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An Infinite Family of 2-Groups with Mixed Beauville Structures

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We construct an infinite family of triples (G_k, H_k, T_k), where G_k are 2-groups of increasing order, H_k are index 2 subgroups of G_k, and T_k are pairs of generators of H_k. We show that the triples $u_k = (G_k, H_k, T_k)$ are mixed Beauville structures if k is not a power of 2. This is the first known infinite family of 2-groups admitting mixed Beauville structures. Moreover, the associated Beauville surface $S(u_3)$ is real and, for $k > 3$ not a power of 2, the Beauville surface $S(u_k)$ is not biholomorphic to $\overline{S(u_k)}$.

1 Introduction

In this article, we construct *infinitely many* 2-groups G_k and show that they admit mixed Beauville structures if k is not a power of 2.

It was mentioned in [3] that *it is rather difficult to find a finite group admitting a mixed Beauville structure*. Computer calculations show that there are no such groups of order $< 2^8$ (see [4, Remark 4.2]). By the definition, if a p-group admits a mixed Beauville structure, then $p = 2$. Until now, only finitely many 2-groups admitting mixed

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Beauville structures are known. There are two examples of order 2^8 in [4], and five more of orders $2^{14}, 2^{16}, 2^{19}, 2^{24}, 2^{27}$ in [1]. The family in this paper is the first known *infinite family* of 2-groups admitting mixed Beauville structures.

A *mixed Beauville structure* of a finite group G is a triple (G, H, T), where H is an index 2 subgroup of G, and $T = (h_1, h_2)$ is a pair of elements $h_1, h_2 \in H$ generating H with particular properties.

Since so little is known about groups admitting mixed Beauville structures, it is generally assumed that they are very rare. Clearly, no simple group can admit a mixed Beauville structure. Fuertes and González-Diez [9] observed that for a mixed Beauville structure (G, H, T), the order of all elements in $G \backslash H$ must be divisible by 4 and, therefore, $G = S_n$ cannot have a mixed Beauville structure. Fairbairn proved that the same holds true for all almost simple groups G whose derived groups $[G, G]$ are sporadic (see [8, Theorem 8]). The only other known construction of groups admitting mixed Beauville structures was given in [3]. These groups are of the form $K_{[4]} = (K \times K) \rtimes (\mathbb{Z}/4\mathbb{Z})$, where K is a group with particular properties listed in [3, Lemma 4.5]. The nature of these other mixed Beauville structures $(K_{[4]}, K_{[2]}, T = (a, c))$ is very different from our family of 2-groups. For example, $\nu(T) = \mathrm{ord}(a)\mathrm{ord}(c)\mathrm{ord}(ac)$ contains necessarily two different primes. Since, for 2-groups, $\nu(T)$ is necessarily a power of 2, this other construction cannot provide examples of 2-*groups* admitting mixed Beauville structures.

Our groups G_k are 2-quotients of a just infinite group G with seven generators $x_0, \ldots, x_6$, acting simply transitively on the vertices of an $\tilde{A}_2$-building. This infinite group first appeared in [7], and then again in [5] in connection with buildings. In [11], we observed that G has an index 2 subgroup H, generated by x_0, x_1, and we used the corresponding index 2 quotients $H_k \lhd G_k$ for explicit Cayley graph expander constructions. The considerations in [11] showed that $|G_3| = 2^8$ and, for $k \geq 3$,

$$|G_{k+1}| \geq \begin{cases} 8|G_k| & \text{if } k \equiv 0, 1 \bmod 3, \\ 4|G_k| & \text{if } k \equiv 2 \bmod 3. \end{cases}$$

For simplicity of notation, we use the same symbols x_i for the generators of G and their images in the finite quotients G_k.

Any mixed Beauville structure $u = (G, H, T)$ gives rise to a Beauville surface $S(u) \cong (C_T \times C_T)/G$ of mixed type. A natural question is whether this Beauville surface

$S(u)$ is real. An algebraic surface S is called *real* if there is a biholomorphism $\sigma : S \to \bar{S}$ with $\sigma^2 = \mathrm{id}$. For the details, we refer, for example, to the papers [3, 4].

Let us now formulate the main result of this paper.

Theorem 1. Let $k \geq 3$ be *not* a power of 2 and $T_k = (x_0, x_1) \in H_k \times H_k$. Then, the triple $u_k = (G_k, H_k, T_k)$ is a mixed Beauville structure. Moreover, the following holds are satisfied.

(i) The mixed Beauville surface $S(u_3)$ is real.

(ii) For every $k > 3$ not a power of 2, the Beauville surface $S(u_k)$ is not biholomorphic to its complex conjugate $\overline{S(u_k)}$. □

For the proof, we realize G as a group of (finite band) upper triangular infinite Toeplitz matrices. The 2-quotients G_k are obtained via truncations of these matrices at their $(k+1)$th upper diagonal, and they have a certain nilpotency structure. Our proof exploits this nilpotency structure as well as subtle periodicity properties of these matrices. It also becomes transparent via these periodicity properties why, in the above theorem, $k \geq 3$ must necessarily avoid the powers of 2.

Let us explain the difference between the results in [1] and in this article: In [1], we used the computational algebra system Magma to check that the *first six groups* of an infinite family of 2-groups admit mixed Beauville structures, which led us to conjecture that this holds true for the full infinite family. In this paper, we provide a *rigorous theoretical proof* that an infinite family of 2-groups admit mixed Beauville structures. In view of the final Remark 7.1, it is very surprising that all our groups (except for G_{2^j} with $j \in \mathbb{N}_0$) admit mixed Beauville structures. Moreover, there is overwhelming evidence that the families of groups in both papers agree, and it has been verified computationally for the first 100 groups in both families that they are pairwise isomorphic (see [11, Conjecture 1]).

Let us finish our introduction with the following question: *For which* 2*-groups* H *does there exist a group* $G \supset H$ *and a choice* $T \in H \times H$ *such that* (G, H, T) *is a mixed Beauville structure?* Both examples of groups of order 2^8 listed in [4, Theorem 0.1] and admitting mixed Beauville structures have the same index 2-subgroup which agrees with our group H_3. The five other examples in [1] agree with our examples $H_5, H_6, H_7, H_9, H_{10}$. It would be interesting to know whether there are any other 2-groups H giving rise to mixed Beauville structures (G, H, T), and which do not agree with one of our groups H_k.

2 Mixed Beauville Structures and Associated Surfaces

The following presentation follows [3] closely. Let G be a finite group and $H \subset G$ be a subgroup of index 2. For $x \in H$ let

$$\Sigma(x) := \{hx^j h^{-1} \mid h \in H,\ j \geq 0\},$$

that is, $\Sigma(x)$ is the union of all conjugates of the cyclic subgroup generated by x. For $T = (x_0, x_1) \in H \times H$, we define

$$\Sigma(T) := \Sigma(x_0) \cup \Sigma(x_1) \cup \Sigma((x_0 x_1)^{-1}).$$

A *mixed Beauville structure* is a triple (G, H, T) with $T = (x_0, x_1)$ satisfying the following properties:

(A) x_0, x_1 generate the group H.
(B) There exists $g_0 \in G \backslash H$ such that $g_0 \Sigma(T) g_0^{-1} \cap \Sigma(T) = \{\mathrm{id}\}$.
(C) For all $g \in G \backslash H$ we have $g^2 \notin \Sigma(T)$.

Next, we explain how to construct the Beauville surface $S = S(u)$ associated to a mixed Beauville structure $u = (G, H, T = (x_0, x_1))$. Let $P_0, P_1, P_2 \in \mathbb{P}^1$ be a sequence of points ordered counterclockwise around a base point $O \in \mathbb{P}^1$ and, for $0 \leq i \leq 2$, let $\gamma_i \in \pi_1(\mathbb{P}^1 \backslash \{P_0, P_1, P_2\}, O)$ be represented by a simple counterclockwise loop around P_i such that $\gamma_0 \gamma_1 \gamma_2 = \mathrm{id}$. By Riemann's existence theorem (see [12, Theorems 4.27 and 4.32] and also [3, (17)]), there exists a surjective homomorphism

$$\Phi : \pi_1(\mathbb{P}^1 \backslash \{P_0, P_1, P_2\}, O) \to H$$

with $\Phi(\gamma_0) = x_0$ and $\Phi(\gamma_1) = x_1$, and a Galois covering $\lambda_T : C_T \to \mathbb{P}^1$, ramified only in $\{P_0, P_1, P_2\}$, with ramification indices equal to the orders of the elements $x_0, x_1, x_0 x_1$. These data induce a well-defined action of H on the curve C_T, and by the Riemann–Hurwitz formula, we have

$$g(C_T) = 1 + \frac{|H|}{2} \left(1 - \frac{1}{\mathrm{ord}(x_0)} - \frac{1}{\mathrm{ord}(x_1)} - \frac{1}{\mathrm{ord}(x_0 x_1)}\right).$$

Let $\varphi_g : H \to H$ be conjugation with g, that is, $\varphi_g(x) = gxg^{-1}$. We then define a G-action on $C_T \times C_T$ by

$$x(z_1, z_2) = (x z_1, \varphi_{g_0}(x) z_2), \quad g_0(z_1, z_2) = (z_2, {g_0}^2 z_1),$$

for all $x \in H$ and $(z_1, z_2) \in C_T \times C_T$. Here $g_0 \in G \backslash H$ is the fixed element from property (B) of the mixed Beauville structure. This action is fixed point-free, and the quotient $(C_T \times C_T)/G$ is the associated mixed Beauville surface S. By the theorem of Zeuthen–Segre, we have for the topological Euler number

$$
\begin{aligned}
e(S) &= \frac{4(g(C_T) - 1)^2}{|H|} \\
&= |H| \left(1 - \frac{1}{\mathrm{ord}(x_0)} - \frac{1}{\mathrm{ord}(x_1)} - \frac{1}{\mathrm{ord}(x_0 x_1)}\right)^2,
\end{aligned}
$$

as well as the relations (see [6, Theorem 3.4]),

$$
\chi(S) = \frac{e(S)}{4} = \frac{{K_S}^2}{8},
$$

where ${K_S}^2$ is the self-intersection number of the canonical divisor and $\chi(S) = 1 + p_g(S) - q(S)$ is the holomorphic Euler–Poincaré characteristic of S.

Let us briefly indicate how we prove the *reality statements* (i),(ii) for the mixed Beauville surfaces in Theorem 1: For $T = (c, a) \in H \times H$ let $T^{-1} = (c^{-1}, a^{-1})$. Every mixed Beauville structure $u = (G, H, T)$ gives rise to another mixed Beauville structure $\iota(u) = (G, H, T^{-1})$, and we have $S(\iota(u)) = \overline{S(u)}$ (see [3, (39)]). Let $\mathbb{M}(G) = \{(G, H, (c, a))\}$ denote the set of all mixed Beauville structures of G. Every automorphism $\psi \in \mathrm{Aut}(G)$ induces a map σ_ψ on $\mathbb{M}(G)$ via

$$
\sigma_\psi(G, H, (c, a)) = (G, \psi(H), (\psi(c), \psi(a))).
$$

Moreover, in accordance with [3, (11) and (32)], let σ_3, σ_4 be maps on $\mathbb{M}(G)$, defined by

$$
\begin{aligned}
\sigma_3(G, H, (c, a)) &= (G, H, (a, c)), \\
\sigma_4(G, H, (c, a)) &= (G, H, (c, c^{-1}a^{-1})),
\end{aligned}
$$

and $A_{\mathbb{M}}(G)$ be the group generated by the maps σ_ψ $(\psi \in \mathrm{Aut}(G))$ and σ_3, σ_4. Then, we have the following facts (see [3, Proposition 4.7]):

(a) $S(u)$ is biholomorphic to $\overline{S(u)}$ iff $\iota(u) \in A_{\mathbb{M}}(G)u$.

(b) $S(u)$ is real iff $\iota(u) = \rho(u)$ for some $\rho \in A_{\mathbb{M}}(G)$ with $\rho(\iota(u)) = u$.

Choosing the mixed Beauville structures u_k from Theorem 1, we find an automorphism $\psi : G_3 \to G_3$, uniquely defined by $\psi(x_0) = x_0^{-1}$, $\psi(x_1) = x_1^{-1}$ and $\psi(x_2) = x_0^{-1}x_2x_0$. This implies $\iota(u_3) = \sigma_\psi(u_3)$ and $\sigma_\psi(\iota(u_3)) = u_3$, and it follows from (b) that $S(u_3)$ is real. On the other hand, for $k > 3$ and not a power of 2, we show that there is no homomorphism $\psi : H_k \to H_k$ satisfying

$$(\psi(x_0), \psi(x_1)) \in \{(x_0^{-1}, x_1^{-1}), (x_1x_0, x_0^{-1}), (x_1^{-1}, x_1x_0),$$
$$(x_1^{-1}, x_0^{-1}), (x_0^{-1}, x_1x_0), (x_1x_0, x_1^{-1})\}.$$

Using [3, Lemma 2.4] and the criterion (a) above, this implies that $S(u_k)$ cannot be biholomorphic to $\overline{S(u_k)}$. (Note that our pair (x_0, x_1) corresponds, in the notation of [3], to the pair (c, a).)

3 The 2-Groups G_k and H_k

Let $\mathcal{K}$ be the simplicial complex constructed from the following seven triangles by identifying sides with the same labels x_i (Figure 1).

It is easily checked that the vertices of all triangles are identified, and that the fundamental group $\pi_1(\mathcal{K})$ is isomorphic to the infinite abstract group

$$G = \langle x_0, \ldots, x_6 \mid x_ix_{i+1}x_{i+3} = \text{ id for } i = 0, \ldots, 6\rangle, \tag{1}$$

where $i, i+1$ and $i+3$ are taken modulo 7. Realizing the triangles as equilateral Euclidean triangles, we can view the universal covering of $\mathcal{K}$ as a thick Euclidean building of type $\widetilde{A}_2$, on which G acts via covering transformations.

Note that the presentation (1) is a presentation of G by seven generators and seven relations. It is easy to see that G is already generated by the three elements x_0, x_1, x_2. Let $H \subset G$ be the subgroup generated by the two elements x_0, x_1. Then, H is an index 2 subgroup of G (see [11, Prop. 2.1]). The groups G_k and H_k will be finite 2-quotients of these groups G and H.

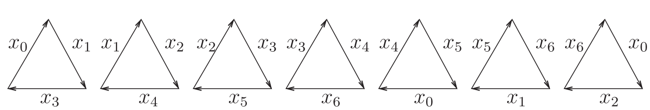

Fig. 1. Labeling scheme for the simplicial complex $\mathcal{K}$.

We now recall the *faithful representation* of G by infinite upper triangular matrices given in [11], where every element $x \in G$ is represented as

$$x = \begin{pmatrix} 1 & a_{11} & a_{21} & \dots & a_{k1} & 0 & 0 & \dots & \dots \\ 0 & 1 & a_{12} & a_{22} & \dots & a_{k2} & 0 & \ddots & \\ 0 & 0 & 1 & a_{13} & a_{23} & \dots & a_{k3} & 0 & \ddots \\ \vdots & \ddots & 0 & 1 & a_{11} & a_{21} & \dots & a_{k1} & \ddots \\ \vdots & & \ddots & \ddots & 1 & a_{12} & a_{22} & \dots & \ddots \\ \vdots & & & \ddots & \ddots & 1 & a_{13} & a_{23} & \ddots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2}$$

and each entry a_{ij} is a matrix in $M(3, \mathbb{F}_2)$ (and 0 and 1 stand for the zero and the identity matrix in $M(3, \mathbb{F}_2)$). Note that the matrix representation (2) has only finitely many nonzero upper diagonals. Moreover, the entries in every diagonal are repeating with period 3.

Let us now introduce a concise notation for these matrices: The entries $a_{j1}, a_{j2}, a_{j3} \in M(3, \mathbb{F}_2)$ of the jth upper diagonal in the matrix representation can be combined to a 3×9 matrix, which we denote by $a_j = [a_{j1}, a_{j2}, a_{j3}]$. (Conversely, we refer to the three 3×3 matrices constituting a 3×9 matrix a_j by $a_j(1), a_j(2), a_j(3) \in M(3, \mathbb{F}_2)$.) We can then write the matrix in (2) as

$$M_0(a_1, \dots, a_k) = M_0([a_{11}, a_{12}, a_{13}], \dots, [a_{k1}, a_{k2}, a_{k3}]).$$

If the first $l \geq 1$ upper diagonals of a matrix $M_0(a_1, \dots, a_k)$ are zero, we use also the notation $M_l(a_{l+1}, \dots, a_k)$. Since the presentation with 3×9 matrices is still not very concise, we translate every matrix $a = (u_{ij}) \in M(3, \mathbb{F}_2)$ into the non-negative integer

$$A = 256u_{11} + 128u_{12} + 64u_{13} + 32u_{21} + 16u_{22} + 8u_{23} + 4u_{31} + 2u_{32} + u_{33},$$

and represent the 3×9 matrix $[a_{j1}, a_{j2}, a_{j3}]$ by the triple $A_j = [A_{j1}, A_{j2}, A_{j3}]$ with $0 \leq A_1, A_2, A_3 \leq 511$. Therefore, another way to write the matrix in (2) is

$$M_0(A_1, \dots, A_k) = M_0([A_{11}, A_{12}, A_{13}], \dots, [A_{k1}, A_{k2}, A_{k3}]).$$

The matrices corresponding to the generators x_0, x_1, x_2 are in this notation:

$$x_0 = M_0([11, 11, 11], [17, 17, 17], [26, 26, 26], [11, 11, 0], [17, 0, 0]),$$

$$x_1 = M_0([23, 224, 138], [59, 136, 495], [26, 488, 227], [23, 224, 0], [59, 0, 0]),$$

$$x_2 = M_0([46, 68, 217], [12, 194, 363], [26, 326, 77], [46, 68, 0], [12, 0, 0]).$$

The proofs of the explicit formulas in the following lemma are straightforward (see [11]). Note that (b) is a refinement of [11, Proposition 2.5]. These formulas are crucial for our later considerations.

Lemma 3.1. Note that in the following formulas all entries j in $a_1(j), a_2(j), b_1(j), c_1(j), c_2(j)$ are taken mod 3 and chosen to be in the range $\{1, 2, 3\}$.

(a) Let $k, j \geq 0$ and $M_1 = M_k(a_1, a_2, \ldots)$ and $M_2 = M_{k+j}(b_1, \ldots)$. Then, both products $M_1 M_2$ and $M_2 M_1$ are of the form

$$M_k(a_1, a_2, \ldots, a_{j-1}, a_j + b_1, \ldots).$$

(b) We have

$$M_k(a_1, a_2, \ldots)^2 = M_{2k+1}(c_1, c_2, \ldots),$$

with $c_1(i) = a_1(i)a_1(k+i+1)$ and

$$c_2(i) = a_1(i)a_2(k+i+1) + a_2(i)a_1(k+i+2).$$

(c) We have

$$M_0(b_1, \ldots)^{-1} M_k(a_1, a_2, \ldots) M_0(b_1, \ldots) = M_k(a_1, c_2, \ldots)$$

with $c_2(i) = a_2(i) + b_1(i)a_1(i+1) + a_1(i)b_1(k+i+1)$. $\square$

Let G^k and H^k be the subgroups of all elements in G and H with vanishing first k upper diagonals (i.e., these elements are of the form $M_k(a_1, \ldots)$). Then, G^k and H^k are normal subgroups of G and H, and our groups G_k and H_k are the quotients G/G^k and H/H^k. We can think of G_k and H_k as truncations of the matrix groups G and H at their $(k+1)$st upper diagonal. The finiteness of these quotients follows then easily from the 3-periodicity of the diagonals.

Remark 3.2. Another way to generate quotients of G (and H) is via the lower-exponent-2 series

$$G = \lambda_0(G) \supset \lambda_1(G) \supset \ldots,$$

where $\lambda_{i+1}(G) = [\lambda_i(G), G](\lambda_i(G))^2$. The quotients $G/\lambda_k(G)$ are finite 2-groups. It follows from [11, Prop 2.5] that $\lambda_k(G) \subset G^k$. Magma computations show for all indices $k \leq 100$ (see [11]) that $\lambda_k(G) \cong G^k$ and

$$\log_2 \left[\gamma_k(G) : \gamma_{k+1}(G)\right] = \begin{cases} 3 & \text{if } k \equiv 0, 1 \bmod 3, \\ 2 & \text{if } k \equiv 2 \bmod 3. \end{cases}$$

We conjecture (see [11, Conjectures 1 and 2]) that these facts hold true for all k, which would mean that the group G *has finite width* 3 (see [10] for definitions). □

4 Powers of the Generators

This and all the following sections are dedicated to the proof that the triple (G_k, H_k, T_k) satisfies the conditions (A), (B) and (C) of a mixed Beauville structure if k is not a power of 2. The explicit calculations were supported by MAPLE procedures which can be found in [2].

Recall that $T_k = (x_0, x_1)$, and x_0, x_1 are here understood as the corresponding elements in the quotient group H_k. The triples

$$[x_0, x_1, x = (x_0 x_1)^{-1}] \quad \text{and} \quad [y_0 = x_2 x_0 x_2^{-1}, y_1 = x_2 x_1 x_2^{-1}, y = (y_0 y_1)^{-1}]$$

are both *spherical systems of generators* of the group H (see, e.g., [4] for this notion). A crucial step towards the proof of Theorem 1 is the explicit determination of the first two nontrivial diagonals of *all powers* of each of the elements x_0, x_1, x, y_0, y_1, y. By the *first two nontrivial diagonals* of a matrix $M_0(a_1, a_2, \ldots) \neq \mathrm{id}$ we mean the pair a_k, a_{k+1} with $a_1 = \cdots = a_{k-1} = 0$ and $a_k \neq 0$. Moreover, we call a_k the *leading diagonal* of this matrix. In fact, it turns out that—in all considerations of this paper—only a good understanding of the first two nontrivial diagonals is needed and that the higher diagonals can be ignored.

Let us focus on the powers of the elements

$$x = (x_0 x_1)^{-1} = M_0([28, 235, 129], [29, 211, 263], \ldots),$$

$$y = x_2 x x_2^{-1} = M_0([28, 235, 129], [58, 3, 445], \ldots)$$

for reasons of illustration (the analogous results for the powers of the pairs $(x_0, y_0 = x_2 x_0 x_2^{-1})$ and $(x_1, y_1 = x_2 x_1 x_2^{-1})$ will be given at the end of this section).

It is remarkable that the first two nontrivial diagonals of the 2-powers of x and y repeat with a *periodicity of* 2. This is the content of the following proposition and can be verified by a straightforward calculation using Lemma 3.1(b).

Proposition 4.1. We have, for all $j \geq 0$,

$$x^{2^{2j+1}} = M_{2^{2j+1}-1}([51, 89, 196], [0, 0, 0], \ldots),$$

$$y^{2^{2j+1}} = M_{2^{2j+1}-1}([51, 89, 196], [0, 157, 106], \ldots),$$

$$x^{2^{2j+2}} = M_{2^{2j}+1}([28, 235, 129], [0, 0, 0], \ldots),$$

$$y^{2^{2j+2}} = M_{2^{2j}+1}([28, 235, 129], [39, 208, 186], \ldots). \qquad \square$$

Remark 4.2. The group G has more remarkable properties. In [11, Proposition 2.6], we present a certain 3-periodicity of commutators. Another interesting property is that the subgroup generated by the squares $x_0^2, x_1^2, \ldots, x_6^2$ of the seven generators is isomorphic to G (see [7, p. 308]). $\square$

The next remark explains why the statement in Theorem 1 cannot hold for powers of 2:

Remark 4.3. Note in Proposition 4.1 that the leading diagonals of the matrix representations of x^{2^n} and y^{2^n} agree for all $n \geq 0$, since both elements are conjugate (see Lemma 3.1(c)). Let $k = 2^n$. Recall that we can think of the elements in H_k as matrices truncated at their $(k+1)$st upper diagonal. Then, the nontrivial group elements x^k and y^k agree in H_k, since their leading diagonals coincide and are the kth upper diagonals. (To separate these two elements in H_k, their first two nontrivial diagonals would have to survive under the truncation procedure.) This implies that

$$x_2 \Sigma(T_k) x_2^{-1} \cap \Sigma(T_k) \supset \{x^k\}. \tag{3}$$

Note that condition (B) in the mixed Beauville structure implies the following property:

(B′) For all $g \in G \backslash H$: $g\Sigma(T)g^{-1} \cap \Sigma(T) = \{\mathrm{id}\}$,

since $\Sigma(T)$ is invariant under conjugation within H. But (3) contradicts to (B′) and we conclude that $(G_k, H_k, (x_0, x_1))$ cannot be a mixed Beauville structure if $k = 2^n$. $\square$

To understand the first two nontrivial diagonals of *all powers* of x and y (not only the 2-powers), we consider the binary presentation of an arbitrary exponent $n \in \mathbb{N}$:

$$n = 2^{k+j}\alpha_{k+j} + \cdots + 2^{k+1}\alpha_{k+1} + 2^k\alpha_k$$

with $\alpha_l \in \{0, 1\}$ for all l and $\alpha_k = 1$. (Note that all coefficients in the binary presentation of n corresponding to 2-powers with exponent $< k$ are assumed to be 0.) Now define

$$t(n) = \begin{cases} 2\alpha_1 + \alpha_0 & \text{if } k = 0, \\ 2^k\alpha_k & \text{if } k \geq 1. \end{cases} \tag{4}$$

Then, x^n is equal to $x^{t(n)}$ multiplied with certain higher 2-powers of x (i.e., the powers $x^{2^{\alpha_l}}$ with $\alpha_l = 1$ and $l \geq \max\{2, k+1\}$). In view of Lemma 3.1(a), this multiplication does not change the first two nontrivial diagonals of $x^{t(n)}$, which shows that the first two nontrivial diagonals of $x^{t(n)}$ and x^n agree. Using the (easily computable) fact that

$$x^3 = M_0([28, 235, 129], [46, 138, 451], \ldots),$$

$$y^3 = M_0([28, 235, 129], [9, 90, 377], \ldots),$$

this leads directly to the following result.

Corollary 4.4. The matrix representation of any power x^n ($n \geq 1$) takes one of the following forms:

$$M_0([28, 235, 129], [29, 211, 263], \ldots), \quad M_0([28, 235, 129], [46, 138, 451], \ldots),$$

$$M_{2^{\text{odd}}-1}([51, 89, 196], [0, 0, 0], \ldots), \quad M_{2^{\text{even}+2}-1}([28, 235, 129], [0, 0, 0], \ldots).$$

The matrix representations of any power y^n ($n \geq 1$) takes one of the following forms:

$$M_0([28, 235, 129], [58, 3, 445], \ldots), \quad M_0([28, 235, 129], [9, 90, 377], \ldots),$$

$$M_{2^{\text{odd}}-1}([51, 89, 196], [0, 157, 106], \ldots), \quad M_{2^{\text{even}+2}-1}([28, 235, 129], [39, 208, 186], \ldots). \qquad \square$$

Analogous results holds for the powers of the other four elements of the two spherical systems of generators.

Proposition 4.5. The matrix representation of any power x_0^n $(n \geq 1)$ takes one of the following forms:

$$M_0([11, 11, 11], [17, 17, 17], \ldots), \quad M_0([11, 11, 11], [11, 11, 11], \ldots),$$

$$M_{2^{\text{odd}}-1}([26, 26, 26], [0, 0, 0], \ldots), \quad M_{2^{\text{even}+2}-1}([11, 11, 11], [0, 0, 0], \ldots).$$

The matrix representations of any power y_0^n $(n \geq 1)$ takes one of the following forms:

$$M_0([11, 11, 11], [44, 219, 177], \ldots), \quad M_0([11, 11, 11], [54, 193, 171], \ldots),$$

$$M_{2^{\text{odd}}-1}([26, 26, 26], [0, 157, 106], \ldots), \quad M_{2^{\text{even}+2}-1}([11, 11, 11], [61, 202, 160], \ldots). \qquad \square$$

Proposition 4.6. The matrix representation of any power x_1^n $(n \geq 1)$ takes one of the following forms:

$$M_0([23, 224, 138], [59, 136, 495], \ldots), \quad M_0([23, 224, 138], [28, 88, 341], \ldots),$$

$$M_{2^{\text{odd}}-1}([39, 208, 186], [0, 0, 0], \ldots), \quad M_{2^{\text{even}+2}-1}([23, 224, 138], [0, 0, 0], \ldots).$$

The matrix representations of any power y_1^n $(n \geq 1)$ takes one of the following forms:

$$M_0([23, 224, 138], [33, 146, 501], \ldots), \quad M_0([23, 224, 138], [6, 66, 335], \ldots),$$

$$M_{2^{\text{odd}}-1}([39, 208, 186], [0, 106, 247], \ldots), \quad M_{2^{\text{even}+2}-1}([23, 224, 138], [26, 26, 26], \ldots). \qquad \square$$

5 Proof of Property (C)

The proof of property (C) for our triple (G_k, H_k, T_k) is relatively easy and follows solely from leading diagonal considerations. Since every element in H is a product of the elements $x_0^{\pm 1}, x_1^{\pm 1}$, we deduce first from Lemma 3.1(a) that the matrix representation of any element $h \in H$ takes one of the following four forms: $M_0([0, 0, 0], \ldots)$, $M_0([11, 11, 11], \ldots)$, $M_0([23, 224, 138], \ldots)$, or $M_0([28, 235, 129], \ldots)$.

Using Lemma 3.1, again, we obtain the following table:

$h \in H$	$h \cdot x_2$	$(h \cdot x_2)^2$
$M_0([0, 0, 0], \ldots)$	$M_0([46, 68, 217], \ldots)$	$M_1([41, 67, 222], \ldots)$
$M_0([11, 11, 11], \ldots)$	$M_0([37, 79, 210], \ldots)$	$M_1([14, 147, 100], \ldots])$
$M_0([23, 224, 138], \ldots)$	$M_0([57, 164, 83], \ldots)$	$M_1([20, 137, 126], \ldots)$
$M_0([28, 235, 129], \ldots)$	$M_0([50, 175, 88], \ldots)$	$M_1([61, 202, 160], \ldots)$

Now assume that $k \geq 2$. Since every element $g \in G_k \backslash H_k$ is of the form $g = hx_2$ with $h \in H_k$, we conclude that g^2 are truncations of matrices of one of the following four forms: $M_1([41, 67, 222], \ldots)$, $M_1([14, 147, 100], \ldots)$, $M_1([20, 137, 126], \ldots)$, or $M_1([61, 202, 160], \ldots)$. Note that the leading diagonal in the matrix of every such element g^2 is the second upper diagonal.

On the other hand, since the leading diagonal of a matrix does not change under conjugation (see Lemma 3.1(c)), we conclude from Corollary 4.4 and Propositions 4.5 and 4.6 that the elements in $\Sigma(T_k)$ are truncations of matrices of one of the following four forms: $M_1([0, 0, 0], \ldots)$, $M_1([51, 89, 196], \ldots)$, $M_1([26, 26, 26], \ldots)$, $M_1([39, 208, 186], \ldots)$.

Since these eight forms are all different, we conclude that $g^2 \notin \Sigma(T_k)$ for all $g \in G_k \backslash H_k$. This shows that property (C) in the definition of a mixed Beauville structure is satisfied for all $k \geq 2$.

6 Proof of Property (B)

In this section, we prove that our triples (G_k, H_k, T_k) satisfy property (B) of a mixed Beauville structure with the choice $g_0 = x_2$, for all k not a power of 2. Recall that $x = (x_0 x_1)^{-1}$ and

$$\Sigma(T) = \Sigma(x_0) \cup \Sigma(x_1) \cup \Sigma(x),$$

and

$$x_2 \Sigma(T) x_2^{-1} = \Sigma(y_0) \cup \Sigma(y_1) \cup \Sigma(y).$$

It follows immediately from inspection of the leading diagonals in Corollary 4.4 and Propositions 4.5 and 4.6 and the fact that these leading diagonals do not change under conjugation (see Lemma 3.1(c)) that, for the pair (x_0, y_1), we have

$$\Sigma(x_0) \cap \Sigma(y_1) = \{\mathrm{id}\},$$

and that the same trivial intersection holds also for all other pairs (x_0, y), (x_1, y_0), (x_1, y), (x, y_0), and (x, y_1). So it only remains to prove the trivial intersection

$$\Sigma(x) \cap \Sigma(y) = \{\mathrm{id}\},$$

and analogous trivial intersection results for the pairs (x_0, y_0) and (x_1, y_1). For this, the consideration of the leading diagonal is not sufficient, and we have to study the behavior of the first two nontrivial diagonals under conjugation. From now on, let k be not a power of 2.

Note that

$$x, y = M_0(A_1 = [28, 235, 129], A_2, \ldots) \bmod G^k$$

with $A_2 = [29, 211, 263]$ or $A_2 = [58, 3, 445]$, respectively. Using Lemma 3.1(c), we see that A_1 does not change under conjugation and that A_2 transforms under conjugation as follows:

$$\begin{tikzcd}
{[29, 211, 263]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[19, 64, 355]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] & {[58, 3, 445]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[52, 144, 473]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] \\
{[19, 64, 355]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[29, 211, 263]} & {[52, 144, 473]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[58, 3, 445]}
\end{tikzcd}$$

Since every element $h \in H_k$ is a product of the generators $x_0^{\pm 1}, x_1^{\pm 1}$, we see that

$$hxh^{-1} \neq h' y(h')^{-1}$$

for any pair $h, h' \in H_k$, since both elements differ in the second of their first two nontrivial diagonals. Similarly, the conjugation scheme for A_2 for the pair $x^3, y^3 = M_0(A_1 = [28, 235, 129], A_2, \ldots) \bmod G^k$ reads as follows:

$$\begin{tikzcd}
{[46, 138, 451]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[32, 25, 423]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] & {[9, 90, 377]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[7, 201, 285]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] \\
{[32, 25, 423]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[46, 138, 451]} & {[7, 201, 285]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[9, 90, 377]}
\end{tikzcd}$$

Comparison of the second nontrivial diagonals shows that we have

$$hx^n h^{-1} \neq h' y^m (h')^{-1} \tag{5}$$

for any pair $h, h' \in H_k$ and every n, m with $t(n), t(m) \in \{1, 3\}$, where $t(n)$ was defined in (4).

Next, let us look at the conjugation scheme for A_2 for any pair $x^{2^r}, y^{2^r} = M_{2^r-1}(A_1 = [51, 89, 196], A_2) \bmod G^k$, where $r \geq 1$ is odd:

$$\begin{tikzcd}
{[0, 0, 0]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[0, 247, 157]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] & {[0, 157, 106]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[0, 106, 247]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] \\
{[0, 247, 157]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[0, 0, 0]} & {[0, 106, 247]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[0, 157, 106]}
\end{tikzcd}$$

Again, this shows that we have (5) for any pair $h, h' \in H_k$ and every n, m with $t(n) = t(m) = 2^r < k$ and odd $r \geq 1$. Moreover, (5) also holds for any choice of n, m such that

(i) one of $t(n)$, $t(m)$ is in $\{1, 3\}$ and the other is of the form 2^r with odd $r \geq 1$, or
(ii) $t(n) = 2^{r_1} < k$ and $t(m) = 2^{r_2} < k$ with $r_1, r_2 \geq 1$ both odd and $r_1 \neq r_2$,

since then the number of first upper vanishing diagonals of $hx^n h^{-1}$ and $h' y^m (h')^{-1}$ do not agree.

Finally, we have the following conjugation scheme for A_2 for any pair $x^{2^r}, y^{2^r} = M_{2^r-1}(A_1 = [28, 235, 129], A_2) \bmod G^k$ with even $r \geq 2$:

$$\begin{tikzcd}
{[0,0,0]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[14,147,100]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] & {[39,208,186]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[41,67,222]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] \\
{[14,147,100]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[0,0,0]} & {[41,67,222]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[39,208,186]}
\end{tikzcd}$$

Combining all above results shows that we have (5) for all $n, m \geq 1$ with $t(n), t(m) \leq k$. Note that $x^n = y^n = \mathrm{id}$ for all $n \geq 1$ with $t(n) > k$, so we conclude that

$$\Sigma(x) \cap \Sigma(y) = \{\mathrm{id}\}.$$

The corresponding commutator schemes for the pairs (x_0, y_0) and (x_1, y_1) are listed in Appendices A and B, finishing the proof of

$$x_2 \Sigma(T) x_2^{-1} \cap \Sigma(T) = \{\mathrm{id}\}.$$

7 Bringing Everything Together

Is is obvious that our triples $u_k = (G_k, H_k, T_k)$ satisfy property (A) of a mixed Beauville structure. Because the previous two sections show the validity of properties (B) and (C) if k is not a power of 2, we conclude that these triples are mixed Beauville structures.

Next, we use the following fact: *Assume that Γ is a finite group with finite presentation*, *that is*,

$$\Gamma = \langle g_0, \ldots, g_k \mid r_1(g_0, \ldots, g_k) = \mathrm{id}, \ldots, r_l(g_0, \ldots, g_k) = \mathrm{id} \rangle.$$

For $0 \le i \le k$, *let*

$$g_i' = w_i(g_0, \dots, g_k)$$

and

$$r_i'(g_0, \dots, g_k) = r_i(w_0(g_0, \dots, g_k), \dots, w_k(g_0, \dots, g_k)).$$

Let Γ' *be the group defined by*

$$\Gamma' = \langle g_0, \dots, g_k \mid r_1(g_0, \dots, g_k) = r_1'(g_0, \dots, g_k) = \mathrm{id}, \dots,$$
$$r_l(g_0, \dots, g_k) = r_l'(g_0, \dots, g_k) = \mathrm{id}\rangle.$$

Then, there exists a unique homomorphism $\psi : \Gamma \to \Gamma$ *with* $\psi(g_i) = g_i'$ $(0 \le i \le k)$ *if and only if* $|\Gamma| = |\Gamma'|$.

In view of [11, p. 2782], it is easily checked that G is canonically isomorphic to

$$\langle x_0, x_1, x_2 \mid r_1(x_0, x_1, x_2) = r_2(x_0, x_1, x_2) = r_3(x_0, x_1, x_2) = \mathrm{id}\rangle,$$

with

$$r_1(x_0, x_1, x_2) = x_2 x_1 x_2 x_0 x_1 x_0,$$
$$r_2(x_0, x_1, x_2) = x_2 x_0^{-1} x_2 x_1^{-1} x_0^{-1} x_1,$$
$$r_3(x_0, x_1, x_2) = x_2^2 x_1^{-1} x_0^{-1} x_1^{-1} x_0,$$

and that the quotient G_3 is canonically isomorphic to

$$\langle x_0, x_1, x_2 \mid r_1(x_0, x_1, x_2) = r_2(x_0, x_1, x_2) = r_3(x_0, x_1, x_2) = \mathrm{id},$$
$$[x_1, x_0, x_0, x_0] = [x_1, x_0, x_0, x_1] = [x_1, x_0, x_0, x_2] = \mathrm{id}\rangle.$$

Using the above criterion, a straightforward MAGMA calculation shows that there exists a unique automorphism $\psi : G_3 \to G_3$ with $\psi(x_0) = x_0^{-1}$, $\psi(x_1) = x_1^{-1}$ and $\psi(x_2) = x_0^{-1} x_2 x_0$. This shows that $S(u_3)$ is a *real* Beauville surface of mixed type.

Finally, recall from [11, p. 2781] that H is canonically isomorphic to

$$\langle x_0, x_1 \mid r_3(x_0, x_1) = r_4(x_0, x_1) = r_5(x_0, x_1) = \mathrm{id}\rangle,$$

with

$$r_3(x_0, x_1) = (x_1x_0)^3 x_1^{-3} x_0^{-3},$$

$$r_4(x_0, x_1) = x_1 x_0^{-1} x_1^{-1} x_0^{-3} x_1^2 x_0^{-1} x_1 x_0 x_1,$$

$$r_5(x_0, x_1) = x_1^3 x_0^{-1} x_1 x_0 x_1 x_0^2 x_1^2 x_0 x_1 x_0,$$

MAGMA calculations show that for any choice

$$(y_0, y_1) \in \{(x_0^{-1}, x_1^{-1}), (x_1x_0, x_0^{-1}), (x_1^{-1}, x_1x_0), (x_1^{-1}, x_0^{-1}), (x_0^{-1}, x_1x_0), (x_1x_0, x_1^{-1})\},$$

we have

$$|\langle x_0, x_1 \mid r_3(x_0, x_1) = r_3(y_0, y_1) = \mathrm{id}, r_4(x_0, x_1) = r_4(y_0, y_1) = \mathrm{id}, r_5(x_0, x_1) = r_5(y_0, y_1) = \mathrm{id}\rangle| = 3072.$$

Since we have $|H_k| \geq 8192$ for $k > 3$ not a power of 2, there cannot be a homomorphism $\psi : H_k \to H_k$ with $\psi(x_0) = y_0$ and $\psi(x_1) = y_1$ by the above criterion, showing that $S(u_k)$ cannot be biholomorphic to $\overline{S(u_k)}$. This finishes the proof of Theorem 1.

Remark 7.1. The fact that the triples (G_k, H_k, T_k) are mixed Beauville structures (for k not a power of 2) is very remarkable. Let us reflect—by looking back at the proof of property (B)—why this result is so surprising:

We know that for indices up to order $k \leq 100$ we have

$$|G_{k+1}| = \begin{cases} 8|G_k| & \text{if } k \equiv 0, 1 \bmod 3, \\ 4|G_k| & \text{if } k \equiv 2 \bmod 3, \end{cases}$$

which gives strong evidence that this should hold for all indices $k \in \mathbb{N}$ (see the *finite width* 3 *conjecture* in [11, Conjecture 1]).

This means that for any $k \leq 99$, $k \equiv 2 \bmod 3$ and A_1, there are *at most four* different choices A_2 such that (A_1, A_2) represent the first two nontrivial diagonals of matrix representations $M_k(A_1, A_2, \ldots)$ of elements in G. On the other hand, it follows from the arguments in the proof of property (B) that we need *at least four* such possibilities to guarantee that $\Sigma(x) \cap \Sigma(y) = \{\mathrm{id}\}$ (and to derive analogous results for the pairs (x_0, y_0)

and (x_1, y_1)). Moreover, these four possibilities must appear in the right combinations in all the conjugation schemes to guarantee the required trivial intersections.

Moreover, for any given A_1, we have *at most eight* choices A_2 such that $M_0(A_1, A_2, \ldots)$ are matrix representations of elements in G. On the other hand, our considerations in the previous section show that we need *at least eight* such choices to guarantee that

$$\{hxh^{-1}, hx^3\,h^{-1}\} \cap \{h'yh^{-1}, hy^3(h')^{-1}\} = \{\mathrm{id}\},$$

for all choices of $h, h' \in H$.

This shows that the conjectured *finite width* 3 property of the infinite group G implies a very tight situation, which leaves "just enough room" to allow the mixed Beauville structures (for k not a power of 2). □

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Appendix A. Conjugation Schemes for the Pairs x_0^n, y_0^n

The notation in the conjugation schemes is the same as in Section 6. The results in this appendix show that

$$\Sigma(x_0) \cap \Sigma(y_0) = \{\mathrm{id}\}.$$

(a) For the pair $x_0, y_0 = M_0(A_1 = [11, 11, 11], A_2, \ldots) \bmod G^k$:

$$\begin{tikzcd}
{[17, 17, 17]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[17, 17, 17]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] & {[44, 219, 177]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"] & {[44, 219, 177]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm 1})"'] \\
{[31, 130, 117]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[31, 130, 117]} & {[34, 72, 213]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm 1})"] & {[34, 72, 213]}
\end{tikzcd}$$

(b) For the pair $x_0^3, y_0^3 = M_0(A_1 = [11, 11, 11], A_2, \ldots) \bmod G^k$:

$$\begin{tikzcd}
{[11,11,11]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"] & {[11,11,11]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"'] \\
{[5,152,111]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] & {[5,152,111]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[54,193,171]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"] & {[54,193,171]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"'] \\
{[56,82,207]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] & {[56,82,207]}
\end{tikzcd}$$

(c) For the pair $x_0^{2^r}, y_0^{2^r} = M_{2^r-1}(A_1 = [26, 26, 26], A_2, \ldots) \bmod G^k$, where $r \geq 1$ is odd:

$$\begin{tikzcd}
{[0,0,0]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"] & {[0,0,0]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"'] \\
{[0,106,247]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] & {[0,106,247]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[0,157,106]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"] & {[0,157,106]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"'] \\
{[0,247,157]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] & {[0,247,157]}
\end{tikzcd}$$

(d) For the pair $x_0^{2^r}, y_0^{2^r} = M_{2^r-1}(A_1 = [11, 11, 11], A_2, \ldots) \bmod G^k$, where $r \geq 2$ is even:

$$\begin{tikzcd}
{[0,0,0]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"] & {[0,0,0]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"'] \\
{[14,147,100]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] & {[14,147,100]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[61,202,160]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"] & {[61,202,160]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"'] \\
{[51,89,196]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] & {[51,89,196]}
\end{tikzcd}$$

Appendix B. Conjugation Schemes for the Pairs x_1^n, y_1^n

The notation in the conjugation schemes is the same as in Section 6. The results in this appendix show that

$$\Sigma(x_1) \cap \Sigma(y_1) = \{\mathrm{id}\}.$$

(a) For the pair $x_1, y_1 = M_0(A_1 = [23, 224, 138], A_2, \ldots) \bmod G^k$:

$$\begin{tikzcd}
{[59,136,495]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"] & {[53,27,395]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"'] \\
{[59,136,495]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] & {[53,27,395]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[33,146,501]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"] & {[47,1,401]} \arrow[d, leftrightarrow, "\mathrm{Conj}(x_1^{\pm1})"'] \\
{[33,146,501]} \arrow[r, leftrightarrow, "\mathrm{Conj}(x_0^{\pm1})"] & {[47,1,401]}
\end{tikzcd}$$

(b) For the pair $x_1^3, y_1^3 = M_0(A_1 = [23, 224, 138], A_2, \ldots) \bmod G^k$:

$$\begin{tikzcd}
{[28, 88, 341]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"] & {[18, 203, 305]} \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"'] & {[6, 66, 335]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"] & {[8, 209, 299]} \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"'] \\
{[28, 88, 341]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] & {[18, 203, 305]} & {[6, 66, 335]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] & {[8, 209, 299]}
\end{tikzcd}$$

(c) For the pair $x_1^{2^r}, y_1^{2^r} = M_{2^r-1}(A_1 = [39, 208, 186], A_2, \ldots) \bmod G^k$, where $r \geq 1$ is odd:

$$\begin{tikzcd}
{[0, 0, 0]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"] & {[0, 157, 106]} \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"'] & {[0, 106, 247]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"] & {[0, 247, 157]} \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"'] \\
{[0, 0, 0]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] & {[0, 157, 106]} & {[0, 106, 247]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] & {[0, 247, 157]}
\end{tikzcd}$$

(d) For the pair $x_1^{2^r}, y_1^{2^r} = M_{2^r-1}(A_1 = [23, 224, 138], A_2, \ldots) \bmod G^k$, where $r \geq 2$ is even:

$$\begin{tikzcd}
{[0, 0, 0]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"] & {[14, 147, 100]} \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"'] & {[26, 26, 26]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"] & {[20, 137, 126]} \arrow[d, leftrightarrow, "\mathtt{Conj}(x_1^{\pm 1})"'] \\
{[0, 0, 0]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] & {[14, 147, 100]} & {[26, 26, 26]} \arrow[r, leftrightarrow, "\mathtt{Conj}(x_0^{\pm 1})"] & {[20, 137, 126]}
\end{tikzcd}$$

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