# THE BERNOULLI PROPERTY FOR EXPANSIVE $\mathbb{Z}^2$ ACTIONS ON COMPACT GROUPS

T. B. Ward

*Israel J. of Math., to appear.*

Abstract. We show that an expansive $\mathbb{Z}^2$ action on a compact abelian group is measurably isomorphic to a two–dimensional Bernoulli shift if and only if it has completely positive entropy. The proof uses the algebraic structure of such actions described by Kitchens and Schmidt and an algebraic characterisation of the $K$ property due to Lind, Schmidt and the author. As a corollary, we note that an expansive $\mathbb{Z}^2$ action on a compact abelian group is measurably isomorphic to a Bernoulli shift relative to the Pinsker algebra. A further corollary applies an argument of Lind to show that an expansive $K$ action of $\mathbb{Z}^2$ on a compact abelian group is exponentially recurrent. Finally an example is given of measurable isomorphism without topological conjugacy for $\mathbb{Z}^2$ actions.

## §1. Introduction

We find conditions under which an expansive $\mathbb{Z}^2$ action by continuous automorphisms on a compact abelian group is measurably isomorphic to a 2–dimensional Bernoulli shift. This is a special case of Conjecture 6.8 of [16].

The correspondence between $\mathbb{Z}^d$ actions on compact abelian groups and modules over the ring of Laurent polynomials in $d$ commuting variables described by Kitchens and Schmidt, [10], will be used extensively. This correspondence enables dynamical properties of the action to be read off from the structure of the module: ergodicity in [10], expansiveness and finiteness of the periodic points in [25], mixing properties in [24], entropy, measures of maximal entropy and the $K$ property in [16]. For a more detailed description of this correspondence, see [10] and [16]. We remark that in the absence of a convenient notion of a "past", a $\mathbb{Z}^2$ action is said to be $K$ if the Pinsker algebra is trivial.

For $\mathbb{Z}$ actions on compact abelian groups (continuous automorphisms), it is well known that ergodicity is equivalent to being measurably isomorphic to a Bernoulli shift. Ergodic automorphisms of compact abelian groups were shown to be $K$ in 1964 by Rokhlin, [21]. In 1971, Katznelson showed that an ergodic automorphism of the torus $\mathbb{T}^n$ is measurably

1991 *Mathematics Subject Classification.* 14A05, 22D40, 28D15, 28D20.

Supported in part by N.S.F. grant No. DMS–88–02593 at the University of Maryland and by N.S.F. grant No. DMS–91–03056 at Ohio State University

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

isomorphic to a Bernoulli shift, [8], and this was extended to ergodic automorphisms of the infinite torus $\mathbb{T}^\infty$ by Chu, [2], and Lind, [12]. Finally Lind, [13], showed that an ergodic automorphism of a compact group is measurably isomorphic to a Bernoulli shift. This was also shown independently by Miles and Thomas, [18].

We will associate to an expansive $\mathbb{Z}^2$ action a module over the ring $\mathcal{R} = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$; the prime filtration of this module expresses the original action as a factor of an iterated skew product of actions whose corresponding modules are cyclic. The case of a cyclic module is dealt with in §3, where the corresponding action is shown to be almost block independent if it is $K$. The primary decomposition and prime filtration is used in §4 to show that an expansive $\mathbb{Z}^2$ action on a compact abelian group is measurably isomorphic to a Bernoulli shift if it is $K$. In §5 we show that expansive systems corresponding to cyclic modules are exponentially recurrent if they are $K$.

I am grateful to Klaus Schmidt for showing me the algebraic reduction used in Lemma 4.1 and for explaining to me the illustration used in §6(5).

## §**2. Notation and preliminaries**

We now define the terms that we will be using, describe the correspondence between $\mathbb{Z}^2$ actions and $\mathcal{R}$–modules of [10], and give some of the basic constructions.

The integers, rationals, real numbers, and complex numbers will be denoted $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ respectively. The additive (resp. multiplicative) circle group will be denoted $\mathbb{T}$ (resp. $\mathbb{S}^1$). The dual group of a locally compact abelian group $G$ will be denoted $\widehat{G}$.

A $\mathbb{Z}^2$ action on the compact abelian group X (all groups will be assumed metrizable) is given by a homomorphism $\alpha : \mathbb{Z}^2 \to Aut(X)$, where $Aut(X)$ is the group of continuous automorphisms of $X$. The dual of the compact metric group $X$ is a discrete countable group $M = \widehat{X}$ and this group inherits the structure of an $\mathcal{R}$–module from the action $\alpha$. This is defined by setting $x \cdot m = \widehat{\alpha}_{(1,0)}(m)$, $y \cdot m = \widehat{\alpha}_{(0,1)}(m)$, and extending to an action of $\mathcal{R}$. Conversely, if $M$ is a countable $\mathcal{R}$–module, then the duals of multiplication by $x$ and $y$ are commuting automorphisms of the compact metric group $X_M = \widehat{M}$, and so they define a $\mathbb{Z}^2$ action $\alpha^M$ on $X_M$.

If $Q : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a homomorphism, denote the action $(n,m) \mapsto \alpha_{Q(n,m)}$ by ${}^Q\alpha$.

If $M$ is a cyclic module, $M = \mathcal{R}/\mathfrak{p}$ for some ideal $\mathfrak{p} \subset \mathcal{R}$, then the dynamical system $(X_M, \alpha^M)$ may be realised explicitly as follows. Since $\widehat{\mathcal{R}} \cong \mathbb{T}^{\mathbb{Z}^2}$, and the duals of multiplication by $x$ and $y$ are the horizontal and vertical shifts respectively, $X_M$ is a closed, shift invariant subgroup of $\mathbb{T}^{\mathbb{Z}^2}$. If $\mathfrak{p} = \langle f^1, \dots, f^l \rangle$, and each $f^j(x,y) = \sum_{\mathbf{n} \in Supp(f^j)} f^j_{\mathbf{n}} \cdot x^{n_1} y^{n_2}$ then

$$X_{\mathcal{R}/\mathfrak{p}} = \{\mathbf{x} \in \mathbb{T}^{\mathbb{Z}^2} \mid \sum_{\mathbf{n} \in Supp(f^j)} f^j_{\mathbf{n}} x_{(k_1+n_1, k_2+n_2)} = 0 \bmod 1,\ i = 1, \dots, l,\ (k_1, k_2) \in \mathbb{Z}^2\}.$$

The action $\alpha^{\mathcal{R}/\mathfrak{p}}$ is the restriction of the natural shift action on $\mathbb{T}^{\mathbb{Z}^2}$ to the shift invariant subgroup $X_{\mathcal{R}/\mathfrak{p}}$.

If $\mathfrak{p}$ is an ideal in $\mathcal{R}$, let $V_{\mathbb{C}}(\mathfrak{p}) = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0 \text{ for any } f \in \mathfrak{p}\}$ denote the set of common zeros of $\mathfrak{p}$; if $\mathfrak{p}$ is a prime ideal, then this is the affine variety associated to $\mathfrak{p}$.

Let $F = \{1, \ldots, s\}$. Given two probability measures $\mu$ and $\nu$ on $F^{\mathbb{Z}^2}$, and $D$ a finite subset of $\mathbb{Z}^2$, define the space of joinings $J_D(\mu, \nu)$ as follows. Write $\mu^D$, $\nu^D$ for the marginal measures induced by $\mu$ and $\nu$ on $\mathbb{T}^D$. If $\lambda$ is a probability measure on $F^D \times F^D$ then write $\lambda^1$, $\lambda^2$ for the two marginals of $\lambda$, and set

$$J_D(\mu, \nu) = \{\lambda \mid \lambda^1 = \mu^D, \lambda^2 = \nu^D\}.$$

The $\bar{d}$ distance between the probability measures $\mu$ and $\nu$ is defined as in [27]. Let $d$ be the usual (trivial) metric on $F$, and let $x = \{x_n\}$, $y = \{y_n\}$ be the processes defined on $F$ by $\mu$ and $\nu$ respectively. Then define

$$\bar{d}_D(\mu, \nu) = \inf_{\lambda \in J_D(\mu,\nu)} \frac{1}{|D|} \sum_{d \in D} \int d(x_d, y_d) \mathrm{d}\lambda,$$

and

$$\bar{d}(\mu, \nu) = \limsup_{D} \bar{d}_D(\mu, \nu). \tag{1.1}$$

We amend the definition of ABI in [27] as follows. Let $\sigma$ be the shift on $F^{\mathbb{Z}^2}$. Say that a stationary $\mathbb{Z}^2$ process $x$ has *almost block independence* if for any $\epsilon > 0$ there is an $N_\epsilon$ such that if $n \geq N_\epsilon$, $R = [0, n-1) \times [0, n-1) \cap \mathbb{Z}^2$, there is another process $y$ with

(1) $\bar{d}_R(\sigma_{n(a,b)}(y), x) = 0$ for all $(a, b) \in \mathbb{Z}^2$ and
(2) $y$ restricted to $n(a, b) + R$ is independent of $y$ restricted to $n(a', b') + R$ if $(a, b) \neq (a', b')$,

and $\bar{d}(x, y) \leq \epsilon$.

Denote the topological entropy of the action $\alpha^M$ on $X_M$ by $h(\alpha^M)$ (this coincides with the entropy with respect to Haar measure by [16], §6).

Let $\Sigma = \widehat{\mathbb{Q}}$, the one–dimensional full solenoid. For an $\mathcal{R}$–module $M$, define $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ and notice that $M_{\mathbb{Q}}$ is still an $\mathcal{R}$–module.

**Lemma 2.1.** *The system $(X_M, \alpha^M)$ is a full entropy factor of $(X_{M_{\mathbb{Q}}}, \alpha^{M_{\mathbb{Q}}})$ if $M$ is torsion free as an additive group.*

*Proof.* Let $\eta : M \to M_{\mathbb{Q}}$ be given by $\eta(m) = m \otimes_{\mathbb{Z}} 1$. Then $\eta$ is an injective homomorphism of additive groups under the assumption that $M$ is torsion free, so $\widehat{\eta} : X_{M_{\mathbb{Q}}} \to X_M$ is a surjective homomorphism of compact groups. Moreover, $\eta$ is a homomorphism of $\mathcal{R}$–modules, so $\widehat{\eta}$ intertwines the actions $\alpha^M$ and $\alpha^{M_{\mathbb{Q}}}$ and realises $(X_M, \alpha^M)$ as a factor of $(X_{M_{\mathbb{Q}}}, \alpha^{M_{\mathbb{Q}}})$.

To see that the factor is of full entropy, we use the argument of [17], Proposition 3.1. For each $n \geq 1$, let $M_n = \frac{1}{n!}M$. Since $M$ is torsion free, the action of $\alpha^{M_{\mathbb{Q}}}$ restricted to the closed invariant subgroup $X_{M_n}$ is isomorphic to the action of $\alpha$ on $X_M$ ($M_n$ and $M$ are isomorphic as $\mathcal{R}$–modules). Thus $h(\alpha^{M_n}) = h(\alpha^M)$ for all $n$. The dual of the direct limit $M_{\mathbb{Q}} = \varinjlim M_n$ is the projective limit $X_{M_{\mathbb{Q}}} = \varprojlim X_{M_n}$ so $h(\alpha^{M_{\mathbb{Q}}}) = \lim h(\alpha^{M_n}) = h(\alpha^M)$. □

As observed in [17], §3, many of the topological dynamical properties of $(X_M, \alpha^M)$ are not reflected in $(X_{M_{\mathbb{Q}}}, \alpha^{M_{\mathbb{Q}}})$; in particular $(X_{M_{\mathbb{Q}}}, \alpha^{M_{\mathbb{Q}}})$ is never expansive and has no periodic points other than the identity. However, the measurable dynamics are preserved; in particular $(X_M, \alpha^M)$ is measurably isomorphic to a Bernoulli shift if $(X_{M_{\mathbb{Q}}}, \alpha^{M_{\mathbb{Q}}})$ is.

Now let $X \subset \Sigma^{\mathbb{Z}^2}$ be any closed shift invariant subgroup. For $F \subset \mathbb{Z}^2$, let $\pi^{(F)} : \Sigma^{\mathbb{Z}^2} \to \Sigma^F$ denote the projection onto the coordinates in $F$; $\pi^{(F)}(X)$ is a closed subgroup of $\Sigma^F$.

By analogy with the Fejér kernel on $\mathbb{T}$, define a positive summability kernel $\{\mathbf{K}_n\}_{n\in\mathbb{N}}$ on the full solenoid $\Sigma$ as follows. For each $n \geq 1$, let

$$C_n = \{r \in \mathbb{Q} \mid \tfrac{1}{n} \leq |r| \leq n \text{ and } r \in \tfrac{1}{n}\mathbb{Z}\} \cup \{0\}.$$

Each $r \in \mathbb{Q}$ defines a character $\chi_r$ on $\Sigma$ under the pairing $<,>: \mathbb{Q} \times \Sigma \to \mathbb{S}^1$ where $< r, x >= \chi_r(x)$.

**Lemma 2.2.** *The family of functions $\{\mathbf{K}_n\}_{n\in\mathbb{N}}$ on $\Sigma$ defined by*

$$\mathbf{K}_n(x) = \sum_{j=-n^2}^{n^2} \big(1 - \frac{j}{n^2+1}\big)\chi_{(j/n)}(x) \tag{1.2}$$

*forms a positive summability kernel on $\Sigma$.*

*Proof.* The sets $C_n$ increase to $\mathbb{Q}$, so the Fourier transform $\widehat{\mathbf{K}}_n$ of $\mathbf{K}_n$ converges pointwise as $n \to \infty$ to the constant function 1. This means that the measure $\lambda_n$ on $\Sigma$ whose Radon–Nikodym derivative with respect to Haar measure on $\Sigma$ is $\mathbf{K}_n$ converges weakly to the point mass at the identity as $n \to \infty$.

In order to see that $\widehat{\mathbf{K}}_n$ is positive definite on $\mathbb{Q}$, recall that the Fejér kernel on $\mathbb{T}$, defined by

$$\mathbf{F}_k(t) = \sum_{i=-n}^{n} (1 - \frac{i}{k+1})e^{2\pi it}$$

is a positive summability kernel. In particular, $\widehat{\mathbf{F}}_k$ is positive definite on $\mathbb{Z}$. Let $k = n^2$ and notice that $\widehat{\mathbf{K}}_n$ is supported on $\frac{1}{n^2}\mathbb{Z}$. The isomorphism $\frac{1}{n^2}\mathbb{Z} \to \mathbb{Z}$ which takes $\widehat{\mathbf{K}}_n$ to $\widehat{\mathbf{F}}_{n^2}$ shows that $\widehat{\mathbf{K}}_n$ is positive definite. □

## §3. Cyclic expansive modules

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{R}$. Since $\mathfrak{p}$ is prime, the collection $Asc(\mathcal{R}/\mathfrak{p})$ of primes associated to the module $\mathcal{R}/\mathfrak{p}$ is exactly $\{\mathfrak{p}\}$. By Theorem 6.5 of [16], this shows that $(X_{\mathcal{R}/\mathfrak{p}}, \alpha^{\mathcal{R}/\mathfrak{p}})$ has completely positive entropy if it has positive entropy.

Following [16], §6, call a prime ideal $\mathfrak{p}$ *positive* if it is principal with generator not a generalised cyclotomic polynomial in the sense of [1]. An ideal which is not positive is therefore either generated by a cyclotomic polynomial or is non–principal; such ideals will be called *null*. Theorem 4.2 and Example 5.4 of [16] show that $h(\alpha^{\mathcal{R}/\mathfrak{p}}) > 0$ if and only if $\mathfrak{p}$ is a positive ideal. By the previous paragraph, we deduce that $(X_{\mathcal{R}/\mathfrak{p}}, \alpha^{\mathcal{R}/\mathfrak{p}})$ is $K$ if and only if $\mathfrak{p}$ is positive.

Theorem 3.8 of [25] shows that the system $(X_{\mathcal{R}/\mathfrak{p}}, \alpha^{\mathcal{R}/\mathfrak{p}})$ is expansive if and only if either $\mathcal{R}/\mathfrak{p}$ has positive characteristic or $V_{\mathbb{C}}(\mathfrak{p}) \cap (\mathbb{S}^1)^2 = \emptyset$. Following [16], §7, call the ideal $\mathfrak{p}$ *expansive* if $(X_{\mathcal{R}/\mathfrak{p}}, \alpha^{\mathcal{R}/\mathfrak{p}})$ is expansive.

We now describe a standard form for principal ideals. Since any monomial is a unit in $\mathcal{R}$, we will without comment multiply by monomials to put polynomials into a convenient form.

**Lemma 3.1.** *For any positive expansive ideal $\mathfrak{p} = \langle f \rangle$, there is a matrix $Q \in GL(2, \mathbb{Z})$ with the following properties:*

(1) $f\big(Q(x,y)\big) = a_0 + a_1 x y^{m_1} + a_2 x^2 y^{m_2} + \cdots + a_d x^d y^{m_d}$ *with* $0 < m_1 < \cdots < m_d$.
(2) *There is a constant $\kappa = \kappa(f) > 1$ such that for $(z,w) \in V_{\mathbb{C}}(fQ)$, $|w| = 1$ implies $|z| > \kappa$ or $|z| < 1/\kappa$ and $|z| = 1$ implies $|w| > \kappa$ or $|w| < 1/\kappa$.*

*Proof.* Let $H(f)$ denote the convex hull of the support of $f$, with $f$ multiplied by a monomial to arrange that $H(f)$ lies in the upper–right–hand quadrant touching each axis. By $fQ$ is meant the polynomial $(fQ)(x,y) = f\big((x,y)Q\big)$, where

$$(x,y)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (x^a y^c, x^b y^d).$$

Notice that $H(fQ) = H(f) \cdot Q$.

An application of the shear $Q_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ fixes the $y^0$ slice of $H(f)$ and moves the horizontal $y^j$ slice $j$ units to the right. After a finite number of applications, the top slice will extend further to the right than any other slice, and each vertical slice will have only one entry. Repeat the process with $Q_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ to move the vertical slice of maximal degree in $x$ far enough up to obtain the form (1).

For (2), expansiveness shows that the sets

$$C_1 = \{z \in \mathbb{C} \mid f(z,w) = 0 \text{ for some } w \in \mathbb{S}^1\}, C_2 = \{w \in \mathbb{C} \mid f(z,w) = 0 \text{ for some } z \in \mathbb{S}^1\}$$

do not meet $\mathbb{S}^1$; compactness shows (2). □

Remarks.
(1) Notice that applying $Q \in GL(2, \mathbb{Z})$ does not alter the dynamics: with the above notation,

$$(X_{\mathcal{R}/\langle f \rangle}, {}^{Q}\alpha^{\mathcal{R}/\langle f \rangle}) \cong (X_{\mathcal{R}/\langle fQ \rangle}, \alpha^{\mathcal{R}/\langle fQ \rangle}).$$

(2) The system $(X_{\mathcal{R}/\langle f\rangle}, \alpha^{\mathcal{R}/\langle f\rangle})$ is measurably isomorphic to a Bernoulli shift if and only if the system $(X_{R/\langle fQ\rangle}, {}^{Q}\alpha^{R/\langle fQ\rangle})$ is.
(3) Since $|\det Q| = 1$, $h(\alpha^{\mathcal{R}/\langle f\rangle}) = h(\alpha^{\mathcal{R}/\langle fQ\rangle})$.
(4) The form of $fQ$ is not unique subject to the stated properties, but every positive expansive ideal does have such a form.
(5) The form of the polynomial in Lemma 3.1 is also in "standard form" with respect to $y$:

$$a_0 + a_1 x y^{m_1} + a_2 x^2 y^{m_2} + \cdots + a_d x^d y^{m_d} = b_0 + b_1 x^{n_1} y + \cdots + b_r x^{n_r} y^r$$

where $r = m_d$, $d = n_r$ and so on.

**Example 3.2.** Let $f(x,y) = 4 + 3x + 2xy + y$. Then $H(f)$ is the unit square $[0,1] \times [0,1]$. Apply $Q_1$ twice to obtain

$$f(Q_1)^2(x,y) = 4 + 3x + x^2 y + 2x^3 y.$$

Now apply $Q_2$:

$$fQ_2(Q_1)^2(x,y) = 4 + 3xy + x^2y^3 + 2x^3y^4.$$

In this case the matrix $Q = Q_2 Q_1^2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ puts $f$ into standard form.

**Theorem 3.3.** *If $\mathfrak{p}$ is expansive and positive, then $(X_{\mathcal{R}/\mathfrak{p}}, \alpha^{\mathcal{R}/\mathfrak{p}})$ is measurably isomorphic to a Bernoulli shift.*

*Proof.* We first deal with some special cases. If $\mathfrak{p} = \langle f\rangle$ where $f(x,y) = F(x^a y^b)$ is a polynomial in a single variable $x^a y^b$ then the matrix $Q = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ transforms $(X_{\mathcal{R}/\mathfrak{p}}, \alpha^{\mathcal{R}/\mathfrak{p}})$ into the system defined by the polynomial $F(x)$. Here $|\det Q| \neq 1$ but Bernoullicity is preserved (this is equivalent to noting that a power or a root of a single Bernoulli transformation is also Bernoulli). The group $X_{\mathcal{R}/\langle F\rangle}$ is isomorphic to $Y^{\mathbb{Z}}$ where $\widehat{Y} = \mathbb{Z}[t^{\pm 1}]/\langle F(t)\rangle$. The automorphism $\alpha^{\mathcal{R}/\langle F\rangle}_{(1,0)}$ corresponds to the automorphism $\prod_{\mathbb{Z}} \beta$ where $\beta$ is dual to multiplication by $t$ on the module $\mathbb{Z}[t^{\pm 1}]/\langle F(t)\rangle$; the automorphism $\alpha^{\mathcal{R}/\langle F\rangle}_{(0,1)}$ corresponds to the full shift $\beta$ with alphabet $Y$. Since the original action was expansive, $\beta$ is an ergodic automorphism of the compact group $Y$. By [13], $\beta_{(1,0)}$ therefore has an independent generator $\xi$ on $Y$. Since $\beta$ acts as a full shift, $\xi_0$ is therefore an independent generator for the $\mathbb{Z}^2$ action $\beta$, where $\xi_0$ is the "time zero" partition defined by $\xi$ for the $\beta$ direction. We deduce that $(X_{\mathcal{R}/\mathfrak{p}}, \alpha^{\mathcal{R}/\mathfrak{p}})$ is measurably isomorphic to a Bernoulli shift.

If $\mathcal{R}/\mathfrak{p}$ has positive characteristic, then $\mathfrak{p}$ contains a constant $s$. Since $\mathfrak{p}$ is positive, we must have $\mathfrak{p} = \langle s\rangle$; in this case $(X_{\mathcal{R}/\mathfrak{p}}, \alpha^{\mathcal{R}/\mathfrak{p}})$ is a full two–dimensional shift on $s$ symbols, which is clearly a Bernoulli shift.

Expansiveness implies that $\mathfrak{p} \neq \{0\}$ (although this case is immediate: if $\mathfrak{p} = \{0\}$ then $(X_{\mathcal{R}/\mathfrak{p}}, \alpha^{\mathcal{R}/\mathfrak{p}})$ is a full two–dimensional shift with alphabet $\mathbb{T}$, which is the two–dimensional Bernoulli shift with infinite entropy).

We are left with the case where $\mathfrak{p} = \langle f \rangle$ for some non–constant polynomial $f$ that is not a polynomial in one variable. Without loss of generality, we may assume $f$ is in the form described by Lemma 3.1. We may also assume that $f$ is not of the form $f(x, y) = g(x^a, y^b)$; if it is of this form then we pass to the action determined by $g$ (the "$(a, b)^{\text{th}}$ root" of the action) and this will be measurably isomorphic to a Bernoulli shift if and only if the original action is. For brevity, let $M = \mathcal{R}/\mathfrak{p}$. We will show that the system $(X_{M_{\mathbb{Q}}}, \alpha^{M_{\mathbb{Q}}})$ has almost block independence and deduce that the factor $(X_M, \alpha^M)$ is measurably isomorphic to a Bernoulli shift.

The $\mathcal{R}$–module $M_{\mathbb{Q}}$ is given by $\mathbb{Q}[x^{\pm1}, y^{\pm1}]/f\mathbb{Q}[x^{\pm1}, y^{\pm1}]$ and so

$$M_{\mathbb{Q}} \cong \mathbb{Q}[y^{\pm1}] \oplus x\mathbb{Q}[y^{\pm1}] \oplus \cdots \oplus x^{d-1}\mathbb{Q}[y^{\pm1}] \tag{3.1}$$

as $\mathbb{Z}[y^{\pm1}]$–modules. Thus the group $X_{M_{\mathbb{Q}}}$ is isomorphic to $(\Sigma^{\mathbb{Z}})^d$, with the action of $\alpha_{(0,1)}$ corresponding to the vertical shift (a full shift with alphabet $\Sigma^d$.) The action of $\alpha_{(1,0)}$ is dual to an automorphism of $M_{\mathbb{Q}}$, given by a non–singular matrix $A \in M_d(\mathbb{Q}[y^{\pm1}])$.

For a fixed $n \in \mathbb{N}$, let $c_n(u)$ be the $n^{\text{th}}$ cyclotomic polynomial, and let $\phi(\cdot)$ be Euler's totient function. Let $X^{(n)}$ be the closed invariant subgroup of $X_{M_{\mathbb{Q}}}$ whose dual is $M_{\mathbb{Q}}/\langle c_n(y)\rangle M_{\mathbb{Q}}$.From (3.1),

$$\frac{M_{\mathbb{Q}}}{\langle c_n(y)\rangle} \cong \frac{\mathbb{Q}[y^{\pm1}]}{\langle c_n(y)\rangle} \oplus x\frac{\mathbb{Q}[y^{\pm1}]}{\langle c_n(y)\rangle} \oplus \cdots \oplus x^{d-1}\frac{\mathbb{Q}[y^{\pm1}]}{\langle c_n(y)\rangle} \tag{3.2}$$

as $\mathbb{Q}[y^{\pm1}]$–modules. Choose an isomorphism $\eta_n : \frac{\mathbb{Q}[y^{\pm1}]}{\langle c_n(y)\rangle} \to \mathbb{Q}^{\phi(n)}$, and let $J_{\phi(n)}$ be the matrix representing the automorphism $\eta_n y \eta_n^{-1}$ of $\mathbb{Q}^{\phi(n)}$. The isomorphism $\eta_n^{(d)} = \eta_n \times \cdots \times \eta_n$ then identifies $M_{\mathbb{Q}}/\langle c_n(y)\rangle$ with $\mathbb{Q}^{d\phi(n)}$; under this isomorphism the matrix $A$ is sent to $\bar{A}(\epsilon_n) \in M_{d\phi(n)}(\mathbb{Q})$.

For $a \le b$ let $S_a^b = \{(i, j) \in \mathbb{Z}^2 \mid a \le i \le b\}$. If $F \subset \mathbb{Z}^2$, let $\mu_F$ denote Haar measure on the group $\pi^{(F)}(X_{M_{\mathbb{Q}}}) \subset \Sigma^F$.

**Lemma 3.4.** *Let*

$$f(\mathbf{x}) = \prod_{(i,j)\in S_{-p}^0} f_{ij}(x_{ij}) \times \prod_{(i,j)\in S_m^{m+q}} f_{ij}(x_{ij})$$

*where each $f_{ij}$ is a trigonometric polynomial on $\widehat{\mathbb{Q}}$ of the form*

$$f_{ij}(x) = \sum_{k=-N^2}^{N^2} c_{ij}^{(k)} \chi_{k/N}(x).$$

*Then, for $m = m(N) \ge 1 + \frac{1}{\log\kappa}\log\left(\frac{2N^2}{\kappa-1}\right)$,*

$$\int f d\mu^{(S_{-p}^0 \cup S_m^{m+q})} = \int f d(\mu^{(S_{-p}^0)} \times \mu^{(S_m^{m+q})}) \tag{3.3}$$

*where $\mu \times \nu$ denotes the independent concatenation of the measures $\mu$ and $\nu$.*

Notice that the above lemma says that if $Z_1$ and $Z_2$ are continuous random variables defined on $\Sigma^{S^0_{-p}}$ and $\Sigma^{S^{m+q}_m}$ respectively, with densities $\rho_1$ and $\rho_2$, and if the Fourier transforms $\widehat{\rho}_1$ and $\widehat{\rho}_2$ are supported on the bounded set of frequencies $C_N$ in each coordinate, then the random variables obtained by restricting $Z_1$ and $Z_2$ to $X_M$ are independent if $m$ is large enough.

*Proof.* It is sufficient to show that for any character $\chi$ on $\pi^{(S^0_{-p})}(X_M) \times \pi^{(S^{m+q}_m)}(X_M)$, with coefficients in $C_N$, and for $m \geq 1 + \frac{1}{\log \kappa} \log\bigl(\frac{2N^2}{\kappa-1}\bigr)$, $\chi$ is trivial on

$$\pi^{(S^0_{-p} \cup S^{m+q}_m)}(X_M) \subset \pi^{(S^0_{-p})}(X_M) \times \pi^{(S^{m+q}_m)}(X_M)$$

if and only if $\chi$ is trivial on $\pi^{(S^0_{-p})}(X_M) \times \pi^{(S^{m+q}_m)}(X_M)$. Equivalently, we need to show that if $E_N$ denotes the set of characters on $(\Sigma^d)^{\mathbb{Z}}$ with frequencies in each coordinate from $C_N$ ($\equiv$ the set of $d$–tuples of Laurent polynomials in $\mathbb{Q}[y^{\pm 1}]$ with coefficients from $C_N$), and

$$H(m,p,q) = A^m\bigl(E_N + AE_N + \cdots + A^p E_N\bigr) \cap \bigl(E_N + A^{-1}E_N + \cdots + A^{-q}E_N\bigr) \qquad (3.4)$$

then $H(m,p,q) = \{0\}$ for $m \geq 1 + \frac{1}{\log \kappa} \log\bigl(\frac{2N^2}{\kappa-1}\bigr)$.

Assume that $\mathbf{h} \in H(m,p,q)$. By Hilbert's ireducibility theorem (see [5]), if $\epsilon_n$ is a primitive $n^{\text{th}}$ root of unity and $n$ is sufficiently large, then $f(x,\epsilon_n)$ is an irreducible polynomial in $\mathbb{Q}(\epsilon_n)[x^{\pm 1}]$. We claim that if $(n,j) = (n,k) = (k,j) = 1$ then $f(x,\epsilon_n^j)$ and $f(x,\epsilon_n^k)$ have no zeros in common. To see this, notice that if $f(x,\epsilon_n^j)$ and $f(x,\epsilon_n^k)$ have a common zero, then by irreducibility they have identical zeros. Since the constant terms of $f(x,\epsilon_n^j)$ and $f(x,\epsilon_n^k)$ coincide, it follows that $f(x,\epsilon_n^j) = f(x,\epsilon_n^k)$. By assumption, $f$ is not a polynomial in $x$ alone, so for large $n$, $f(x,\epsilon_n^j) = f(x,\epsilon_n^k)$ implies that $\epsilon_n^j = \epsilon_n^k$. We deduce that the polynomial

$$f_n(x) = \prod_{(j,n)=1} f(x,\epsilon_n^j)$$

has $d\phi(n)$ distinct zeros.

The characteristic equation of $\bar{A}(\epsilon_n)$ is given by

$$\det\bigl(\bar{A}(\epsilon_n) - \lambda I_{d\phi(n)}\bigr) = \prod_{(j,n)=1} f(\lambda,\epsilon_n^j) = f_n(\lambda).$$

By Lemma 3.1, any eigenvalue $\xi$ of $\bar{A}(\epsilon_n)$ has $|\xi| > \kappa$ or $|\xi| < \kappa^{-1}$. Moreover, $\bar{A}(\epsilon_n)$ has distinct eigenvalues so there is a matrix $Q_n \in M_{d\phi(n)}(\mathbb{C})$ with $\bar{A}(\epsilon_n) = Q_n \Lambda_n Q_n^{-1}$, where $\Lambda_n = \operatorname{diag}(\xi_1, \ldots, \xi_{d\phi(n)})$ and the spectrum of $\bar{A}(\epsilon_n)$ comprises $\{\xi_1, \ldots, \xi_{d\phi(n)}\}$.

Let $\bar{\mathbf{h}}(\epsilon_n) = \eta_n^{(d)}(\mathbf{h})$. For any $n$, we have

$$\bar{\mathbf{h}}(\epsilon_n) = \bar{A}^m(\epsilon_n)\bigl(\bar{\mathbf{a}}_0(\epsilon_n) + \bar{A}(\epsilon_n)\bar{\mathbf{a}}_1(\epsilon_n) + \cdots + \bar{A}^p(\epsilon_n)\bar{\mathbf{a}}_p(\epsilon_n)\bigr)$$

$$= \big(\bar{\mathbf{c}}_0(\epsilon_n) + \bar{A}^{-1}(\epsilon_n)\bar{\mathbf{c}}_1(\epsilon_n) + \dots + \bar{A}^{-q}(\epsilon_n)\bar{\mathbf{c}}_q(\epsilon_n)\big) \quad (3.5)$$

where each $\bar{\mathbf{a}}, \bar{\mathbf{c}} \in \eta_n^{(d)}(E_N)$. Let $\bar{\mathbf{a}}' = Q_n^{-1}\bar{\mathbf{a}}$, and notice that if $\phi(n)$ exceeds the degree of any of the polynomials in $A^m\big(E_N + AE_N + \dots + A^pE_N\big) \cup \big(E_N + A^{-1}E_N + \dots + A^{-q}E_N\big)$ then the coefficients of $\bar{h}$ are terms in the rational vector $\bar{\mathbf{h}}(\epsilon_n) \in \mathbb{Q}^{d\phi(n)}$, so each entry of $\bar{\mathbf{h}}(\epsilon_n)$ and of each $\bar{\mathbf{a}}$, $\bar{\mathbf{c}}$ lies in $C_N$. Let $z_i \times C_N$ be the projection of $Q_N^{-1}\bar{E}_N^d$ onto the $i^{\text{th}}$ axis in $\mathbb{C}^{d\phi(n)}$, and look at the $i^{\text{th}}$ coordinate of (3.5):

$$\xi_i^m\big((\bar{a}_0')_i + \xi_i(\bar{a}_1')_i + \dots \xi_i^p(\bar{a}_p')_i\big) = \big((\bar{c}_0')_i + \xi_i^{-1}(\bar{c}_1')_i + \dots \xi_i^{-q}(\bar{c}_q')_i\big). \quad (3.6)$$

We may assume that $|\xi_i| > \kappa$; if $|\xi_i| < \kappa^{-1}$ then apply the same argument to the inverse map.

Since each $(\bar{a}_0')_i, (\bar{c}_0')_i \in \frac{1}{N}z_1 \times \{0, \pm 1, \dots, \pm N^2\}$, (3.6) is equivalent to

$$\xi^m(n_0 + n_1\xi + \dots + n_p\xi^p) = (m_0 + m_1\xi^{-1} + \dots + m_q\xi^{-q}) \quad (3.7)$$

where each $n_i, m_i \in \{0, \pm 1, \dots, \pm N^2\}$, $n_0 \neq 0$ and $\xi = \xi_i$. Write

$$B = \xi^{m+q}(n_0 + n_1\xi + \dots + n_p\xi^p) \quad \text{and} \quad C = (m_0\xi^q + m_1\xi^{q-1} + \dots + m_q).$$

Then

$$|C| \leq N^2(1 + |\xi| + |\xi|^2 + \dots + |\xi|^q) \leq N^2\Big(\frac{\kappa^{q+1} - 1}{\kappa - 1}\Big). \quad (3.8)$$

Let $\Omega$ denote the closed path traced out in $\mathbb{C}$ by $\xi$, the $i^{\text{th}}$ zero of $f(x, y)$ for $y = \epsilon_n$, as $y$ moves around $\mathbb{S}^1$ from $\epsilon_n$ back to $\epsilon_n$. By the maximum modulus principle,

$$\min \max_{\zeta \in \Omega}\{k_0 + \zeta k_1 + \dots + \zeta^p k_p\} \geq 1$$

where the minimum is taken over all $k_i \in \{0, \pm 1, \dots, \pm N^2\}$ with $k_0 \neq 0$. Since the primitive unit roots are dense, we deduce that we may choose $\epsilon_n$ to have

$$|B| \geq \tfrac{1}{2}|\xi|^{m+q} \geq \tfrac{1}{2}\kappa^{m+q}. \quad (3.9)$$

Comparing (3.8) and (3.9), and noting that $\big(\frac{\kappa^{q+1}-1}{\kappa-1}\big)\kappa^{-q} \leq \frac{\kappa}{\kappa-1}$, we see that $H(m, p, q) = \{0\}$ for $m \geq 1 + \frac{1}{\log \kappa}\log\big(\frac{2N^2}{\kappa-1}\big)$ as required. □

Let $\mathcal{P}$ be a finite measurable partition of $X_{M_{\mathbb{Q}}}$ at "time zero", that is an atom of this partition is a set of the form $P_i = \{x \in X_{M_{\mathbb{Q}}} \mid x_{(0,0)} \in Q_i\}$ where $\{Q_1, \dots, Q_r\}$ is a finite partition of the alphabet $\Sigma$. The space of $\mathcal{P}$–names determines a finite state $\mathbb{Z}^2$ process denoted $(X_{M_{\mathbb{Q}}}, \mathcal{P})$.

Define a standard family of finite partitions of $\Sigma$ analogous to the partition into intervals of length $\frac{1}{n}$ on $\mathbb{T}$. First recall that $\Sigma$ is homeomorphic to the direct product $\mathbb{T} \times \prod_p \mathbb{Z}_p$ where the product is taken over all rational primes $p$ (see [29], §IV.2). Let us say that a set of the form $[\frac{j}{k}, \frac{j+1}{k}) \subset \mathbb{T}$ is a $k$–interval, and on $\mathbb{Z}_p$ call a cylinder set defined by specifying the first $k$ $p$–adic digits a $p$–adic $k$–interval. The standard $k^{\text{th}}$ partition, $\mathcal{P}_k$, of $\Sigma$ is defined to be that partition each of whose atoms comprises a set of the form $A_\infty \times A_2 \times \cdots \times A_{p(k)} \times \prod \mathbb{Z}_p$ where $A_\infty$ is a $k$–interval, each $A_p$ is a $p$–adic $k$–interval, and the product is taken over all primes exceeding the $k^{\text{th}}$ prime $p(k)$. Given two points $z \neq w$ in $\Sigma$, there is some $k$ for which the partition $\mathcal{P}_k$ separates them.

**Corollary 3.5.** *The finite state system $(X_{M_\mathbb{Q}}, \mathcal{P}_k)$ is almost block independent for any $k$.*

*Proof.* Fix $k$ throughout. Given $\epsilon > 0$ and $n \in \mathbb{N}$, choose $N(n,\epsilon)$ as follows. Let $T = [0,n] \times [0,n]$ be an $n \times n$ tile of coordinates in $\mathbb{Z}^2$, and let $\mathcal{P}_k^T = \bigvee_T \alpha_{ij}\mathcal{P}_k$ denote the join over $T$ of the partition $\mathcal{P}_k$. Let the atoms of $\mathcal{P}_k^T$ be $\{P_1,\dots,P_r\}$. Now let $N(n,\epsilon)$ be the least integer for which there is a family of trigonometric polynomials $\{f_1,\dots,f_r\}$ where each $f_s$ is a product of the form $f_s(x) = \prod_T f_{ij}^s(x_{ij})$ and each $f_{ij}^s$ is a trigonometric polynomial of the form

$$f_{ij}^s(t) = \sum_{k=-N^2}^{N^2} c_{(ij,s)}^{(k)} \chi_{k/N}(t)$$

(where $N = N(n,\epsilon)$) and the functions $\{f_1,\dots,f_r\}$ approximate $\mathcal{P}_k^T$ well in the following sense: there is an error set $E$, $\mu(E) < \epsilon^2$, $f_s \geq 1$ on $P_s \backslash E$, and $\sum_{s=1}^r \int f_s < 1 + \epsilon^2$. Notice that the size of $N(n,\epsilon)$ is no larger than a polynomial in $n$ and $\frac{1}{\epsilon}$. Say that the functions $\epsilon$–approximate $\mathcal{P}_k$ over $T$ in this case.

Let $\epsilon_j = \frac{\epsilon}{j^2}\frac{6}{\pi^2}$. Now choose $n(\epsilon)$ so that if $n \geq n(\epsilon)$, then $\frac{4m(N(3n,\epsilon_1),\epsilon_1)}{n} < \epsilon$ and $m(N((2r+1)n,\epsilon_r),\epsilon_r) < \sum_{j=1}^r m(N((2(r-j)+1)n,\epsilon_{(r-j)}),\epsilon_{(r-j)}) + n$ for all $r$. This is possible because $m(N,\epsilon)$ grows as the logarithm of a polynomial in $N$ and $\frac{1}{\epsilon}$.

Assume $n \geq n(\epsilon)$. Tile all of $\mathbb{Z}^2$ with $n \times n$ tiles $T_1, T_2, \dots$ spaced $m(N(3n,\epsilon_1),\epsilon_1)$ apart. From Lemma 3.4 and the choice of $m$, we may choose functions that $\epsilon_1$–approximate $\mathcal{P}_k$ over each $T_j$, and these functions are independent if they are built on different tiles $T_i$, $T_j$. Moreover, we claim that they are independent in the following stronger sense: given a tile $T_i$ and some collection of tiles $\{T_j\}$ disjoint from $T_i$, we may build a function that $\epsilon_1$ approximates $\mathcal{P}_k$ over $T_i$ and then functions on the other tiles $\{T_j\}$ that $\epsilon_2$–approximate $\mathcal{P}_k$ over the tiles nearest to $T_i$ (i.e. over a bigger tile of side $3n$ a distance $m(N(3n,\epsilon_1),\epsilon_1)$ away), that $\epsilon_3$–approximate $\mathcal{P}_k$ over the next nearest tiles (i.e. over a bigger tile of side no more than $5n$ a distance $m(N(3n,\epsilon_1),\epsilon_1) + n > m(N(3n,\epsilon_2),\epsilon_2)$ away) and so on. By construction, these functions are all independent of the functions approximating $\mathcal{P}_k$ on $T_i$, and so we may make the following claim. The join of the partition $\mathcal{P}_k$ can be $\epsilon$–approximated over any of the tiles with functions independent of those $\epsilon$–approximating $\mathcal{P}_k$ over some other tile.

It follows by a basic Lemma of Katznelson (Lemma 1 of [8]) that the partition $\mathcal{P}_k^{\mathbb{Z}^2}$ (the join over the whole of $\mathbb{Z}^2$) is $11\epsilon$–block–independent when restricted to the $n \times n$ tiles $\{T_i\}$. It follows (Lemma 6.3 of [26] extends in a straightforward manner from $\mathbb{Z}$ to $\mathbb{Z}^2$) that

$(X_{M_\mathbb{Q}}, \mathcal{P}_k)$ is within $44\epsilon$ $(\bar{d})$ of an $n$–block independent process when restricted to the tiles, which is all but $\epsilon$ of the time.

Hence $(X_{M_\mathbb{Q}}, \mathcal{P}_k)$ is within $45\epsilon$ of a $n$–block independent process, and hence is almost block independent. The block independent process to which it is close may be though of as the process obtained by independently concatenating $(X_{M_\mathbb{Q}}, \mathcal{P}_k)$ restricted to each of the tiles. □

For each partition $\mathcal{P}$ of $X_{M_\mathbb{Q}}$ arising from a standard partition of $\Sigma$ at time zero, the finite state process $(X_{M_\mathbb{Q}}, \mathcal{P}_k)$ is almost block independent by Corollary 3.5, and hence is finitely determined by a standard argument (see Appendix B of [28] for this extension of the argument of [27] from $\mathbb{Z}$ to $\mathbb{Z}^2$). It follows from [7], Theorem 1.1 that $(X_{M_\mathbb{Q}}, \mathcal{P}_k)$ is measurably isomorphic to a Bernoulli shift.

If $\mathbf{x} \neq \mathbf{y}$ are distinct points in $X_{M_\mathbb{Q}}$ then they differ in some position, so for some $k$ they lie in different atoms of $\mathcal{P}_k^{(\mathbb{Z}^2)}$. Thus the algebra generated by $\mathcal{P}_k^{(\mathbb{Z}^2)}$ increases to the whole $\sigma$–algebra $\mathcal{B}$ modulo null sets. By [20], §III, Theorem 5 (the Monotone Theorem for amenable group actions), we conclude that $(X_{M_\mathbb{Q}}, \alpha^{M_\mathbb{Q}})$ is measurably isomorphic to a Bernoulli shift, which is Theorem 3.3. □

Let $(B, U)$ be some $\mathbb{Z}^2$ action on a Lebesgue space $B$. If $\phi : B \times \mathbb{Z}^2 \to X_M$ is a measurable skewing function, then the skew product $(B \times X_M, U \rtimes_\phi \alpha^M)$ is the $\mathbb{Z}^2$ action defined by

$$(U \rtimes_\phi \alpha^M)_{(n,m)}(b, x) = (U_{(n,m)}(b), \alpha^M_{(n,m)}(x) + \phi_{(n,m)}(b)). \tag{3.10}$$

In order that this define a $\mathbb{Z}^2$ action, $\phi$ must satisfy certain consistency conditions; these may be thought of as follows. The generators $\phi_{(1,0)}$ and $\phi_{(0,1)}$ can be any measurable functions; $\phi_{(0,0)} = 1_{X_M}$. Then the whole cocycle $\phi$ is generated by the following relations:

(1) $\phi_{(n+1,0)} = \alpha^M_{(n,0)}\phi_{(1,0)} + \phi_{(n,0)}$,
(2) $\phi_{(0,m+1)} = \alpha^M_{(0,m)}\phi_{(0,1)} + \phi_{(0,m)}$,
(3) $\phi_{(n,m)} = \alpha^M_{(0,m)}\phi_{(n,0)} + \phi_{(0,m)}$,

**Lemma 3.6.** *Under the assumptions of Theorem 3.3, $(B \times X_M, U \rtimes_\phi \alpha^M)$ is almost block independent relative to the base $(B, U)$.*

*Proof.* We prove this in two ways for the case where $f$ is not constant (the case of $f$ constant is a full group shift and may be dealt with directly). Firstly, we use a technique due to Rudolph (see [4]) to split the skew product. The second method is to notice a rotation invariance in the proof of Theorem 3.3.

Define a new $\mathbb{Z}^2$ action, denoted $U \rtimes_\phi (\alpha^M \times \alpha^M)$ on the product space $B \times X_M \times X_M$ by setting

$$\big(U \rtimes_\phi (\alpha^M \times \alpha^M)\big)_{(n,m)}(b, x_1, x_2) = (U_{(n,m)}(b), \alpha^M_{(n,m)}(x_1) + \phi_{(n,m)}(b), \alpha^M_{(n,m)}(x_2) + \phi_{(n,m)}(b)).$$

The automorphism of the measure space $B \times X_M \times X_M$ given by

$$\theta : (b, x_1, x_2) \mapsto (b, x_1, x_2 - x_1)$$

shows that $U \rtimes_\phi (\alpha^M \times \alpha^M) \cong (U \rtimes_\phi \alpha^M) \times \alpha^M$, since $\theta(U \rtimes_\phi (\alpha^M \times \alpha^M)) = ((U \rtimes_\phi \alpha^M) \times \alpha^M)\theta$. Now $(X_M, \alpha^M)$ is by assumption almost block independent, so the third component $(X_M, \alpha^M)$ is certainly almost block independent relative to the first two components $(B \times X_M, U \rtimes_\phi \alpha^M)$. Now apply the isomorphism $\theta$ to see that in $(B \times X_M \times X_M, U \rtimes_\phi (\alpha^M \times \alpha^M))$, the third component is almost block independent relative to the other two. However, the transformation $x_2 \mapsto \alpha^M(x_2) + \phi(b)$ is completely independent relative to the base $(B, U)$ of the transformation $x_1 \mapsto \alpha^M(x_1) + \phi(b)$. So the third component is almost block independent relative to $(B, U)$. Thus, in $(B \times X_M, U \rtimes_\phi \alpha^M)$, $(X_M, \alpha^M)$ is almost block independent relative to the base $(B, U)$.

For the second argument, let $\mathcal{M} = \mathcal{R}/\langle f\rangle$ where $f$ is a non–constant polynomial, and let $c : X_{M_\mathbb{Q}} \to X_M$ be a measurable section of the quotient map $X_M \to X_{M_\mathbb{Q}}$. Define $\phi' : B \to X_{M_\mathbb{Q}}$ by $\phi'(b) = c(\phi(b))$. Then $\phi'$ determines a skew product action $(B \times X_{M_\mathbb{Q}}, U \rtimes_{\phi'} \alpha^{M_\mathbb{Q}}$ and if this is almost block independent relative to the base then so too is $(B \times X_M, U \rtimes_\phi \alpha^M)$.

Notice that the action on the fibre $X_{M_\mathbb{Q}}$ is affine, so it is sufficient to show the following. If $(s_{(n,m)})$ is any $\mathbb{Z}^2$ sequence of elements of $X_{M_\mathbb{Q}}$, then the process defined by $(\alpha^{M_\mathbb{Q}}_{(n,m)}(x) + s_{(n,m)})$ is almost block independent. From the proof of Theorem 3.3 it is clear that rotation by elements of $X_{M_\mathbb{Q}}$ does not affect the independence of the frequencies $C_{p(\epsilon)}$ across the gap of size $m(\epsilon)$. The same argument as that used above then shows that the skew product $(B \times X_{M_\mathbb{Q}}, U \rtimes_{\phi'} \alpha^{M_\mathbb{Q}}$ is almost block independent relative to the base.

If the polynomial $f$ is constant, say $f = s$, then $(X_M, \alpha^M)$ is a $\mathbb{Z}^2$ group shift on $\mathbb{Z}/s\mathbb{Z}$ (see §3 of [13]). In this case we can show directly that the skew product $(B \times X_M, U \rtimes_\phi \alpha^M)$ measurably splits in the sense of [13], that is, it is isomorphic to the direct product $(B \times X_M, U \times \alpha^M)$ via an isomorphism preserving the base factor. This shows Lemma 3.6 *a fortiori* for this case. The argument is an immediate extension of the single automorphism case described in [13] and [18]; we describe it briefly for completeness.

Let $\omega : B \times X_M \to B \times X_M$ be a map of the form $\omega(b, x) = (b, x + \beta(b))$ where $\beta : B \to X_M$ is measurable. The map $\omega$ is a measurable isomorphism between $(U \rtimes_\phi \alpha^M)$ and the direct product $(U \times \alpha^M)$ if and only if $\beta$ is a measurable solution of the functional equation

$$\phi_{(n,m)}(b) = \alpha^M_{(n,m)}\beta(b) - \beta(U_{(n,m)}(b)) \tag{3.11}$$

for all $b \in B$ and $(n, m) \in \mathbb{Z}^2$.

Now $X_M \cong \widehat{\mathcal{R}/\langle s\rangle} \cong (\mathbb{Z}/s\mathbb{Z})^{\mathbb{Z}^2}$, so $\beta(b) = (\beta^{(i,j)}(b))_{(i,j)\in\mathbb{Z}^2}$ with each $\beta^{(i,j)}$ a measurable map from $B$ to $\mathbb{Z}/s\mathbb{Z}$. Moreover, the action of $\alpha^M$ is the group shift, so the equation (3.11) is equivalent to

$$\phi^{(i,j)}_{(n,m)}(b) = \beta^{(i,j)+(n,m)}(b) - \beta^{(i,j)}(U_{(n,m)}(b)) \tag{3.12}$$

for all $i, j, n, m \in \mathbb{Z}$ and $b \in B$. This equation is readily solved: let $\beta^{(0,0)} = 0$ and inductively define $\beta^{(i,j)}$ for all $i, j \in \mathbb{Z}$. The consistency relations satisfied by $\phi$ ensure that $\beta$ will be consistently defined by (3.12). □

## §4. Expansive actions on compact groups

We now use the algebraic description of all pairs $(X, \alpha)$ where $X$ is a compact abelian group and $\alpha$ is an expansive action of $\mathbb{Z}^d$ on $X$ provided by [10], together with the relativized isomorphism theorem for Bernoulli $\mathbb{Z}^d$ actions ([7]), and a characterisation of the $K$ property from [16] to extend §3 to the general expansive situation. The algebraic techniques used here are closely related to those of [25] and [16].

For a given prime ideal $\mathfrak{q} \subset \mathcal{R}$, an $\mathcal{R}$–module $N$ will be called $\mathfrak{q}$–*elementary* if it has a prime filtration of the form

$$N = N_s \supset N_{s-1} \supset \cdots \supset N_0 = \{0\} \tag{4.1}$$

with succesive quotients $\frac{N_j}{N_{j-1}} \cong \frac{\mathcal{R}}{\mathfrak{q}}$ for each $j = 1, \ldots, s$. Notice that this is a stronger property than being $\mathfrak{q}$–primary. In general, a $\mathfrak{q}$–primary module has a prime filtration with succesive quotients of the form $\frac{\mathcal{R}}{\mathfrak{p}_j}$ where each $\mathfrak{p}_j \supseteq \mathfrak{q}$. This distinction lies at the heart of the topological structure of group automorphisms, but will be seen not to affect the Bernoulli property if it is present. We will return to this point in §6(5).

**Lemma 4.1.** *If $X = X_M$ is a compact abelian group and $\alpha = \alpha^M$ is an expansive action of $\mathbb{Z}^2$ on $X$ with completely positive entropy, then $(X, \alpha)$ is a factor of a finite cartesian product of expansive $\mathbb{Z}^2$ systems with completely positive entropy $(X_{N_j}, \alpha^{N_j})$, $j = 1, \ldots, k$, where each $N_j$ is $\mathfrak{p}_j$–elementary for some prime ideal $\mathfrak{p}_j$ associated to $M$.*

*Proof.* Since $(X, \alpha)$ is expansive, the corresponding $\mathcal{R}$–module $M$ is Noetherian by Theorem 5.2 of [10]. Let the associated primes of $M$ be $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$; each of these is expansive and positive in the sense of [16] by Theorem 3.9 of [25] and Theorem 6.5 of [16]. By the primary decomposition theorem for $M$, there are $\mathfrak{p}_j$–primary modules $L_j \cong \frac{M}{M_j}$, $j = 1, \ldots, n$, with $\bigcap_{j=1}^n L_j = \{0\}$. It follows that the canonical map

$$\phi : M \to L_1 \oplus \cdots \oplus L_n \tag{4.2}$$

is an injective $\mathcal{R}$–module homomorphism. The dual map is therefore a surjective map commuting with the actions:

$$\widehat{\phi} : X_{L_1} \times \cdots \times X_{L_n} \to X_M \tag{4.3}$$

realises $(X_M, \alpha^M)$ as a factor of a direct product of $\mathfrak{p}_j$–primary modules.

To reduce to $\mathfrak{p}_j$–elementary modules, consider a fixed $j$. Choose a prime filtration of $L_j$,

$$L_j = L_j^s \supset L_j^{s-1} \supset \cdots \supset L_j^t \supset \cdots \supset L_j^0 = \{0\} \tag{4.4}$$

with succesive quotients $\frac{L_j^r}{L_j^{r-1}} \cong \frac{\mathcal{R}}{\mathfrak{q}_r}$ for $r = s, \ldots, 1$ where $\mathfrak{q}_r \supsetneq \mathfrak{p}_j$ for $r = s, \ldots, t+1$ and $\mathfrak{q}_r = \mathfrak{p}_j$ for $r = t, \ldots, 1$. Choose a polynomial $h = h_s \ldots h_{t+1}$ with each $h_k \in \mathfrak{q}_k \backslash \mathfrak{p}_j$. The map $\phi_j : L_j \to L_j^t$ given by $\phi_j(a) = ha$ is injective, so the dual map

$$\widehat{\phi}_j : X_{L_j^t} \to X_{L_j} \tag{4.5}$$

realises $(X_{L_j}, \alpha^{L_j})$ as a factor of $(X_{L_j^t}, \alpha^{L_j^t})$. Let $N_j = L_j^t$.

By composing $\widehat{\phi}_1 \times \cdots \times \widehat{\phi}_n$ with $\widehat{\phi}$, we get a factor map

$$\theta : X_{N_1} \times \cdots \times X_{N_n} \to X_M. \tag{4.6}$$

Each of the associated primes of $M$ is expansive and positive so we are done. □

**Theorem 4.2.** *If $X = X_M$ is a compact abelian group and $\alpha = \alpha^M$ is an expansive action of $\mathbb{Z}^2$ on $X$, then $(X, \alpha)$ is measurably isomorphic to a Bernoulli shift if and only if $(X, \alpha)$ has completely positive entropy.*

*Proof.* By Lemma 4.1 we may assume without loss that $M$ is a $\mathfrak{p}$–elementary module for some expansive positive prime ideal $\mathfrak{p}$. Form the prime filtration

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M \tag{4.7}$$

with $\frac{M_i}{M_{i-1}} \cong \frac{\mathcal{R}}{\mathfrak{p}}$ for $i = 1, \dots, s$.

Let $X_i = M_i^{\perp} \subset \widehat{M}$. Then the annihilator of the chain (4.7) is

$$X_M = M_0^{\perp} \supset X_1 \supset X_2 \supset \cdots \supset X_{s-1} \supset X_s = \{0\} \tag{4.8}$$

with $\frac{X_{i-1}}{X_i} \cong \frac{\widehat{\mathcal{R}}}{\mathfrak{p}}$ for $i = 1, \dots, s$. For brevity, say that a closed invariant subgroup $Y$ of $X$ is $\mathcal{B}$ if the action of $\mathbb{Z}^2$ obtained by restricting $\alpha^M$ to $Y$ is measurably isomorphic to a Bernoulli shift. Assume that $X_{l+1}$ is $\mathcal{B}$. Then $X_{l+1} \subset X_l$ is a closed invariant subgroup, so $(X_l, \alpha^M|_{X_l})$ is measurably isomorphic to the skew product of $(X_{l+1}, \alpha^M|_{X_{l+1}})$ with $(\frac{X_l}{X_{l+1}}, \bar{\alpha})$, where $\bar{\alpha}$ is the $\mathbb{Z}^2$ action induced by $\alpha^M|_{X_l}$ on the quotient group $\frac{X_l}{X_{l+1}}$. Now by Lemma 3.6, the skew product $(X_{l+1}, \alpha^M|_{X_{l+1}}) \rtimes (\frac{X_l}{X_{l+1}}, \bar{\alpha})$ is almost block independent relative to the base, so by the relative isomorphism theorem ([7]), it is measurably isomorphic to the direct product $(X_{l+1}, \alpha^M|_{X_{l+1}}) \times (\frac{X_l}{X_{l+1}}, \bar{\alpha})$. Thus, $(X_l, \alpha^M|_{X_l})$ is measurably isomorphic to the direct product of two Bernoulli shifts, and we are done. □

Let $\pi(\alpha)$ denote the Pinsker partition of the $\mathbb{Z}^2$ action $\alpha$: $\pi(\alpha)$ is the supremum of the collection of all finite measurable partitions $\xi$ with $h(\alpha, \xi) = 0$. For a discussion of the Pinsker partition and proof of the result used below, see §6 of [16].

**Corollary 4.3.** *If $X$ is a compact abelian group and $\alpha$ is an expansive action of $\mathbb{Z}^2$ on $X$, then $(X, \alpha)$ is measurably isomorphic to a Bernoulli shift relative to $\pi(\alpha)$.*

*Proof.* As in Theorem 4.2, we may assume that there is a finitely–generated $\mathcal{R}$–module $M$ with $(X, \alpha)$ algebraically conjugate to $(X_M, \alpha^M)$. By Theorem 6.5 of [16], there is a unique submodule $N$ of $M$ with the property that $\pi(\alpha^M)$ is the partition into cosets of the closed invariant subgroup $N^\perp$ of $X_M$. Notice that the dual of $N^\perp$ is $M/N$, while the dual of $X_M/N^\perp$ is $N$. The short exact sequence

$$0 \to N^\perp \cong X_{M/N} \to X_M \cong \widehat{M} \to X_M/N^\perp \cong X_N \to 0$$

shows that $(X_M, \alpha^M)$ is measurably isomorphic to a skew product of $(X_{M/N}, \alpha^{M/N})$ with $(X_N, \alpha^N)$. Since $X_N$ is the factor (in the usual sense) of $X_M$ corresponding to the Pinsker algebra $\pi(\alpha^M)$, the action on each fibre $(X_{M/N}, \alpha^{M/N})$ is $K$. By assumption, the action of $\alpha^M$ on $X_M$ is expansive, so the restriction $\alpha^{M/N}$ acts expansively on the closed invariant subgroup $X_{M/N}$. Now $(X_M, \alpha^M)$ can be broken down as above into a factor of a succesion of affine skew products with $K$ expansive fibre action, so by Lemma 3.6 and the relative isomorphism theorem, $(X_M, \alpha^M)$ is Bernoulli relative to $(X_N, \alpha^N)$. □

## §5. Exponential recurrence

The methods of [15] may be applied directly here to show that the independence property of Lemma 3.4 implies that expansive $K$ systems corresponding to cyclic modules are exponentially recurrent.

Consider a $\mathbb{Z}^2$ action $\alpha$ by measure preserving transformations on the Lebesgue space $(X, \mathcal{B}, \mu)$ and let $R_k = \{(n,m) \in \mathbb{Z}^2 \mid \max\{|n|,|m|\} = k\}$. Let $U \in \mathcal{B}$ have $\mu(U) > 0$, and let $r_U(x)$ denote the least positive integer $k$ with the property that there is a point $(n,m) \in R_k$ with $\alpha_{(n,m)}(x) \in U$. Notice that $r_U(x)$ is finite almost everywhere on $U$ by Poincaré recurrence applied to $\alpha_{(1,0)}$ say. The action $\alpha$ is *exponentially recurrent* if $r_n(U) = \mu\{x \in U \mid r_U(x) = n\}$ decays at a rate $e^{-\lambda n^2}$ for some $\lambda > 0$.

Let $T_1$ and $T_2$ be two actions of $\mathbb{Z}^2$ by homeomorphisms of compact metric spaces $M_1$ and $M_2$, preserving Borel probabilities $\nu_1$ and $\nu_2$ respectively. The two actions are *finitarily isomorphic* if there exist null sets $N_1 \subset M_1$, $N_2 \subset M_2$ and a homeomorphism $\phi : M_1 \backslash N_1 \to M_2 \backslash N_2$ with $\phi T_1 = T_2 \phi$. It is clear that exponential recurrence is an invariant of finitary isomorphism, and an example due to Smorodinsky show that it is not an invariant of measurable isomorphism (see [15] for a discussion of this). The argument of [15], Proposition 1, may be easily modified to show the following.

**Lemma 5.1.** *The Bernoulli $\mathbb{Z}^2$ action $\beta$, defined by the shift on $Y = \{1, \dots, s\}^{\mathbb{Z}^2}$, with measure $m = p^{\mathbb{Z}^2}$, (where $p$ is the $(p_1, \dots, p_s)$ measure, $p_i > 0$, $\sum p_i = 1$) and metric*

$$\rho(\mathbf{a}, \mathbf{b}) = \sum_{(n,m) \in \mathbb{Z}^2} |a(n,m) - b(n,m)| 2^{-\max\{|n|,|m|\}}$$

*has exponential recurrence.*

*Proof.* If $U \subset Y$ is open, then $U$ contains a cylinder set

$$V = \{\mathbf{a} \in Y \mid a(i,j) = b(i,j) \text{ for } (i,j) \in (k_1, k_2) + B_k\}$$

where $B_k = [0, k-1] \times [0, k-1] \cap \mathbb{Z}^2$. Then

$$m\{\mathbf{a} \in Y \mid r_U(\mathbf{a}) = n+1\} \leq m(\bigcap_{i,j=1}^{[\frac{n}{k}]} \beta_{(i,j)}(Y \backslash V) = \prod_{i,j=1}^{[\frac{n}{k}]} m(\beta_{(i,j)}(Y \backslash V)) \leq [m(Y \backslash V)^{1/k}]^{n^2}$$

as required. □

**Theorem 5.2.** *If $\alpha^{\mathcal{R}/\mathfrak{p}}$ is expansive and has completely positive entropy, then it has exponential recurrence.*

*Proof.* This is a straightforward extension of the proof of Theorem 1 in Lind's paper [15]. Let $X = X_{\mathcal{R}/\mathfrak{p}}$, and let $\mu$ denote Haar measure on $X$. Let $U \subset X$ be an open subset with $\mu(U) \in (0,1)$ (if $\mu(U) = 1$ then $r_n(U)$ decays to zero immediatly). Since trigonometric polynomials are dense in $L^2(X)$, we may find a function $f_U : X \to \mathbb{R}$ of the form

$$f_U(x) = \prod_{i=-l}^{l} \prod_{j=-l}^{j=l} f_{ij}(x_{ij})$$

where each $f_{ij}$ is a trigonometric polynomial on $\Sigma$,

$$f_{ij}(x) = \sum_{k=-N^2}^{N^2} c_{ij}^{(k)} \chi_{k/N}(x),$$

having the property that $f_U(x) \geq 1$ if $x \notin U$ and $\int_X f_U d\mu = \lambda < 1$. Let $m \geq 1 + \frac{1}{\log \kappa} \log\bigl(\frac{2N^2}{\kappa - 1}\bigr)$ where $\kappa = \kappa(\mathfrak{p})$ is defined as in Lemma 3.1. Then, putting $p = [\frac{n}{2l+m+1}]$,

$$\begin{aligned}
\mu\{x \in X \mid r_U(x) = n+1\} &\leq \mu\bigl(\{ \bigcap_{a,b=1}^{p} \alpha_{(2l+m+1)(a,b)}(X \backslash U)\bigr) \\
&= \int_X \prod_{a,b=1}^{p} \alpha_{(2l+m+1)(a,b)} 1_{X \backslash U} d\mu \\
&\leq \int_X \prod_{a,b=1}^{p} \alpha_{(2l+m+1)(a,b)} f_U d\mu \\
&= \prod_{a,b=1}^{p} \int_X \alpha_{(2l+m+1)(a,b)} f_U d\mu \\
&= \lambda^{p^2} \\
&\leq \bigl(\lambda^{(1/(2l+m+1))^2}\bigr)^{n^2},
\end{aligned}$$

showing exponential recurrence. □

The above shows that the finitary invariant of exponential recurrence does not provide an obstruction to the following conjecture.

**Conjecture 5.3.** *If $\alpha^{\mathcal{R}/\mathfrak{p}}$ is expansive and has completely positive entropy, then it is finitarily isomorphic to a Bernoulli shift.*

For $\mathbb{Z}$ actions, hyperbolic toral automorphisms are finitarily isomorphic to Bernoulli shifts since they have Markov partitions: mixing Markov shifts of the same entropy are finitarily isomorphic by [9], and a Bernoulli shift is a mixing Markov shift.

## §6. Remarks

(1) The Bernoullicity of $(X_l, \alpha^M|_{X_l})$ shows that the skew product

$$(X_{l+1}, \alpha^M|_{X_{l+1}}) \rtimes (\frac{X_l}{X_{l+1}}, \bar{\alpha})$$

*measurably splits*: it is Bernoulli and has the same entropy as the direct product (by the addition formula for $\mathbb{Z}^d$ actions, [16] Appendix B), which is a direct product of two Bernoulli shifts, so it is measurably isomorphic to the direct product. The proof that skew products with ergodic group automorphisms satisfying weak specification algebraically split (Theorem 4.2 of [14]) extends to $\mathbb{Z}^2$ actions with weak specification (see [23] for the definition of weak specification in the $\mathbb{Z}^2$ setting for finite state space). Does an expansive $\mathbb{Z}^2$ action corresponding to a cyclic module satisfy weak specification? If not, is the probabilistic version of weak specification afforded by Lemma 3.4 sufficient to show that skew products split?

(2) In [16] it is conjectured that for $\mathbb{Z}^d$ actions on compact abelian groups, the $K$ property is equivalent to being measurably isomorphic to a Bernoulli $\mathbb{Z}^d$ shift. Our methods readily extend from $\mathbb{Z}^2$ to $\mathbb{Z}^d$, but it should be emphasised that the assumption of expansiveness is an enormous simplification. Firstly, the class of compact groups that can carry an expansive action is very limited: in particular, expansiveness implies that the action has the Descending Chain Condition (see Definition 3.1 and Theorem 5.2 of [10]). Secondly, expansiveness provides a uniform hyperbolicity, expressed in our setting by the number $\kappa > 1$. A special case of this conjecture, namely the non–expansive action corresponding to the module $\mathcal{R} \,/\, \langle\, 1 \,+\, x \,+\, y \,\rangle$, is shown to be measurably isomorphic to a Bernoulli shift in [28], by an *ad hoc* argument. For the general non–expansive case, more delicate arguments will be needed, as in the case of a single ergodic automorphism ([13], [8], [18]). Beyond this conjecture there is the following question: for $\mathbb{Z}^d$ Markov shifts, is the $K$ property equivalent to being measurably isomorphic to a Bernoulli shift? In the case $d = 1$ this is well known to be so ([19]). For $d > 1$, a partial result has been shown by Rosenthal ([22]): a higher dimensional $K$ Markov shift on a finite alphabet has the weak Pinsker property.

(3) The familiar phenomenon of measurable isomorphism without topological conjugacy for toral automorphisms (see for instance [10], Examples 6.6(3)) has a higher dimensional analogue. The only difference is that there will typically be infinitely many topological conjugacy classes in a fixed measurable isomorphism class, because the relevant ideal class

group is infinite. Let $L = \mathcal{R}^2/A\mathcal{R}^2$, $M = \mathcal{R}^2/B\mathcal{R}^2$ and $N = \mathcal{R}^2/C\mathcal{R}^2$ where

$$A = \begin{bmatrix} 3+x & 0 \\ 0 & 3+y \end{bmatrix}, \quad B = \begin{bmatrix} 3+x & 1 \\ 0 & 3+y \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3+x & 2 \\ 0 & 3+y \end{bmatrix}.$$

By [16] (Example 5.6), $h(\alpha^L) = h(\alpha^M) = h(\alpha^N) = \log 9$. Moreover, the three systems are expansive and $K$. We deduce that they are measurably isomorphic to each other. In order to see that they are not topologically conjugate, it is enough to notice that

$$\frac{L}{\langle 1-x, 1-y\rangle L} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad \frac{M}{\langle 1-x, 1-y\rangle M} \cong \mathbb{Z}/16\mathbb{Z}$$

and

$$\frac{N}{\langle 1-x, 1-y\rangle N} \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

which shows that $L$, $M$ and $N$ are not isomorphic as $\mathcal{R}$–modules. By [25], Corollary 4.3, $(X_L, \alpha^L)$, $(X_M, \alpha^M)$ and $(X_N, \alpha^N)$ are therefore not topologically conjugate.

(4) The first part of the proof of Theorem 3.3, dealing with the case where $\mathfrak{p} = \langle f \rangle$ and $f(x, y) = F(x^a y^b)$ is a polynomial in a single monomial, may be seen in a much more general setting. Let $(Y, \mathcal{C}, \nu)$ be a Lebesgue space and let $S_0$ be an invertible measure preserving transformation of $Y$. Let $(X, \mathcal{B}, \mu) = \prod_{i\in\mathbb{Z}}(Y_i, \mathcal{C}_i, \nu_i)$ where $Y_i = Y$, $\mathcal{C}_i = \mathcal{C}$ and $\nu_i = \nu$ for all $i \in \mathbb{Z}$. Now define a pair of commuting invertible measure preserving transformations $S$, $T$ on $(X, \mathcal{B}, \mu)$ by $(Tx)_n = x_{n+1}$, the full shift, and $(Sx)_n = S_0(x_n)$. The $\mathbb{Z}^2$ action $\beta$ generated by $S$ and $T$ has been studied by Conze [3] and Kaminski [6]. We observed that $\beta$ is Bernoulli when $S_0$ is Bernoulli. Conze (example (3) after Theorem 2.3 in [3]) has shown that $h(S_0) = h(\beta)$. Kaminski has shown that $\beta$ is $K$ if and only if $S_0$ is $K$ (Theorem 3 in [6]), and that $\beta$ is Bernoulli if and only if $S_0$ is (Theorem 5 in [6]).

(5) The distinction between $\mathfrak{p}$–primary and $\mathfrak{p}$–elementary modules can be seen in a very familiar dynamical setting; (3) above provides an illustration of this. Here we describe the simplest possible situation where this phenomenon arises, namely Williams' example (see Examples 6.6(4) of [10]). Let $\alpha_A$ and $\alpha_B$ be the automorphisms of the 2–torus dual to the matrices $A = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ respectively. Then $\alpha_A$ and $\alpha_B$ are measurably isomorphic but not topologically conjugate because $A$ and $B$ are not conjugate in $GL(2, \mathbb{Z})$. The corresponding $S = \mathbb{Z}[x^{\pm 1}]$–modules are

$$M_A = \frac{S^2}{(A - xI)S^2} \text{ and } M_B = \frac{S^2}{(B - xI)S^2}.$$

Both modules are $\langle x^2 - 4x - 1\rangle$–primary, but only the first is $\langle x^2 - 4x - 1\rangle$–elementary. In fact $M_A \cong S/\langle x^2 - 4x - 1\rangle$. The module $M_B$ has a prime filtration of the form $M_B =$

$M \supset N \supset \{0\}$, with first quotient $N/\{0\} \cong S/\langle x^2 - 4x - 1\rangle$ and second quotient $M/N \cong S/\langle x^2 - 4x - 1\rangle + \mathfrak{p}$ for some ideal $\mathfrak{p} \neq \langle x^2 - 4x - 1\rangle$. Corresponding to this filtration, there is a measurable realisation of $(\mathbb{T}^2, \alpha_B)$ as an extension of $(\mathbb{T}^2, \alpha_A)$ by the system corresponding to $S/\langle x^2 - 4x - 1\rangle + \mathfrak{p}$. It follows that this extension cannot measurably split, since if it did the Bernoulli dynamical system $(\mathbb{T}^2, \alpha_B)$ would have a zero entropy factor. This contrasts with extensions in the reverse order: a fundamental result of Lind (the Splitting Theorem of [13], §2) states that an extension of anything by an ergodic compact group automorphism splits.

## References


[1] D. Boyd, *Kronecker's theorem and Lehmer's problem for polynomials in several variables*, Journal of Number Theory **13** (1981), 116–121.

[2] H. Chu, *Some results on affine transformations of compact groups*, Inventiones mathematicæ **28** (1975), 161–183.

[3] J. P. Conze, *Entropie d'un groupe abélien de transformations*, Z. Wahrsch. verw. Geb. **25** (1972), 11–30.

[4] J. Feldman, D. J. Rudolph & C. C. Moore, *Affine extensions of a Bernoulli shift*, Trans. Amer. Math. Soc. **257** (1980), 171–191.

[5] M. Fried, *On Hilbert's irreducibility theorem*, Journal of Number Theory **6** (1974), 211–231.

[6] B. Kaminski, *Mixing properties of two–dimensional dynamical systems with completely positive entropy*, Bull. L'Acad. Polonaise des Sci. **27** (1980), 453–463.

[7] J. Kammeyer, *The isomorphism theorem for relatively finitely determined $\mathbb{Z}^n$ actions*, Israel Journal of Mathematics **69** (1990), 117–127.

[8] Y. Katznelson, *Ergodic automorphisms of $\mathbb{T}^n$ are Bernoulli shifts*, Israel Journal of Mathematics **10** (1971), 186–95.

[9] M. Keane & M. Smorodinsky, *The finitary isomorphism theorem for Markov shifts*, Bull. Amer. Math. Soc. **1** (1979), 436–438.

[10] B. Kitchens & K. Schmidt, *Automorphisms of compact groups*, Ergodic Theory & Dynamical Systems **9** (1989), 691–735.

[11] S. Lang, *Algebra*, Addison–Wesley, 2nd. ed., Reading, 1984.

[12] D. A. Lind, *Ergodic automorphisms of the infinite torus are Bernoulli*, Israel Journal of Mathematics **17** (1974), 162–168.

[13] ———, *The structure of skew products with ergodic group automorphisms*, Israel Journal of Mathematics **28** (1977), 205–248.

[14] ———, *Split skew products, a related functional equation, and specification*, Israel Journal of Mathematics **30** (1978), 236–254.

[15] ———, *Ergodic group automorphisms are exponentially recurrent*, Israel Journal of Mathematics **41** (1982), 313–320.

[16] D. Lind, K. Schmidt & T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Inventiones mathematicæ **101** (1990), 593–629.

[17] D. A. Lind & T. Ward, *Automorphisms of solenoids and p–adic entropy*, Ergodic Theory & Dynamical Systems **8** (1988), 411–419.

[18] G. Miles & R. K. Thomas, *Generalised torus automorphisms are Bernoullian*, Studies in Probability and Ergodic Theory, Adv. in Math. Supp. Studies **2** (1978), 231–249.

[19] D. S. Ornstein, *Mixing Markov shifts of kernel type are Bernoulli*, Advances in Math. **10** (1973), 143–146.

[20] D. S. Ornstein & B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, Journal d'Analyse Mathématique **48** (1987), 1–141.

[21] V. A. Rokhlin, *Metric properties of endomorphisms of compact commutative groups*, Amer. Math. Soc. Transl.(3) **64** (1967), 244–252.
[22] A. Rosenthal, *Weak Pinsker property and Markov processes*, Ann. Inst. Henri Poincaré **22** (1986), 347–369.
[23] D. Ruelle, *Statistical mechanics on a compact set with $\mathbb{Z}^v$ action satisfying expansiveness and specification*, Trans. Amer. Math. Soc. **185** (1973), 237–251.
[24] K. Schmidt, *Mixing automorphisms of compact groups and a theorem by Kurt Mahler*, Pacific Journal of Mathematics **137** (1989), 371–385.
[25] ———, *Automorphisms of compact abelian groups and affine varieties*, Proc. London Math. Soc. **61** (1990), 480–496.
[26] P. Shields, *The theory of Bernoulli shifts*, Univ. of Chicago Press, 1973.
[27] ———, *Almost block independence*, Z. Wahrsch. verw. Geb. **49** (1979), 119–123.
[28] T. Ward, *Almost block independence for the three dot $\mathbb{Z}^2$ dynamical system*, Israel Journal of Math., to appear (1992).
[29] A. Weil, *Basic Number Theory*, 3rd ed., Springer, New York, 1982.

Department of Mathematics
Ohio State University
Columbus
OH 43210
USA.
e–mail: tomward@ function.mps.ohio–state.edu