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Mixing actions of the rationals

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Abstract. We study mixing properties of algebraic actions of $\mathbb{Q}^d$, showing in particular that prime mixing $\mathbb{Q}^d$ actions on connected groups are mixing of all orders, as is the case for $\mathbb{Z}^d$-actions. This is shown using a uniform result on the solution of S-unit equations in characteristic zero fields due to Evertse, Schlickewei and W. Schmidt. In contrast, algebraic actions of the much larger group $\mathbb{Q}^*$ are shown to behave quite differently, with finite order of mixing possible on connected groups.

Mixing properties of $\mathbb{Z}^d$-actions by automorphisms of a compact metrizable abelian group are quite well understood. Roughly speaking, the picture has three facets. First, the one-to-one correspondence between such actions and countably generated modules over the integral group ring $R_d = \mathbb{Z}[\mathbb{Z}^d]$ of the acting group $\mathbb{Z}^d$ due to Kitchens and Schmidt [**6**] allows any mixing problem to be reduced to the case corresponding to a cyclic module of the form R_d/P for a prime ideal $P \subset R_d$. Second, in the connected case $P \cap \mathbb{Z} = \{0\}$, Schmidt and Ward [**13**] showed that mixing implies mixing of all orders by relating the mixing property to S-unit equations and exploiting a deep result of Schlickewei on solutions of such equations [**11**] (see also [**4**] and [**14**]). Finally, in the totally disconnected case $P \cap \mathbb{Z} = p\mathbb{Z}$ for some rational prime p, Masser [**9**] has shown that the order of mixing is determined by the mixing behaviour of shapes, reducing the problem—in principle—to an algebraic one.

Our purpose here is to show how some of this changes for algebraic actions of infinitely generated abelian groups. The algebra is more involved, so for simplicity we restrict attention to the simplest extreme examples: actions of $\mathbb{Q}^{\times}_{>0}$ (isomorphic to the direct sum of countably many copies of $\mathbb{Z}$) and actions of $\mathbb{Q}^d$ (which is a torsion extension of $\mathbb{Z}^d$). These groups are the simplest non-trivial examples chosen from the 'dual' categories of free abelian and infinitely divisible groups in the sense of MacLane [**8**]. The algebraic difficulties mean we cannot present the complete picture found for $\mathbb{Z}^d$-actions, and the emphasis is partly on revealing or suggestive examples. Some topological properties (expansiveness and closed invariant sets) for actions of infinitely generated abelian groups have been studied by Berend [**1**] and Miles [**10**].

Let α be an action of a countable abelian group Γ on a probability space $(X, \mathcal{B}, \mu)$. For a sequence (γ_n) in Γ, write $\gamma_n \to \infty$ if, for every finite set $F \subset \Gamma$, there is an N for which $n > N$ implies that $\gamma_n \notin F$. The action α is said to be *mixing on r sets* if, for any sets

$$A_1, \ldots, A_r \in \mathcal{B},$$
$$\mu(\alpha_{\gamma_1} A_1 \cap \cdots \cap \alpha_{\gamma_r} A_r) \to \mu(A_1) \cdots \mu(A_r) \quad \text{as } \gamma_s - \gamma_t \to \infty \text{ for } s \neq t.$$

The *order of mixing* $\mathcal{M}(\alpha)$ of α is the largest value of r for which α is mixing on r sets, and α is said to be *mixing of all orders*, denoted $\mathcal{M}(\alpha) = \infty$, if it is mixing on r sets for all r.

1. *Algebraic actions*

Just as for algebraic $\mathbb{Z}^d$-actions (see Schmidt's monograph [**12**]), Pontryagin duality gives a description of Γ-actions by automorphisms of compact abelian groups in terms of modules over the ring $\mathbb{Z}[\Gamma]$. If α is a Γ-action by automorphisms of X, then the character group $M = \widehat{X}$ inherits the structure of a $\mathbb{Z}[\Gamma]$-module via the dual automorphisms $\widehat{\alpha_\gamma}$ for $\gamma \in \Gamma$; conversely any $\mathbb{Z}[\Gamma]$-module M defines a compact abelian group $X_M = \widehat{M}$ carrying a dual Γ-action α_M. Write $\lambda = \lambda_X$ for the Haar measure on X.

A module is said to be *cyclic* if it is singly generated as a module, so takes the form $\mathbb{Z}[\Gamma]/I$ for some ideal $I \subset \mathbb{Z}[\Gamma]$, and the dual Γ-action will be said to be *prime* (or *radical*) if the module takes the form $\mathbb{Z}[\Gamma]/P$ for some prime (respectively radical) ideal $P \subset \mathbb{Z}[\Gamma]$.

The rings that arise here are $R_\infty = \mathbb{Z}[\mathbb{Q}^\times_{>0}]$, corresponding to actions of $\mathbb{Q}^\times_{>0}$, and $R_{\mathbb{Q}^d} = \mathbb{Z}[\mathbb{Q}^d]$, corresponding to actions of $\mathbb{Q}^d$. Notice that these are wildly different rings; for example, R_∞ has infinite Krull dimension, while $R_{\mathbb{Q}^d}$ has Krull dimension $d+1$. Both are non-Noetherian rings.

2. *Actions of* $\mathbb{Q}^d$

The main result of [**13**] says that, for an algebraic $\mathbb{Z}^d$-action α on a *connected* group,

$$\mathcal{M}(\alpha) > 1 \implies \mathcal{M}(\alpha) = \infty.$$

The same property turns out to also hold for the simplest actions of $\mathbb{Q}^d$. This is shown in Theorem 2.1 below, which is stated in a slightly more general setting. The *rational rank* of an abelian group is the maximal number of elements which are linearly independent over $\mathbb{Z}$. Thus, $\mathbb{Q}^d$ and $\mathbb{Z}^d$ have rational rank d, while $\mathbb{Q}^\times$ does not have finite rational rank. If Γ has rational rank d, then $R_\Gamma = \mathbb{Z}[\Gamma]$ has Krull dimension $d + 1$, and it may or may not be Noetherian depending on the divisibility properties of Γ.

THEOREM 2.1. *Let* α *be an algebraic action of a countable torsion-free group* Γ *of finite rational rank corresponding to a cyclic module* R_Γ/I *with* $I \cap \mathbb{Z} = \{0\}$ *and* I *a radical ideal. Then*

$$\mathcal{M}(\alpha) > 1 \implies \mathcal{M}(\alpha) = \infty.$$

Proof. The result in [**13**] depended on a bound for the number of solutions to S-unit equations over number fields [**11**] with certain uniformity properties; as shown in [**12**, §VIII.27] it is enough to have a qualitative bound for any characteristic zero field. Here we use instead the following deep result from [**5**]. □

THEOREM. (Evertse, Schlickewei and W. Schmidt) *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and let Γ be a finitely generated multiplicative subgroup of $(\mathbb{K}^\times)^r$ with rank d. For fixed*

$$a_1, \dots, a_r \in \mathbb{K}^\times,$$

the number of solutions $(x_1, \dots, x_r) \in \Gamma$ of the equation

$$a_1 x_1 + \cdots + a_r x_r = 1$$

for which no proper subsum vanishes is bounded above by

$$\exp((6r)^{3r}(d+1)).$$

Returning to the proof of Theorem 2.1, write $\alpha = \alpha_{R_\Gamma/I}$, $X = X_{R_\Gamma/I}$ and $\lambda = \lambda_X$. Assume that the action is not mixing on r sets for some $r > 1$. It follows that there are measurable sets $A_1, \dots, A_r \subset X$ and a sequence of r-tuples

$$(q_1^{(j)}, \dots, q_r^{(j)})_{j \geq 1} \in \Gamma^r$$

such that $q_s^{(j)} - q_t^{(j)} \to \infty$ as $j \to \infty$ for $s \neq t$, for which

$$\lambda(\alpha_{q_1^{(j)}} A_1 \cap \cdots \cap \alpha_{q_r^{(j)}} A_r) \not\to \prod_{s=1}^{r} \lambda(A_s) \quad \text{as } j \to \infty. \tag{1}$$

By approximating the indicator functions of the sets appearing in (1) and applying the orthogonality relations for characters on X, it follows that there are non-zero elements $a_1, \dots, a_r \in R_\Gamma/I$ with the property that

$$\widehat{\alpha_{q_1^{(j)}}}(a_1) + \cdots + \widehat{\alpha_{q_r^{(j)}}}(a_r) = 0 \quad \text{for infinitely many } j. \tag{2}$$

First assume that the ideal I is a prime ideal P. Embed R_Γ/P into a field $\mathbb{K}$ of characteristic zero (this is possible because P is prime and $P \cap \mathbb{Z} = \{0\}$); denote by

$$x \mapsto u_1^{q_{s,1}^{(j)}} \cdots u_d^{q_{s,d}^{(j)}} x = \boldsymbol{u}^{\boldsymbol{q}_s^{(j)}} x$$

the automorphism of $\mathbb{K}$ defined by the automorphism $\widehat{\alpha_{\boldsymbol{q}_s^{(j)}}}$ of R_Γ/P (writing $\boldsymbol{u}^{\boldsymbol{q}}$ for $u_1^{q_1} \cdots u_d^{q_d}$, where $\boldsymbol{q} = (q_1, \dots, q_d) \in \Gamma^d$). Then equation (2) implies that

$$\boldsymbol{u}^{\boldsymbol{q}_1^{(j)}} a_1 + \cdots + \boldsymbol{u}^{\boldsymbol{q}_r^{(j)}} a_r = 0 \quad \text{for infinitely many } j \tag{3}$$

holds in $\mathbb{K}$. Rearranging, this gives an equation

$$(-a_2/a_1)\boldsymbol{u}^{\boldsymbol{q}_2^{(j)} - \boldsymbol{q}_1^{(j)}} + \cdots + (-a_r/a_1)\boldsymbol{u}^{\boldsymbol{q}_r^{(j)} - \boldsymbol{q}_1^{(j)}} = 1 \quad \text{for infinitely many } j. \tag{4}$$

Assume that the rational rank of Γ is $d < \infty$ and let (Γ_n) denote a sequence of subgroups with the following properties:

- for each n, $\Gamma_n \cong \mathbb{Z}^d$;
- $\Gamma_1 \subset \Gamma_2 \subset \cdots$;
- $\Gamma = \bigcup_{n \geq 1} \Gamma_n$.

Let A_n denote the set of solutions to (4) with each $\boldsymbol{q}_s^{(j)} \in \Gamma_n$ for which no subsum vanishes (thus A_n is a subset of the set of values of j for which (4) holds, $A_1 \subset A_2 \subset \cdots$ and $\bigcup_{n\geq 1} A_n$ is the set of all j for which (4) holds). We may assume without loss of generality that the map

$$j \mapsto (\boldsymbol{q}_2^{(j)} - \boldsymbol{q}_1^{(j)}, \dots, \boldsymbol{q}_r^{(j)} - \boldsymbol{q}_1^{(j)})$$

is injective. Since Γ_n is isomorphic to $\mathbb{Z}^d$, the theorem of Evertse, Schlickewei and W. Schmidt applies to show that

$$|A_n| \leq \exp((6r)^{3r}(d+1)). \tag{5}$$

Any finite bound in (5) would suffice to prove the main result in [**13**], namely Theorem 2.1 for $\mathbb{Z}^d$-actions. Here the additional uniformity in the theorem of Evertse, Schlickewei and Schmidt is needed: the bound in (5) is independent of n, so it follows that equation (4) holds without the vanishing subsum for only finitely many j. Thus, there exists a set $S \subsetneq \{2, \dots, r\}$ such that

$$\sum_{s\in S}(-a_s/a_1)\boldsymbol{u}^{\boldsymbol{q}_s^{(j)} - \boldsymbol{q}_1^{(j)}} = 0 \quad \text{for infinitely many } j. \tag{6}$$

The identity (6) shows that α is not mixing on $|S| < r$ sets. Thus, for any r, $1 < r < \infty$,

$$\mathcal{M}(\alpha) \leq r \implies \mathcal{M}(\alpha) < r,$$

so

$$\mathcal{M}(\alpha) > 1 \implies \mathcal{M}(\alpha) = \infty.$$

This proves Theorem 2.1 when $I = P$ is prime. Assume now that I is a radical ideal and that the system corresponding to the module R_Γ/I is not mixing on r sets for some $r > 1$ but is mixing. As before, this means there is a sequence of r-tuples

$$(\boldsymbol{q}_1^{(j)}, \dots, \boldsymbol{q}_r^{(j)})_{j\geq 1} \in \Gamma^r$$

such that $\boldsymbol{q}_s^{(j)} - \boldsymbol{q}_t^{(j)} \to \infty$ as $j \to \infty$ for $s \neq t$, for which the equation

$$\boldsymbol{u}^{\boldsymbol{q}_1^{(j)}} a_1 + \cdots + \boldsymbol{u}^{\boldsymbol{q}_r^{(j)}} a_r = 0 \quad \text{for infinitely many } j \tag{7}$$

holds in the ring R_Γ/I. Let

$$U = \langle\!\langle c, a_1, \boldsymbol{u}^{\boldsymbol{q}} - 1 \mid c \in \mathbb{Z}\backslash\{0\}, \boldsymbol{q} \in \Gamma^d \rangle\!\rangle$$

where $\langle\!\langle A \rangle\!\rangle$ denotes the multiplicative group generated by A. This is a multiplicative set, and we claim that $U \cap I = \emptyset$. If, for some $\boldsymbol{q} \in \Gamma^d$, $\boldsymbol{u}^{\boldsymbol{q}} - 1 \in I$, then $\alpha_{R_\Gamma/I}$ is not mixing, which is excluded by hypothesis. Since I is radical, it follows that $(\boldsymbol{u}^{\boldsymbol{q}} - 1)^m \notin I$ for all $m \geq 1$. If

$$(\boldsymbol{u}^{\boldsymbol{q}} - 1)^m b \in I \text{ for some } \boldsymbol{q} \in \Gamma^d, m \in \mathbb{Z} \text{ and } b \notin I,$$

then we must have $(\boldsymbol{u}^{\boldsymbol{q}} - 1)^m b^m \in I$. Since I is radical, this implies that $(\boldsymbol{u}^{\boldsymbol{q}} - 1)b \in I$, so in R_Γ/I

$$b + I = \boldsymbol{u}^{j\boldsymbol{q}} b + I \quad \text{for all } j \geq 1,$$

which again contradicts the assumption that $\alpha_{R_\Gamma/I}$ is mixing. By induction, no product of the form

$$a_1^{\ell}(\boldsymbol{u}^{\boldsymbol{q}_1}-1)^{m_1}\cdots(\boldsymbol{u}^{\boldsymbol{q}_k}-1)^{m_k}$$

can be in I. Since R_Γ/I has no additive torsion, it follows that $U\cap I=\emptyset$. By finding a maximal ideal above I in the localization of R_Γ at U, we may find a prime ideal $P\supset I$ with the property that $P\cap U=\emptyset$ (see [**3**, Proposition 2.11]). Equation (7) drops via the map $x\to x+P=\overline{x}$ to a non-trivial equation

$$\boldsymbol{u}^{\boldsymbol{q}_1^{(j)}}\overline{a_1}+\cdots+\boldsymbol{u}^{\boldsymbol{q}_r^{(j)}}\overline{a_r}=0 \quad \text{for infinitely many } j, \tag{8}$$

in which not all the coefficients have vanished. It follows that the sequence

$$(\boldsymbol{q}_1^{(j)},\ldots,\boldsymbol{q}_r^{(j)})_{j\geq 1}\in\Gamma^r$$

witnesses non-mixing on r sets in the prime system corresponding to R_Γ/P, and the argument above shows that this is only possible if $\mathcal{M}(\alpha_{R_\Gamma/P})=1$. By [**12**, Theorem 1.6], this would require that there be a non-mixing element in Γ, which is impossible by the choice of U. □

Theorem 2.1 does not hold without the assumption that the rational rank is finite—see Theorem 2.3. It also cannot hold without the assumption of connectedness. In the notation of the proof of Theorem 2.1, if the cyclic module R_Γ/I has additive torsion, then there is an element $a+I$ and an integer $k>0$ with $ka\in I$. For a sufficiently large n, $a\in R_{\Gamma_n}$, and setting $J=I\cap R_{\Gamma_n}$ induces an inclusion

$$R_{\Gamma_n}/J\subset R_\Gamma/I,$$

which dualizes to show that the original action restricted to Γ_n has a factor corresponding to $\mathbb{Z}[\Gamma_n]/J$ for some ideal J with $J\cap\mathbb{Z}\neq\{0\}$. Since $\Gamma_n\cong\mathbb{Z}^d$, finite non-trivial order of mixing is possible unless there are additional conditions on the ideal (see Schmidt [**12**, Ch. VIII]).

One of the most striking features of $\mathbb{Z}^d$-actions for $d>1$ (as opposed to actions of $\mathbb{Z}$) is that simple examples may have order of mixing satisfying

$$1<\mathcal{M}<\infty.$$

This was first pointed out by Ledrappier [**7**], who showed that the $\mathbb{Z}^2$-action α corresponding to the module $\mathbb{Z}[u_1^{\pm1},u_2^{\pm1}]/\langle 2,1+u_1+u_2\rangle$ has

$$\mathcal{M}(\alpha)=2;$$

the papers [**2**] and [**15**] give related constructions for any specified order of mixing. The same construction will give algebraic $\mathbb{Q}^d$-actions for any $d>1$ with any specified order of mixing.

Theorem 2.1 shows that $\mathbb{Q}^d$-actions on connected groups also behave much like $\mathbb{Z}^d$-actions. Further evidence for the essential similarity of algebraic $\mathbb{Q}$-actions and $\mathbb{Z}$-actions is provided by the next result.

PROPOSITION 2.2. *Any mixing prime algebraic $\mathbb{Q}$-action is mixing of all orders.*

Proof. Let the action correspond to the module $R_{\mathbb{Q}}/P$ for some prime ideal P. If the action is not mixing on r sets for some $r \geq 2$ then as before we find an equation

$$u^{q_1^{(j)}} a_1 + \cdots + u^{q_r^{(j)}} a_r = 0 \quad \text{for infinitely many } j \tag{9}$$

which holds in some field $\mathbb{K}$ (not necessarily of characteristic zero) with

$$q_s^{(j)} - q_t^{(j)} \to \infty \text{ in } \mathbb{Q} \text{ as } j \to \infty \text{ for } s \neq t.$$

If $P \cap \mathbb{Z} \neq \{0\}$ and P contains a non-trivial polynomial, then the variable u satisfies an algebraic equation over a finite field, so must be a root of unity in $\mathbb{K}$. This precludes mixing. So $P = \langle p \rangle$ for some rational prime p, and the $\mathbb{Q}$-action is a full $\mathbb{Q}$ shift on p symbols, so is mixing of all orders.

If $P \cap \mathbb{Z} = \{0\}$ then we are in the setting of Theorem 2.1, which shows that the system is mixing of all orders. □

Actions of the much larger group $\mathbb{Q}^{\times}_{>0}$ behave quite differently. A simple argument using similar ideas shows the following.

THEOREM 2.3. *Let α be an algebraic action of $\mathbb{Q}^{\times}_{>0}$ corresponding to a module R_∞/P with $P \cap \mathbb{Z} = \{0\}$ and P a* finitely generated *prime ideal. Then $\mathcal{M}(\alpha) > 1 \implies \mathcal{M} = \infty$.*

The ring R_∞ has ideals that are not finitely generated, and the next example shows that these behave differently.

Example 2.4. Consider the natural action of $\mathbb{Q}^{\times}_{>0}$ on $\widehat{\mathbb{Q}}$ (that is, the rational $r \in \mathbb{Q}^{\times}_{>0}$ acts via the automorphism dual to $x \mapsto rx$ on $\mathbb{Q}$). This is mixing, since the equation $ax+b=0$ in $\mathbb{Q}$ determines x. On the other hand, the action is not mixing on three sets since

$$1 \cdot (1) + r \cdot (-1) + (r-1) \cdot (1) = 0$$

for all r in $\mathbb{Q}$, and as elements of $\mathbb{Q}^{\times}_{>0}$, r, $r-1$ and $r/(r-1)$ all go to infinity as $r \to \infty$.

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