$ for $p \notin P$.

Lemma 4.3. $h(A; \mathbb{Q}_{\mathbb{A}}^d) = h(A; \mathbb{Q}_{\mathbb{A}}(P)^d)$.

Proof. Since $A \in GL(d, \mathbb{Z}_p)$ for $p \notin P$, it follows that $\mathbb{Q}_{\mathbb{A}}(P)^d$ is an A-invariant neighborhood of the identity. Since Harr measure on $\mathbb{Q}_{\mathbb{A}}$, which is the restriction of Haar measure on $\mathbb{Q}_{\mathbb{A}}(P)$, is A-homogeneous, it follows from [**B**, Prop. 7] that $h(A; \mathbb{Q}_{\mathbb{A}}^d) = h(A; \mathbb{Q}_{\mathbb{A}}(P)^d)$. □

Lemma 4.4. $h(A; \mathbb{Q}_{\mathbb{A}}(P)^d) = h(A; \prod_{p \in P} \mathbb{Q}_p^d)$.

Proof. Since $\mathbb{Q}_{\mathbb{A}}(P)^d = \prod_{p \in P} \mathbb{Q}_p^d \times \prod_{p \notin P} \mathbb{Z}_p^d$, with the second factor compact, it follows from [**Wa**, Thm. 7.10] that

$$h(A; \mathbb{Q}_{\mathbb{A}}(P)^d) = h\left(A; \prod_{p \in P} \mathbb{Q}_p^d\right) + h\left(A; \prod_{p \notin P} \mathbb{Z}_p^d\right).$$

If F is any finite set of primes in P^c, and $m > 0$, then

$$\prod_{p \in F} p^m \mathbb{Z}_p^d \times \prod_{p \in P^c \setminus F} \mathbb{Z}_p^d$$

is an A-invariant neighborhood of 0 since $A \in GL(d, \mathbb{Z}_p)$ for $p \in P^c$. Such neighborhoods form a basis, and again using [**B**, Prop. 7] with the A-homogeneous Haar measure, we find $h(A; \prod_{p \notin P} \mathbb{Z}_p^d) = 0$. □

Lemma 4.5. $h(A; \prod_{p \in P} \mathbb{Q}_p^d) = \sum_{p \in P} h(A; \mathbb{Q}_p^d)$.

Proof. An argument is required, since Bowen's definition of entropy is not in general additive over products [**Wa**, p. 176]. Let μ_p be Haar measure on $\mathbb{Q}_p$, and $\mu = \prod_{p\in P} \mu_p$. Then A is μ_p^d-homogeneous. We will show in the proof of Theorem 2 that if B is a compact open subring in $\mathbb{Z}_p$, then

$$-\frac{1}{n}\log \mu_p^d\left(\bigcap_{k=0}^{n-1} A^{-k}(B^d)\right) \to h(A; \mathbb{Q}_p^d) \qquad \text{as} \quad n\to\infty,$$

the essential point being that in this case lim sup's are actually limits. Additivity of entropy then follows. □

Proof of Theorem 1. From Lemmas 4.2-4.5, we conclude that

$$h(A; \Sigma^d) = \sum_{p\in P} h(A; \mathbb{Q}_p^d).$$

An argument as in the proof of Lemma 4.4 shows $h(A; \mathbb{Q}_p^d) = 0$ for $p \notin P$. This completes the proof. □

5. *Calculation of p-adic entropy*

It remains to compute the p-adic entropy of the action of A on $\mathbb{Q}_p^d$. The formula and arguments are similar to the euclidean case, with p-adic eigenvalues replacing complex ones.

THEOREM 2. *If A has p-adic eigenvalues $\lambda_1, \ldots, \lambda_d$ in some finite extension of $\mathbb{Q}_p$, then*

$$h(A; \mathbb{Q}_p^d) = \sum_{|\lambda_j|_p > 1} \log |\lambda_j|_p,$$

where eigenvalues are counted with multiplicity.

Proof. Let K be a finite extension of $\mathbb{Q}_p$ containing all roots of $\chi_A(t)$, and set $r = [K : \mathbb{Q}_p]$. Then $\mathbb{Q}_p^d \otimes_{\mathbb{Q}_p} K \cong K^d$, and A extends to $A\otimes 1_K$ acting on K^d. Since K is a vector space of dimension r over $\mathbb{Q}_p$, and A has entries in $\mathbb{Q}_p$, it follows that $A\otimes 1_K$ is isomorphic to the direct sum of r copies of A acting on $\mathbb{Q}_p^d$. Thus $h(A\otimes 1_K; K^d) = rh(A; \mathbb{Q}_p^d)$.

Since K contains the eigenvalues of $A\otimes 1_K$, we can put $A\otimes 1_K$ into its Jordan form

$$A\otimes 1_K \cong \bigoplus_{i=1}^{k} J(\lambda_i, d_i),$$

where $J(\lambda_i, d_i)$ denotes the Jordan block of size d_i corresponding to λ_i. To compute the entropy of Jordan blocks, we require the following lemma. Define as before $\log^+ x = \max\{\log x, 0\}$.

LEMMA 5.1. *With the above notations, $h(J(\lambda, m); K^m) = rm \log^+ |\lambda|_p$.*

Proof. We first treat the case $m = 1$. Let J denote $J(\lambda, 1) = [\lambda]$. If μ is Haar measure on K, recall that $\text{mod}_K(\lambda)$ is the number defined by $\mu(\lambda E) = \text{mod}_K(\lambda)\mu(E)$ for measurable $E \subset K$ [**We**, p. 3]. Now $\text{mod}_K(\lambda) = \text{mod}_{\mathbb{Q}_p}(N_{K/\mathbb{Q}_p}(\lambda))$ [**We**, p. 7]. For $x\in \mathbb{Q}_p$ we have $\text{mod}_{\mathbb{Q}_p}(x) = |x|_p$, and also $|\lambda|_p = |N_{K/\mathbb{Q}_p}(\lambda)|_p^{1/[K:\mathbb{Q}_p]}$ [**K**, p. 61]. If C is a compact ball centered at 0, then

$$\bigcap_{j=0}^{n-1} J^{-j}C = \begin{cases} J^{-(n-1)}C & \text{if } |\lambda|_p > 1, \\ C & \text{if } |\lambda|_p \le 1. \end{cases}$$

It follows that

$$\lim_{n\to\infty} -\frac{1}{n}\log \mu\left(\bigcap_{j=0}^{n-1} J^{-j}C\right) = k(\mu, J)$$

exists, and equals $\log^+ \operatorname{mod}_K(\lambda) = r\log^+|\lambda|_p$. Since this also equals $h(J; K)$ by [**B**, Prop. 7], the case $m = 1$ is completed.

Next consider $J = J(\lambda, m)$ acting on K^m equipped with the sup norm. We may assume without loss that $|\lambda|_p > 1$, since otherwise both sides are 0. Since J commutes with multiplication by a power of p, by [**B**, Prop. 7] it is enough to show that

$$\lim_{n\to\infty} -\frac{1}{n}\log \mu\left(\bigcap_{j=0}^{n-1} J^{-j}C\right) = rm\log^+|\lambda|_p,$$

where C is the unit ball in K^m. Use of the non-archimedean nature of $|\cdot|_p$ makes the following p-adic computations about Jordan blocks simpler than the corresponding euclidean ones (see [**Wa**, Thm. 8.14] for the latter). Expansion of J^j shows that $J^jC \subset \lambda^{n-1}C$ for $0 \le j \le n-1$, so $\lambda^{-(n-1)}C \subset \bigcap_{j=0}^{n-1} J^{-j}C$. Expansion of J^{-j} shows that $J^{-(n-1)}C \subset \lambda^{-(n-1)}C$. We conclude that $\bigcap_{j=0}^{n-1} J^{-j}C = \lambda^{-(n-1)}C$. This proves the existence of the required limit, while its value is computed exactly as in the case $m = 1$. □

To complete the proof of Theorem 2, we note

$$h(A:\mathbb{Q}_p^d) = \frac{1}{r}h(A\otimes 1_K; K^d) = \frac{1}{r}\sum_{i=1}^{k} h(J(\lambda_i, d_i), K^{d_i})$$

$$= \frac{1}{r}\sum_{i=1}^{k} rd_i\log^+|\lambda_i|_p = \sum_{|\lambda_j|_p>1}\log^+|\lambda_j|_p. \qquad \square$$

6. *Derivation of Yuzvinskii's formula*

We use Theorems 1 and 2 to deduce Yuzvinskii's formula (2). Let $A \in GL(d, \mathbb{Q})$ have complex eigenvalues $\lambda_1, \ldots, \lambda_d$, and let s be the least common multiple of the denominators of the coefficients of $\chi_A(t)$.

THEOREM 3 (Yuzvinskii). $h(A; \Sigma^d) = \log s + \sum_{|\lambda_j|>1}\log|\lambda_j|$.

Proof. Let $\chi_A(t) = t^d + a_1t^{d-1} + \cdots + a_d$. If p^e is the highest power of p dividing s, then

$$p^e = \max\{|a_1|_p, \ldots, |a_d|_p, 1\}.$$

We will prove that $\log p^e = h(A; \mathbb{Q}_p^d)$. Then Theorem 1 shows that $\log s$ is just the sum over $p < \infty$ of the p-adic contributions to entropy, and this will complete the proof.

Factor $\chi_A(t) = \prod_{j=1}^{d}(t - \lambda_j^{(p)})$ over a finite extension of $\mathbb{Q}_p$, and order the eigenvalues so

$$|\lambda_1^{(p)}|_p \ge |\lambda_2^{(p)}|_p \ge \cdots \ge |\lambda_m^{(p)}|_p > 1 \ge |\lambda_{m+1}^{(p)}|_p \ge \cdots \ge |\lambda_d^{(p)}|_p.$$

If $|\lambda_j^{(p)}|_p \le 1$ for all j, then clearly $e = 0$ and $h(A; \mathbb{Q}_p^d) = 0$ as well. Thus we may suppose $|\lambda_1^{(p)}|_p > 1$. By the non-archimedean property of $|\cdot|_p$, we have by the specified ordering that

$$|a_m|_p = \left| \sum_{i_1<\cdots<i_m} \lambda_{i_1}^{(p)} \cdots \lambda_{i_m}^{(p)} \right|_p$$

$$= |\lambda_1^{(p)} \cdots \lambda_m^{(p)} + \text{smaller terms}|_p$$

$$= |\lambda_1^{(p)} \cdots \lambda_m^{(p)}|_p,$$

and by a similar calculation that all $|a_j|_p \le |a_m|_p$. Thus

$$p^e = \max_{1\le j\le d} \{|a_j|_p = \prod_{|\lambda_j^{(p)}|_p>1} |\lambda_j^{(p)}|_p.$$

By Theorem 2,

$$\log p^e = \sum_{|\lambda_j^{(p)}|_p>1} \log|\lambda_j^{(p)}|_p = h(A; \mathbb{Q}_p^d),$$

completing the proof. □

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