

A NOTE ON MIXING PROPERTIES OF INVERTIBLE EXTENSIONS

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ABSTRACT. The natural invertible extension \tilde{T} of an \mathbb{N}^d -action T has been studied by Lacroix. He showed that \tilde{T} may fail to be mixing even if T is mixing for $d \geq 2$. We extend this observation by showing that if T is mixing on $(k+1)$ sets then \tilde{T} is in general mixing on no more than k sets, simply because \mathbb{N}^d has a corner. Several examples are constructed when $d = 2$: (i) a mixing T for which $\tilde{T}^{(n,m)}$ has an identity factor whenever $n \cdot m < 0$; (ii) a mixing T for which \tilde{T} is rigid but $\tilde{T}^{(n,m)}$ is mixing for all $(n, m) \neq (0, 0)$; (iii) a T mixing on 3 sets for which \tilde{T} is not mixing on 3 sets.

1. INVERTIBLE EXTENSIONS

Let T be a measure-preserving \mathbb{N}^d -action on the probability space (X, \mathcal{B}, μ) . Such an action may be thought of as the natural shift-action on the space

$$\left\{ (x_{\mathbf{n}}) \in X^{\mathbb{N}^d} \mid x_{\mathbf{n}} = T^{\mathbf{n}}x_0 \ \forall \ \mathbf{n} \in \mathbb{N}^d \right\};$$

the projection π_0 onto the zero coordinate shows that T is isomorphic to the shift action, so we identify them. The natural invertible extension of T is constructed in [3], and may be thought of as the natural shift action \tilde{T} on

$$\tilde{X} = \left\{ (x_{\mathbf{n}}) \in X^{\mathbb{Z}^d} \mid x_{\mathbf{n}+\mathbf{m}} = T^{\mathbf{n}}x_{\mathbf{m}} \ \forall \ \mathbf{m} \in \mathbb{Z}^d, \ \mathbf{n} \in \mathbb{N}^d \right\}.$$

For any sets $F \subset \mathbb{Z}^d$, $G \subset \mathbb{N}^d$ let $\tilde{\pi}_F : \tilde{X} \rightarrow X^F$, $\pi_G : X \rightarrow X^G$ denote the projections. The set \tilde{X} is a probability space with σ -algebra $\tilde{\mathcal{B}}$ and measure $\tilde{\mu}$ defined as follows. The σ -algebra $\tilde{\mathcal{B}}$ is the smallest one containing all sets of the form

$$A_{\mathbf{m}, C} = \left\{ (x_{\mathbf{n}}) \in \tilde{X} \mid x_{\mathbf{m}} \in C \right\}$$

for $\mathbf{m} \in \mathbb{Z}^d$ and $C \in \mathcal{B}$, and $\tilde{\mu}$ is defined via the Daniell-Kolmogorov consistency theorem (see [1, Theorem 1, Chapter IV.6]) from the requirement that $\tilde{\mu}(A_{\mathbf{m}, C}) =$

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$\mu(C)$. Notice that for $\{\mathbf{m}_1, \dots, \mathbf{m}_s\} \subset \mathbb{Z}^d$ and sets $C_1, \dots, C_s \in \mathcal{B}$, if $\ell \in \mathbb{N}^d$ has $\ell + \mathbf{m}_j \in \mathbb{N}^d$ for all j , then

$$\tilde{\mu} \left(\{(x_{\mathbf{n}}) \in \tilde{X} \mid x_{\mathbf{m}_j} \in C_j \text{ for } j = 1, \dots, s\} \right)$$

and

$$\mu \left(T^{-(\ell + \mathbf{m}_1)}(C_1) \cap \dots \cap T^{-(\ell + \mathbf{m}_s)}(C_s) \right)$$

coincide. We shall use the following notation: if $\tilde{B} \subset \tilde{X}$ is measurable with respect to $\tilde{\pi}_{\mathbb{N}^d}^{-1}(\mathcal{B})$ then let $B = \pi_{\mathbb{N}^d}(\tilde{B}) \subset X$. Let $\tilde{T}_+ = \tilde{T}|_{\mathbb{N}^d}$ be the \mathbb{N}^d -action obtained by restricting the invertible extension to $\mathbb{N}^d \subset \mathbb{Z}^d$. The projection $\tilde{\pi}_{\mathbb{N}^d} : \tilde{X} \rightarrow X^{\mathbb{N}^d}$ realizes T as a factor of \tilde{T}_+ . If the generators of the original \mathbb{N}^d -action are invertible, then $\tilde{\pi}_{\mathbb{N}^d}$ is an isomorphism.

Definition. The \mathbb{N}^d -action T is mixing on $(k + 1)$ sets if for any $A_0, A_1, \dots, A_k \in \mathcal{B}$,

$$(1) \quad \mu \left(A_0 \cap T^{-\mathbf{n}_1} A_1 \cap \dots \cap T^{-\mathbf{n}_k} A_k \right) \longrightarrow \mu(A_0) \dots \mu(A_k)$$

as $\mathbf{n}_i \rightarrow \infty$, $\mathbf{n}_i - \mathbf{n}_j \rightarrow \infty$ for $i \neq j$. Here $\rightarrow \infty$ means leaving finite subsets of \mathbb{N}^d , and $\mathbf{n}_i - \mathbf{n}_j \rightarrow \infty$ means that if $\mathbf{n}_i + \ell = \mathbf{n}_j + \mathbf{m}$ for $\ell, \mathbf{m} \in \mathbb{N}^d$ then ℓ or $\mathbf{m} \rightarrow \infty$.

If $k = 1$ then mixing on $(k + 1)$ sets is called mixing. A \mathbb{Z}^d -action T is said to be mixing on $(k + 1)$ sets if (1) holds with the vectors \mathbf{n}_j now allowed to lie in \mathbb{Z}^d .

Lacroix [3] has shown, **inter alia**, that T mixing does not imply that \tilde{T} will be mixing, with an example in which $\tilde{T}^{\mathbf{n}}$ has an identity factor for some $\mathbf{n} \in \mathbb{Z}^d \setminus \mathbb{N}^d$. We extend this by proving the following theorem and illustrating it with several examples in $d = 2$, including one in which T is mixing but $\tilde{T}^{\mathbf{n}}$ has an identity factor for every $\mathbf{n} \in \mathbb{Z}^2 \setminus (\mathbb{N}^2 \cup -\mathbb{N}^2)$.

The ‘‘corner’’ $0 \in \mathbb{N}^d$ is distinguished because it must (unlike the \mathbb{Z}^d case) appear in the expression (1) above. This forces the order of mixing to drop.

Theorem. *If the \mathbb{N}^d -action T is mixing on $(k + 1)$ sets, then the invertible extension \tilde{T} is mixing on k sets.*

Proof. Assume T is mixing on $(k + 1)$ sets for some $k \geq 1$. Let $\tilde{B}_1, \dots, \tilde{B}_k$ be sets measurable with respect to $\tilde{\pi}_{S(N)}^{-1}(\mathcal{B})$ where $S(N) = [-N, N]^d \cap \mathbb{Z}^d$. Write $\mathbf{N} = (N, N, \dots, N)$. Let $\mathbf{m}_2(n), \dots, \mathbf{m}_k(n)$ be integer vectors with $\mathbf{m}_i(n) \rightarrow \infty$ and $\mathbf{m}_i(n) - \mathbf{m}_j(n) \rightarrow \infty$ as $n \rightarrow \infty$ for each $i \neq j$. For each $n = 1, 2, \dots$ let $\ell(n) \in \mathbb{N}^d$ be chosen so that $\ell(n) \rightarrow \infty$, $\mathbf{n}_j(n) = \mathbf{m}_j(n) + \ell(n) \rightarrow \infty$ as $n \rightarrow \infty$, and $\mathbf{n}_j(n) \in \mathbb{N}^d$ for all n .

Notice by construction we have $\ell(n) \rightarrow \infty$, $\mathbf{n}_j(n) \rightarrow \infty$, $\ell(n) - \mathbf{n}_j(n) \rightarrow \infty$, and for each $i \neq j$, $\mathbf{n}_j(n) - \mathbf{n}_i(n) \rightarrow \infty$. It follows that if n is large enough to ensure

that $\ell(n) - \mathbf{N} \in \mathbb{N}^d$, then we have

$$\begin{aligned}
 & \tilde{\mu} \left(\tilde{B}_1 \cap \tilde{T}^{-\mathbf{m}_2(n)} \tilde{B}_2 \cap \dots \cap \tilde{T}^{-\mathbf{m}_k(n)} \tilde{B}_k \right) \\
 &= \tilde{\mu} \left(\tilde{T}^{-\ell(n)} \tilde{B}_1 \cap \tilde{T}^{-\mathbf{n}_2(n)} \tilde{B}_2 \cap \dots \cap \tilde{T}^{-\mathbf{n}_k(n)} \tilde{B}_k \right) \\
 &= \tilde{\mu} \left(\tilde{X} \cap \tilde{T}^{-\ell(n)} \tilde{B}_1 \cap \tilde{T}^{-\mathbf{n}_2(n)} \tilde{B}_2 \cap \dots \cap \tilde{T}^{-\mathbf{n}_k(n)} \tilde{B}_k \right) \\
 &= \tilde{\mu} \left(\tilde{X} \cap \tilde{T}^{-(\ell(n)-\mathbf{N})} \left(\tilde{T}^{-\mathbf{N}} \tilde{B}_1 \right) \cap \tilde{T}^{-(\mathbf{n}_2(n)-\mathbf{N})} \left(\tilde{T}^{-\mathbf{N}} \tilde{B}_2 \right) \cap \dots \right. \\
 &\quad \left. \cap \tilde{T}^{-(\mathbf{n}_k(n)-\mathbf{N})} \left(\tilde{T}^{-\mathbf{N}} \tilde{B}_k \right) \right) \\
 &= \mu \left(X \cap T^{-(\ell(n)-\mathbf{N})} C_1 \cap T^{-(\mathbf{n}_2(n)-\mathbf{N})} C_2 \cap \dots \cap T^{-(\mathbf{n}_k(n)-\mathbf{N})} C_k \right) \\
 &\rightarrow \mu(C_1) \dots \mu(C_k) \\
 &= \tilde{\mu}(\tilde{T}^{-\mathbf{N}} \tilde{B}_1) \dots \tilde{\mu}(\tilde{T}^{-\mathbf{N}} \tilde{B}_k) = \tilde{\mu}(\tilde{B}_1) \dots \mu(\tilde{B}_k),
 \end{aligned}$$

where $C_j = \tilde{\pi}_{\mathbb{N}^d}(\tilde{T}^{-\mathbf{N}} \tilde{B}_j)$ for each j . It follows that \tilde{T} is mixing on k sets. □

2. EXAMPLES

Example 1. If $X = \mathbb{T}$, the additive group, and the \mathbb{N}^2 -action T is generated by $T^{(1,0)}x = T^{(0,1)}x = 2x \pmod 1$, then it is clear that T is mixing while \tilde{T} cannot be mixing since $\tilde{T}^{(1,-1)}$ is the identity map on $\tilde{X} = \widehat{\mathbb{Z}[\frac{1}{2}]}$.

This example is of course not a faithful action — in [3] a faithful example is given, generated by the toral endomorphisms dual to the matrices $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$.

Example 2. The previous example may be refined to produce a mixing \mathbb{N}^2 -action T with the property that $\tilde{T}^{(n,m)}$ has an identity factor for every pair n, m with opposite signs. Let X be the infinite torus $\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \times \dots$. Let $S_a : \mathbb{T} \rightarrow \mathbb{T}$ denote the map $S_a(x) = ax \pmod 1$, and let

$$S^* = S_2 \times S_4 \times S_8 \times S_{16} \times \dots,$$

and

$$S_a^\infty = S_a \times S_a \times S_a \times S_a \times \dots$$

Throughout the indicated correspondence between positions in infinite products holds. Define the \mathbb{N}^2 -action T by the two generators

$$T^{(1,0)} = S^* \times S^* \times S^* \times S^* \times \dots$$

and

$$T^{(0,1)} = S_2^\infty \times S_4^\infty \times S_8^\infty \times S_{16}^\infty \times \dots$$

Then T is a mixing \mathbb{N}^2 -action on X : it is enough to check that for any pair of non-trivial characters $\chi_0, \chi_1 \in \widehat{X}$ the character $\chi_0 + \widehat{T}^{(n,m)}\chi_1$ is non-trivial for large $(n, m) \in \mathbb{N}^2$ and this is clear since each character is finitely supported.

The invertible extension \tilde{T} is obtained as follows. Let $\Sigma = \widehat{\mathbb{Z}[\frac{1}{2}]}$ be the solenoid, and $\tilde{S}_a : \Sigma \rightarrow \Sigma$ the endomorphism dual to multiplication by a , invertible if a is a power of 2. Then the generators of \tilde{T} are simply given by placing tildes on the definition of the generators of T , and they act on $\tilde{X} = \Sigma^\infty$. For any pair $(n, m) \in \mathbb{Z}^2 \setminus (\mathbb{N}^2 \cup -\mathbb{N}^2)$, the map $\tilde{T}^{(n,m)}$ has a non-trivial identity factor and therefore cannot be mixing: to see this, notice that $\tilde{T}^{(|n|,0)}$ acts in the $|m|$ th position in each of the indicated factors as $\times 2^{|nm|}$, while $\tilde{T}^{(0,|m|)}$ acts in the $|n|$ th position in the $S_{2^{|m|}}^\infty$ factor as $\times 2^{|nm|}$ in each copy of Σ .

Example 3. The opposite extreme to the previous example is given by the Gaussian construction of Ferenczi and Kaminski [2]: for numbers $\alpha > 0, \beta > 0$ with $1, \alpha, \beta$ rationally independent they construct a two-dimensional Gaussian action T with covariance function

$$R(n, m) = \frac{\sin(2\pi(n\alpha + m\beta))}{2\pi(n\alpha + m\beta)}.$$

If (n_j, m_j) is a sequence with $n_j\alpha + m_j\beta \rightarrow 0$ as $j \rightarrow \infty$ then for large j we must have $n_j \cdot m_j < 0$. Along such a sequence $R(n_j, m_j) \rightarrow 1$ so the action is rigid, showing that the \mathbb{Z}^2 -action is not mixing. On the other hand, if $(n_j, m_j) \rightarrow \infty$ in \mathbb{N}^2 or $-\mathbb{N}^2$ then it is clear that $R(n_j, m_j) \rightarrow 0$ showing that the \mathbb{N}^2 -action T_+ is mixing.

For the next example, recall that a finite set F with $(0, 0) \in F \subset \mathbb{Z}^2$ (or \mathbb{N}^2) is a **mixing shape** for a λ -preserving \mathbb{Z}^2 -action \tilde{T} (resp. \mathbb{N}^2 -action T) if

$$\lim_{k \rightarrow \infty} \lambda \left(\bigcap_{\mathbf{n} \in F} T^{-k\mathbf{n}} B_{\mathbf{n}} \right) = \prod_{\mathbf{n} \in F} \lambda(B_{\mathbf{n}})$$

for all measurable sets $B_{\mathbf{n}}$.

Example 4. Using ideas from algebraic dynamical systems, as described for example in [5], we exhibit an \mathbb{N}^2 -action T which is mixing on three sets for which the extension \tilde{T} is not mixing on three sets. The example is a modification of Ledrappier’s original example, [4]. Let \mathbb{F}_2 denote the field with two elements, let

$$X = \left\{ \mathbf{x} \in \mathbb{F}_2^{\mathbb{N}^2} \mid x_{(n-1,m+1)} + x_{(n,m)} + x_{(n+1,m)} = 0 \ \forall (n, m) \in \mathbb{N}^2 \right\},$$

and define the \mathbb{N}^2 -action T to be the shift action on X . We claim that T is mixing on three sets. To see this, work in the dual group $\widehat{X} = \mathbb{F}_2/\langle y + x + x^2 \rangle$, with the \mathbb{N}^2 action being generated by the endomorphisms dual to multiplication by x and y . The map $f(x, y) \mapsto f(x, x + x^2)$ identifies \widehat{X} with $\mathbb{F}_2[x]$, with the generators now being multiplication by x and by $x + x^2$. Using Fourier analysis on the group X (see for example [5, Section 27]) it is enough to show that for any $a, b, c \in \mathbb{N}$ and $\epsilon_a, \epsilon_b, \epsilon_c \in \mathbb{F}_2$ the equation

$$\epsilon_a x^a + \epsilon_b x^{n_1} y^{m_1} x^b + \epsilon_c x^{n_2} y^{m_2} = 0$$

for $(n_1, m_1), (n_2, m_2) \in \mathbb{N}^2$ requires that the points $(n_1, m_1), (n_2, m_2), (0, 0)$ cannot be far apart or the coefficients $\epsilon_a, \epsilon_b, \epsilon_c$ are zero. Using the identity $y = x + x^2$, the equation becomes

$$\epsilon_a x^a + \epsilon_b (x^{b+n_1+m_1} + \dots + x^{b+n_1+2m_1}) + \epsilon_c (x^{b+n_2+m_2} + \dots + x^{b+n_2+2m_2}) = 0.$$

If (n_1, m_1) and (n_2, m_2) are far from the origin then we see that $\epsilon_a = 0$, and if (n_1, m_1) and (n_2, m_2) are far from each other then we see that $\epsilon_b = \epsilon_c = 0$.

The natural extension \tilde{T} has $\{(-1, 1), (0, 0), (1, 0)\}$ as a non-mixing shape since in the group

$$\tilde{X} = \left\{ \mathbf{x} \in \mathbb{F}_2^{\mathbb{N}^2} \mid x_{(n-1, m+1)} + x_{(n, m)} + x_{(n+1, m)} = 0 \ \forall (n, m) \in \mathbb{Z}^2 \right\}$$

the relation $x_{(-2^n, 2^n)} = x_{(0, 0)} + x_{(2^n, 0)}$ holds for all n .

It is not clear how to construct examples along the lines of Example 4 with the property that T is mixing on k sets while \tilde{T} is not mixing on k sets for each $k \geq 1$: see Remark 28.12 in [5] for what is known.

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