REPRESENTATION ZETA FUNCTIONS OF NILPOTENT GROUPS AND GENERATING FUNCTIONS FOR WEYL GROUPS OF TYPE B

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Abstract. We study representation zeta functions of finitely generated, torsion-free nilpotent groups which are groups of rational points of unipotent group schemes over rings of integers of number fields. Using the Kirillov orbit method and $\mathfrak{p}$-adic integration, we prove rationality and functional equations for almost all local factors of the Euler products of these zeta functions. We further give explicit formulae, in terms of Dedekind zeta functions, for the zeta functions of class-2-nilpotent groups obtained from three infinite families of group schemes, generalizing the integral Heisenberg group. As an immediate corollary, we obtain precise asymptotics for the representation growth of these groups, and key analytic properties of their zeta functions, such as meromorphic continuation. We express the local factors of these zeta functions in terms of generating functions for finite Weyl groups of type B. This allows us to establish a formula for the joint distribution of three functions, or "statistics", on such Weyl groups. Finally, we compare our explicit formulae to $\mathfrak{p}$-adic integrals associated to relative invariants of three infinite families of prehomogeneous vector spaces.

1. Introduction and statement of main results.

1.1. Background and summary. Let G be a group and denote, for $n \in \mathbb{N}$, by $r_n(G)$ the number of isomorphism classes of n-dimensional irreducible complex representations of G. The group G is called (*representation*) *rigid* if $r_n(G)$ is finite for all $n \in \mathbb{N}$. If G is rigid and the numbers $r_n(G)$ grow at most polynomially, a fruitful approach to the study of these numbers is to encode them into a Dirichlet generating function, which is called the representation zeta function of G. A variety of tools from complex analysis, algebraic geometry, model theory and combinatorics is available to investigate these zeta functions, and thus to shed light on the arithmetic and asymptotic properties of the sequence $(r_n(G))$; see, for instance, [2, 4, 29]. Throughout this paper we use the term "representations" to refer to complex representations. In the context of topological groups, we only consider continuous representations.

In the current paper we study representation zeta functions associated to finitely generated, torsion-free nilpotent groups (so-called $\mathcal{T}$-groups). Such groups are not rigid. Indeed, a non-trivial $\mathcal{T}$-group has infinitely many representations of dimension 1. However, $\mathcal{T}$-groups are "rigid up to twisting" by 1-dimensional representations. More precisely, let G be a group and let ρ and σ be irreducible representations of G. One calls ρ and σ *twist-equivalent* if there exists a 1-dimensional

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representation χ of G such that $\rho \cong \chi \otimes \sigma$. If G is a topological group, we demand in addition that χ be continuous. This defines an equivalence relation on the set of irreducible representations of G, whose classes are called *twist-isoclasses*. For a group G and $n \in \mathbb{N}$ we denote by $\widetilde{r}_n(G)$ the number of twist-isoclasses of G of dimension n. It is known that if G is a $\mathcal{T}$-group and $n \in \mathbb{N}$ then there exists a finite quotient $G(n)$ of G such that every n-dimensional representation of G is twist-equivalent to one that factors through $G(n)$. In particular, the number $\widetilde{r}_n(G)$ is finite for all $n \in \mathbb{N}$; see [30, Theorem 6.6]. The *representation zeta function* of a $\mathcal{T}$-group G is defined to be the Dirichlet generating function

$$\zeta_G(s) := \sum_{n=1}^{\infty} \widetilde{r}_n(G) n^{-s}, \tag{1.1}$$

where s is a complex variable. The sequence $(\widetilde{r}_n(G))$ grows polynomially, and thus this series converges on a complex right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$, for some $\alpha \in \mathbb{R}$; see Lemma 2.1. The *abscissa of convergence* $\alpha(G)$ of $\zeta_G(s)$, that is, the infimum of such α, gives the precise degree of polynomial growth. More precisely, if G is non-trivial, $\alpha(G)$ is the smallest value such that $\sum_{n=1}^{N} \widetilde{r}_n(G) = O(N^{\alpha(G)+\varepsilon})$ for every $\varepsilon \in \mathbb{R}_{>0}$.

The zeta function $\zeta_G(s)$ has an Euler product, indexed by the rational primes. Indeed, one has

$$\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{G,p}(s), \tag{1.2}$$

where $\zeta_{G,p}(s) := \sum_{i=0}^{\infty} \widetilde{r}_{p^i}(G) p^{-is}$, for each prime number p. This Euler product simply reflects the facts that every representation of G is twist-equivalent to one that factors through a finite quotient, and that finite nilpotent groups are direct products of their Sylow p-subgroups. Much deeper lies the fact, proved by Hrushovski and Martin, that all the factors in (1.2) are rational functions in the parameter p^{-s}; see [20, Theorem 8.4]. Another deep result establishes functional equations of almost all of the local factors "upon inversion of p"; see [43, Theorem D].

The only explicitly computed examples of representation zeta functions of $\mathcal{T}$-groups in print prior to the current paper are formulae for the zeta function of the Heisenberg group $\mathbf{H}(\mathcal{O})$ of upper-unitriangular 3×3-matrices over the ring of integers $\mathcal{O}$ of a number field K of degree at most 2; cf. [33] for $K = \mathbb{Q}$, and [14, Theorem 1.1] for K a quadratic number field. These examples agree with the formula

$$\zeta_{\mathbf{H}(\mathcal{O})}(s) = \frac{\zeta_K(s-1)}{\zeta_K(s)} = \prod_{\mathfrak{p}} \frac{1-q^{-s}}{1-q^{1-s}}, \tag{1.3}$$

where $\zeta_K(s)$ is the Dedekind zeta function of K, $\mathfrak{p}$ ranges over the non-zero prime ideals of $\mathcal{O}$, and $q = |\mathcal{O}/\mathfrak{p}|$. In particular, we have $\zeta_{\mathbf{H}(\mathbb{Z})}(s) = \sum_{n=1}^{\infty} \varphi(n) n^{-s}$, where φ is the Euler totient function. Ezzat conjectured in [14] that (1.3) holds for arbitrary number fields K. This is in fact implied by one of our main results; cf. Theorem B. Note that for $G = \mathbf{H}(\mathcal{O})$ the Euler product in (1.3) is finer than the product (1.2), and that all the local factors are rational in q^{-s}. We will show that these facts, too, are special cases of more general phenomena.

The $\mathcal{T}$-groups studied in the current paper are obtained from unipotent group schemes over rings of integers of number fields. From now on, let $\mathcal{O}$ be the ring of integers of a number field K. Let $\mathbf{G}$ be a unipotent group scheme defined over $\mathcal{O}$; see Section 2.1. Then $\mathbf{G}(\mathcal{O})$ is a $\mathcal{T}$-group. Given a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, we denote by $\mathcal{O}_\mathfrak{p}$ the completion of $\mathcal{O}$ at $\mathfrak{p}$, and by p the residue field characteristic of $\mathcal{O}_\mathfrak{p}$. In Proposition 2.2 we establish the Euler factorization

$$\zeta_{\mathbf{G}(\mathcal{O})}(s) = \prod_{\mathfrak{p}} \zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s), \tag{1.4}$$

indexed by the non-zero prime ideals of $\mathcal{O}$, with local factors given by

$$\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s) = \sum_{i=0}^{\infty} \widetilde{r}_{p^i}(\mathbf{G}(\mathcal{O}_\mathfrak{p})) p^{-is}.$$

The factorization (1.4) reflects the fact that unipotent groups have the Congruence Subgroup Property and the strong approximation property. Note that it refines (1.2).

The Euler product (1.4) is similar to the Euler product satisfied by representation zeta functions of arithmetic subgroups of semisimple algebraic groups; cf. [29, Proposition 1.3]. The archimedean factors present in the latter are absent in the realm of nilpotent groups, reflecting the fact that every representation of a $\mathcal{T}$-group is twist-equivalent to one which factors over a finite quotient. The non-archimedean factors in the context of semisimple groups have been studied using techniques from model theory and $\mathfrak{p}$-adic integration; see, for instance, [1, 2, 22].

The following summarizes the main results of the paper and some of its motivation. On the one hand, we set up a general framework for studying representation zeta functions of $\mathcal{T}$-groups obtained from group schemes over $\mathcal{O}$, which in turn are associated to $\mathcal{O}$-Lie lattices. The principal tools here are the Kirillov orbit method and $\mathfrak{p}$-adic integration, which enable us to analyze the local factors of Euler products of the form (1.4). In particular, we derive formulae which are uniform under extensions of $\mathcal{O}$, and prove rationality and functional equations for almost all local factors; see Theorem A.

On the other hand, we study three infinite families of $\mathcal{T}$-groups of nilpotency class 2 for which we carry out this general analysis explicitly and prove more precise results. Our approach allows us to compute the zeta functions of these groups as finite products involving translates of Dedekind zeta functions; see Theorem B.

This shows, in particular, that in these cases all of the local factors in (1.4) are rational both in q and in q^{-s}, and satisfy the functional equations which Theorem A asserts only for almost all local factors. Using our formulae it is easy to read off the zeta functions' key analytic properties, such as abscissae of convergence, meromorphic continuation, and location and order of the poles; see Corollary 1.3. The formulae for the zeta functions also imply precise asymptotic formulae for the representation growth of the relevant groups.

We do not expect the strong regularity properties displayed by the zeta functions in Theorem B to hold in general, not even in nilpotency class 2. Nevertheless, the interesting problem arises to characterize the groups for which they do hold.

Originally our interest in the three specific families of $\mathcal{T}$-groups studied in Theorem B arose out of an analogy with $\mathfrak{p}$-adic integrals associated to reduced irreducible prehomogeneous vector spaces with relative invariants; see Section 6. These are complex vector spaces on which algebraic groups act with Zariski dense orbits. Three infinite families of such prehomogeneous vector spaces are given by $n \times n$-matrices, symmetric matrices and antisymmetric matrices, respectively. The relative invariants in these cases are the determinant or, in the case of antisymmetric matrices, the Pfaffian; the associated $\mathfrak{p}$-adic integrals are Igusa's local zeta functions associated to these polynomials. These well-known integrals are of particular interest as they are cases in which the Bernstein-Sato polynomial conjecture is known to hold. This conjecture connects the real parts of the poles of Igusa's local zeta function with the zeros of the integrand's Bernstein-Sato polynomial. We record in this paper an intriguing analogy between the representation zeta functions computed in Theorem B and the $\mathfrak{p}$-adic integrals associated to the above-mentioned prehomogeneous vector spaces. In particular, the local pole spectra of the former are obtained from the pole spectra of the latter by a simple translation by integers which turn out to be the (global) abscissae of convergence of the relevant zeta functions. Our observations give rise to the general question to what degree the local pole spectra of zeta functions of groups reflect geometric invariants like zeros of Bernstein-Sato polynomials. This points to a possible analogue of the Bernstein-Sato polynomial conjecture for zeta functions of groups.

Our explicit "multiplicative" formulae given in Theorem B are proved using "additive" formulae given in Theorem C. The bridge between the two is given by an identity which is reminiscent of the q-multinomial theorem in the theory of basic hypergeometric series; see Proposition 1.5. The present paper provides a natural motivation for this identity in the context of zeta functions of groups; see also Remark 4.3. We show that the polynomials appearing in Theorem C have a rich combinatorial structure: we express them as generating functions for statistics on finite Weyl groups of type B, also known as hyperoctahedral groups. As an application, we prove a formula for the joint distribution of three statistics on Weyl groups of type B; see Proposition 1.7. It would be interesting to know whether other types of Weyl groups occur in this framework, and to study the resulting analogues of our results.

1.2. Uniformity, rationality and functional equations. In Section 2.1 we describe a class of group schemes defined by nilpotent $\mathcal{O}$-Lie lattices. By an $\mathcal{O}$-Lie lattice we mean a free and finitely generated $\mathcal{O}$-module, together with an antisymmetric, bi-additive form satisfying the Jacobi identity. Let Λ be a nilpotent $\mathcal{O}$-Lie lattice of nilpotency class c, and write Λ' for the derived Lie lattice $[\Lambda,\Lambda]$. If Λ satisfies the condition $\Lambda' \subseteq c!\Lambda$ it gives rise to a unipotent group scheme $\mathbf{G}_\Lambda$ over $\mathcal{O}$ via the Hausdorff series. When the residue characteristic p is odd or when $p=2$ and $c \geq 4$, there exists a Kirillov orbit method to describe the irreducible representations of groups of the form $\mathbf{G}_\Lambda(\mathcal{O}_\mathfrak{p})$ in terms of co-adjoint orbits; see Section 2.2. In the case where $c=2$ we give an unconditional construction of a unipotent group scheme associated to Λ which coincides with $\mathbf{G}_\Lambda$ if $\Lambda' \subseteq 2\Lambda$, and describe a Kirillov orbit method for $\mathbf{G}_\Lambda(\mathcal{O}_\mathfrak{p})$ which holds for all primes; see Section 2.4. In any case, whenever the Kirillov orbit method applies it allows us to describe local representation zeta functions in terms of Poincaré series and suitable $\mathfrak{p}$-adic integrals. The first main result of the paper establishes universal formulae for the generic factors in Euler products of the form (1.4) for groups of the form $\mathbf{G}_\Lambda(\mathcal{O}_L)$, where $\mathcal{O}_L$ is the ring of integers in a finite extension L of the number field K. Let $d = \dim_K(\Lambda' \otimes_\mathcal{O} K)$.

THEOREM A. *There exist a finite set S of prime ideals of $\mathcal{O}$, $t \in \mathbb{N}$, and a rational function $R(X_1,\dots,X_t,Y) \in \mathbb{Q}(X_1,\dots,X_t,Y)$ such that, for every prime ideal $\mathfrak{p}$ of $\mathcal{O}$ with $\mathfrak{p} \notin S$, the following is true. There exist algebraic integers $\lambda_1,\dots,\lambda_t$, depending on $\mathfrak{p}$, such that, for all finite extensions $\mathfrak{O}$ of $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$ one has*

$$\zeta_{\mathbf{G}_\Lambda(\mathfrak{O})}(s) = R(\lambda_1^f,\dots,\lambda_t^f,q^{-fs}),$$

where q denotes the residue field cardinality of $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$, and $f = f(\mathfrak{O},\mathfrak{o})$ is the relative degree of inertia. In particular, the local factor $\zeta_{\mathbf{G}_\Lambda(\mathfrak{O})}(s)$ is a rational function in q^{-fs}. Furthermore, the following functional equation holds:

$$\zeta_{\mathbf{G}_\Lambda(\mathfrak{O})}(s)\big|_{\substack{q\to q^{-1}\\ \lambda_i\to\lambda_i^{-1}}} = q^{fd}\zeta_{\mathbf{G}_\Lambda(\mathfrak{O})}(s). \tag{1.5}$$

Remark 1.1. The statement of Theorem A is analogous to [3, Theorem A]. In fact, the proof of A leans heavily on the methods developed in [3]. We note, however, that [3, Theorem A] applies only to certain pro-p subgroups of the groups featuring in the Euler factors, whereas Theorem A yields a formula for almost all factors in (1.4). The functional equation (1.5) refines the statement of [43, Theorem D].

The proof of Theorem A is found in Section 2.3.

1.3. Groups of types F, G and H, and multiplicative formulae. Much of the present paper is concerned with the representation zeta functions of $\mathcal{T}$-groups obtained from three specific infinite families of group schemes. These arise from class-2-nilpotent $\mathbb{Z}$-Lie lattices, and each family generalizes a different aspect of the Heisenberg group scheme $\mathbf{H}$.

Definition 1.2. Let $n \in \mathbb{N}$ and $\delta \in \{0,1\}$. We define the following nilpotent $\mathbb{Z}$-Lie lattices of class 2:

$$\begin{aligned}\mathcal{F}_{n,\delta} &= \big\langle x_k, y_{ij} \mid [x_k, y_{ij}], [x_i, x_j] - y_{ij},\ 1 \leq k \leq 2n+\delta, 1 \leq i < j \leq 2n+\delta \big\rangle,\\ \mathcal{G}_n &= \big\langle x_k, y_{ij} \mid [x_k, y_{ij}], [x_i, x_{n+j}] - y_{ij},\ 1 \leq k \leq 2n,\ 1 \leq i,j \leq n \big\rangle,\\ \mathcal{H}_n &= \big\langle x_k, y_{ij} \mid [x_k, y_{ij}], [x_i, x_{n+j}] - y_{ij}, [x_j, x_{n+i}] - y_{ij},\\ &\qquad\qquad 1 \leq k \leq 2n,\ 1 \leq i \leq j \leq n \big\rangle.\end{aligned}$$

Note that all these Lie lattices are isomorphic to quotients of the free class-2-nilpotent Lie rings generated by the x_k. In fact, $\mathcal{F}_{n,\delta}$ is isomorphic to the free class-2-nilpotent Lie ring on $x_1, \dots, x_{2n+\delta}$. In any case, the elements x_k, y_{ij} yield $\mathbb{Z}$-bases for the respective Lie lattices.

Let $F_{n,\delta}$, G_n and H_n denote the unipotent group schemes over $\mathbb{Z}$ associated to the Lie lattices $\mathcal{F}_{n,\delta}$, $\mathcal{G}_n$, and $\mathcal{H}_n$, respectively. We call groups of the form $F_{n,\delta}(\mathcal{O})$, $G_n(\mathcal{O})$ and $H_n(\mathcal{O})$ groups of type F, G and H, respectively. The groups $F_{n,\delta}(\mathbb{Z})$ are the free class-2-nilpotent groups on $2n+\delta$ generators. Note that $F_{1,0}(\mathcal{O}) = G_1(\mathcal{O}) = H_1(\mathcal{O}) = \mathbf{H}(\mathcal{O})$, the Heisenberg group over $\mathcal{O}$.

Apart from being natural generalizations of $\mathbf{H}(\mathcal{O})$, the groups of type F, G and H are of interest as their zeta functions are close analogues of zeta integrals associated to relative invariants of irreducible prehomogeneous vector spaces. This connection is explored in Section 6. In our second main result we give explicit formulae for the zeta functions of groups of type F, G and H, in terms of Dedekind zeta functions. For $n \in \mathbb{N}$, we set $m = \lfloor n/2 \rfloor$ and $\varepsilon = n - 2m \in \{0,1\}$, so that $n = 2m + \varepsilon$.

THEOREM B. *Let $n = 2m + \varepsilon \in \mathbb{N}$, $\delta \in \{0,1\}$ and let K be a number field with ring of integers $\mathcal{O}$. Then*

$$\zeta_{F_{n,\delta}(\mathcal{O})}(s) = \prod_{i=0}^{n-1} \frac{\zeta_K(s - 2(n+i+\delta)+1)}{\zeta_K(s-2i)}, \tag{1.6}$$

$$\zeta_{G_n(\mathcal{O})}(s) = \prod_{i=0}^{n-1} \frac{\zeta_K(s-n-i)}{\zeta_K(s-i)}, \tag{1.7}$$

$$\zeta_{H_n(\mathcal{O})}(s) = \frac{\zeta_K(s-n)}{\zeta_K(s)} \prod_{i=0}^{m-1} \frac{\zeta_K(2(s-m-i-\varepsilon)-1)}{\zeta_K(2(s-i-1))}. \tag{1.8}$$

COROLLARY 1.3. *Let* $\mathbf{G} \in \{F_{n,\delta}, G_n, H_n\}$.

(1) *For all non-zero prime ideals* $\mathfrak{p}$ *of* $\mathcal{O}$, *the following functional equation holds:*

$$\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s)|_{q\to q^{-1}} = q^{d(\mathbf{G})}\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s), \tag{1.9}$$

where

$$d(F_{n,\delta}) = \binom{2n+\delta}{2}, \quad d(G_n) = n^2, \quad d(H_n) = \binom{n+1}{2}$$

is the $\mathbb{Z}$*-rank of the corresponding derived Lie lattice.*

(2) *The abscissa of convergence* $\alpha(\mathbf{G}(\mathcal{O}))$ *of* $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ *is an integer. More precisely, we have*

$$\alpha(F_{n,\delta}(\mathcal{O})) = 2(2n+\delta-1), \quad \alpha(G_n(\mathcal{O})) = 2n, \quad \alpha(H_n(\mathcal{O})) = n+1. \tag{1.10}$$

In particular, $\alpha(\mathbf{G}) := \alpha(\mathbf{G}(\mathcal{O}))$ *is independent of* $\mathcal{O}$.

(3) *The zeta function* $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ *has meromorphic continuation to the whole complex plane. The continued zeta function has no singularities on the line* $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = \alpha(\mathbf{G})\}$, *apart from a simple pole at* $s = \alpha(\mathbf{G})$.

(4) *There exists a constant* $c(\mathbf{G}(\mathcal{O}))$, *given explicitly in terms of special values of the Dedekind zeta function* $\zeta_K(s)$, *such that*

$$\sum_{n\le N} \widetilde{r}_n(\mathbf{G}(\mathcal{O})) \sim c(\mathbf{G}(\mathcal{O}))\cdot N^{\alpha(\mathbf{G})} \quad \text{as } N \longrightarrow \infty.$$

Theorem B and its corollary are proved in Section 5.

We remark that the functional equations (1.9) illustrate (1.5), which Theorem A only asserts for almost all $\mathfrak{p}$. It is of interest to what extent the assertions of Corollary 1.3 generalize to more general group schemes. Specifically, we ask the following:

Question 1.4. Let $\mathbf{G}$ be a unipotent group scheme defined over the ring of integers $\mathcal{O}$ of a number field K, and let L be a finite extension of K, with ring of integers $\mathcal{O}_L$. Is the abscissa of convergence $\alpha(\mathbf{G}(\mathcal{O}_L))$ independent of L? Is it a rational number? Does the zeta function $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ allow for analytic continuation beyond its abscissa of convergence?

The last two questions would have an affirmative answer if local representation zeta functions of $\mathcal{T}$-groups were "cone integrals" in the sense of [13].

1.4. Additive formulae and Weyl group generating functions. In our third main result we prove "additive formulae" for the local factors of the zeta functions of groups of type F, G and H. To state this result we introduce some notation. Throughout the paper, X, Y and Z will denote indeterminates in the field $\mathbb{Q}(X,Y,Z)$. For $N \in \mathbb{N}$, we set $[N] = \{1,\dots,N\}$ and $[N]_0 = \{0,1,\dots,N\}$. We write $(\underline{N})_X$ for the polynomial $1 - X^N$. We set $(\underline{0})_X = 1$ and write $(\underline{N})_X!$ for $(\underline{1})_X(\underline{2})_X \cdots (\underline{N})_X$. Given $n \in \mathbb{N}$ and $I = \{i_1,\dots,i_l\}_< \subseteq [n-1]_0$, we set $i_0 = 0$, and $i_{l+1} = n$, respectively. Here the subscript $<$ on $\{i_1,\dots,i_l\}$ indicates that $i_1 < \cdots < i_l$. Note that $i_1 = n$ when $I = \varnothing$. For $j \in [l]_0$ we define $\mu_j = i_{j+1} - i_j$. For $a,b \in \mathbb{N}_0$ such that $a \geq b$, we have the "X-binomial coefficient", also known as the Gaussian polynomial

$$\binom{a}{b}_X = \frac{(\underline{a})_X!}{(\underline{a-b})_X!(\underline{b})_X!}.$$

Furthermore, we have the "X-multinomial coefficient"

$$\binom{n}{I}_X = \binom{n}{i_l}_X \binom{i_l}{i_{l-1}}_X \cdots \binom{i_2}{i_1}_X.$$

We also define the Y-Pochhammer symbol, or Y-shifted factorial, as

$$(X;Y)_n = \prod_{i=0}^{n-1} (1 - XY^i). \tag{1.11}$$

In the literature, the above symbols are often defined in terms of a formal variable q, and thus one often encounters the q-binomial and q-multinomial coefficients, and q-Pochhammer symbol, respectively. In our context, however, q is always a prime power, so we choose X and Y as formal variables.

Given a fixed prime power $q = p^f$ we write $(\underline{N})$ for $(\underline{N})_{q^{-1}}$.

THEOREM C. *Let $n \in \mathbb{N}$, $\delta \in \{0,1\}$ and let K be a number field with ring of integers $\mathcal{O}$. Let $\mathbf{G} \in \{F_{n,\delta}, G_n, H_n\}$. There exist polynomials $f_{\mathbf{G},I}(X) \in \mathbb{Z}[X]$, $I \subseteq [n-1]_0$, and natural numbers $a(\mathbf{G},i)$, $i \in [n-1]_0$, such that, for all non-zero prime ideals $\mathfrak{p}$ of $\mathcal{O}$, one has*

$$\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s) = \sum_{I \subseteq [n-1]_0} f_{\mathbf{G},I}(q^{-1}) \prod_{i \in I} \frac{q^{a(\mathbf{G},i)-(n-i)s}}{1 - q^{a(\mathbf{G},i)-(n-i)s}}, \tag{1.12}$$

where $q = |\mathcal{O}/\mathfrak{p}|$. The data $f_{\mathbf{G},I}(X)$ and $a(\mathbf{G},i)$ are given in the following table.

$\mathbf{G}$	$f_{\mathbf{G},I}(X)$	$a(\mathbf{G},i)$
$F_{n,\delta}$	$\binom{n}{I}_{X^2}(X^{2(i_1+\delta)+1};X^2)_{n-i_1}$	$\binom{2n+\delta}{2} - \binom{2i+\delta}{2}$
G_n	$\binom{n}{I}_X(X^{i_1+1};X)_{n-i_1}$	$n^2 - i^2$
H_n	$\left(\prod_{j=1}^{l}(X^2;X^2)^{-1}_{\lfloor \mu_j/2 \rfloor}\right)(X^{i_1+1};X)_{n-i_1}$	$\binom{n+1}{2} - \binom{i+1}{2}$

Note that $f_{\mathbf{G},I}(X)$ and $a(\mathbf{G},i)$ are independent of $\mathcal{O}$ and $\mathfrak{p}$. Theorem C is proved in Section 3.

The proof of the "multiplicative" Theorem B relies on the "additive" Theorem C, together with the following identity, which we prove in Section 4.1.

PROPOSITION 1.5. *For $n \in \mathbb{N}$ we have*

(1.13)
$$\sum_{I \subseteq [n-1]_0} \binom{n}{I}_{X^{-1}} (YX^{-i_1-1}; X^{-1})_{n-i_1} \prod_{i \in I} \mathrm{gp}((X^i Z)^{n-i}) = \frac{(X^{-n}YZ;X)_n}{(Z;X)_n}.$$

We call the identity (1.13) of "multinomial type" due to its analogy with the multinomial theorem. For groups of type F and G, Theorem B is a formal consequence of Theorem C and Proposition 1.5. Further combinatorial arguments are needed to deal with groups of type H, and these are treated in Section 5.1.

Proposition 1.5 has applications to generating functions of statistics on Weyl groups of type B; for details, see Section 4.2. Let $n \in \mathbb{N}$, and consider the group B_n of permutations w of the set $[\pm n]_0 := \{-n, \dots, n\}$ such that, for all $i \in [\pm n]_0$, $w(-i) = -w(i)$. We may identify B_n with the group of "signed permutation matrices", that is, monomial matrices whose non-zero entries are in $\{1, -1\}$. Interesting statistics on B_n include the usual Coxeter length function l with respect to the standard Coxeter generating set $S = \{s_0, \dots, s_{n-1}\}$ and the statistic "neg" which keeps track of the number of negative entries of a signed permutation. A result of Reiner in [35] allows us to express the polynomials $f_{F_{n,\delta},I}(X)$ and $f_{G_n,I}(X)$ given in Theorem C in terms of the joint distribution of the statistics l and neg over descent classes in B_n. For groups of type H we present a conjectural formula of this kind. To state it, we introduce a new statistic L on B_n. For $w \in B_n$ we define

$$L(w) = \frac{1}{2}\#\{(x,y) \in [\pm n]_0^2 \mid x < y,\ w(x) > w(y),\ x \not\equiv y \bmod (2)\} \in \mathbb{N}_0 \tag{1.14}$$

and write $D(w) = \{s \in S \mid l(ws) < l(w)\}$ for the (right) descent set of w. From now on, we identify S with $[n-1]_0$ in the obvious way. For $I \subseteq S$, let $I^c = [n-1]_0 \setminus I$ denote the complement of I, and let $B_n^{I^c} = \{w \in B_n \mid D(w) \subseteq I\}$.

CONJECTURE 1.6. *For $n \in \mathbb{N}$ and $I \subseteq [n-1]_0$ we have*

$$f_{H_n,I}(X) = \sum_{w \in B_n^{I^c}} (-1)^{l(w)} X^{L(w)}.$$

In [41] we prove Conjecture 1.6 for arbitrary $n \in \mathbb{N}$ and $I \in \{\{0\}, [n-1]_0\}$, as well as in the case where n is even and $I \subseteq [n-1]_0 \cap 2\mathbb{N}_0$. We remark that the statistic L is a natural extension of a statistic on the symmetric group S_n defined by Klopsch and the second author; cf. [26, Lemma 5.2].

Combining our Weyl-group theoretical interpretations of the polynomials $f_{\mathbf{G},I}(X)$ with Proposition 1.5 allows us to describe the joint distribution of three statistics on Weyl groups of type B, namely $\sigma - l$, neg and rmaj. The statistics l and neg have already been introduced. We now give the definitions of σ and rmaj. For a general finite Weyl group W, with root system Φ and simple roots $\alpha_0,\dots,\alpha_{n-1}$, let $b_0,\dots,b_{n-1}$ denote the simple root coordinates for half the sum of all positive roots, that is $\sum_{\alpha\in\Phi}\alpha = 2\sum_{i=0}^{n-1} b_i\alpha_i$. Specifically, for B_n with generating set S as above, the simple root coordinates b_i, $i\in[n-1]_0$ are given by $b_i = n^2-i^2$; cf. [8, Plate II] (note that we use the reverse ordering of the simple roots). For $w\in B_n$, let

$$\sigma(w) = \sum_{i\in D(w)} b_i = \sum_{i\in D(w)} (n^2-i^2)$$

and define the *reverse major index* by $\mathrm{rmaj}(w) = \sum_{i\in D(w)}(n-i)$.

PROPOSITION 1.7. *For $n\in\mathbb{N}$ we have*

$$\sum_{w\in B_n} X^{(\sigma-l)(w)}Y^{\mathrm{neg}(w)}Z^{\mathrm{rmaj}(w)} = \prod_{i=0}^{n-1}\frac{(1+X^iYZ)(1-(X^{n+i}Z)^{n-i})}{1-X^{n+i}Z}.$$

The generating function for the statistic $\sigma - l$ over a general finite Weyl group was studied in [42]. The geometric relevance of the statistic $\sigma - l$ is explained in [42, Lemma 2.2]. Proposition 1.7 generalizes [42, Theorem 1.1] in the case of Weyl groups of type B. A similar result, pertaining to the statistic L defined in (1.14), exploits the two different expressions of the local zeta functions of groups of type H given by Theorems B and C; see Proposition 5.5.

1.5. Notation. We record some of our recurrent notation. Throughout, we denote by K a number field with ring of integers $\mathcal{O}=\mathcal{O}_K$. We denote by $\mathfrak{p}$ a non-zero prime ideal of $\mathcal{O}$, and sometimes write $\mathfrak{o}$ for the completion $\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}$ at $\mathfrak{p}$. We write q for the residue field cardinality $|\mathfrak{o}/\mathfrak{p}|$ and p for the residue field characteristic of $\mathfrak{o}$. The Dedekind zeta function $\zeta_K(s)$ of K is

$$\zeta_K(s) = \sum_{I\lhd\mathcal{O}} |\mathcal{O}:I|^{-s} = \prod_{\mathfrak{p}} \zeta_{K,\mathfrak{p}}(s), \tag{1.15}$$

where the product is indexed by the non-zero prime ideals of $\mathcal{O}$, and $\zeta_{K,\mathfrak{p}}(s) = 1/(1-q^{-s})$. For a non-trivial $\mathfrak{o}$-module M, we write $M^* := M\setminus\mathfrak{p}M$, and for the trivial $\mathfrak{o}$-module $\{0\}$ we set $\{0\}^*=\{0\}$. Given a ring R, we write $\mathrm{rk}_R(M)$ to denote the rank of a free R-module M. We also write $\mathrm{rk}(x)$ for the rank of a matrix x. For a fixed $d\in\mathbb{N}$, we write $W(\mathfrak{o}) = (\mathfrak{o}^d)^*$ and, given $N\in\mathbb{N}$, we set $W_N(\mathfrak{o}) = ((\mathfrak{o}/\mathfrak{p}^N)^d)^*$.

For any compact abelian group $\mathfrak{a}$ we write $\widehat{\mathfrak{a}}$ for its Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}^{\mathrm{cont}}(\mathfrak{a},\mathbb{C}^\times)$. In the context of nilpotent groups we use the notation $\widehat{G}$ to denote the profinite completion of the $\mathcal{T}$-group G. We write $\mathrm{Irr}(G)$ for the collection of isomorphism classes of continuous, irreducible complex representations of a topological group G.

Given a subset $I \subseteq \mathbb{N}$ we write I_0 for $I \cup \{0\}$. For $a, b \in \mathbb{Z}$, we use the notation $aI + b = \{ai + b \mid i \in I\}$. Given a term X different from 1, we often write $\mathrm{gp}(X)$ for the "geometric progression" $X/(1 - X)$.

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2. Representation zeta functions of nilpotent groups. In this section we develop some general machinery to study representation zeta functions of finitely generated nilpotent groups obtained from unipotent group schemes associated to nilpotent Lie lattices over rings of integers of number fields. We first give a general description of the construction of $\mathcal{T}$-groups from group schemes, the Kirillov orbit method and tools from $\mathfrak{p}$-adic integration in Sections 2.1 and 2.2. We prove Theorem A in Section 2.3. Section 2.4 is dedicated to a more explicit analysis in the case of nilpotency class 2, affording slightly stronger results in this case. The results in this section also prepare the ground for the subsequent proof of Theorem C.

Let G be a $\mathcal{T}$-group. Recall that, for $n \in \mathbb{N}$, we denote by $\widetilde{r}_n(G)$ the number of twist-isoclasses of n-dimensional irreducible representations of G. The following lemma establishes that the sequence $(\widetilde{r}_n(G))$ has polynomial growth, so that the Dirichlet series $\zeta_G(s)$ has non-empty domain of convergence.

LEMMA 2.1. *The series $\zeta_G(s) = \sum_{n=1}^{\infty} \widetilde{r}_n(G) n^{-s}$ converges on a complex half-plane.*

Proof. We need to show that the sequence $(\widetilde{r}_n(G))$ is bounded by a polynomial in n. It is well-known that every finite-dimensional irreducible representation of a $\mathcal{T}$-group is monomial, that is induced from a 1-dimensional representation of some subgroup. $\mathcal{T}$-groups are further known to have polynomial subgroup growth, that is the sequence of the numbers $a_n(G)$ of subgroups of G of index n is bounded by a polynomial in n; see, for instance, [31, Theorem 5.1]. It thus suffices to show that the sequence of the numbers of twist-isoclasses of representations of G obtained by inducing to G a 1-dimensional representation of an index-n-subgroup of G is bounded by a polynomial in n. This follows from the fact that, given a subgroup H of G of index n, a 1-dimensional representation χ of G and a 1-dimensional representation ψ of H, we have that

$$\chi \otimes \mathrm{Ind}_H^G(\psi) = \mathrm{Ind}_H^G(\mathrm{Res}_H^G(\chi) \otimes \psi), \tag{2.1}$$

and that the index $|G' \cap H : H'|$ is bounded by a polynomial in n. To see the latter note that, since G is nilpotent, each subgroup of index n in G contains G^n, and that the index of $(G^n)'$ in G' is bounded by a power of n, which only depends on the number of generators and the nilpotency class of G. $\square$

2.1. Unipotent group schemes, $\mathcal{T}$-groups and nilpotent Lie lattices.

2.1.1. $\mathcal{T}$-groups from group schemes. Let **G** be an affine smooth group scheme over $\mathcal{O}$, the ring of integers of a number field K. We say that **G** is *unipotent* (*over* $\mathcal{O}$) if the geometric fibre $\mathbf{G} \times_{\mathcal{O}} \overline{k(\mathfrak{p})}$ is a connected unipotent algebraic group for all $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$. Here $k(\mathfrak{p})$ is the residue field $\mathcal{O}/\mathfrak{p}$ when $\mathfrak{p} \neq (0)$, and $k((0)) = K$. Note that, if **G** is unipotent, it is automatically finitely presented over $\mathcal{O}$, by the definition of smoothness; see e.g., [40, Section 2]. It is well-known that if k is any field and **G** is unipotent over k then $\mathbf{G}(k)$ embeds as a subgroup of the group of upper-unitriangular matrices in $\mathrm{GL}_N(k)$, for some N. Thus $\mathbf{G}(\mathcal{O})$ is nilpotent and torsion-free, by virtue of being a subgroup of $\mathbf{G}(K)$. Moreover, since **G** is affine and finitely presented over $\mathcal{O}$, and $\mathcal{O}$ is free of finite rank over $\mathbb{Z}$, the Weil restriction $\mathrm{Res}_{\mathcal{O}/\mathbb{Z}}\mathbf{G}$ is an affine finitely presented group scheme over $\mathbb{Z}$; cf. [37, Proposition 4.4]. By a result of Borel and Harish-Chandra, the group of $\mathbb{Z}$-points of an affine group scheme of finite type over $\mathbb{Z}$ is finitely generated; cf. [7, Theorem 6.12]. Therefore $\mathbf{G}(\mathcal{O}) = (\mathrm{Res}_{\mathcal{O}/\mathbb{Z}}\mathbf{G})(\mathbb{Z})$ is finitely generated, and thus a $\mathcal{T}$-group.

For a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, we denote by $\mathcal{O}_{\mathfrak{p}}$ the completion of $\mathcal{O}$ at $\mathfrak{p}$, with maximal ideal $\mathfrak{p}$, residue field cardinality q and residue field characteristic p. Let **G** be a unipotent group scheme over $\mathcal{O}$. By the Congruence Subgroup Property for unipotent groups (see, for instance, [10]), and the strong approximation property for unipotent groups (cf. [34, Lemma 5.5]), the profinite completion of the $\mathcal{T}$-group $\mathbf{G}(\mathcal{O})$ satisfies

$$\widehat{\mathbf{G}(\mathcal{O})} = \prod_{\mathfrak{p}} \mathbf{G}(\mathcal{O}_{\mathfrak{p}}), \tag{2.2}$$

where $\mathfrak{p}$ ranges over the non-zero prime ideals of $\mathcal{O}$. The factorization (2.2) implies that the zeta function $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ satisfies an Euler product, indexed by the non-zero prime ideals on $\mathcal{O}$. Indeed, every finite-dimensional irreducible complex representation of $\mathbf{G}(\mathcal{O})$ is twist-equivalent to one with finite image; see [30, Theorem 6.6]. We thus have a dimension-preserving bijection between the twist-isoclasses of representations of $\mathbf{G}(\mathcal{O})$ on the one hand and continuous irreducible complex representations of $\widehat{\mathbf{G}(\mathcal{O})}$ up to twists by continuous 1-dimensional representations on the other. Owing to the product (2.2) and the resulting fact that

$$\mathrm{Hom}^{\mathrm{cts}}(\widehat{\mathbf{G}(\mathcal{O})}, \mathbb{C}^{\times}) = \prod_{\mathfrak{p}} \mathrm{Hom}^{\mathrm{cts}}(\widehat{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}, \mathbb{C}^{\times}),$$

the zeta function $\zeta_{\mathbf{G}(\mathcal{O})}(s)$ therefore is the Euler product of the local zeta functions

$$\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s) := \sum_{i=0}^{\infty} \widetilde{r}_{p^i}(\mathbf{G}(\mathcal{O}_{\mathfrak{p}}))p^{-is}. \tag{2.3}$$

Note that $\widetilde{r}_n(\mathbf{G}(\mathcal{O}_\mathfrak{p})) = 0$ unless n is a power of p, as $\mathbf{G}(\mathcal{O}_\mathfrak{p})$ is a pro-p group. The above discussion is summarized in the following result.

PROPOSITION 2.2. *We have the Euler product*

$$\zeta_{\mathbf{G}(\mathcal{O})}(s) = \prod_{\mathfrak{p}} \zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s).$$

Remark 2.3. We will show that the local factor $\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s)$ is in fact a rational function in q^{-s} whenever the group scheme $\mathbf{G}$ is obtained from a Lie lattice in the sense defined in Section 2.1.2 and p is odd or $p = 2$ and $c \neq 3$; cf. Corollaries 2.11 and 2.19. We do not know whether these conditions are also necessary. In any case, it follows from work of Hrushovski and Martin that, for all rational primes p, the "mini Euler product" $\zeta_{\mathbf{G}(\mathcal{O}),p}(s) = \prod_{\mathfrak{p}|p} \zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s)$ is rational in p^{-s}; see [20, Theorem 8.4].

2.1.2. Group schemes from Lie lattices. Recall the notion of Lie lattice from Section 1.2. Let $(\Lambda, [\cdot,\cdot])$ be a nilpotent $\mathcal{O}$-Lie lattice of $\mathcal{O}$-rank h and nilpotency class c. Choose an $\mathcal{O}$-basis $(x_1, \dots, x_h)$ for Λ. For any $\mathcal{O}$-algebra R, let $\Lambda(R) := \Lambda \otimes_\mathcal{O} R$. Then $(x_1 \otimes 1, \dots, x_h \otimes 1)$ is an R-basis for $\Lambda(R)$; cf. [28, XVI, Proposition 2.3]. We write Λ' for the derived Lie lattice $[\Lambda, \Lambda]$.

Assume that $\Lambda' \subseteq c!\Lambda$. By means of the Hausdorff series one may define a group structure on $\Lambda(R)$ by setting, for $x, y \in \Lambda(R)$,

$$\begin{aligned} x * y &= x + y + \frac{1}{2}[x,y] + \frac{1}{12}[[x,y],y] + \cdots, \\ x^{-1} &= -x; \end{aligned}$$

see [24, Chapter 9.2]. The group $(\Lambda(R), *)$ is nilpotent of class c. In co-ordinates with respect to the R-basis $(x_1 \otimes 1, \dots, x_h \otimes 1)$ for $\Lambda(R)$, the group operations are given by polynomials over $\mathcal{O}$ which are independent of R. This defines a unipotent group scheme $\mathbf{G}_\Lambda$ over $\mathcal{O}$, isomorphic as a scheme to affine h-space over $\mathcal{O}$, representing the group functor

$$R \longmapsto (\Lambda(R), *).$$

The group $\mathbf{G}_\Lambda(\mathcal{O})$ is a $\mathcal{T}$-group of nilpotency class c and Hirsch length $h(\mathbf{G}_\Lambda(\mathcal{O})) = \mathrm{rk}_\mathbb{Z}(\mathcal{O})h = [K : \mathbb{Q}]h$. If R is a finitely generated pro-p ring, such as $\mathcal{O}_\mathfrak{p}$, the group $\mathbf{G}_\Lambda(R)$ is a finitely generated class-c-nilpotent pro-p group.

Remark 2.4. Let G be a $\mathcal{T}$-group of nilpotency class c. It is well-known that there exists a $\mathbb{Q}$-Lie algebra $\mathcal{L}_G(\mathbb{Q})$ of $\mathbb{Q}$-dimension $h(G)$, and an injective mapping $\log : G \to \mathcal{L}_G(\mathbb{Q})$ such that $\log(G)$ spans $\mathcal{L}_G(\mathbb{Q})$ over $\mathbb{Q}$; cf., for instance, [38, Chapter 6]. It is further known that there exists a subgroup H of G of finite index such that $\log(H)$ is a $\mathbb{Z}$-Lie lattice inside $\mathcal{L}_G(\mathbb{Q})$ and $\log(H)' \subseteq c!\log(H)$. Thus H

may be recovered as the group of $\mathbb{Z}$-points of the group scheme defined by $\log(H)$, and it makes sense to study the $\mathcal{O}$-points of this group scheme for extensions $\mathcal{O}$ of $\mathbb{Z}$; cf. Remark 2.12 and [18, Sections 1 and 5]. In particular, for all p which do not divide the index $|G:H|$, we have that $\zeta_{G,p}(s) = \zeta_{H,p}(s)$.

We close this section with a simple lemma which we will need in later computations with coordinates. Let $Z(\Lambda) = \{x \in \Lambda \mid [x,\Lambda] = 0\}$ be the center of Λ. Let R be either $\mathcal{O}$ or $\mathfrak{o}$, and let M be a finitely generated R-module, and N an R-submodule of M. We write $\iota(N)$ for the *isolator* of N in M, that is the smallest submodule L of M containing N such that M/L is torsion-free. We say that N is *isolated* in M if $\iota(N) = N$, that is if M/N is torsion-free. Note that if M is torsion-free then N is isolated in M if and only if N is a pure submodule of M.

LEMMA 2.5. *The center $Z(\Lambda)$ is isolated in Λ. Moreover, suppose that M is a free $\mathcal{O}$-module of finite rank and N an isolated submodule of M. Then there exists a free finite index submodule N_0 of N and a free finite index submodule M_0 of M containing N_0 such that there exists a basis for N_0 which can be extended to a basis for M_0.*

Proof. Let $x \in \Lambda$. If $x + Z(\Lambda) \in \Lambda/Z(\Lambda)$ is torsion then there exists a non-zero element $a \in \mathcal{O}$ such that $ax \in Z(\Lambda)$. This implies that $[ax,\Lambda] = a[x,\Lambda] = 0$, but since Λ is torsion-free, this means that $[x,\Lambda] = 0$, that is, $x \in Z(\Lambda)$. We now prove the second claim. Note that, if $\mathcal{O}$ is a principal ideal domain, the claim follows in a straightforward way from the structure theory of modules over such rings. In general, we use well-known facts from the structure theory of finitely generated modules over Dedekind domains; see, for instance, [11, Chapter 10.6]. Since torsion-free modules over a Dedekind domain are projective [11, Proposition 10.6.6], the short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

splits, and so $M \cong N \oplus M/N$. By [11, Theorem 10.6.11] there exist non-zero ideals I_1 and I_2 of $\mathcal{O}$ such that $N \cong \mathcal{O}^{r-1} \oplus I_1$ and $M/N \cong \mathcal{O}^{s-1} \oplus I_2$, where r and s are the K-ranks of N and M/N, respectively. The K-rank of M is then $r+s$ and $M \cong \mathcal{O}^{r+s}$. Thus $\mathcal{O}^{r+s} \cong \mathcal{O}^{r-1} \oplus I_1 \oplus \mathcal{O}^{s-1} \oplus I_2 \cong \mathcal{O}^{r+s-2} \oplus I_1 \oplus I_2 \cong \mathcal{O}^{r+s-1} \oplus I_1 I_2$, where for the last isomorphism we have used [11, (10.6.8)]. Invoking [11, Theorem 10.6.11] again yields that $\mathcal{O} \cong I_1 I_2$, so in particular $I_1 I_2$ is a free $\mathcal{O}$-module which is contained in both I_1 and I_2. Hence $\mathcal{O}^{r-1} \oplus I_1 I_2$ is a free submodule of N and $\mathcal{O}^{s-1} \oplus I_1 I_2$ is a free submodule of M/N. Let

$$N_0 := \mathcal{O}^{r-1} \oplus I_1 I_2 \quad \text{and} \quad M_0 := N_0 \oplus (\mathcal{O}^{s-1} \oplus I_1 I_2).$$

Then $N/N_0 \cong I_1/I_1 I_2$ and $M/M_0 \cong N/N_0 \oplus I_2/I_1 I_2$ are finite as sets because every non-zero ideal of $\mathcal{O}$ is of finite index. Any basis of N_0 can obviously be extended to M_0, and the lemma is proved. $\square$

2.2. Kirillov orbit method, Poincaré series and $\mathfrak{p}$-adic integration. Let Λ be a nilpotent $\mathcal{O}$-Lie lattice such that $\Lambda' \subseteq c!\Lambda$, and let $\mathbf{G} = \mathbf{G}_\Lambda$ be the unipotent group scheme over $\mathcal{O}$ associated to Λ as in Section 2.1.2. Our aim in the current section is to provide tools to study and compute the local factors $\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s)$.

2.2.1. Kirillov orbit method. A key technical tool in our analysis is the Kirillov orbit method for groups of the form $\mathbf{G}(\mathcal{O}_\mathfrak{p})$. Where it is applicable, it provides a way to construct the irreducible representations of a group in terms of co-adjoint orbits. For the class of $\mathcal{T}$-groups, this method was pioneered by Howe in [19]. A treatment of the Kirillov orbit method for pro-p groups can be found in [17].

We now fix a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ and write $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$. Consider the $\mathfrak{o}$-Lie lattice $\mathfrak{g} := \Lambda(\mathfrak{o}) = \Lambda \otimes_\mathcal{O} \mathfrak{o}$, and its Pontryagin dual $\widehat{\mathfrak{g}} = \mathrm{Hom}_\mathbb{Z}^{\mathrm{cont}}(\mathfrak{g}, \mathbb{C}^\times)$. For any $\psi \in \widehat{\mathfrak{g}}$ we have an associated alternating bi-additive form

$$B_\psi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}^\times, \quad (x,y) \longmapsto \psi([x,y]).$$

The form B_ψ clearly only depends on the restriction of ψ to $\mathfrak{g}'$, and if $\varphi \in \widehat{\mathfrak{g}'}$ we simply write B_φ for $B_{\tilde{\varphi}}$, where $\tilde{\varphi}$ is any element in $\widehat{\mathfrak{g}}$ such that $\tilde{\varphi}|_{\mathfrak{g}'} = \varphi$. The radical of the form B_ψ is $\mathrm{Rad}(B_\psi) := \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g} : B_\psi(x,y) = 1\} = \{x \in \mathfrak{g} \mid \psi([x,\mathfrak{g}]) = 1\}$. The following is a refinement of [43, Corollary 3.1].

THEOREM 2.6. *If p is odd or $p = 2$ and $c \geq 4$, we have*

$$\zeta_{\mathbf{G}(\mathfrak{o})}(s) = \sum_{\psi \in \widehat{\mathfrak{g}'}} |\mathfrak{g} : \mathrm{Rad}(B_\psi)|^{-s/2} |\mathfrak{g} : \mathfrak{g}_{\psi,2}|^{-1}, \tag{2.4}$$

where $\mathfrak{g}_{\psi,2} = \{x \in \mathfrak{g} \mid \psi([x,\mathfrak{g}']) = 1\}$.

Proof. Assume first that $p > c$. The pro-p group $\mathbf{G}(\mathfrak{o})$ is then saturable and there exists a Kirillov correspondence between the finite co-adjoint orbits in the dual of the Lie algebra $\mathfrak{g}$ and the continuous irreducible representations of $\mathbf{G}(\mathfrak{o})$; cf. [17, Theorem 4.4].

Assume now that $p \leq c$. The condition $\Lambda' \subseteq c!\Lambda$ implies that $\mathfrak{g}' \subseteq c!\mathfrak{g} \subseteq p\mathfrak{g}$. Furthermore, if $c \geq 4$ the condition implies that $\mathfrak{g}' \subseteq 4\mathfrak{g}$. Applying the exponential map we obtain $\mathbf{G}(\mathfrak{o})' \subseteq \mathbf{G}(\mathfrak{o})^p$ and, if $c \geq 4$, $\mathbf{G}(\mathfrak{o})' \subseteq \mathbf{G}(\mathfrak{o})^4$. Thus, if p is odd or if $p = 2$ and $c \geq 4$, we have that $\mathbf{G}(\mathfrak{o})$ is a uniform pro-p group. For such groups there is again a Kirillov correspondence between the finite co-adjoint orbits in the dual of the Lie algebra $\mathfrak{g}$ and the continuous irreducible representations of $\mathbf{G}(\mathfrak{o})$; cf. [22, Section 2].

Moreover, if $\Omega \subset \widehat{\mathfrak{g}}$ is a finite co-adjoint orbit and $\psi \in \Omega$ then the dimension of the corresponding representation $\pi(\Omega)$ is given by

$$\dim(\pi(\Omega)) = |\Omega|^{1/2} = |\mathbf{G}(\mathfrak{o}) : \mathrm{Stab}_{\mathbf{G}(\mathfrak{o})}(\psi)|^{1/2} = |\mathfrak{g} : \mathrm{Rad}(B_\psi)|^{1/2}.$$

The representation $\pi(\Omega)$ is obtained by inducing to $\mathbf{G}(\mathfrak{o})$ the restriction of ψ to a finite-index subgroup of $\mathbf{G}(\mathfrak{o})$. It follows that the twist-isoclass of the representation $\pi(\Omega)$ determines and is determined by the multiset of restrictions of the elements of Ω to $\mathfrak{g}'$. The number of distinct restrictions to $\mathfrak{g}'$ in the orbit Ω containing ψ is $|\mathfrak{g} : \mathfrak{g}_{\psi,2}|$. $\square$

2.2.2. Poincaré series. In order to effectively compute the generating function (2.4) we express it in terms of Poincaré series. To this end, we compute its terms in explicit coordinates. Write $\mathfrak{z}$ for the center of $\mathfrak{g}$. By Lemma 2.5, $\mathfrak{z}$ is isolated, that is, $\mathfrak{z} = \iota(\mathfrak{z})$. Recall that $h = \mathrm{rk}_\mathfrak{o}(\mathfrak{g})$ and set, in addition,

$$d = \mathrm{rk}_\mathfrak{o}(\mathfrak{g}'), \quad k = \mathrm{rk}_\mathfrak{o}(\iota(\mathfrak{g}')/\iota(\mathfrak{g}' \cap \mathfrak{z})) = \mathrm{rk}_\mathfrak{o}(\iota(\mathfrak{g}' + \mathfrak{z})/\mathfrak{z}),$$
$$r - k = \mathrm{rk}_\mathfrak{o}(\mathfrak{g}/\iota(\mathfrak{g}' + \mathfrak{z})),$$

so that $r = \mathrm{rk}_\mathfrak{o}(\mathfrak{g}/\mathfrak{z})$. Note that the nilpotency class c of $\mathfrak{g}$ is at most 2 if and only if $k = 0$. We choose a uniformizer π of $\mathfrak{o}$, write $\overline{}$ for the natural surjection $\mathfrak{g} \mapsto \mathfrak{g}/\mathfrak{z}$, and choose an $\mathfrak{o}$-basis

$$\mathbf{e} = (e_1, \ldots, e_{r-k}, \underbrace{e_{r-k+1}, \ldots, e_r}_{\iota(\overline{\mathfrak{g}'+\mathfrak{z}})}, \overbrace{\underbrace{e_{r+1}, \ldots, e_{r-k+d}}_{\iota(\mathfrak{g}' \cap \mathfrak{z})}, e_{r-k+d+1}, \ldots, e_h}^{\mathfrak{z}}) \tag{2.5}$$

for $\mathfrak{g}$, as well as nonnegative integers $b_1, \ldots, b_d$, such that the following hold:

$$\mathfrak{z} = \langle e_{r+1}, \ldots, e_h \rangle_\mathfrak{o}$$
$$\overline{\mathfrak{g}' + \mathfrak{z}} = \langle \overline{\pi^{b_1} e_{r-k+1}}, \ldots, \overline{\pi^{b_k} e_r} \rangle_\mathfrak{o} \qquad \iota(\overline{\mathfrak{g}' + \mathfrak{z}}) = \langle \overline{e_{r-k+1}}, \ldots, \overline{e_r} \rangle_\mathfrak{o}$$
$$\mathfrak{g}' \cap \mathfrak{z} = \langle \pi^{b_{k+1}} e_{r+1}, \ldots, \pi^{b_d} e_{r-k+d} \rangle_\mathfrak{o} \quad \iota(\mathfrak{g}' \cap \mathfrak{z}) = \langle e_{r+1}, \ldots, e_{r-k+d} \rangle_\mathfrak{o}.$$

The existence of such integers is a consequence of the elementary divisor theorem. We choose an $\mathfrak{o}$-basis $\mathbf{f} = (f_1, \ldots, f_d)$ for $\mathfrak{g}'$ such that

$$(\overline{f_1}, \ldots, \overline{f_k}) = (\overline{\pi^{b_1} e_{r-k+1}}, \ldots, \overline{\pi^{b_k} e_r}),$$
$$(f_{k+1}, \ldots, f_d) = (\pi^{b_{k+1}} e_{r+1}, \ldots, \pi^{b_d} e_{r-k+d}).$$

The structure constants $\lambda_{ij}^l \in \mathfrak{o}$ for $\mathfrak{g}$, where $i, j \in [r]$ and $l \in [d]$, with respect to these bases are defined by $[e_i, e_j] = \sum_{l=1}^d \lambda_{ij}^l f_l$ and are encoded in the *commutator matrix*

$$\mathcal{R}(\mathbf{Y}) = \left(\sum_{l=1}^d \lambda_{ij}^l Y_l \right)_{ij} \in \mathrm{Mat}_r(\mathfrak{o}[\mathbf{Y}]).$$

We define the submatrix

$$\mathcal{S}(\mathbf{Y}) = (\mathcal{R}(\mathbf{Y})_{ij})_{i \in [r],\, j \in \{r-k+1, \ldots, r\}} \in \mathrm{Mat}_{r \times k}(\mathfrak{o}[\mathbf{Y}]),$$

comprising the last k columns of $\mathcal{R}(\mathbf{Y})$. As in [3, Lemma 2.4], we write

$$\widehat{\mathfrak{g}'} \cong \bigcup_{N \in \mathbb{N}_0} \mathrm{Irr}_N(\mathfrak{g}'),$$

with $\mathrm{Irr}_N(\mathfrak{g}') = \widehat{\mathfrak{g}'/\mathfrak{p}^N\mathfrak{g}'} \cong \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}', \mathfrak{o}/\mathfrak{p}^N)^*$. Let $N \in \mathbb{N}_0$. We say that $w \in \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}', \mathfrak{o})^*$ is a representative of $\psi \in \mathrm{Irr}_N(\mathfrak{g}')$ if ψ is the image of w under the natural surjection $\mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}', \mathfrak{o})^* \to \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}', \mathfrak{o}/\mathfrak{p}^N)^*$. The $\mathfrak{o}$-basis $\mathbf{e}$ yields a co-ordinate system $\overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z} \cong \mathfrak{o}^r, z \mapsto \underline{z} = (z_1, \ldots, z_r)$. The dual basis $\mathbf{f}^\vee$, on the other hand, gives a co-ordinate system $\mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}', \mathfrak{o})^* \cong (\mathfrak{o}^d)^*, w \mapsto \underline{w} = (w_1, \ldots, w_d)$. The following is proved in a way similar to [3, Lemma 3.3], and generalizes the analysis in [43, Section 3.4].

LEMMA 2.7. *Let $w \in \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}', \mathfrak{o})^*$ and $N \in \mathbb{N}_0$. Consider the element $\psi \in \mathrm{Irr}_N(\mathfrak{g}')$ represented by w. Then, for every $z \in \mathfrak{g}/\mathfrak{z}$, we have*

$$\begin{aligned} z \in \overline{\mathrm{Rad}(B_\psi)} &\iff \underline{z} \cdot \mathcal{R}(\underline{w}) \equiv 0 \bmod \mathfrak{p}^N \text{ and} \\ z \in \overline{\mathfrak{g}_{\psi,2}} &\iff \underline{z} \cdot \mathcal{S}(\underline{w}) \cdot \mathrm{diag}(\pi^{b_1}, \ldots, \pi^{b_k}) \equiv 0 \bmod \mathfrak{p}^N. \end{aligned}$$

We say that a matrix $S \in \mathrm{Mat}_{r \times k}(\mathfrak{o})$ has (elementary divisor) type $\mathbf{c} = (c_1, \ldots, c_k) \in (\mathbb{N}_0 \cup \{\infty\})^k$—written $\widetilde{\nu}(S) = \mathbf{c}$—if S is equivalent by elementary row and column operations to the $r \times k$-matrix

$$\begin{pmatrix} \pi^{c_1} & & \\ & \ddots & \\ & & \pi^{c_k} \\ & & \end{pmatrix},$$

where $0 \le c_1 \le \cdots \le c_k$. This is a variant of the definition of the type $\nu(R)$ of an antisymmetric matrix $R \in \mathrm{Mat}_r(\mathfrak{o})$ given in [3, Section 3.1]. By definition, we have $\nu(R) = (a_1, \ldots, a_{\lfloor r/2 \rfloor}) \in (\mathbb{N}_0 \cup \{\infty\})^{\lfloor r/2 \rfloor}$, where $0 \le a_1 \le \cdots \le a_{\lfloor r/2 \rfloor}$ if

$$\widetilde{\nu}(R) = \begin{cases} (a_1, a_1, a_2, a_2, \ldots, a_{r/2}, a_{r/2}) & \text{if } r \text{ is even,} \\ (a_1, a_1, a_2, a_2, \ldots, a_{(r-1)/2}, a_{(r-1)/2}, \infty) & \text{if } r \text{ is odd.} \end{cases}$$

The definition of ν takes into account the fact that the elementary divisors of an antisymmetric matrix of even size come in pairs. Analogously to [3, Lemma 3.4] we have the following:

LEMMA 2.8. *Let $w \in \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{g}', \mathfrak{o})^*$ and $N \in \mathbb{N}_0$. Consider the element $\psi \in \mathrm{Irr}_N(\mathfrak{g}')$ represented by w. Let $\mathbf{a} := (a_1, \ldots, a_{\lfloor r/2 \rfloor}) = \nu(\mathcal{R}(\underline{w}))$ and*

$\mathbf{c} := (c_1, \ldots, c_k) = \widetilde{\nu}(\mathcal{S}(\underline{w}) \cdot \mathrm{diag}(\pi^{b_1}, \ldots, \pi^{b_k}))$. *Then*

$$|\mathfrak{g} : \mathrm{Rad}(B_\psi)| = q^{2\sum_{i=1}^{\lfloor r/2 \rfloor}(N-\min\{a_i,N\})} \text{ and}$$
$$|\mathfrak{g} : \mathfrak{g}_{\psi,2}| = q^{\sum_{i=1}^{k}(N-\min\{c_i,N\})}.$$

Let $N \in \mathbb{N}_0$. Given an antisymmetric matrix $\overline{R} \in \mathrm{Mat}_r(\mathfrak{o}/\mathfrak{p}^N)$, we set $\nu(\overline{R}) := (\min\{a_i, N\})_{i\in[\lfloor r/2 \rfloor]} \in ([N]_0)^{\lfloor r/2 \rfloor}$, where $\mathbf{a} = \nu(R)$ is the type of any lift R of $\overline{R}$ under the natural surjection $\mathrm{Mat}_r(\mathfrak{o}) \to \mathrm{Mat}_r(\mathfrak{o}/\mathfrak{p}^N)$. Given $\overline{S} \in \mathrm{Mat}_{r\times k}(\mathfrak{o}/\mathfrak{p}^N)$, the vector $\widetilde{\nu}(\overline{S}) \in ([N]_0)^k$ is defined similarly. We set $W_N(\mathfrak{o}) := \left((\mathfrak{o}/\mathfrak{p}^N)^d\right)^*$.

Given $N \in \mathbb{N}_0$, $\mathbf{a} \in \mathbb{N}_0^{\lfloor r/2 \rfloor}$, $\mathbf{c} \in \mathbb{N}_0^k$, we set

$$(2.6)\quad \mathcal{N}^{\mathfrak{o}}_{N,\mathbf{a},\mathbf{c}} := \#\left\{\mathbf{y} \in W_N(\mathfrak{o}) \mid \nu\big(\mathcal{R}(\mathbf{y})\big) = \mathbf{a},\ \widetilde{\nu}\big(\mathcal{S}(\mathbf{y}) \cdot \mathrm{diag}\left(\pi^{b_1}, \ldots, \pi^{b_k}\right)\big) = \mathbf{c}\right\}.$$

In analogy with [3, Proposition 3.1] we have the following:

PROPOSITION 2.9. *If p is odd or if $p = 2$ and $c \geq 4$, then*

$$(2.7)\quad \zeta_{\mathbf{G}(\mathfrak{o})}(s) = \sum_{\substack{N\in\mathbb{N}_0,\\ \mathbf{a}\in\mathbb{N}_0^{\lfloor r/2\rfloor},\, \mathbf{c}\in\mathbb{N}_0^k}} \mathcal{N}^{\mathfrak{o}}_{N,\mathbf{a},\mathbf{c}} q^{-\sum_{i=1}^{\lfloor r/2\rfloor}(N-a_i)s - \sum_{i=1}^{k}(N-c_i)} =: \mathcal{P}_{\mathcal{R},\mathcal{S},\mathfrak{o}}(s).$$

COROLLARY 2.10. *Under the hypothesis of Proposition 2.9 the zeta function $\zeta_{\mathbf{G}(\mathfrak{o})}(s)$ is a power series in q^{-s}, that is the degrees of continuous irreducible complex representations of $\mathbf{G}(\mathfrak{o})$ are powers of q.*

It is interesting to ask whether the conclusion of Corollary 2.10 holds without the hypothesis of Proposition 2.9.

2.2.3. $\mathfrak{p}$-Adic integration. As explained in [43, Section 2.2], we can express the Poincaré series $\mathcal{P}_{\mathcal{R},\mathcal{S},\mathfrak{o}}(s)$ defined in (2.7) in terms of a $\mathfrak{p}$-adic integral. We define

$$(2.8)\quad \begin{aligned} \mathcal{Z}_{\mathfrak{o}}(\rho,\sigma,\tau) := \int_{(x,\mathbf{y})\in\mathfrak{p}\times W(\mathfrak{o})} |x|_{\mathfrak{p}}^{\tau} \prod_{j=1}^{u} \frac{\|F_j(\mathbf{y}) \cup F_{j-1}(\mathbf{y})x^2\|_{\mathfrak{p}}^{\rho}}{\|F_{j-1}(\mathbf{y})\|_{\mathfrak{p}}^{\rho}} \\ \times \prod_{l=1}^{v} \frac{\|G_l(\mathbf{y}) \cup G_{l-1}(\mathbf{y})x\|_{\mathfrak{p}}^{\sigma}}{\|G_{l-1}(\mathbf{y})\|_{\mathfrak{p}}^{\sigma}} d\mu(x,\mathbf{y}), \end{aligned}$$

where $W(\mathfrak{o}) = (\mathfrak{o}^d)^*$, the additive Haar measure μ on $\mathfrak{o}^{d+1}$ is normalized so that $\mu(\mathfrak{o}^{d+1}) = 1$, and

$$2u = \max\left\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})}(\mathcal{R}(\mathbf{z})) \mid \mathbf{z} \in \mathfrak{o}^d\right\},$$
$$v = \max\left\{\mathrm{rk}_{\mathrm{Frac}(\mathfrak{o})}(\mathcal{S}(\mathbf{z})) \mid \mathbf{z} \in \mathfrak{o}^d\right\},$$

$$F_j(\mathbf{Y}) = \{f \mid f = f(\mathbf{Y}) \text{ a principal } 2j \times 2j \text{ minor of } \mathcal{R}(\mathbf{Y})\},$$
$$G_l(\mathbf{Y}) = \{g \mid g = g(\mathbf{Y}) \text{ an } l \times l \text{ minor of } \mathcal{S}(\mathbf{Y}) \cdot \operatorname{diag}(\pi^{b_1}, \ldots, \pi^{b_k})\},$$
$$\|H(X,\mathbf{Y})\|_{\mathfrak{p}} = \max\{|h(X,\mathbf{Y})|_{\mathfrak{p}} \mid h \in H\} \text{ for a finite set } H \subset \mathfrak{o}[X,\mathbf{Y}].$$

As in [43, Section 2.2] one shows that

$$\mathcal{P}_{\mathcal{R},\mathcal{S},\mathfrak{o}}(s) = 1 + (1-q^{-1})^{-1}\mathcal{Z}_{\mathfrak{o}}(-s/2, -1, us+v-d-1).$$

This yields the following corollary to Proposition 2.9.

COROLLARY 2.11. *Under the hypothesis of Proposition 2.9 we have*

$$\zeta_{\mathbf{G}(\mathfrak{o})}(s) = 1 + (1-q^{-1})^{-1}\mathcal{Z}_{\mathfrak{o}}(-s/2, -1, us+v-d-1).$$

In particular, the zeta function $\zeta_{\mathbf{G}(\mathfrak{o})}(s)$ is a rational function in q^{-s}.

Indeed, the rationality in q^{-s} of integrals like (2.8) is a well known fact in the theory of $\mathfrak{p}$-adic integration; cf., for instance, [12].

2.3. Proof of Theorem A. We now return to the global setup of Theorem A. Recall that $\mathbf{G} = \mathbf{G}_\Lambda$ is a unipotent group scheme, defined by a nilpotent $\mathcal{O}$-Lie lattice Λ of nilpotency class c, where $\mathcal{O} = \mathcal{O}_K$ is the ring of integers of a number field K. For a finite extension L of K, with ring of integers $\mathcal{O}_L$, we wish to describe the zeta function of the $\mathcal{T}$-group $\mathbf{G}(\mathcal{O}_L)$. By Proposition 2.2 we have

$$\zeta_{\mathbf{G}(\mathcal{O}_L)}(s) = \prod_{\mathfrak{P}} \zeta_{\mathbf{G}(\mathcal{O}_{L,\mathfrak{P}})}(s), \tag{2.9}$$

where the product ranges over the non-zero prime ideals of $\mathcal{O}_L$. For such a prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$, dividing the prime ideal $\mathfrak{p}$ of $\mathcal{O}$, we write $\mathfrak{O}$ for the local ring $\mathcal{O}_{L,\mathfrak{P}}$ and $\mathfrak{o}$ for $\mathcal{O}_{K,\mathfrak{p}}$. Further we write $f = f(\mathfrak{O},\mathfrak{o})$ for the relative degree of inertia. We continue to write q for the residue field cardinality of $\mathfrak{o}$, and p for its residue field characteristic, so that $|\mathfrak{O}/\mathfrak{P}| = q^f$.

Assume that p is odd or that $p = 2$ and $c \geq 4$. The $\mathfrak{p}$-adic formalism developed in Sections 2.2 is applicable to the factor $\zeta_{\mathbf{G}(\mathfrak{O})}(s)$ in (2.9). Two facts are key to proving Theorem A: Firstly, we observe that the polynomials occurring in the integrand of the $\mathfrak{p}$-adic integral (2.8) are defined over $\mathcal{O}$, so that effectively only the domain of integration depends on $\mathfrak{O}$. Secondly, we exploit that there is, as we shall explain, a uniform formula for (2.8) in which only the residue field of $\mathfrak{O}$ enters.

A priori, the $\mathfrak{O}$-bases $\mathbf{e}$ and $\mathbf{f}$ defined in Section 2.2.2—and thus the matrices $\mathcal{R}(\mathbf{Y})$ and $\mathcal{S}(\mathbf{Y})$ and the data $\mathbf{b} = (b_1, \ldots, b_d)$—are defined only locally. As we do not assume that $\mathcal{O}$ or $\mathcal{O}_L$ are principal ideal domains, we may not hope for a global analogue of the construction of the bases $\mathbf{e}$ and $\mathbf{f}$. Instead, we choose an $\mathcal{O}$-basis $\mathbf{f} = (e_{r-k+1}, \ldots, e_{r-k+d})$ for a free finite index $\mathcal{O}$-submodule of the $\mathcal{O}$-isolator $\iota(\Lambda'(\mathcal{O}))$, and extend it to an $\mathcal{O}$-basis $\mathbf{e}$ for a free finite index $\mathcal{O}$-submodule of $\Lambda(\mathcal{O})$; cf. Lemma 2.5. Provided p does not divide the index of the latter in $\Lambda(\mathcal{O})$, we may use this basis **e** to obtain an $\mathfrak{O}$-basis for $\Lambda(\mathfrak{O})$ in the analysis of Section 2.2, with $\mathbf{b} = (0,\dots,0)$. This ensures that the polynomials occurring in (2.8) are defined over $\mathcal{O}$. Theorem A now follows formally by the arguments given in [3, Section 4], as the integral (2.8) can be expressed in terms of an integral of the form [3, Equation (4.1)].

2.4. Nilpotency class 2. In nilpotency class 2, many of the constructions given in the previous sections can be made more directly, allowing us to deduce slightly stronger results. While some of these modifications will be known to the experts, we record them here for completeness. Throughout this section, let Λ be a class-2-nilpotent $\mathcal{O}$-Lie lattice, with $\mathrm{rk}_{\mathcal{O}}(\Lambda) = h$, say. Note that we do not assume that $\Lambda' \subseteq 2\Lambda$ here.

2.4.1. We start by constructing a group scheme $\mathbf{G}_\Lambda$ associated to Λ, which coincides with the group scheme defined in Section 2.1.2 if $\Lambda' \subseteq 2\Lambda$. We fix an $\mathcal{O}$-basis $(x_1,\dots,x_h)$ for Λ. Let R be an $\mathcal{O}$-algebra and consider $\Lambda(R) = \Lambda \otimes_{\mathcal{O}} R$. We also write $(x_1,\dots,x_h)$ for the R-basis $(x_1 \otimes 1,\dots,x_h \otimes 1)$ for $\Lambda(R)$.

Any element $g \in \Lambda(R)$ can be expressed uniquely as $g = \sum_{i=1}^{h} a_i x_i$, for some $\mathbf{a} = (a_1,\dots,a_h) \in R^h$. Adopting multiplicative notation, we identify g with the formal monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_h^{a_h}$, and define a group multiplication $*$ on the set of all such monomials by defining group commutators via the Lie bracket on $\Lambda(R)$. More precisely, we define, for $1 \leq i < j \leq r$ and $a_i, a_j \in R$,

$$x_i^{a_i} * x_j^{a_j} = x_i^{a_i} x_j^{a_j}, \quad x_j^{a_j} * x_i^{a_i} = x_i^{a_i} x_j^{a_j} \mathbf{x}^{a_i a_j \boldsymbol{\lambda}_{ij}},$$

where $\boldsymbol{\lambda}_{ij} = (\lambda_{ij}^1,\dots,\lambda_{ij}^h) \in \mathcal{O}^h$ is defined via the identity $[x_i, x_j] = \sum_{k=1}^{h} \lambda_{ij}^k x_k$ in $\Lambda(R)$. Extending this to the set of all monomials in the obvious way, we obtain polynomials $M_i(X_1,\dots,X_h,\tilde{X}_1,\dots,\tilde{X}_h)$, for $i = 1,\dots,h$, over $\mathcal{O}$ such that

$$\mathbf{x}^{\mathbf{a}} * \mathbf{x}^{\mathbf{a}'} = \mathbf{x}^{\mathbf{a}+\mathbf{a}'+(M_i(\mathbf{a},\mathbf{a}'))_i}.$$

Similarly, there are polynomials $I_i(X_1,\dots,X_h)$, for $i = 1,\dots,h$, over $\mathcal{O}$ such that

$$(\mathbf{x}^{\mathbf{a}})^{-1} = \mathbf{x}^{-\mathbf{a}+(I_i(\mathbf{a}))_i}.$$

This defines a unipotent group scheme $\mathbf{G}_\Lambda$ over $\mathcal{O}$, isomorphic as a scheme to affine h-space over $\mathcal{O}$, representing the group functor

$$R \longmapsto (\{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in R^h\}, *).$$

Note that when $c = 2$ and $\Lambda' \subseteq 2\Lambda$, the group scheme $\mathbf{G}_\Lambda$ is isomorphic to the one constructed in Section 2.1.2 by means of the Hausdorff series.

Remark 2.12. The construction sketched here mirrors, of course, the classical construction of a unipotent group scheme $\mathbf{G}$ over $\mathbb{Z}$ associated to a $\mathcal{T}$-group G of nilpotency class 2 by means of a Mal'cev-basis. This allows one to recover G as $\mathbf{G}(\mathbb{Z})$ and to define the group $G^R := \mathbf{G}(R)$, for any $\mathbb{Z}$-module R; cf. [18, Sections 1 and 5]. The point of view taken in the present paper allows us to study group schemes defined over proper extensions of $\mathbb{Z}$.

2.4.2. The main results in Section 2.2 all assume that p is odd or that $p = 2$ and $c \geq 4$. In the current section we formulate a Kirillov orbit formalism for group schemes of nilpotency class 2 which is valid for all primes p, without restriction. We indicate how the analysis of Sections 2.2.2 and 2.2.3 simplifies in this case. Note that the proof of Theorem A may still require the exclusion of finitely many places.

Let $(x_1,\dots,x_h)$ be an $\mathcal{O}$-basis for Λ, and let $\mathbf{G} = \mathbf{G}_\Lambda$ be the group scheme defined in Section 2.4.1. For any $\mathcal{O}$-algebra R, we have $\mathbf{G}(R) = \{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in R^h\}$, and an obvious bijection

$$\lambda_R : \mathbf{G}(R) \longrightarrow \Lambda(R), \quad \mathbf{x}^{\mathbf{a}} \longmapsto \sum_{i=1}^{h} a_i x_i.$$

We now assume that $R = \mathfrak{o}$, a compact discrete valuation ring of characteristic zero, and write λ for $\lambda_{\mathfrak{o}}$. We further write $\mathfrak{g}$ for the $\mathfrak{o}$-Lie lattice $\Lambda(\mathfrak{o})$, and $\widehat{\mathfrak{g}}$ for its Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}^{\mathrm{cts}}(\mathfrak{g}, \mathbb{C}^\times)$. For any $\psi \in \widehat{\mathfrak{g}}$ we consider the form

$$B_\psi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}^\times, \quad (x,y) \longmapsto \psi([x,y])$$

with radical $\mathrm{Rad}(B_\psi) := \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g} : B_\psi(x,y) = 1\}$. For $g \in \mathbf{G}(\mathfrak{o})$ and $x \in \mathfrak{g}$, we define the co-adjoint action $\mathbf{G}(\mathfrak{o}) \times \widehat{\mathfrak{g}} \mapsto \widehat{\mathfrak{g}}$ by $(g,\psi) \mapsto \mathrm{ad}^*(g)\psi$, where $\mathrm{ad}^*(g)\psi$ is the map given by

$$x \longmapsto \psi(x + [\lambda(g), x]).$$

Let $\mathrm{Stab}(\psi)$ denote the stabilizer in $\mathbf{G}(\mathfrak{o})$ of ψ under the co-adjoint action, and write $\Omega(\psi)$ for the co-adjoint orbit of ψ.

LEMMA 2.13. *For any $\psi \in \widehat{\mathfrak{g}}$, the group* $\mathrm{Stab}(\psi)$ *contains $Z(\mathbf{G}(\mathfrak{o}))$, and*

$$\mathrm{Stab}(\psi) = \lambda^{-1}(\mathrm{Rad}(B_\psi)).$$

In particular,

$$|\Omega(\psi)| = |\mathbf{G}(\mathfrak{o}) : \mathrm{Stab}(\psi)| = |\mathfrak{g} : \mathrm{Rad}(B_\psi)| < \infty. \tag{2.10}$$

Proof. We have

$$\begin{aligned}\mathrm{Stab}(\psi) &= \{g \in \mathbf{G}(\mathfrak{o}) \mid \forall x \in \mathfrak{g} : \psi(x + [\lambda(g), x]) = \psi(x)\} \\ &= \{g \in \mathbf{G}(\mathfrak{o}) \mid \forall x \in \mathfrak{g} : \psi([\lambda(g), x]) = 1\} \\ &= \lambda^{-1}(\mathrm{Rad}(B_\psi)).\end{aligned}$$

This proves the first two assertions. The claims regarding the orbit size follow from the orbit stabilizer theorem and the fact that ψ is continuous. $\square$

A key role in the Kirillov orbit method is played by polarizing subalgebras for characters $\psi \in \widehat{\mathfrak{g}}$, that is subalgebras P of $\mathfrak{g}$ with the property that $B_\psi|_{P\times P} = 1$, and which are maximal with respect to this property.

LEMMA 2.14. *Let $\psi \in \widehat{\mathfrak{g}}$. Then there exists a polarizing subalgebra for ψ.*

Proof. There exists $n \in \mathbb{N}$ and a homomorphism $\overline{\psi} : \Lambda(\mathfrak{o}/\mathfrak{p}^n) \to \mathbb{C}^\times$ such that ψ factors through $\overline{\psi}$ via the natural surjection $p_n : \mathfrak{g} \to \Lambda(\mathfrak{o}/\mathfrak{p}^n)$, that is $\psi = \overline{\psi} \circ p_n$. Thus the form B_ψ factors through the form

$$B_{\overline{\psi}} : \Lambda(\mathfrak{o}/\mathfrak{p}^n) \times \Lambda(\mathfrak{o}/\mathfrak{p}^n) \longrightarrow \mathbb{C}^\times, \quad (x,y) \longmapsto \overline{\psi}([x,y]).$$

Let $P_{\overline{\psi}}$ be a polarising subalgebra for $\overline{\psi}$ in $\Lambda(\mathfrak{o}/\mathfrak{p}^n)$; see, for instance, [19, Lemma 4]. Clearly $P_\psi := p_n^{-1}(P_{\overline{\psi}})$ has the desired properties. $\square$

LEMMA 2.15. *Let $P \subseteq \mathfrak{g}$ be an $\mathfrak{o}$-subalgebra. Let $\psi \in \widehat{\mathfrak{g}}$ be such that $\psi(P') =$ 1. Then $H := \lambda^{-1}(P)$ is a subgroup of $\mathbf{G}(\mathfrak{o})$ and the restriction $\psi \circ \lambda|_H$ is a 1-dimensional representation of H.*

Proof. Let $h_1 = \mathbf{x}^{\mathbf{a}_1}$ and $h_2 = \mathbf{x}^{\mathbf{a}_2} \in H$. It is easy to check that

$$h_1 h_2 = \mathbf{x}^{\mathbf{a}_1}\mathbf{x}^{\mathbf{a}_2} = \mathbf{x}^{\mathbf{a}_1+\mathbf{a}_2+\mathbf{u}},$$

for some $\mathbf{u} \in \mathfrak{o}^h$ such that $\mathbf{x}^{\mathbf{u}} \in H'$. Clearly $\mathbf{u} \in P'$, and so

$$\lambda(h_1 h_2) = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{u} \in P, \tag{2.11}$$

and hence $h_1 h_2 \in H$. Similarly, given $h = \mathbf{x}^{\mathbf{a}} \in H$, we have

$$h^{-1} = (\mathbf{x}^{\mathbf{a}})^{-1} = \mathbf{x}^{-\mathbf{a}+\mathbf{v}},$$

for some $\mathbf{v} \in \mathfrak{o}^h$ such that $\mathbf{y}^{\mathbf{v}} \in H'$. Then $\mathbf{v} \in P'$, and so

$$\lambda(h^{-1}) = -\mathbf{a} + \mathbf{v} \in P.$$

Thus $h^{-1} \in H$, and so H is a subgroup of $\mathbf{G}(\mathfrak{o})$. Using (2.11) and the fact that $\psi(P') = 1$, we obtain

$$\psi(\lambda(h_1h_2)) = \psi(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{u}) = \psi(\mathbf{a}_1)\psi(\mathbf{a}_2) = \psi(\lambda(h_1))\psi(\lambda(h_2)),$$

so $\psi \circ \lambda|_H$ is a homomorphism of H, as asserted. □

For every $\psi \in \widehat{\mathfrak{g}}$, we choose a polarizing subalgebra P_ψ for ψ in $\mathfrak{g}$; cf. Lemma 2.14. Lemma 2.15 asserts that $H_\psi := \lambda^{-1}(P_\psi)$ is a subgroup of $\mathbf{G}(\mathfrak{o})$, and that $\psi \circ \lambda|_{H_\psi}$ is a 1-dimensional representation of H_ψ. We define the representation

$$\pi(\psi) = \operatorname{Ind}_{H_\psi}^{\mathbf{G}(\mathfrak{o})}(\psi \circ \lambda|_{H_\psi}).$$

Recall that $\operatorname{Irr}(\mathbf{G}(\mathfrak{o}))$ denotes the set of isomorphism classes of continuous, irreducible complex representations of $\mathbf{G}(\mathfrak{o})$. The following result establishes a Kirillov orbit method for the group $\mathbf{G}(\mathfrak{o})$.

PROPOSITION 2.16. *For every $\psi \in \widehat{\mathfrak{g}}$ the representation $\pi(\psi)$ is irreducible, of dimension $|\Omega(\psi)|^{1/2}$, and all representations in* $\operatorname{Irr}(\mathbf{G}(\mathfrak{o}))$ *arise in this way. Given $\varphi, \psi \in \widehat{\mathfrak{g}}$, the representations $\pi(\varphi)$ and $\pi(\psi)$ are isomorphic if and only if $\Omega(\varphi) = \Omega(\psi)$. They are twist-equivalent if and only if $\varphi|_{\mathfrak{g}'} = \psi|_{\mathfrak{g}'}$.*

Proof. The character $\chi_{\pi(\psi)}$ of the representation $\pi(\psi)$ is given by

$$\chi_{\pi(\psi)}(g) = |\Omega(\psi)|^{-1/2} \sum_{\omega \in \Omega(\psi)} \omega(\lambda(g)), \quad \text{for } g \in \mathbf{G}(\mathfrak{o});$$

see, for instance, [23, Proposition 1], with log replaced by λ. This character formula immediately shows that the isomorphism class of $\pi(\psi)$ only depends on $\Omega(\psi)$, that $\pi(\psi)$ is irreducible, and that $\langle \pi(\varphi), \pi(\psi) \rangle = 0$ if $\Omega(\varphi) \neq \Omega(\psi)$. With (2.10) it also implies that $\dim \pi(\psi) = |\Omega(\psi)|^{1/2} = |\mathfrak{g} : \operatorname{Rad}(B_\psi)|^{1/2}$. For $n \in \mathbb{N}_0$, the set of continuous additive characters of $\mathfrak{g}$ which factor through $\Lambda(\mathfrak{o}/\mathfrak{p}^n)$ is a union of co-adjoint orbits. The representations of $\mathbf{G}(\mathfrak{o})$ associated to these orbits all factor through the finite group $\mathbf{G}(\mathfrak{o}/\mathfrak{p}^n)$, of order $q^{nh} = |\Lambda(\mathfrak{o}/\mathfrak{p}^n)|$. But q^{nh} is also the sum of the squares of the dimensions of the irreducible representations of $\mathbf{G}(\mathfrak{o}/\mathfrak{p}^n)$, so every irreducible representation of $\mathbf{G}(\mathfrak{o}/\mathfrak{p}^n)$ must be of the form $\pi(\psi)$, for some ψ.

Finally, let χ be a continuous 1-dimensional representation of $\mathbf{G}(\mathfrak{o})$. We have

$$\chi \otimes \pi(\psi) = \operatorname{Ind}_{H_\psi}^{\mathbf{G}(\mathfrak{o})} \left(\chi|_{H_\psi} \otimes (\psi \circ \lambda|_{H_\psi}) \right). \tag{2.12}$$

Since $\mathbf{G}(\mathfrak{o})' \leq H_\psi$, this implies that two representations $\pi(\varphi)$ and $\pi(\psi)$ are twist-equivalent if and only if $\varphi|_{\mathfrak{g}'} = \psi|_{\mathfrak{g}'}$. □

We record the following immediate consequence of Proposition 2.16.

COROLLARY 2.17. *We have*

$$\zeta_{\mathbf{G}(\mathfrak{o})}(s) = \sum_{\psi \in \widehat{\mathfrak{g}'}} |\mathfrak{g} : \mathrm{Rad}(B_\psi)|^{-s/2}.$$

We note that, in contrast to Theorem 2.6, the formula given in Corollary 2.17 is valid for all primes p, and is somewhat simpler than (2.4). The formalism developed in Sections 2.2.1 and 2.2.3 applies, without the assumption that $p > 2$, and simplifies as follows. We have $k = 0$, as $\mathfrak{g}' \leq \mathfrak{z}$, so the matrix $\mathcal{S}$ makes no appearance. In analogy with the number $\mathcal{N}^{\mathfrak{o}}_{N,\mathbf{a},\mathbf{c}}$ defined in (2.6), we therefore set, for $N \in \mathbb{N}_0$, $\mathbf{a} \in \mathbb{N}_0^{\lfloor r/2 \rfloor}$,

$$\mathcal{N}^{\mathfrak{o}}_{N,\mathbf{a}} := \#\{\mathbf{y} \in W_N(\mathfrak{o}) \mid \nu(\mathcal{R}(\mathbf{y})) = \mathbf{a}\}.$$

The class-2-analogue of Proposition 2.9 is the following:

PROPOSITION 2.18. *We have*

$$\zeta_{\mathbf{G}(\mathfrak{o})}(s) = \sum_{N \in \mathbb{N}_0, \mathbf{a} \in \mathbb{N}_0^{\lfloor r/2 \rfloor}} \mathcal{N}^{\mathfrak{o}}_{N,\mathbf{a}} q^{-\sum_{i=1}^{\lfloor r/2 \rfloor}(N-a_i)s} =: \mathcal{P}_{\mathcal{R},\mathfrak{o}}(s). \tag{2.13}$$

The Poincaré series $\mathcal{P}_{\mathcal{R},\mathfrak{o}}(s)$ may be expressed in terms of the $\mathfrak{p}$-adic integral (2.8), simplified by the fact that $v = 0$. The following is analogous to Corollary 2.11.

COROLLARY 2.19. *The zeta function $\zeta_{\mathbf{G}(\mathfrak{o})}(s)$ is a rational function in q^{-s}.*

In the next section we compute the Poincaré series $\mathcal{P}_{\mathcal{R},\mathfrak{o}}(s)$ directly for groups of type F, G and H, bypassing the need to evaluate $\mathfrak{p}$-adic integrals like (2.8).

3. Proof of Theorem C. Recall that $n \in \mathbb{N}$ and $\delta \in \{0,1\}$. Let $\Lambda \in \{\mathcal{F}_{n,\delta}, \mathcal{G}_n, \mathcal{H}_n\}$ be one of the Lie rings defined in Definition 1.2, and $\mathbf{G} = \mathbf{G}_\Lambda \in \{F_{n,\delta}, G_n, H_n\}$ the associated group scheme. As before, given a number field K with ring of integers $\mathcal{O}$, and a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, we write $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$ for the completion of $\mathcal{O}$ at $\mathfrak{p}$, and q for the residue field cardinality $|\mathfrak{o}/\mathfrak{p}|$. We write $\mathfrak{g}$ for $\Lambda(\mathfrak{o})$ and set $r = \mathrm{rk}_\mathfrak{o}(\mathfrak{g}/\mathfrak{g}')$, $d = \mathrm{rk}_\mathfrak{o}(\mathfrak{g}')$, in accordance with the notation introduced in Section 2.2.2. Note that, in all three cases, we have $n = \lfloor r/2 \rfloor$. We are looking to compute the zeta function $\zeta_{\mathbf{G}(\mathfrak{o})}(s)$. By Proposition 2.18 it suffices to study the Poincaré series $\mathcal{P}_{\mathcal{R},\mathfrak{o}}(s)$ associated to the commutator matrix $\mathcal{R}(\mathbf{Y}) = \mathcal{R}_\Lambda(\mathbf{Y})$ of the relevant Lie lattice with respect to the $\mathbb{Z}$-bases given in the presentations in Definition 1.2. These are as follows:

- $\mathcal{R}_{\mathcal{F}_{n,\delta}}(\mathbf{Y})$ is the generic antisymmetric $(2n+\delta) \times (2n+\delta)$-matrix in the variables Y_{ij}, $1 \leq i < j \leq 2n+\delta$. We have $r = 2n+\delta$ and $d = \binom{2n+\delta}{2}$.
- $\mathcal{R}_{\mathcal{G}_n}(\mathbf{Y}) = \begin{pmatrix} & \mathrm{M}(Y_{ij}) \\ -\mathrm{M}(Y_{ij})^{\mathrm{t}} & \end{pmatrix}$, where $\mathrm{M}(Y_{ij})$ is the generic $n \times n$-matrix in the variables Y_{ij}, $1 \leq i,j \leq n$. We have $r = 2n$ and $d = n^2$.

- $\mathcal{R}_{\mathcal{H}_n}(\mathbf{Y}) = \begin{pmatrix} & \mathrm{S}(Y_{ij}) \\ -\mathrm{S}(Y_{ij}) & \end{pmatrix}$, where $\mathrm{S}(Y_{ij})$ is the generic symmetric $n \times n$-matrix in the variables Y_{ij}, $1 \leq i \leq j \leq n$. We have $r = 2n$ and $d = \binom{n+1}{2}$.

It is advantageous to re-organize the respective series $\mathcal{P}_{\mathcal{R},\mathfrak{o}}(s)$ in the following way. Let $I = \{i_1, \ldots, i_l\}_< \subseteq [n-1]_0$, and recall that, for $j \in [n]_0$, we write $\mu_j = i_{j+1} - i_j$, with $i_0 = 0$ and $i_{l+1} = n$. For $\mathbf{r}_I = (r_{i_1}, \ldots, r_{i_l}) \in \mathbb{N}^I$, we set $N = \sum_{i \in I} r_i$, $W_N(\mathfrak{o}) = \left((\mathfrak{o}/\mathfrak{p}^N)^d\right)^*$ and define

$$\begin{aligned}\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathbf{G}) := \{\underline{w} \in W_N(\mathfrak{o}) \mid \nu(\mathcal{R}_\Lambda(\underline{w}))\\ = (\underbrace{0, \ldots, 0}_{\mu_l}, \underbrace{r_{i_l}, \ldots, r_{i_l}}_{\mu_{l-1}}, \underbrace{r_{i_l} + r_{i_{l-1}}, \ldots, r_{i_l} + r_{i_{l-1}}}_{\mu_{l-2}}, \ldots, \underbrace{N, \ldots, N}_{\mu_0}) \in \mathbb{N}_0^{\lfloor r/2 \rfloor}\}.\end{aligned}$$

Note that $W_N(\mathfrak{o})$ is partitioned by such sets. In particular, for every $\underline{w} \in W_N(\mathfrak{o})$ the type $\nu(\mathcal{R}_\Lambda(\underline{w}))$ always contains at least one zero, as $\mathcal{R}_\Lambda(\underline{w}) \not\equiv 0 \bmod \mathfrak{p}$.

We can now rewrite (2.13) as

$$\zeta_{\mathbf{G}(\mathfrak{o})}(s) = \mathcal{P}_{\mathcal{R},\mathfrak{o}}(s) = \sum_{I \subseteq [n-1]_0} \sum_{\mathbf{r}_I \in \mathbb{N}^I} |\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathbf{G})| q^{-s \sum_{i \in I} r_i (n-i)}. \tag{3.1}$$

We prove Theorem C by computing the quantities $|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathbf{G})|$ explicitly in the three cases; see Proposition 3.4. We start with a few preliminary definitions and a lemma. Given $j \in \mathbb{N}$ and a ring R, we denote by $\mathrm{Alt}_j(R)$ and $\mathrm{Sym}_j(R)$ the antisymmetric and symmetric matrices in $\mathrm{Mat}_j(R)$, respectively. Let $i \in [n]_0$, and define

$$\begin{aligned}\mathrm{Alt}_{2n+\delta, 2(n-i)}(\mathbb{F}_q) &= \{x \in \mathrm{Alt}_{2n+\delta}(\mathbb{F}_q) \mid \mathrm{rk}(x) = 2(n-i)\},\\ \mathrm{Mat}_{n,n-i}(\mathbb{F}_q) &= \{x \in \mathrm{Mat}_n(\mathbb{F}_q) \mid \mathrm{rk}(x) = n-i\},\\ \mathrm{Sym}_{n,n-i}(\mathbb{F}_q) &= \{x \in \mathrm{Sym}_n(\mathbb{F}_q) \mid \mathrm{rk}(x) = n-i\}.\end{aligned}$$

LEMMA 3.1. *For $i \in [n]_0$ we have*

$$|\mathrm{Alt}_{2n+\delta,2(n-i)}(\mathbb{F}_q)| = \binom{n}{i}_{q^{-2}} (q^{-2(i+\delta)-1}; q^{-2})_{n-i} \cdot q^{\binom{2n+\delta}{2} - \binom{2i+\delta}{2}}, \tag{3.2}$$

$$|\mathrm{Mat}_{n,n-i}(\mathbb{F}_q)| = \binom{n}{i}_{q^{-1}} (q^{-i-1}; q^{-1})_{n-i} \cdot q^{n^2 - i^2}, \tag{3.3}$$

$$|\mathrm{Sym}_{n,n-i}(\mathbb{F}_q)| = (q^{-2}; q^{-2})^{-1}_{\lfloor (n-i)/2 \rfloor} (q^{-i-1}; q^{-1})_{n-i} \cdot q^{\binom{n+1}{2} - \binom{i+1}{2}}. \tag{3.4}$$

Proof. These are all well-known; see, for instance, [9, Section 7] for (3.2), [27, Proposition 3.1] for (3.3) and [21, Lemma 10.3.1] for (3.4). $\square$

Definition 3.2. For $I = \{i_1, \dots, i_l\}_< \subseteq [n-1]_0$ set

$$f_{F_{n,\delta},I}(X) = \binom{n}{I}_{X^2} (X^{2(i_1+\delta)+1}; X^2)_{n-i_1}, \tag{3.5}$$

$$f_{G_n,I}(X) = \binom{n}{I}_{X} (X^{i_1+1}; X)_{n-i_1}, \tag{3.6}$$

$$f_{H_n,I}(X) = \left(\prod_{j=1}^{l} (X^2; X^2)^{-1}_{\lfloor \mu_j/2 \rfloor} \right) (X^{i_1+1}; X)_{n-i_1}.$$

Remark 3.3. We note that comparison with Lemma 3.1 shows that the polynomials $f_{\mathbf{G},\{i\}}(X)$, where $\mathbf{G} \in \{F_{n,\delta}, G_n, H_n\}$ and $i \in [n-1]_0$, give, in effect, the Poincaré polynomials of the determinantal varieties of (symmetric or antisymmetric) matrices of given rank. In Proposition 4.6 we give an interpretation of the polynomials $f_{F_{n,\delta},I}(X)$ and $f_{G_n,I}(X)$ in terms of generating functions over descent classes in Weyl groups of type B. In Conjecture 1.6 we record a conjectural formula of this type for the polynomials $f_{H_n,I}(X)$.

PROPOSITION 3.4. *Let $I \subseteq [n-1]_0$ and $\mathbf{r}_I \in \mathbb{N}^I$. Then we have*

$$|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(F_{n,\delta})| = f_{F_{n,\delta},I}(q^{-1})\, q^{\sum_{i \in I} r_i \left(\binom{2n+\delta}{2} - \binom{2i+\delta}{2} \right)}, \tag{3.7}$$

$$|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(G_n)| = f_{G_n,I}(q^{-1})\, q^{\sum_{i \in I} r_i (n^2 - i^2)}, \tag{3.8}$$

$$|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(H_n)| = f_{H_n,I}(q^{-1})\, q^{\sum_{i \in I} r_i \left(\binom{n+1}{2} - \binom{i+1}{2} \right)}. \tag{3.9}$$

Proof. We write $\rho_{i_l} : \mathrm{N}^{\mathfrak{o}}_{I,r_I}(\mathbf{G}) \to W_{r_{i_l}}(\mathfrak{o})$ for the map given by reduction of entries modulo $q^{r_{i_l}}$. We first prove (3.7). There are

$$\left| \mathrm{Alt}_{2n+\delta, 2(n-i_l)}(\mathbb{F}_q) \right| q^{(r_{i_l}-1)(\binom{2n+\delta}{2} - \binom{2i_l+\delta}{2})}$$

elements in $\rho_{i_l}(\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(F_{n,\delta}))$. Each such element has

$$q^{\left(\sum_{i \in I, i < i_l} r_i \right) \left(\binom{2n+\delta}{2} - \binom{2i_l+\delta}{2} \right)} |\mathrm{N}^{\mathfrak{o}}_{I \setminus \{i_l\}, r_{I \setminus \{i_l\}}}(F_{i_l,\delta})|$$

lifts to an element in $\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(F_{n,\delta})$. By (3.2) we thus get

$$\begin{aligned} |\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(F_{n,\delta})| &= |\mathrm{Alt}_{2n+\delta, 2(n-i_l)}(\mathbb{F}_q)| q^{(N-1)\left(\binom{2n+\delta}{2} - \binom{2i_l+\delta}{2} \right)} |\mathrm{N}^{\mathfrak{o}}_{I \setminus \{i_l\}, r_{I \setminus \{i_l\}}}(F_{i_l,\delta})| \\ &= \binom{n}{i_l}_{q^{-2}} (q^{-2(i_l+\delta)-1}; q^{-2})_{n-i_l} q^{N\left(\binom{2n+\delta}{2} - \binom{2i_l+\delta}{2} \right)} |\mathrm{N}^{\mathfrak{o}}_{I \setminus \{i_l\}, r_{I \setminus \{i_l\}}}(F_{i_l,\delta})|. \end{aligned}$$

Working recursively in this way, we obtain

$$
\begin{aligned}
&|\mathrm{N}^{\mathrm{o}}_{I,\mathbf{r}_I}(F_{n,\delta})| \\
&\quad = \prod_{j=1}^{l} \binom{i_{j+1}}{i_j}_{q^{-2}} (q^{-2(i_j+\delta)-1}; q^{-2})_{i_{j+1}-i_j} \cdot q^{\left(\sum_{i\in I, i\leq i_j} r_i\right)\left(\binom{2i_{j+1}+\delta}{2}-\binom{2i_j+\delta}{2}\right)} \\
&\quad = \binom{n}{I}_{q^{-2}} (q^{-2(i_1+\delta)-1}; q^{-2})_{n-i_1} \cdot q^{\sum_{i\in I} r_i\left(\binom{2n+\delta}{2}-\binom{2i+\delta}{2}\right)}.
\end{aligned}
$$

Next we prove (3.8). There are $|\mathrm{Mat}_{n,n-i_l}(\mathbb{F}_q)|q^{(r_{i_l}-1)(n^2-i_l^2)}$ elements in the set $\rho_{i_l}(\mathrm{N}^{\mathrm{o}}_{I,\mathbf{r}_I}(G_n))$. Each such element has

$$
q^{\left(\sum_{i\in I, i<i_l} r_i\right)(n^2-i_l^2)} |\mathrm{N}^{\mathrm{o}}_{I\setminus\{i_l\}, r_{I\setminus\{i_l\}}}(G_{i_l})|
$$

lifts to an element in $\mathrm{N}^{\mathrm{o}}_{I,\mathbf{r}_I}(G_n)$. By (3.3) we thus get

$$
\begin{aligned}
|\mathrm{N}^{\mathrm{o}}_{I,\mathbf{r}_I}(G_n)| &= |\mathrm{Mat}_{n,n-i_l}(\mathbb{F}_q)|q^{(N-1)(n^2-i_l^2)} |\mathrm{N}^{\mathrm{o}}_{I\setminus\{i_l\}, r_{I\setminus\{i_l\}}}(G_{i_l})| \\
&= \binom{n}{i_l}_{q^{-1}} (q^{-i_l-1}; q^{-1})_{n-i_l} q^{N(n^2-i_l^2)} |\mathrm{N}^{\mathrm{o}}_{I\setminus\{i_l\}, r_{I\setminus\{i_l\}}}(G_{i_l})|.
\end{aligned}
$$

Working recursively in this way, we obtain

$$
\begin{aligned}
|\mathrm{N}^{\mathrm{o}}_{I,\mathbf{r}_I}(G_n)| &= \prod_{j=1}^{l} \binom{i_{j+1}}{i_j}_{q^{-1}} (q^{-i_j-1}; q^{-1})_{i_{j+1}-i_j} \cdot q^{\left(\sum_{i\in I, i\leq i_j} r_i\right)(i_{j+1}^2-i_j^2)} \\
&= \binom{n}{I}_{q^{-1}} (q^{-i_1-1}; q^{-1})_{n-i_1} \cdot q^{\sum_{i\in I} r_i(n^2-i^2)}.
\end{aligned}
$$

Finally we prove (3.9). There are $|\mathrm{Sym}_{n,n-i_l}(\mathbb{F}_q)|q^{(r_{i_l}-1)\left(\binom{n+1}{2}-\binom{i_l+1}{2}\right)}$ elements in $\rho_{i_l}(\mathrm{N}^{\mathrm{o}}_{I,\mathbf{r}_I}(H_n))$. Each such element has

$$
q^{\left(\sum_{i\in I, i<i_l} r_i\right)\left(\binom{n+1}{2}-\binom{i_l+1}{2}\right)} |\mathrm{N}^{\mathrm{o}}_{I\setminus\{i_l\}, r_{I\setminus\{i_l\}}}(H_{i_l})|
$$

lifts to an element in $\mathrm{N}^{\mathrm{o}}_{I,\mathbf{r}_I}(H_n)$. By (3.4) we thus get

$$
\begin{aligned}
&|\mathrm{N}^{\mathrm{o}}_{I,\mathbf{r}_I}(H_n)| \\
&\quad = |\mathrm{Sym}_{n,n-i_l}(\mathbb{F}_q)|q^{(N-1)\left(\binom{n+1}{2}-\binom{i_l+1}{2}\right)} |\mathrm{N}^{\mathrm{o}}_{I\setminus\{i_l\}, r_{I\setminus\{i_l\}}}(H_{i_l})| \\
&\quad = (q^{-2}; q^{-2})^{-1}_{\lfloor (n-i_l)/2\rfloor} (q^{-i_l-1}; q^{-1})_{n-i_l} q^{N\left(\binom{n+1}{2}-\binom{i_l+1}{2}\right)} |\mathrm{N}^{\mathrm{o}}_{I\setminus\{i_l\}, r_{I\setminus\{i_l\}}}(H_{i_l})|.
\end{aligned}
$$

Working recursively in this way, we obtain

$$
\begin{aligned}
|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(H_n)| &= \prod_{j=1}^{l} (q^{-2};q^{-2})^{-1}_{\lfloor \mu_j/2 \rfloor} (q^{-i_j-1};q^{-1})_{i_{j+1}-i_j} \cdot q^{\left(\sum_{i\le i_j} r_i\right)\left(\binom{i_{j+1}+1}{2}-\binom{i_j+1}{2}\right)} \\
&= \left(\prod_{j=1}^{l} (q^{-2};q^{-2})^{-1}_{\lfloor \mu_j/2 \rfloor}\right) (q^{-i_1-1};q^{-1})_{n-i_1} \cdot q^{\sum_{i\in I} r_i \left(\binom{n+1}{2}-\binom{i+1}{2}\right)},
\end{aligned}
$$

and the proposition is proved. □

Remark 3.5. An alternative approach to the proof of Proposition 3.4 is to observe that suitable groups act on the sets $\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(\mathbf{G})$—viewed as subsets of $\mathrm{Mat}_r(\mathfrak{o}/\mathfrak{p}^N)$—with few orbits. For example, the group $\mathrm{GL}_{2n+\delta}(\mathfrak{o})$ acts transitively on each of the sets $\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(F_{n,\delta})$, viewed as sets of antisymmetric $(2n+\delta)\times(2n+\delta)$-matrices, via simultaneous row- and column-operations, that is via the action $(g,x)\mapsto gxg^{\mathrm{t}}$, reducing the computations of the numbers $|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(F_{n,\delta})|$ to stabilizer computations. A similar argument works for the groups of type G. For groups of type H, however, this approach leads one to consider equivalence classes of quadratic forms over compact discrete valuation rings of characteristic zero. This is straightforward if the residue field characteristic is odd, but much more complicated if $p=2$, obscuring the fact that the resulting formula (3.9) holds uniformly for all p. A similar phenomenon seems to occur in the computation of the integral (6.3) over the relative invariant of the prehomogeneous vector space of symmetric matrices; cf. the remark on the bottom of p. 177 in [21].

We now finish the proof of Theorem C. For $\zeta_{F_{n,\delta}(\mathfrak{o})}(s)$ we obtain, by (3.1) and (3.7),

$$
\begin{aligned}
\zeta_{F_{n,\delta}(\mathfrak{o})}(s) &= \sum_{I\subseteq[n-1]_0} \sum_{\mathbf{r}_I\in\mathbb{N}^I} |\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(F_{n,\delta})| q^{-s\sum_{i\in I} r_i(n-i)} \\
&= \sum_{I\subseteq[n-1]_0} f_{F_{n,\delta},I}(q^{-1}) \sum_{\mathbf{r}_I\in\mathbb{N}^I} q^{\sum_{i\in I} r_i\left(\binom{2n+\delta}{2}-\binom{2i+\delta}{2}-(n-i)s\right)} \\
&= \sum_{I\subseteq[n-1]_0} f_{F_{n,\delta},I}(q^{-1}) \prod_{i\in I} \frac{q^{\binom{2n+\delta}{2}-\binom{2i+\delta}{2}-(n-i)s}}{1-q^{\binom{2n+\delta}{2}-\binom{2i+\delta}{2}-(n-i)s}}.
\end{aligned}
$$

Similarly we obtain the following formulae for $\zeta_{G_n(\mathfrak{o})}(s)$ and $\zeta_{H_n(\mathfrak{o})}(s)$ by combining (3.1) with (3.8) and (3.9), respectively:

$$
\zeta_{G_n(\mathfrak{o})}(s) = \sum_{I\subseteq[n-1]_0} f_{G_n,I}(q^{-1}) \prod_{i\in I} \frac{q^{n^2-i^2-(n-i)s}}{1-q^{n^2-i^2-(n-i)s}},
$$

$$\zeta_{H_n(\mathfrak{o})}(s) = \sum_{I \subseteq [n-1]_0} f_{H_n,I}(q^{-1}) \prod_{i \in I} \frac{q^{\binom{n+1}{2}-\binom{i+1}{2}-(n-i)s}}{1-q^{\binom{n+1}{2}-\binom{i+1}{2}-(n-i)s}}.$$

This concludes the proof of Theorem C.

4. A multinomial-type identity and signed permutation statistics.

In this section we prove Proposition 1.5, express the polynomials $f_{F_{n,\delta},I}(X)$ and $f_{G_n,I}(X)$ defined in (3.5) and (3.6) in terms of generating functions over descent classes in Weyl groups of type B, and compute a number of joint distribution of statistics on Weyl groups of types B and A. We remark that Proposition 1.5 may well have a proof in the context of basic hypergeometric series. It resembles, for instance, the q-multinomial theorem; cf. [16, Exercise 1.3(ii)]. Lacking a suitable reference, we prove it here directly.

4.1. Proof of Proposition 1.5.

Recall from Section 1.5 that given a subset $I \subseteq \mathbb{N}$ we write I_0 for $I \cup \{0\}$ and for $a, b \in \mathbb{Z}$ we write $aI + b = b + aI$ for the set $\{ai + b \mid i \in I\}$. On several occasions we will use the bijections

$$\begin{aligned} \{I \mid I \subseteq [n-1-j]\} &\longleftrightarrow \{I \subseteq [n-1]_0 \mid \min\{I \cup \{n\}\} = j\} \\ I &\longmapsto I_0 + j, \end{aligned} \tag{4.1}$$

for $j \in [n-1]_0$. We will also make use of the following, easily verifiable identities:

$$\binom{n}{I_0+j}_X = \binom{n}{j}_X \binom{n-j}{I}_X, \quad \text{for } j \in [n-1]_0,\ I \subseteq [n-1-j], \tag{4.2}$$

$$\binom{n}{n-I}_X = \binom{n}{I}_X, \quad \text{for } I \subseteq [n]_0, \tag{4.3}$$

$$\binom{n}{j}_X = \binom{n}{j}_{X^{-1}} X^{j(n-j)}, \quad \text{for } j \in [n-1]_0. \tag{4.4}$$

The following is known as the q-binomial theorem.

LEMMA 4.1. *For $n \in \mathbb{N}$ we have*

$$(ZY;X)_n = \sum_{j=0}^{n} \binom{n}{j}_X Z^j (Z;X)_{n-j} (Y;X)_j.$$

Proof. See, for example, [15, Formula 1.16]. □

We begin by proving a special case of Proposition 1.5. We set, for $n \in \mathbb{N}$,

$$\mathcal{A}_n(X,Z) := \sum_{I \subseteq [n-1]} \binom{n}{I}_{X^{-1}} \prod_{i \in I} \mathrm{gp}((X^i Z)^{n-i}),$$

$$\mathcal{B}_n(X,Y,Z) := \sum_{I \subseteq [n-1]_0} \binom{n}{I}_{X^{-1}} (YX^{-i_1-1}; X^{-1})_{n-i_1} \prod_{i \in I} \mathrm{gp}((X^i Z)^{n-i}).$$

Note that $\mathcal{A}_n(X,Z) := \mathcal{B}_n(X,0,Z)(1-Z^n)$.

PROPOSITION 4.2. *For $n \in \mathbb{N}$ we have*

$$\mathcal{A}_n(X,Z) = \frac{1-Z^n}{(Z;X)_n} = \sum_{I \subseteq [n-1]} \binom{n}{I}_{X^{-1}} \prod_{i \in I} \mathrm{gp}((X^{n-i} Z)^{i}). \tag{4.5}$$

Proof. We prove the first equation by induction on n. For $n = 1$, we have

$$\mathcal{A}_1(X,Z) = \binom{1}{\varnothing}_{X^{-1}} = \frac{1-Z}{(Z;X)_1} = 1.$$

Suppose now that $n > 1$ and that the assertion holds for all $m < n$. The key idea for the proof is to re-organize the sum defining $\mathcal{A}_n$ according to the minima of the indexing subsets. Using the bijections (4.1), identity (4.2), the induction hypothesis and identity (4.4) we obtain

$$\begin{aligned}
\mathcal{A}_n(X,Z) &= \sum_{j=1}^{n} \sum_{\substack{I \subseteq [n-1] \\ \min\{I \cup \{n\}\} = j}} \binom{n}{I}_{X^{-1}} \prod_{i \in I} \mathrm{gp}((X^i Z)^{n-i}) \\
&= 1 + \sum_{j=1}^{n-1} \sum_{I \subseteq [n-1-j]} \binom{n}{I_0 + j}_{X^{-1}} \prod_{i \in I_0 + j} \mathrm{gp}((X^i Z)^{n-i}) \\
&= 1 + \sum_{j=1}^{n-1} \binom{n}{j}_{X^{-1}} \sum_{I \subseteq [n-1-j]} \binom{n-j}{I}_{X^{-1}} \prod_{i \in I_0 + j} \mathrm{gp}((X^i Z)^{n-i}) \\
&= 1 + \sum_{j=1}^{n-1} \binom{n}{j}_{X^{-1}} \mathrm{gp}((X^j Z)^{n-j}) \\
&\quad \times \sum_{I \subseteq [n-1-j]} \binom{n-j}{I}_{X^{-1}} \prod_{i \in I} \mathrm{gp}((X^{i+j} Z)^{n-i-j}) \\
&= 1 + \sum_{j=1}^{n-1} \binom{n}{j}_{X^{-1}} \mathrm{gp}((X^j Z)^{n-j}) \mathcal{A}_{n-j}(X, X^j Z)
\end{aligned}$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n}{j}_X X^{-j(n-j)} \mathrm{gp}((X^j Z)^{n-j}) \frac{1-(X^j Z)^{n-j}}{(X^j Z; X)_{n-j}}$$
$$= \sum_{j=1}^{n} \binom{n}{j}_X \frac{Z^{n-j}}{(X^j Z; X)_{n-j}}.$$

Writing this expression for $\mathcal{A}_n(X,Z)$ on the common denominator $(Z;X)_n$, we obtain

$$\mathcal{A}_n(X,Z)(Z;X)_n = \sum_{j=1}^{n} \binom{n}{j}_X Z^{n-j}(Z;X)_j = -Z^n + \sum_{j=0}^{n} \binom{n}{j}_X Z^{n-j}(Z;X)_j.$$

Changing j to $n-j$ and applying Lemma 4.1 with $Y=0$ yields

$$\sum_{j=0}^{n} \binom{n}{j}_X Z^{n-j}(Z;X)_j = \sum_{j=0}^{n} \binom{n}{j}_X Z^{j}(Z;X)_{n-j} = 1.$$

Thus

$$\mathcal{A}_n(X,Z) = \frac{1-Z^n}{(Z;X)_n}.$$

The second equation in (4.5) follows from the first equation, by changing i to $n-i$ and using (4.3). $\square$

We now prove Proposition 1.5 in general. We organize the sum defining $\mathcal{B}_n$ according to the minima of the indexing subsets. For $j \in [n]_0$, let

$$S_{n,j}(X,Z) := \sum_{\substack{I \subseteq [n-1]_0 \\ \min\{I \cup \{n\}\} = j}} \binom{n}{I}_{X^{-1}} \prod_{i \in I} \mathrm{gp}((X^i Z)^{n-i}).$$

We claim that, for all $j \in [n]_0$,

$$S_{n,j}(X,Z) = \binom{n}{j}_X \frac{Z^{n-j}}{(X^j Z; X)_{n-j}}. \tag{4.6}$$

This clearly holds for $j=n$, so assume $j \in [n-1]_0$. Due to the bijections (4.1) and the identities (4.2), (4.3), (4.5) and (4.4), we have

$$S_{n,j}(X,Z) = \sum_{I \subseteq [n-1-j]} \binom{n}{I_0 + j}_{X^{-1}} \prod_{i \in I_0 + j} \mathrm{gp}((X^i Z)^{n-i})$$
$$= \binom{n}{j}_{X^{-1}} \sum_{I \subseteq [n-1-j]} \binom{n-j}{I}_{X^{-1}} \prod_{i \in I_0 + j} \mathrm{gp}((X^i Z)^{n-i})$$

$$
\begin{aligned}
&= \binom{n}{j}_{X^{-1}} \mathrm{gp}((X^j Z)^{n-j}) \mathcal{A}_{n-j}(X, X^j Z) \\
&= \binom{n}{j}_{X^{-1}} \mathrm{gp}((X^j Z)^{n-j}) \frac{1-(X^j Z)^{n-j}}{(X^j Z;X)_{n-j}} \\
&= \binom{n}{j}_{X} \frac{Z^{n-j}}{(X^j Z;X)_{n-j}},
\end{aligned}
$$

establishing (4.6). This yields

$$
\begin{aligned}
\mathcal{B}_n(X,Y,Z) &= \sum_{j=0}^{n} (YX^{-j-1};X^{-1})_{n-j} S_{n,j}(X,Z) \\
&= \sum_{j=0}^{n} \binom{n}{j}_X (YX^{-j-1};X^{-1})_{n-j} \frac{Z^{n-j}}{(X^j Z;X)_{n-j}}.
\end{aligned}
\tag{4.7}
$$

Writing this expression for $\mathcal{B}_n(X,Y,Z)$ on the common denominator $(Z;X)_n$, we obtain

$$
\mathcal{B}_n(X,Y,Z)(Z;X)_n = \sum_{j=0}^{n} \binom{n}{j}_X (YX^{-j-1};X^{-1})_{n-j} Z^{n-j} (Z;X)_j. \tag{4.8}
$$

Using the identities

$$
(YX^{-j-1};X^{-1})_{n-j} = (X^{-n}Y;X)_{n-j},
$$

for $j \in [n-1]_0$, changing j to $n-j$ and applying Lemma 4.1 with Y replaced by $X^{-n}Y$, we can rewrite the right-hand side of (4.8) as

$$
\sum_{j=0}^{n} \binom{n}{j}_X Z^j (Z;X)_{n-j} (X^{-n}Y;X)_j = (X^{-n}YZ;X)_n.
$$

This proves Proposition 1.5.

Remark 4.3. Given a $\mathcal{T}$-group G, its *subgroup zeta function* is defined as the Dirichlet series

$$
\zeta_G^{<}(s) := \sum_{H \leq_f G} |G:H|^{-s},
$$

where s is a complex variable and the sum ranges over the subgroups of G of finite index; cf. [31, Chapter 15]. It is well known that the zeta function of $G = \mathbb{Z}^n$ equals

$$
\zeta_{\mathbb{Z}^n}^{<}(s) = \prod_{i=0}^{n-1} \zeta(s-i) = \prod_{p \text{ prime}} \frac{1}{(p^{-s};p)_n} = \prod_{p \text{ prime}} \mathcal{B}_n(p,0,p^{-s}),
$$

where $\zeta(s)$ is the Riemann zeta function; see, for instance, [31, Theorem 51.1]. The expression of the local zeta function of $\zeta^{<}_{\mathbb{Z}^n}(s)$ in terms of a sum, like the one defining the function $\mathcal{B}_n$, illustrates a general approach to the study of local (subgroup and representation) zeta functions of $\mathcal{T}$-groups developed in [43].

4.2. Some Weyl group generating functions. Our main source for background material on Coxeter groups is [6]. Let (W,S) be a finite Coxeter system, consisting of a finite Coxeter group W and a set S of Coxeter generators for W. For $w \in W$, the length of w, denoted by $l(w)$, is the minimal length of a word in elements of S representing w. Recall that the (right) descent set of w is defined as

$$D(w) := \{s \in S \mid l(ws) < l(w)\}.$$

For $I \subseteq S$, we denote by $W_I = \langle I \rangle$ the corresponding standard parabolic subgroup of W. We also have the so-called quotient

$$W^I := \{w \in W \mid D(w) \subseteq I^c\}. \tag{4.9}$$

The quotient W^I is the collection of the unique coset representatives of W_I of shortest length.

Consider now, specifically, Weyl groups of type B. Let $n \in \mathbb{N}$. Recall that we defined the group B_n as the group of all bijections w of the set $[\pm n]_0$ such that, for all $a \in [\pm n]_0$, $w(-a) = -w(a)$. Such bijections are determined by their values on the positive integers up to n, and thus B_n may be viewed as the group of signed permutations, that is monomial matrices with non-zero entries in $\{-1,1\}$. We write $w = [a_1,\dots,a_n]$ to mean that, for $i \in [n]$, $w(i) = a_i$. By $S = \{s_0, s_1, \dots, s_{n-1}\}$ we denote the set of standard Coxeter generators of B_n, that is $s_i = [1,\dots,i-1,i+1,i,i+2,\dots,n]$ for $i \in [n-1]$ and $s_0 = [-1,2,\dots,n]$. We frequently identify S with the interval $[n-1]_0$ in the obvious way. Given $w \in B_n$, we define $\operatorname{neg}(w) := \#\{i \in [n] \mid w(i) < 0\}$.

LEMMA 4.4. *Let (W,S) be a finite Coxeter system, $r \in \mathbb{N}$, and let $Y_1,\dots,Y_r$ and $Z_i, i \in S$, be independent variables. Let $h_w(\mathbf{Y}) = h_w(Y_1,\dots,Y_r)$, $w \in W$, be polynomials in $\mathbb{Q}[Y_1,\dots,Y_r]$. Then the following identity holds:*

$$\sum_{I\subseteq S}\left(\sum_{w\in W^{I^c}} h_w(\mathbf{Y})\right)\prod_{i\in I}\frac{Z_i}{1-Z_i} = \frac{\sum_{w\in W} h_w(\mathbf{Y})\prod_{i\in D_W(w)} Z_i}{\prod_{i\in S}(1-Z_i)}.$$

Proof. This is an easy application of the inclusion-exclusion principle. $\square$

4.2.1. Joint distribution of (l,neg) over descent classes of B_n.

LEMMA 4.5 (Reiner). *For $n\in\mathbb{N}$ and $I=\{i_1,\dots,i_l\}_<\subseteq[n-1]_0$ we have*

$$\sum_{w\in B_n^{I^c}} X^{l(w)}Y^{\mathrm{neg}(w)}=\binom{n}{I}_X(-YX^{i_1+1};X)_{n-i_1}. \tag{4.10}$$

Proof. This is proved by Reiner in [35, Lemma 3.1]; we just need to translate between Reiner's notation and ours. First, note that Reiner uses 'inv' to denote the length function l on B_n. Recall that we set $i_0=0$, and, for $j\in\{0,1,\dots,l\}$,

$$\mu_j=i_{j+1}-i_j. \tag{4.11}$$

The relations (4.11) provide the transition between Reiner's sets

$$\mathcal{S}=\{\mu_k,\mu_k+\mu_{k-1},\dots,\mu_k+\dots+\mu_1\}\subseteq[n]$$

and our sets $I\subseteq[n-1]_0$. Moreover, $\mu_0=i_1$. Given $\mathcal{S}\subseteq[n]$, an element $w\in B_n$ satisfies Reiner's relation "$D(w)\subseteq\mathcal{S}$" if and only if it satisfies $D(w)\subseteq I$, which is, by (4.9), equivalent to $w\in B_n^{I^c}$. From [35, Lemma 3.1] and formula (2) in its proof we thus obtain, partly in the notation of [35],

$$\begin{aligned}\sum_{w\in B_n^{I^c}} X^{l(w)}Y^{\mathrm{neg}(w)}&=\frac{[\hat{n}]!_{Y,X}}{[\hat{\mu}_0]!_{Y,X}[\mu_1]!_X\cdots[\mu_l]!_X}\\&=\frac{(-XY;X)_n[n]!_X}{(-XY;X)_{\mu_0}[\mu_0]_X![\mu_1]!_X\cdots[\mu_l]!_X}\\&=(-YX^{i_1+1};X)_{n-i_1}\frac{[n]!_X}{[i_1]_X![i_2-i_1]!_X\cdots[n-i_l]!_X}\\&=\binom{n}{I}_X(-YX^{i_1+1};X)_{n-i_1}.\end{aligned}$$ □

PROPOSITION 4.6. *Let $n\in\mathbb{N}$, $\delta\in\{0,1\}$ and let $I\subseteq[n-1]_0$. The polynomials $f_{F_{n,\delta},I}(X)$ and $f_{G_n,I}(X)$ defined in* (3.5) *and* (3.6) *satisfy the following identities:*

$$\begin{aligned}f_{F_{n,\delta},I}(X)&=\sum_{w\in B_n^{I^c}}(-1)^{\mathrm{neg}(w)}X^{(2l+(2\delta-1)\,\mathrm{neg})(w)},\\f_{G_n,I}(X)&=\sum_{w\in B_n^{I^c}}(-1)^{\mathrm{neg}(w)}X^{l(w)}.\end{aligned}$$

Proof. Replace (X,Y) in (4.10) by $(X^2,-X^{2\delta-1})$ in type F and by $(X,-1)$ in type G. □

4.2.2. Proof and discussion of Proposition 1.7. We recall that by Proposition 1.5 we have

$$\begin{aligned}\mathcal{B}_n(X,Y,Z) &:= \sum_{I\subseteq[n-1]_0} \binom{n}{I}_{X^{-1}} (YX^{-i_1-1};X^{-1})_{n-i_1} \prod_{i\in I} \mathrm{gp}((X^iZ)^{n-i}) \\ &= \frac{(X^{-n}YZ;X)_n}{(Z;X)_n}\end{aligned}$$

and by Lemma 4.5 we have, for all $I \subseteq [n-1]_0$,

$$\binom{n}{I}_{X^{-1}} (-YX^{-i_1-1};X^{-1})_{n-i_1} = \sum_{w\in B_n^{I^c}} X^{-l(w)}Y^{\mathrm{neg}(w)}.$$

Therefore Lemma 4.4, with $(W,S) = (B_n,\{s_0,\dots,s_{n-1}\})$, implies that

$$\begin{aligned}\mathcal{B}_n(X,-Y,X^nZ) &= \frac{\sum_{w\in B_n} X^{-l(w)}Y^{\mathrm{neg}(w)} \prod_{i\in D(w)} (X^{n+i}Z)^{n-i}}{\prod_{i=0}^{n-1}(1-(X^{n+i}Z)^{n-i})} \\ &= \prod_{i=0}^{n-1} \frac{1+X^iYZ}{1-X^{n+i}Z}.\end{aligned}$$

Hence

$$\sum_{w\in B_n} X^{(\sigma-l)(w)}Y^{\mathrm{neg}(w)}Z^{\mathrm{rmaj}(w)} = \prod_{i=0}^{n-1} \frac{(1+X^iYZ)(1-(X^{n+i}Z)^{n-i})}{1-X^{n+i}Z}, \tag{4.12}$$

concluding the proof of Proposition 1.7.

We see Proposition 1.7 in the context of a number of results in the literature which establish multivariate generating functions describing the joint distributions of various statistics on finite Weyl groups, sometimes “twisted” by 1-dimensional representations; see, for example, [35, 36, 5]. For instance we observe that setting $X = 1$ in (4.12) yields a special case of [36, Theorem 3.2]. Upon setting $Y = Z = 1$ in (4.12) we recover [42, Theorem 1.1] for Weyl groups of type B. In its generality, the latter result describes the generating function $\sum_{w\in W} X^{(\sigma-l)(w)}$ for a finite Weyl group W in terms of the simple root coordinates b_i and the Weyl group’s exponents. A twisted version of this result for Weyl groups of type B is the following:

COROLLARY 4.7.

$$\sum_{w\in B_n} (-1)^{\mathrm{neg}(w)} X^{(\sigma-l)(w)} = 0.$$

Proof. Set $Y = -1$ and $Z = 1$ in (4.12). $\square$

Analyzing our formulae for $\mathcal{A}_n(X,Z)$ yields formulae over Weyl groups of type A which are similar to (4.12). In the case of the Weyl group $W = S_n$, with Coxeter generating set $S = (s_1, \dots, s_{n-1})$ comprising the standard transpositions, the simple root coordinates b_i, $i \in [n-1]$, defined in Section 1 are given by $b_i = i(n-i)$; cf. [42, Remark 1.5] or [8, Plate I]. Therefore

$$\sigma(w) = \sum_{i \in D(w)} b_i = \sum_{i \in D(w)} i(n-i), \quad \text{for } w \in S_n. \tag{4.13}$$

Here we identified the generating set S with the interval $[n-1]$ in the obvious way. The statistics maj and rmaj on S_n are defined by setting, for $w \in S_n$, $\mathrm{maj}(w) = \sum_{i \in D(w)} i$ and $\mathrm{rmaj}(w) = \sum_{i \in D(w)} (n-i)$, respectively.

PROPOSITION 4.8.

$$\sum_{w \in S_n} X^{(\sigma - l)(w)} Z^{\mathrm{maj}(w)} = \sum_{w \in S_n} X^{(\sigma - l)(w)} Z^{\mathrm{rmaj}(w)} = \prod_{i=0}^{n-1} \frac{1 - (X^i Z)^{n-i}}{1 - X^i Z}. \tag{4.14}$$

Proof. By (4.5) we have

$$\mathcal{A}_n(X,Z) = \sum_{I \subseteq [n-1]} \binom{n}{I}_{X^{-1}} \prod_{i \in I} \mathrm{gp}(X^{i(n-i)} Z^i) = \frac{1 - Z^n}{(Z;X)_n},$$

and by [39, Proposition 1.3.17] we have, for $I \subseteq [n-1]$,

$$\binom{n}{I}_{X^{-1}} = \sum_{w \in S_n^{I^c}} X^{-l(w)}.$$

Therefore Lemma 4.4, with $(W,S) = (S_n, \{s_1, \dots, s_{n-1}\})$, implies that

$$\mathcal{A}_n(X,Z) = \frac{\sum_{w \in S_n} X^{-l(w)} \prod_{i \in D(w)} X^{i(n-i)} Z^i}{\prod_{i=1}^{n-1} (1 - (X^{n-i} Z)^i)} = \frac{\sum_{w \in S_n} X^{(\sigma - l)(w)} Z^{\mathrm{maj}(w)}}{\prod_{i=1}^{n-1} (1 - (X^{n-i} Z)^i)},$$

and so

$$\sum_{w \in S_n} X^{(\sigma - l)(w)} Z^{\mathrm{maj}(w)} = \mathcal{A}_n(X,Z) \prod_{i=1}^{n-1} (1 - (X^{n-i} Z)^i) = \prod_{i=0}^{n-1} \frac{1 - (X^i Z)^{n-i}}{1 - X^i Z}.$$

The equality involving rmaj follows similarly, using the second equality in (4.5). $\square$

Note that setting $X = 1$ in (4.14) yields the Poincaré polynomial of S_n, reflecting the well-known facts that the statistics maj and rmaj on S_n are Mahonian, that is, equidistributed with the length function l. Setting $Z = 1$ reproduces [42, Remark 1.5].

5. Proof of Theorem B.

5.1. Proof of Theorem B. We start by proving the formulae for the zeta functions of groups of type F and G given in (1.6) and (1.7). Considering the Euler product (1.4), it clearly suffices to establish the following result.

PROPOSITION 5.1. *For every non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, with $|\mathcal{O} : \mathfrak{p}| = q$, say, we have*

$$\zeta_{F_{n,\delta}(\mathcal{O}_\mathfrak{p})}(s) = \frac{(q^{-s}; q^2)_n}{(q^{2(n+\delta)-1-s}; q^2)_n}, \tag{5.1}$$

$$\zeta_{G_n(\mathcal{O}_\mathfrak{p})}(s) = \frac{(q^{-s}; q)_n}{(q^{n-s}; q)_n}. \tag{5.2}$$

Proof. By Theorem C, we have

$$\zeta_{F_{n,\delta}(\mathcal{O}_\mathfrak{p})}(s) = \sum_{I \subseteq [n-1]_0} \binom{n}{I}_{q^{-2}} (q^{-2(i_1+\delta)-1}; q^{-2})_{n-i_1} \prod_{i \in I} \mathrm{gp}(q^{(2(n+i+\delta)-1-s)(n-i)}),$$

$$\zeta_{G_n(\mathcal{O}_\mathfrak{p})}(s) = \sum_{I \subseteq [n-1]_0} \binom{n}{I}_{q^{-1}} (q^{-i_1-1}; q^{-1})_{n-i_1} \prod_{i \in I} \mathrm{gp}(q^{(n+i-s)(n-i)}).$$

Thus $\zeta_{F_{n,\delta}(\mathcal{O}_\mathfrak{p})}(s) = \mathcal{B}_n(q^2, q^{-2\delta+1}, q^{2(n+\delta)-1-s})$ and $\zeta_{G_n(\mathcal{O}_\mathfrak{p})}(s) = \mathcal{B}_n(q, 1, q^{n-s})$, and the claim follows from Proposition 1.5. □

The rest of this section is dedicated to proving the formulae for the zeta functions of groups of type H given in (1.8). Short of direct proof akin to the proof of Proposition 5.1, we reduce type H to type F; cf. Proposition 5.3. For this result we need some preparation.

Recall that $n = 2m + \varepsilon \in \mathbb{N}$ with $\varepsilon \in \{0, 1\}$, that we write $I = \{i_1, \dots, i_l\}_< \subseteq [n-1]_0$ and $J = \{j_1, \dots, j_k\} \subseteq [m-1]_0$ for subsets of $[n-1]_0$ and $[m-1]_0$, respectively, and the conventions that $i_0 = 0$ and $i_{l+1} = n$. We set

$$\begin{aligned} f_{n,I} &:= f_{H_n,I}(q^{-1}) \\ &:= \left(\prod_{j=1}^{l} (q^{-2}; q^{-2})^{-1}_{\lfloor (i_{j+1}-i_j)/2 \rfloor} \right) (q^{-i_1-1}; q^{-1})_{n-i_1} \quad \text{for } I \subseteq [n-1]_0, \end{aligned}$$

and

$$X_i := X_i(H_n) := q^{\binom{n+1}{2} - \binom{i+1}{2} - (n-i)s} \quad \text{for } i \in [n-1]_0. \tag{5.3}$$

Given $I \subset [n-1]_0$ we write Π_I for $\prod_{i \in I} \mathrm{gp}(X_i(H_n))$.

Theorem C represents the local factor $\zeta_{H_n(\mathcal{O}_\mathfrak{p})}(s)$ as a sum, indexed by the subsets of $[n-1]_0$. It is advantageous to organise this sum according to the fibres of the surjective "bisection map" φ, defined as follows. We set

$$\varphi : 2^{[n-1]_0} \longrightarrow 2^{[m-1]_0}, \quad I \longmapsto \left\{ \lfloor \frac{i+1-\varepsilon}{2} \rfloor \mid i \in I \right\} \setminus \{m\}.$$

We note that the "doubling map" $2^{[m-1]_0} \to 2^{[n-1]_0}, J \mapsto 2J+\varepsilon$, is a section of this map, so each fibre $\varphi^{-1}(J)$ contains the set $2J+\varepsilon \subseteq [n-1]_0$. The proof of Proposition 5.3 hinges on the following technical result. Its proof will occupy the bulk of this section.

LEMMA 5.2. *For all $J \subseteq [m-1]_0$ we have*

$$\sum_{I \in \varphi^{-1}(J)} f_I \Pi_I = (1 + (\underline{n})\mathrm{gp}(X_{n-1}))\, f_{2J+\varepsilon}\Pi_{2J+\varepsilon} = \frac{1-q^{-s}}{1-q^{n-s}} f_{2J+\varepsilon}\Pi_{2J+\varepsilon}. \tag{5.4}$$

Informally speaking, Lemma 5.2 "eliminates" occurrences of the terms X_i, where $i \in [n-2]_0 \setminus (2[m-1]_0 + \varepsilon)$, and simplifies the sum on the left-hand side of (5.4), which has roughly $3^{|J|}$ terms.

Proof. The second equation in (5.4) is clear. The proof of the first equation requires slightly different arguments in the cases $\varepsilon = 0$ and $\varepsilon = 1$. Nevertheless we treat both cases in parallel. We start with an observation in the case $\varepsilon = 0$. Let $J \subseteq [m-1]_0$. We note that, if $\varepsilon = 0$, we have $0 \in J$ if and only if $0 \in I$ for all $I \in \varphi^{-1}(J)$. We claim that we may, without loss of generality, assume in this case that $0 \notin J$. Indeed, if $0 \in J$, we write $J = \{0\} \cup J'$, where $J' = J \setminus \{0\}$, and similarly, for each $I \in \varphi^{-1}(J)$, we write $I = \{0\} \cup I'$, where $I' = I \setminus \{0\}$. Set $j_1' := \min\{J' \cup \{m\}\}$. It now suffices to observe that for all $I \subseteq \varphi^{-1}(J)$ (including the set $I = 2J$) one has $f_I = (q^{-1}; q^{-2})_{j_1'} f_{I'}$. Therefore if (5.4) holds for J' it also holds for J. Indeed, we then have

$$\begin{aligned} \sum_{I \in \varphi^{-1}(J)} f_I \Pi_I &= (q^{-1}; q^{-2})_{j_1'}\, \mathrm{gp}(X_0) \sum_{I \in \varphi^{-1}(J')} f_I \Pi_I \\ &= (q^{-1}; q^{-2})_{j_1'}\, \mathrm{gp}(X_0)\, (1 + (\underline{n})\mathrm{gp}(X_{n-1}))\, f_{2J'}\Pi_{2J'} \\ &= (1 + (\underline{n})\mathrm{gp}(X_{n-1}))\, f_{2J}\Pi_{2J}. \end{aligned} \tag{5.5}$$

We thus assume henceforth that $0 \notin J$ if $\varepsilon = 0$.

We return to the general situation with $\varepsilon \in \{0, 1\}$. A key role in the proof is played by the relations

$$X_{2j+\varepsilon-1} = q^{-2(m-j)} X_{2j+\varepsilon} X_{n-1}, \quad j \in \{1-\varepsilon, \dots, m-1\}, \tag{5.6}$$

which are immediate from the definitions (5.3). The validity of Lemma 5.2 depends only on these relations, and not on the particular "numerical data" (X_i). The first equation of (5.4) is thus equivalent to an equality in the quotient of the ring

$\mathbb{Z}[q^{-1}, X_1, \dots, X_{n-1}]$ by the ideal generated by the relations (5.6). The independence from the precise numerical data (X_i) is used in an inductive argument later in the proof.

We prove Lemma 5.2 by induction on $|J|$. We first deal with the special case $J = \varnothing$, the base for our induction. It is clear that $\varphi^{-1}(\varnothing) = \{\varnothing, \{n-1\}\}$ and easily checked that $f_\varnothing = 1$ and $f_{\{n-1\}} = (\underline{n})$, so that

$$\sum_{I \in \varphi^{-1}(\varnothing)} f_I \Pi_I = f_\varnothing + f_{\{n-1\}} \mathrm{gp}(X_{n-1}) = 1 + (\underline{n}) \mathrm{gp}(X_{n-1}) \tag{5.7}$$

as claimed.

Assume now that $k = |J| \geq 1$ and write $J = \{j_1, \dots, j_k\}_<$. Note that, by assumption, $j_1 > 0$ if $\varepsilon = 0$. Our strategy is to split up the fibre $\varphi^{-1}(J)$ into three disjoint sets of equal size $2 \cdot 3^{k-1}$, according to the intersection with $T_1 := \{2j_1 + \varepsilon - 1, 2j_1 + \varepsilon\}$. We define

$$\begin{aligned}
\mathcal{I}_1 &= \{I \in \varphi^{-1}(J) \mid I \cap T_1 = \{2j_1 + \varepsilon\}\}, \\
\mathcal{I}_2 &= \{I \in \varphi^{-1}(J) \mid I \cap T_1 = \{2j_1 + \varepsilon - 1, 2j_1 + \varepsilon\}\}, \\
\mathcal{I}_3 &= \{I \in \varphi^{-1}(J) \mid I \cap T_1 = \{2j_1 + \varepsilon - 1\}\}.
\end{aligned}$$

Hence $\varphi^{-1}(J) = \mathcal{I}_1 \,\dot\cup\, \mathcal{I}_2 \,\dot\cup\, \mathcal{I}_3$, as $I \cap T_1 \neq \varnothing$ for all $I \in \varphi^{-1}(J)$. For $r \in \{1,2,3\}$ we set $\mathcal{S}_r = \sum_{I \in \mathcal{I}_r} f_I \Pi_I$ so that $\sum_{I \in \varphi^{-1}(J)} f_I \Pi_I = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3$. We claim that

$$\mathcal{S}_1 = f_{2J+\varepsilon} \Pi_{2J+\varepsilon} \big(1 + (\underline{2(m-j_1)}) \mathrm{gp}(X_{n-1})\big), \tag{5.8}$$

$$\mathcal{S}_2 = (\underline{2j_1 + \varepsilon}) \mathrm{gp}(X_{2j_1+\varepsilon-1}) \mathcal{S}_1, \tag{5.9}$$

$$\mathcal{S}_3 = f_{2J+\varepsilon} (\underline{2j_1 + \varepsilon}) \mathrm{gp}(X_{2j_1+\varepsilon-1}) \Pi_{2(J \setminus \{j_1\})+\varepsilon} \big(1 + \mathrm{gp}(X_{n-1})\big). \tag{5.10}$$

To prove (5.8) we note that for all $I \in \mathcal{I}_1$ we have

$$f_{n,I} = \binom{n}{2j_1 + \varepsilon}_{q^{-1}} f_{2(m-j_1), I - 2j_1 - \varepsilon},$$

and hence

$$\begin{aligned}
\mathcal{S}_1 &= \binom{n}{2j_1 + \varepsilon}_{q^{-1}} \sum_{I \in \mathcal{I}_1} f_{2(m-j_1), I - 2j_1 - \varepsilon} \Pi_I \\
&= \binom{n}{2j_1 + \varepsilon}_{q^{-1}} f_{2(m-j_1), 2J - 2j_1} \Pi_{2J+\varepsilon} (1 + (\underline{2(m-j_1)}) \mathrm{gp}(X_{n-1})) \\
&= f_{2J+\varepsilon} \Pi_{2J+\varepsilon} \big(1 + (\underline{2(m-j_1)}) \mathrm{gp}(X_{n-1})\big).
\end{aligned}$$

Here, the second equality uses the induction hypothesis for $J - j_1 \subseteq [m - j_1 - 1]_0$, $\varepsilon = 0$. This is justified as the terms $X_i(H_n)$ for $2j_1 + \varepsilon \leq i < n$ satisfy the same relations given by (5.6) as the terms $X_i(H_{2(m-j_1)})$ for $0 \leq i < 2(m - j_1)$, and because $|(J - j_1) \cap \mathbb{N}| < k$.

To prove (5.9) it suffices to observe that $f_I = \underline{(2j_1+\varepsilon)} f_{I\setminus\{2j_1+\varepsilon-1\}}$ for all $I \in \mathcal{I}_2$.

To prove (5.10) we proceed by a second induction on $|J| = k$, the induction base $k = 1$ being a straightforward computation which we leave to the reader. If $k > 1$ we partition the set $\mathcal{I}_3$. We let $T_2 = \{2j_2+\varepsilon-1, 2j_2+\varepsilon\}$ and define

$$\begin{aligned}
\mathcal{I}_{3,1} &= \{I \in \mathcal{I}_3 \mid I \cap T_2 = \{2j_2+\varepsilon-1\}\},\\
\mathcal{I}_{3,2} &= \{I \in \mathcal{I}_3 \mid I \cap T_2 = \{2j_2+\varepsilon-1, 2j_2+\varepsilon\}\},\\
\mathcal{I}_{3,3} &= \{I \in \mathcal{I}_3 \mid I \cap T_2 = \{2j_2+\varepsilon\}\}.
\end{aligned}$$

Note that $I \cap T_2 \neq \varnothing$ for all $I \in \mathcal{I}_3$. For $r \in \{1,2,3\}$ we set $\mathcal{S}_{3,r} = \sum_{I\in\mathcal{I}_{3,r}} f_I \Pi_I$ so that $\mathcal{S}_3 = \mathcal{S}_{3,1} + \mathcal{S}_{3,2} + \mathcal{S}_{3,3}$. We claim that

(5.11)
$$\mathcal{S}_{3,1} = f_{2J+\varepsilon}\underline{(2j_1+\varepsilon)}\mathrm{gp}(X_{2j_1+\varepsilon-1})\Pi_{2(J\setminus\{j_1\})+\varepsilon}\big(1+\underline{(2(m-j_2))}\mathrm{gp}(X_{n-1})\big),$$

(5.12)
$$\mathcal{S}_{3,2} = \mathrm{gp}(X_{2j_2+\varepsilon-1})\mathcal{S}_{3,1},$$

(5.13)
$$\mathcal{S}_{3,3} = f_{2J+\varepsilon}\underline{(2j_1+\varepsilon)}\mathrm{gp}(X_{2j_1+\varepsilon-1})\,\mathrm{gp}(X_{2j_2+\varepsilon-1})\Pi_{2(J\setminus\{j_1,j_2\})+\varepsilon}\big(1+\mathrm{gp}(X_{n-1})\big).$$

To prove (5.11) we observe that for all $I \in \mathcal{I}_{3,1}$ we have

$$f_I = f_{I\setminus\{2j_1-1\}}(q^{-2j_1-1};q^{-2})_{j_2-j_1}\underline{(2j_1)}\binom{j_2}{j_1}_{q^{-2}} \quad \text{if } \varepsilon = 0, \tag{5.14}$$

$$f_I = f_{I\setminus\{2j_1\}}(q^{-2j_1-1};q^{-2})_{j_2-j_1}\underline{(2j_2+1)}\binom{j_2}{j_1}_{q^{-2}} \quad \text{if } \varepsilon = 1. \tag{5.15}$$

Furthermore, for all $J \subseteq [m-1]_0$ we have

$$f_{2J} = f_{2J\setminus 2\{j_1\}}(q^{-2j_1-1};q^{-2})_{j_2-j_1}\binom{j_2}{j_1}_{q^{-2}}, \tag{5.16}$$

$$f_{2J+1} = f_{2(J\setminus 2\{j_1\})+1}(q^{-2j_1-1};q^{-2})_{j_2-j_1}\binom{j_2}{j_1}_{q^{-2}}\frac{\underline{(2j_2+1)}}{\underline{(2j_1+1)}}, \tag{5.17}$$

so that if $\varepsilon = 0$ we have, using (5.14), (5.8) for $J\setminus\{j_1\}$, and (5.16),

$$\begin{aligned}
\mathcal{S}_{3,1} &= (q^{-2j_1-1};q^{-2})_{j_2-j_1}\underline{(2j_1)}\binom{j_2}{j_1}_{q^{-2}}\mathrm{gp}(X_{2j_1-1})\sum_{I\in\mathcal{I}_{3,1}} f_{I\setminus\{2j_1-1\}}\Pi_{I\setminus\{2j_1-1\}}\\
&= (q^{-2j_1-1};q^{-2})_{j_2-j_1}\underline{(2j_1)}\binom{j_2}{j_1}_{q^{-2}}\mathrm{gp}(X_{2j_1-1})f_{2J\setminus 2\{j_1\}}\Pi_{2(J\setminus\{j_1\})}\\
&\quad\cdot\big(1+\underline{(2(m-j_2))}\mathrm{gp}(X_{n-1})\big)\\
&= f_{2J}\underline{(2j_1)}\mathrm{gp}(X_{2j_1-1})\Pi_{2(J\setminus\{j_1\})}\big(1+\underline{(2(m-j_2))}\mathrm{gp}(X_{n-1})\big),
\end{aligned}$$

as claimed. If $\varepsilon = 1$ we have, using (5.15), (5.8) for $J \setminus \{j_1\}$, and (5.17),

$$\begin{aligned}
\mathcal{S}_{3,1} &= (q^{-2j_1-1}; q^{-2})_{j_2-j_1} \underline{(2j_2+1)} \binom{j_2}{j_1}_{q^{-2}} \mathrm{gp}(X_{2j_1}) \sum_{I \in \mathcal{I}_{3,1}} f_{I\setminus\{2j_1\}} \Pi_{I\setminus\{2j_1\}} \\
&= (q^{-2j_1-1}; q^{-2})_{j_2-j_1} \underline{(2j_2+1)} \binom{j_2}{j_1}_{q^{-2}} \mathrm{gp}(X_{2j_1}) f_{2J\setminus 2\{j_1\}+1} \Pi_{2(J\setminus\{j_1\})+1} \\
&\quad \cdot \big(1 + (\underline{2(m-j_2)})\mathrm{gp}(X_{n-1})\big) \\
&= f_{2J+1} \underline{(2j_1+1)} \mathrm{gp}(X_{2j_1}) \Pi_{2(J\setminus\{j_1\})+1} \big(1 + (\underline{2(m-j_2)})\mathrm{gp}(X_{n-1})\big),
\end{aligned}$$

as claimed. This establishes (5.11).

To prove (5.12) it suffices to observe that for all $I \in \mathcal{I}_{3,2}$ we have $f_I = f_{I\setminus\{2j_2+\varepsilon-1\}}$.

To prove (5.13) we note that for all $I \in \mathcal{I}_{3,3}$ we have

$$f_I = f_{I\setminus\{2j_1-1\}} (q^{-2j_1-1}; q^{-2})_{j_2-j_1} \binom{j_2}{j_1}_{q^{-2}} \frac{\underline{(2j_1)}}{\underline{(2j_2)}} \quad \text{if } \varepsilon = 0, \tag{5.18}$$

$$f_I = f_{I\setminus\{2j_1\}} (q^{-2j_1-1}; q^{-2})_{j_2-j_1} \binom{j_2}{j_1}_{q^{-2}} \quad \text{if } \varepsilon = 1, \tag{5.19}$$

Hence, if $\varepsilon = 0$ we have, using (5.18), the second induction hypothesis for the formula (5.10) for $\mathcal{S}_3$ (for $J \setminus \{j_1\}$, $\varepsilon = 0$), and (5.16),

$$\begin{aligned}
\mathcal{S}_{3,3} &= (q^{-2j_1-1}; q^{-2})_{j_2-j_1} \binom{j_2}{j_1}_{q^{-2}} \frac{\underline{(2j_1)}}{\underline{(2j_2)}} \mathrm{gp}(X_{2j_1-1}) \sum_{I \in \mathcal{I}_{3,3}} f_{I\setminus\{2j_1-1\}} \Pi_{I\setminus\{2j_1-1\}} \\
&= (q^{-2j_1-1}; q^{-2})_{j_2-j_1} \binom{j_2}{j_1}_{q^{-2}} \frac{\underline{(2j_1)}}{\underline{(2j_2)}} \\
&\quad \cdot \mathrm{gp}(X_{2j_1-1}) f_{2J\setminus 2\{j_1\}} \underline{(2j_2)} \mathrm{gp}(X_{2j_2-1}) \Pi_{2(J\setminus\{j_1,j_2\})} \big(1 + \mathrm{gp}(X_{n-1})\big) \\
&= f_{2J} \underline{(2j_1)} \mathrm{gp}(X_{2j_1-1})\, \mathrm{gp}(X_{2j_2-1}) \Pi_{2(J\setminus\{j_1,j_2\})} \big(1 + \mathrm{gp}(X_{n-1})\big),
\end{aligned}$$

as claimed. If $\varepsilon = 1$ we have, using (5.19), the second induction hypothesis for the formula (5.10) for $\mathcal{S}_3$ (for $J \setminus \{j_1\}$, $\varepsilon = 1$), and (5.17),

$$\begin{aligned}
\mathcal{S}_{3,3} &= (q^{-2j_1-1}; q^{-2})_{j_2-j_1} \binom{j_2}{j_1}_{q^{-2}} \mathrm{gp}(X_{2j_1}) \sum_{I \in \mathcal{I}_{3,3}} f_{I\setminus\{2j_1\}} \Pi_{I\setminus\{2j_1\}} \\
&= (q^{-2j_1-1}; q^{-2})_{j_2-j_1} \binom{j_2}{j_1}_{q^{-2}} \mathrm{gp}(X_{2j_1}) \\
&\quad \cdot f_{2(J\setminus\{j_1\})+1} \underline{(2j_2+1)} \mathrm{gp}(X_{2j_2}) \Pi_{2(J\setminus\{j_1,j_2\})+1} \big(1 + \mathrm{gp}(X_{n-1})\big) \\
&= f_{2J+1} \underline{(2j_1+1)} \mathrm{gp}(X_{2j_1})\, \mathrm{gp}(X_{2j_2}) \Pi_{2(J\setminus\{j_1,j_2\})+1} \big(1 + \mathrm{gp}(X_{n-1})\big),
\end{aligned}$$

as claimed. This establishes (5.13).

It remains to simplify the sums $\mathcal{S}_{31}+\mathcal{S}_{32}+\mathcal{S}_{33}$ and $\mathcal{S}=\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3$. For both calculations we use the following direct consequence of the relations (5.6): for all $j\in\{1+\varepsilon,\dots,m-1\}$ we have

(5.20)
$$\begin{aligned}&\mathrm{gp}(X_{2j+\varepsilon-1})\big(1+\mathrm{gp}(X_{2j+\varepsilon})+\mathrm{gp}(X_{n-1})+(\underline{2(m-j)})\mathrm{gp}(X_{2j+\varepsilon})\mathrm{gp}(X_{n-1})\big)\\&\qquad=q^{-2(m-j)}\mathrm{gp}(X_{2j+\varepsilon})\mathrm{gp}(X_{n-1}).\end{aligned}$$

Thus, using (5.20) for $j=j_2$, we have

$$\begin{aligned}\mathcal{S}_3&=\mathcal{S}_{3,1}+\mathcal{S}_{3,2}+\mathcal{S}_{3,3}\\&=f_{2J+\varepsilon}(\underline{2j_1+\varepsilon})\mathrm{gp}(X_{2j_1+\varepsilon-1})\Pi_{2(J\setminus\{j_1,j_2\})+\varepsilon}\\&\quad\times\big(\mathrm{gp}(X_{2j_2+\varepsilon})(1+(\underline{2(m-j_2)})\mathrm{gp}(X_{n-1}))+\mathrm{gp}(X_{2j_2+\varepsilon-1})\\&\quad\times\big(1+\mathrm{gp}(X_{2j_2+\varepsilon})+\mathrm{gp}(X_{n-1})+(\underline{2(m-j_2)})\mathrm{gp}(X_{2j_2+\varepsilon})\mathrm{gp}(X_{n-1})\big)\big)\\&=f_{2J+\varepsilon}(\underline{2j_1+\varepsilon})\mathrm{gp}(X_{2j_1+\varepsilon-1})\Pi_{2(J\setminus\{j_1\})+\varepsilon}\\&\quad\cdot\big(1+(\underline{2(m-j_2)})\mathrm{gp}(X_{n-1})+q^{-2(m-j_2)}\mathrm{gp}(X_{n-1})\big)\\&=f_{2J+\varepsilon}(\underline{2j_1+\varepsilon})\mathrm{gp}(X_{2j_1+\varepsilon-1})\Pi_{2(J\setminus\{j_1\})+\varepsilon}\big(1+\mathrm{gp}(X_{n-1})\big)\end{aligned}$$

as claimed in (5.10).

It remains to simplify the sum $\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3$. Using (5.20) for $j=j_1$, we obtain

$$\begin{aligned}\mathcal{S}&=\mathcal{S}_1+\mathcal{S}_2+\mathcal{S}_3\\&=f_{2J+\varepsilon}\Pi_{2(J\setminus\{j_1\})+\varepsilon}\big(\mathrm{gp}(X_{2j_1+\varepsilon})(1+(\underline{2(m-j_1)})\mathrm{gp}(X_{n-1}))\\&\quad+(\underline{2j_1+\varepsilon})\mathrm{gp}(X_{2j_1+\varepsilon-1})\big(1+\mathrm{gp}(X_{2j_1+\varepsilon})+\mathrm{gp}(X_{n-1})\\&\quad+(\underline{2(m-j_1)})\mathrm{gp}(X_{2j_1+\varepsilon})\mathrm{gp}(X_{n-1})\big)\big)\\&=f_{2J+\varepsilon}\Pi_{2J+\varepsilon}\big(1+(\underline{2(m-j_1)})\mathrm{gp}(X_{n-1})+(\underline{2j_1+\varepsilon})q^{-2(m-j_1)}\mathrm{gp}(X_{n-1})\big)\\&=f_{2J+\varepsilon}\Pi_{2J+\varepsilon}\big(1+(\underline{n})\mathrm{gp}(X_{n-1})\big).\end{aligned}$$

This concludes the proof of the lemma. □

Given the Euler product (1.4), the multiplicative formulae (1.8) for $\zeta_{H_n(\mathcal{O}_\mathfrak{p})}(s)$ given in Theorem B now follow easily from the following result.

PROPOSITION 5.3. *For $n=2m+\varepsilon\in\mathbb{N}$ with $\varepsilon\in\{0,1\}$, we have*

$$\zeta_{H_n(\mathcal{O}_\mathfrak{p})}(s)=\frac{\zeta_{K,\mathfrak{p}}(s-n)}{\zeta_{K,\mathfrak{p}}(s)}\zeta_{F_{m,\varepsilon}(\mathcal{O}_\mathfrak{p})}(2s-2)=\frac{1-q^{-s}}{1-q^{n-s}}\cdot\frac{(q^{2-2s};q^2)_m}{(q^{2(m+\varepsilon)+1-2s};q^2)_m}.$$

Proof. The second equality is, of course, Proposition 5.1, with (n,δ) replaced by (m,ε). We prove the first equality. By Theorem C, with (n,δ) replaced by

(m,ε), we have

$$\zeta_{F_{m,\varepsilon}(\mathcal{O}_{\mathfrak{p}})}(s) = \sum_{J\subseteq[m-1]_0} f_{F_{m,\varepsilon},J}(q^{-1}) \prod_{j\in J} \mathrm{gp}(X_j(F_{m,\varepsilon})),$$

where $X_j(F_{m,\varepsilon}) = q^{\binom{2m+\varepsilon}{2}-\binom{2j+\varepsilon}{2}-(n-j)s}$. It follows from Definition 3.2 that, for all $J \subseteq [m-1]_0$,

$$f_{H_n,2J+\varepsilon}(q^{-1}) = \binom{m}{J}_{q^{-2}} (q^{-2(j_1+\varepsilon)-1}; q^{-2})_{m-j_1} = f_{F_{m,\varepsilon},J}(q^{-1})$$

and, for all $j \in [m-1]_0$,

$$\begin{aligned}
X_j(F_{m,\varepsilon})|_{s\to 2s-2} &= q^{\binom{2m+\varepsilon}{2}-\binom{2j+\varepsilon}{2}-(2s-2)(m-j)} \\
&= q^{\binom{2m+\varepsilon}{2}+2m+\varepsilon-\left(\binom{2j+\varepsilon}{2}+2j+\varepsilon\right)-2s(m-j)} \\
&= q^{\binom{2m+1+\varepsilon}{2}-\binom{2j+1+\varepsilon}{2}-2s(m-j)} \\
&= X_{2j+\varepsilon}(H_n).
\end{aligned}$$

We thus have, by Theorem C and Lemma 5.2,

$$\begin{aligned}
\zeta_{H_n(\mathcal{O}_{\mathfrak{p}})}(s) &= \sum_{I\subseteq[n-1]_0} f_{H_n,I}(q^{-1}) \prod_{i\in I} \mathrm{gp}(X_i(H_n)) \\
&= \sum_{J\subseteq[m-1]_0} \sum_{I\in\varphi^{-1}(J)} f_{H_n,I}(q^{-1}) \prod_{i\in I} \mathrm{gp}(X_i(H_n)) \\
&= (1+(\underline{n})\mathrm{gp}(X_{n-1}(H_n))) \sum_{J\subseteq[m-1]_0} f_{H_{n,2J+\varepsilon}}(q^{-1}) \prod_{j\in J} \mathrm{gp}(X_{2j+\varepsilon}(H_n)) \\
&= \frac{1-q^{-s}}{1-q^{n-s}} \sum_{J\subseteq[m-1]_0} f_{F_{m,\varepsilon},J}(q^{-1}) \prod_{j\in J} \mathrm{gp}(X_j(F_{m,\varepsilon})|_{s\to 2s-2}) \\
&= \zeta_{K,\mathfrak{p}}(s-n)\zeta_{K,\mathfrak{p}}(s)^{-1}\zeta_{F_{m,\varepsilon}(\mathcal{O}_{\mathfrak{p}})}(2s-2).
\end{aligned}$$

This proves the proposition. □

This concludes the proof of Theorem B.

COROLLARY 5.4. *For $n = 2m+\varepsilon \in \mathbb{N}$ with $\varepsilon \in \{0,1\}$, we have*

$$\sum_{I\subseteq[n-1]_0} \left(\prod_{j=1}^{l} (X^{-4};X^{-4})^{-1}_{\lfloor \mu_j/2\rfloor}\right) \left(X^{-2(i_1+1)};X^{-2}\right)_{n-i_1} \prod_{i\in I} \mathrm{gp}\left((X^iZ)^{n-i}\right)$$
$$= \frac{1-X^{-n-1}Z}{1-X^{n-1}Z} \cdot \frac{\left(X^{2(-n+1)}Z^2;X^4\right)_m}{\left(X^{\varepsilon}Z^2;X^4\right)_m}.$$

Proof. For $X = q^{1/2}$ and $Z = q^{(n+1)/2-s}$ the identity follows from Theorem C together with Proposition 5.3. As it holds for infinitely many values of q and s, it therefore holds as a formal identity. $\square$

Corollary 5.4 is analogous to Proposition 1.5. The identity was found by working backwards from Proposition 5.3 and Theorem C. An independent proof of Corollary 5.4 would yield an alternative proof of Proposition 5.3, similar to the proof of Proposition 5.1.

5.2. Another generating function on B_n. We record a corollary of the existence of both "additive" and "multiplicative" expressions for the local zeta functions of groups of type H, together with Conjecture 1.6. Recall that $n = 2m+\varepsilon \in \mathbb{N}$, with $\varepsilon \in \{0,1\}$.

PROPOSITION 5.5. *If Conjecture 1.6 holds then*

$$\begin{aligned}
&\sum_{w\in B_n} (-1)^{l(w)} X^{((\sigma+\mathrm{rmaj})/2-L)(w)} Z^{\mathrm{rmaj}(w)} \\
&\qquad = \frac{(1-Z)(X^2Z^2;X^2)_m}{(X^{2(m+\varepsilon)+1}Z^2;X^2)_m} \prod_{i=0}^{n-2} (1-(X^{(n+i+1)/2}Z)^{n-i}).
\end{aligned} \tag{5.21}$$

Proof. Let $\mathfrak{p}$ be a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, with $|\mathcal{O}:\mathfrak{p}| = q$, say. By Theorem C and Proposition 5.3 we have

$$\begin{aligned}
\zeta_{H_n(\mathcal{O}_\mathfrak{p})}(s) &= \sum_{I\subseteq[n-1]_0} f_{H_n,I}(q^{-1}) \prod_{i\in I} \mathrm{gp}(q^{\binom{n+1}{2}-\binom{i+1}{2}-(n-i)s}) \\
&= \frac{(1-q^{-s})}{(1-q^{n-s})} \frac{(q^{2-2s};q^2)_m}{(q^{2(m+\varepsilon)+1-2s};q^2)_m},
\end{aligned}$$

and, assuming Conjecture 1.6, we have

$$f_{H_n,I}(q^{-1}) = \sum_{w\in B_n^{I^c}} (-1)^{l(w)} q^{-L(w)}.$$

Therefore Lemma 4.4 with $(W,S) = (B_n, \{s_0,\dots,s_{n-1}\})$ implies that

$$\zeta_{H_n(\mathcal{O}_\mathfrak{p})}(s) = \frac{\sum_{w\in B_n} (-1)^{l(w)} q^{-L(w)} \prod_{i\in D(w)} q^{((n+i+1)/2-s)(n-i)}}{\prod_{i=0}^{n-1}(1-q^{((n+i+1)/2-s)(n-i)})},$$

and so

$$\begin{aligned}
&\sum_{w\in B_n} (-1)^{l(w)} q^{((\sigma+\mathrm{rmaj})/2-L)(w)} (q^{-s})^{\mathrm{rmaj}(w)} \\
&\qquad = \sum_{w\in B_n} (-1)^{l(w)} q^{-L(w)+\sum_{i\in D(w)}((n+i)(n-i)+n-i)/2} (q^{-s})^{\sum_{i\in D(w)}(n-i)}
\end{aligned}$$

$$= \sum_{w \in B_n} (-1)^{l(w)} q^{-L(w)} \prod_{i \in D(w)} q^{((n+i+1)/2-s)(n-i)}$$
$$= \frac{1-q^{-s}}{1-q^{n-s}} \cdot \frac{(q^{2-2s};q^2)_m}{(q^{2(m+\varepsilon)+1-2s};q^2)_m} \cdot \prod_{i=0}^{n-1} \left(1-q^{((n+i+1)/2-s)(n-i)}\right)$$
$$= \frac{(1-q^{-s})(q^{2-2s};q^2)_m}{(q^{2(m+\varepsilon)+1-2s};q^2)_m} \prod_{i=0}^{n-2} \left(1-q^{((n+i+1)/2-s)(n-i)}\right).$$

This identity holds for infinitely many values of q and s, and hence yields a formal identity in variables $X = q$ and $Z = q^{-s}$. □

5.3. Proof of Corollary 1.3. The functional equations in (1) follow directly from the formulae given in Theorem B, and the Euler product for the Dedekind zeta function $\zeta_K(s)$; cf. (1.15). The abscissae of convergence in (2) and the analytical statements in (3) reflect classical facts about $\zeta_K(s)$, viz. its abscissa of convergence 1 and meromorphic continuation to the whole complex plane with a simple pole at $s = 1$. The asymptotic statements in (4) follow from standard Tauberian theorems; cf., for instance, [13, Theorem 4.20].

5.4. Jordan's totient functions. We record an interpretation of the zeta functions of groups of type F, G and H, described in Theorem B, in terms of Jordan's totient functions. Given $b,n \in \mathbb{N}$, let $J_b(n)$ be the number of b-tuples $(a_1,a_2,\dots,a_b)$ of integers satisfying $1 \leq a_i \leq n$, for all i, such that $\gcd(a_1,\dots,a_b,n) = 1$. The function J_b is called the b-th Jordan totient function; cf. [32, 1.5.2]. Clearly $J_1 = \varphi$, the Euler totient function.

LEMMA 5.6. *Let $a \in \mathbb{N}_0$, $b \in \mathbb{N}$. The Dirichlet generating series for the arithmetic function $J_{a,b} : \mathbb{N} \to \mathbb{N}, n \mapsto n^a J_b(n)$ is $\sum_{n=1}^{\infty} J_{a,b}(n) n^{-s} = \zeta(s-a-b)/\zeta(s-a)$, where $\zeta(s)$ is the Riemann zeta function.*

Proof. This follows easily from the fact that, for a prime power p^e, $e \in \mathbb{N}$, one has $J_b(p^e) = p^{eb}(1-p^{-b})$. □

We observe that, for $\mathbf{G} \in \{F_{n,\delta}, G_n, H_n\}$ the zeta function of the group $\mathbf{G}(\mathbb{Z})$ is a finite product of factors of the form $\zeta(s-a-b)/\zeta(s-a)$, for suitable pairs of integers (a,b). By Lemma 5.6, the arithmetic function $n \mapsto \widetilde{r}_{\mathbf{G}(\mathbb{Z})}(n)$ may thus be described as a finite convolution product of functions of the form $J_{a,b}$. Over number rings, one may define analogues to the functions $J_{a,b}$ in terms of tuples of coprime ideals, and hence obtain analogous expressions for the zeta functions of groups of the form $\mathbf{G}(\mathcal{O})$ as suitable convolution products.

6. Analogy with prehomogeneous vector spaces. The local factors of representation zeta functions of groups of type F, G and H studied in this paper bear a striking resemblance to the zeta integrals associated to Igusa local zeta functions of certain prehomogeneous vector spaces (PVS).

An irreducible PVS is a pair (V,G), comprising an n-dimensional complex vector space V and a connected algebraic subgroup G of $\mathrm{GL}(V)$, acting irreducibly on V with a Zariski-dense G-orbit. Irreducible PVS were classified, up to a certain equivalence relation, by Kimura and Sato in terms of irreducible, so-called reduced PVS. Associated with an irreducible reduced PVS (G,V) there is a "relatively invariant" irreducible polynomial $f(\mathbf{x}) \in \mathbb{C}[x_1,\dots,x_n]$ with the property that $V \setminus f^{-1}(0)$ is the Zariski-dense G-orbit in V. If $f \in \mathfrak{o}[x_1,\dots,x_n]$ for a compact discrete valuation ring $\mathfrak{o}$ of characteristic zero then the integral

$$Z_f^{\mathfrak{o}}(s) = \int_{\mathfrak{o}^n} |f(x)|^s \mathrm{d}\mu$$

is known as Igusa's local zeta function attached to f. The real parts of the poles of $Z_f^{\mathfrak{o}}(s)$ are known to be among the zeros of the Bernstein-Sato polynomial $b_f(s)$ associated to f. The polynomial $b_f(s)$ provides a measure of the complexity of the singularities of the hypersurface $f^{-1}(0)$. This intriguing interpretation of the real parts of poles of the integral $Z_f^{\mathfrak{o}}(s)$ is conjectured to hold in a much more general context: If $f(x) \in K[x_1,\dots,x_n]$, where K is a number field, the Bernstein-Sato polynomial conjecture states that, for almost all non-archimedean completions $\mathfrak{k}$ of K with valuation ring $\mathfrak{o}$, say, the real parts of the poles of $Z_f^{\mathfrak{o}}(s)$ should be among the zeros of $b_f(s)$; cf. [25, 21] for further details on PVS, and [12, Section 7] for details on the Bernstein-Sato polynomial conjecture.

The list of irreducible reduced PVS in the appendix of [25] starts off with three infinite families of prehomogeneous vector spaces of generic matrices, viz. the vector spaces $\mathrm{Mat}_n(\mathbb{C})$, $\mathrm{Sym}_n(\mathbb{C})$ and $\mathrm{Alt}_{2n}(\mathbb{C})$, respectively. The associated relative invariants are $f(X) = \det(X)$, $f(X) = \det(X)$ and $f(X) = \mathrm{Pf}(X)$, respectively, where $\mathrm{Pf}(X)$ denotes the Pfaffian of an antisymmetric matrix X. Let $\mathfrak{o}$ be a complete discrete valuation ring with residue field cardinality q. We continue to write $n = 2m+\varepsilon \in \mathbb{N}$ with $\varepsilon \in \{0,1\}$. The following formulae for the associated Igusa zeta functions are well known; see, for instance, [21, pp. 164, 163, 177].

(6.1)
$$Z_{\mathrm{Alt}_{2n}(\mathfrak{o})}(s) := \int_{X \in \mathrm{Alt}_{2n}(\mathfrak{o})} |\mathrm{Pf}(X)|^s \mathrm{d}\mu = \prod_{i=0}^{n-1} \frac{1-q^{-1-2i}}{1-q^{-s-1-2i}},$$

(6.2)
$$Z_{\mathrm{Mat}_n(\mathfrak{o})}(s) := \int_{X \in \mathrm{Mat}_n(\mathfrak{o})} |\det(X)|^s \mathrm{d}\mu = \prod_{i=0}^{n-1} \frac{1-q^{-1-i}}{1-q^{-s-1-i}},$$

(6.3)
$$Z_{\mathrm{Sym}_n(\mathfrak{o})}(s) := \int_{X \in \mathrm{Sym}_n(\mathfrak{o})} |\det(X)|^s \mathrm{d}\mu = \frac{1-q^{-(1-\varepsilon)(s+1)-n}}{1-q^{-s-1}} \prod_{i=0}^{m-1} \frac{1-q^{-1-2i}}{1-q^{-2s-3-2i}}.$$

We observe that, in analogy to Proposition 5.3, we have

$$Z_{\mathrm{Sym}_{2n}(\mathfrak{o})}(s) = \frac{1-q^{-s-2n-1}}{1-q^{-s-1}} Z_{\mathrm{Alt}_{2n}(\mathfrak{o})}(2s+2). \tag{6.4}$$

We are not aware of a conceptional explanation for this identity.

The group schemes $F_{n,\delta}$, G_n and H_n studied in this paper are designed so that the commutator matrices associated to their respective Lie rings reflect the prehomogeneous vector spaces $\mathrm{Alt}_{2n}(\mathbb{C})$, $\mathrm{Mat}_n(\mathbb{C})$ and $\mathrm{Sym}_n(\mathbb{C})$, respectively. Theorem B shows that the local representation zeta functions associated to groups of type F, G and H closely resemble the $\mathfrak{p}$-adic integrals (6.1), (6.2) and (6.3), without being obtainable from these integrals by simple transformations of variables. We record an immediate consequence of Theorem B regarding the poles of the local zeta functions.

COROLLARY 6.1. *Let $\mathbf{G} \in \{F_{n,\delta}, G_n, H_n\}$. There exists a finite set $P(\mathbf{G})$ of rational numbers such that the following holds. Given a ring of integers $\mathcal{O}$ of a number field K, and for any non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, we have*

$$P(\mathbf{G}) = \{\mathrm{Re}(s) \mid s \in \mathbb{C} \textit{ a pole of } \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)\}.$$

More precisely, we have

$$\begin{aligned} P(F_{n,\delta}) &= \{2(n+i+\delta)-1 \mid i \in [n-1]_0\}, \\ P(G_n) &= \{n+i \mid i \in [n-1]_0\}, \\ P(H_n) &= \{n, m+i+\varepsilon+1/2 \mid i \in [m-1]_0\}. \end{aligned}$$

In other words, the set of real parts of poles of $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s)$ is independent of $\mathcal{O}$ and $\mathfrak{p}$. We note further that the sets $P(\mathbf{G})$ defined in Corollary 6.1 are obtained from the sets of real parts of the poles of $\mathfrak{p}$-adic integral associated to the corresponding PVS by translation by the global abscissa of convergence of the relevant representation zeta function. Indeed we have, by (6.1), (6.2) and (6.3), Corollary 6.1 and (1.10), that for all $\mathfrak{o}$

$$\begin{aligned} \{\mathrm{Re}(s) \mid s \in \mathbb{C} \text{ a pole of } Z_{\mathrm{Alt}_{2n}(\mathfrak{o})}(s)\} &= P(F_{n,\delta}) - \alpha(F_{n,\delta}), \\ \{\mathrm{Re}(s) \mid s \in \mathbb{C} \text{ a pole of } Z_{\mathrm{Mat}_{n}(\mathfrak{o})}(s)\} &= P(G_n) - \alpha(G_n), \\ \{\mathrm{Re}(s) \mid s \in \mathbb{C} \text{ a pole of } Z_{\mathrm{Sym}_{n}(\mathfrak{o})}(s)\} &= P(H_n) - \alpha(H_n). \end{aligned}$$

As mentioned above, the zeta integrals (6.3), (6.2) and (6.3) are examples of Igusa zeta functions which are known to satisfy the Bernstein-Sato-polynomial conjecture. More precisely, they are Igusa zeta functions with the property that the real parts of the poles of $Z(s)$ are among the zeros of the Bernstein-Sato polynomial $b_f(s)$, with pole multiplicities not exceeding the multiplicities of the corresponding zeros. In the cases (6.1) and (6.2) these sets coincide. In the case (6.3), however, the set of Bernstein-Sato zeros is strictly larger: indeed, in this case we have $b_f(s) = \prod_{i=0}^{n-1}(s+(i+2)/2)$. It strikes us as remarkable that

$$\{s \in \mathbb{C} \mid b_f(s) = 0\} = \{a(H_n,i)/(n-i) \mid i \in [n-1]_0\} - \alpha(H_n).$$

The rational numbers $a(H_n,i)/(n-i)$ arise as candidate real parts coming from terms of the form $X_i/(1-X_i)$, with $X_i = q^{a(H_n,i)-(n-i)s}$. We remark that the numbers that do not give rise to real parts of poles come from the variables which cancel in the transition from the additive to the multiplicative formulae; cf. Lemma 5.2.

It would be interesting to associate other irreducible prehomogeneous vector spaces with finitely generated nilpotent groups. We are presently not aware of any other irreducible reduced prehomogeneous vector spaces whose geometry is reflected in this way in representation zeta functions of $\mathcal{T}$-groups.

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