MODULI VIA DOUBLE PANTS DECOMPOSITIONS

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Abstract: We consider (local) parametrizations of Teichmüller space $\mathcal{T}_{g,n}$ (of genus g hyperbolic surfaces with n boundary components) by lengths of $6g-6+3n$ geodesics. We find a large family of suitable sets of $6g-6+3n$ geodesics, each set forming a special structure called "admissible double pants decomposition". For admissible double pants decompositions containing no double curves we show that the lengths of curves contained in the decomposition determine the point of $\mathcal{T}_{g,n}$ up to finitely many choices. Moreover, these lengths provide a local coordinate in a neighborhood of all points of $\mathcal{T}_{g,n} \setminus X$ where X is a union of $3g-3+n$ hypersurfaces. Furthermore, there exists a groupoid acting transitively on admissible double pants decompositions and generated by transformations exchanging only one curve of the decomposition. The local charts arising from different double pants decompositions compose an atlas covering the Teichmüller space. The gluings of the adjacent charts are coming from the elementary transformations of the decompositions, the gluing functions are algebraic. The same charts provide an atlas for a large part of the boundary strata in Deligne-Mumford compactification of the moduli space $\mathcal{M}_{g,n}$.

Keywords: Moduli spaces, Hyperbolic surfaces, Pants decompositions

MSC: 32G15, 57M50

Introduction

Consider a hyperbolic structure on a closed oriented surface $S_{g,n}$, $2g+n>2$, of genus g with n boundary components. In [5], Fricke and Klein proved that in case $n=0$ the Teichmüller space $\mathcal{T}=\mathcal{T}_{g,n}$ for such a surface is homeomorphic to $(6g-6)$-dimensional Euclidean space. Moreover, they specified a point of Teichmüller space by the lengths of closed geodesics contained in some (rather large) set.

After Fricke and Klein many authors investigated various sets of global parameters on the Teichmüller space. Fenchel and Nielsen [2] introduced "length-twists" coordinates which in

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case of closed surface consist of $3g-3$ lengths of mutually non-intersecting geodesics and $3g-3$ twist parameters along them. Natanzon [10] described a convenient set of parameters (including both lengths of geodesics and parameters of other nature), allowing to recover the Fuchsian group of the surface. A lot of efforts were spent on descriptions of purely length global parameters, especially, for the question of minimal possible number of geodesics whose lengths are sufficient to serve as a global coordinate on the Teichmüller space. First, it was shown that $9g-9$ length of geodesics may serve as global parameters in $\mathcal{T}_{g,0}$ (see [15]). Later, Wolpert [16] used the construction of Fricke and Klein to show that $6g-6$ lengths are sufficient for a local coordinate in $\mathcal{T}_{g,0}$ (but not for a global one). It was natural to expect that $6g-6$ lengths of geodesics can serve as a global coordinate on $\mathcal{T}_{g,0}$, however, Wolpert [17] showed that $\mathcal{T}_{g,0}$ can not be parametrized globally by lengths of $6g-6$ geodesics. Seppälä and Sorvali [13] presented a global parameterization of $\mathcal{T}_{g,0}$ by $6g-4$ length functions (as a by-product they also gave an example of $6g-6$ length parameters defining the surface up to at most 4 possibilities). Finally, in [12] Schmutz obtained a global parameterization by $6g-5$ lengths of geodesics, which is due to [17] is minimal possible. Another example of such a minimal parameterization is given in [6] by Hamenstädt. In the case of surfaces with cusps or holes the situation is easier: the $(6g-6+2m+3n)$-dimensional Teichmüller space of surfaces with m cusps and n holes may be globally parametrized by $(6g-6+2m+3n)$ length parameters (see [13], [12] and [6]). Hamenstädt [7] also showed that such a parametrization may be extended to the Thurston boundary of $\mathcal{T}$.

In this paper, we consider the Teichmüller space $\mathcal{T}=\mathcal{T}_{g,n}$ of marked hyperbolic structures on an oriented surface $S=S_{g,n}$, $2g+n>2$ of genus g with n geodesic boundary components. The dimension of this space is $6g-6+3n$, so we are interested in sets of $6g-6+3n$ curves on S whose lengths parametrize $\mathcal{T}$. We build a large family of the sets of $6g-6+3n$ curves such that the lengths of curves from each set determine a point of $\mathcal{T}$ up to finitely many possibilities and provide a local coordinate in neighborhoods of most points of $\mathcal{T}$, the local charts of this type compose an atlas on $\mathcal{T}$, the transition functions between the charts are algebraic. Moreover, the same atlas works for regular points of the moduli the space $\mathcal{M}=\mathcal{T}/Mod$ (where Mod is a modular group) and covers also a large part of the Deligne-Mumford compactification of $\mathcal{M}$.

In more details, we build a large family of the sets of $6g-6+3n$ curves on S satisfying the following properties:

1. (Parametrizing property). The lengths of the curves of each set determine a point of $\mathcal{T}$ up to finitely many choices; they provide a local coordinate in the neighborhoods of almost all points of $\mathcal{T}$.

2. (Double pants decomposition property). Each set compose an *admissible double pants decomposition* defined and studied recently in [1]; it consists of two pants decompositions (where a *pants decomposition* is a set of curves decomposing the surface into "pairs of pants", i.e. into spheres with 3 holes). Each pants decomposition defines a handlebody with S as the boundary, so, two pants decompositions define a Heegaard splitting of some 3-manifold M^3. The *admissible* double pants decompositions are ones corresponding to Heegaard splittings of the 3-sphere (there exists also an equivalent combinatorial definition which is used throughout the proofs).

3. (Groupoid action). There exists a groupoid acting on admissible double pants decompositions transitively and generated by simple transformations of two types (called "flips" and "handle-twists"), each of the generating transformations changes exactly one curve of a double pants decomposition. The length of the new curve is an algebraic function of the lengths of the initial curves.

4. (Atlas on $\mathcal{T}$ with algebraic transition functions). The charts arising from admissible double pants decompositions compose an atlas on $\mathcal{T}$; the transition functions between the charts are algebraic.

5. (Extension to most strata of Deligne-Mumford compactification). Let Mod be a modular group of S and let $\mathcal{M} = \mathcal{T}/Mod$ be the corresponding moduli space. Each point of the Deligne-Mumford compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$ is a boundary point for some chart coming from a double pants decomposition. Moreover, for most points of $\overline{\mathcal{M}}$ (including almost all points of the strata of minimal codimension) there exists a chart coming from a double pants decomposition and covering a neighborhood of the point in the corresponding stratum as well as covering almost all point in the neighborhood of the point in $\overline{\mathcal{M}}$.

More precisely, let DP be an admissible double pants decomposition whose curves are closed geodesics in S. In principle, two pants decompositions contained in DP may have a common curve (called a *double curve*), we will be interested in double pants decompositions containing no double curves. Let $l(DP)$ be the ordered set of lengths of curves composing DP. Then we prove the following:

Theorem A. (see Theorem 4.11 below). Let DP be an admissible double pants decomposition without double curves. Then DP together with the ordered set of lengths $l(DP) = \{l(c_i) | c_i \in DP\}$ is a local coordinate in $\mathcal{T} \setminus Z$ where Z is a union of finitely many codimension 1 subsurfaces in $\mathcal{T}$ (each homeomorphic to a codimension 1 disk).

Moreover, we show also

Theorem B. (see Theorem 5.1 below). Let DP be an admissible double pants decomposition containing no double curves. Then $l(DP)$ determines a point of $\mathcal{T}$ up to finitely many choices.

Composing Theorems A and B with the fact (see [1]) that there exists a groupoid acting on admissible double pants decompositions transitively, we derive the following

Theorem C. (see Theorem 6.8 below). (1) The charts with coordinates $l(DP)$, where DP is an admissible double pants decomposition without double curves, provide an atlas on Teichmüller space $\mathcal{T}$.

(2) The elementary transition functions of these charts are induced by elementary transformations of double pants decompositions, each elementary transition function change only one coordinate. This unique non-trivial transition function is algebraic.

(3) The compositions of elementary transition functions act transitively on the charts.

The entire construction looks similar to one in the theory of cluster algebras arising from triangulated surfaces ([11], [3], [4]). In the latter, the hyperbolic metrics are encoded by suitably defined lengths of the arcs of triangulations (called lambda lengths), the set of lengths of arcs in one triangulation composing together Penner coordinates on the decorated Teihmüller space. The groupoid generated by flips of triangulations acts transitively on the triangulations, each of the flips changes exactly one of the arcs in the triangulation, so, changing exactly one component of the coordinates. In addition, in case of cluster algebras there are explicit formulas expressing changes in combinatorics of the triangulation (exchange matrices) and transition functions between the charts (exchange relations). For the case of pants decompositions explicit formulas are not known.

It is worth to mention that a single set of $6g-6+3n$ length (local) coordinates on $\mathcal{T}$ may be easily obtained from Fenchel-Nielsen coordinates in the following way: for each curve c_i in a pants decomposition one takes the lengths of c_i and the length of c_i', where c' is obtained from c_i by a flip. However, the coordinates obtained are not symmetric (have distinct geometrical origin), and the are not subject to transitive action of a groupoid.

The structure of double pants decomposition is convenient to work with Deligne-Mumford compactification of the moduli space. Let C be a set of mutually disjoint simple curves on S. Contracting the curves contained in C we obtain a point of the compactification, on the other hand, we stay in any chart arising from a double pants decomposition DP such that $C \in DP$ (more precisely, the limit point belongs to the boundary of the chart), see Theorem 7.1 and Corollary 7.2.

Furthermore, contraction of the curves of C turns a conveniently chosen double pants decomposition DP into a double pants decomposition of the obtained surface with nodal singularities (provided that $C \in DP$ and each curve of C is intersected by a unique other curve of DP). There are some cases when such a convenient decomposition does not exist, however, for the most configuration of curves C we show that it does exist. In this case we say that the set C is *good* and the stratum $\mathcal{S}_C \in \overline{\mathcal{M}}$ is *good* (here S_C is the set of nodal surfaces obtained by shrinking all curves of C, $\mathcal{M}$ is the moduli space and $\overline{\mathcal{M}}$ is its Deligne-Mumford compactification). In particular, all strata of minimal codimension (i.e. of codimension 2) are good strata. For a good set of curves C we define another length-type coordinates as $\tilde{l}(DP, C) = \{l(c_i), \frac{1}{l(c_j)} \mid c_i \in C, c_j \in DP \setminus C\}$. We show that the functions $\tilde{l}(DP, C)$ produce *almost charts* covering the good strata of $\overline{\mathcal{M}}$, i.e. given a point $\tau' \in \mathcal{S}_C$ in a good stratum $\mathcal{S}_C$ there exists an admissible double pants decomposition DP and a neighborhood $O(\tau') \subset \overline{\mathcal{M}}$ in a natural topology such that $\tilde{l}(DP, C)$ produce a local coordinate in $O(\tau') \cap \mathcal{S}_C$ and give a local coordinate in some set $O(\tau') \setminus Z \in \overline{\mathcal{M}}$, where Z is a union of finitely many codimension 1 subsurfaces in $\mathcal{M}$. More precisely, we prove the following

Theorem D. (see Theorem 7.13 below). Let S be a nodal surface, let $\mathcal{M}(S)$ be its moduli space and let $\overline{\mathcal{M}}(S)$ be Deligne-Mumford compactification of $\mathcal{M}$. Let $\mathcal{S}^{\mathcal{M}}_{good} = \mathcal{S}_{good}/Mod$ be the union of good strata in $\mathcal{M}$. Let O be a locus of orbifold points of $\mathcal{M}$, let $\overline{O}$ be the closure of O in $\overline{\mathcal{M}}$. Then

(1) the charts with coordinates $\tilde{l}(DP, C)$ provide an atlas on $\mathcal{M} \setminus O$ and on $\mathcal{S}^{\mathcal{M}}_{good} \setminus \overline{O}$, (here C is a good set and DP is an admissible double pants decomposition without double curves);
(2) each point $\tau' \in S^{\mathcal{M}}_{good} \setminus \overline{O}$ is covered by some almost chart $(O'(\tau'), \tilde{l}(DP, C))$;
(3) the elementary transition functions of these charts (almost charts) change only one coordinate, this unique non-trivial transition function is algebraic;
(4) the compositions of elementary transition functions act transitively on the union of charts and almost charts.

The paper is organized as follows. In Section 1, we recall from [1] the definition of double pants decompositions and their properties. In Sections 2 and 3, we discuss Fenchel-Nielsen coordinates on $\mathcal{T}$, and use them to prove some technical lemmas. In Section 4, we prove Theorem A, i.e. we prove that double pants decompositions induce some local charts on $\mathcal{T}$ (see Theorem 4.11). Section 5 is devoted to the proof of Theorem B (see Theorem 5.1). In

Section 6, we collect the above mentioned local charts into an atlas on $\mathcal{T}$, this leads to Theorem C (see Theorem 6.8). Finally, in Section 7 we consider Deligne-Mumford compactification of the moduli space and prove Theorem D (see Theorem 7.13).

1. Preliminaries I: double pants decompositions

In this section we recall from [1] the definition of double pants decompositions and their properties.

1.1. **Pants decompositions.** Let $S = S_{g,n}$ be an oriented closed surface of genus $g \geq 0$ with n boundary components. We assume $2g + n > 2$, which excludes spheres with less than 3 holes and the torus. The surface S is fixed throughout the paper.

A *curve* c on S is an embedded closed non-contractible non-selfintersecting curve considered up to a homotopy of S.

Given a set of curves we always assume that there are no "unnecessary intersections", so that if two curves of this set intersect each other in k points then there are no homotopy equivalent pair of curves intersecting in less than k points.

For a pair of curves c_1 and c_2 we denote by $|c_1 \cap c_2|$ the number of (geometric) intersections of c_1 with c_2.

Definition 1.1 (*Pants decomposition*)**.** A *pants decomposition* of S is a set of (non-oriented) mutually disjoint curves $P = \{c_1, \dots, c_k\}$ decomposing S into pairs of pants (i.e. into spheres with 3 holes). In this paper, all boundary curves of S are considered as a part of each pants decomposition of S.

It is easy to see that any pants decomposition of $S_{g,n}$ consists of $3g-3+2n$ (where $3g-3+n$ curves decompose S and n curves are boundary curves). Note, that we do allow self-folded pants, two of whose boundary components are identified in S. A surface which consists of one self-folded pair of pants will be called *handle*.

A curve $c \in P$, is *regular* if $c \notin \partial S$ and c is not a self-identified boundary curve of the self-folded pair of pants (i.e. if it is not lying inside a handle cut out by a curve $c' \in P$).

Definition 1.2 (*Flip*)**.** Let $P = \{c_1, \dots, c_{3g-3+2n}\}$ be a pants decomposition. Define a *flip of P in a regular curve c_i* as a replacing of $c_i \subset P$ by any curve c_i' satisfying the following properties:

- c_i' does not coincide with any of $c_1, \dots, c_{3g-3+2n}$;
- $|c_i' \cap c_i| = 2$;
- $c_i' \cap c_j = \emptyset$ for all $j \neq i$.

See Fig. 1.1 for an example of a flip. Clearly, an inverse operation to a flip is also a flip (so that the set of flips compose a groupoid acting on pants decompositions).

Definition 1.3 (*Standard decomposition*)**.** A decomposition P of $S_{g,n}$ is *standard* if P contains g curves $c_1, \dots, c_g$ such that c_i, $i = 1, \dots, n$, cuts out a handle.

1.2. **Double pants decompositions.** Let $P = \{c_1, \dots, c_{3g-3+2n}\}$ be a pants decomposition. A *Lagrangian plane* $\mathcal{L}(P) \subset H_1(S, \mathbb{Z})$ is a subspace spanned by the homology classes $h(c_i)$, $i = 1, \dots, 3g-3+2n$ (here c_i is taken with any orientation).

Two Lagrangian planes $\mathcal{L}(P_1)$ and $\mathcal{L}(P_2)$ are *in general position* if $\mathcal{L}_1 \cap \mathcal{L}_2 = 0$ and $H_1(S, \mathbb{Z}) = \langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ (where $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ denotes the sublattice of $H_1(S, \mathbb{Z})$ spanned by $\mathcal{L}_1$ and $\mathcal{L}_2$).

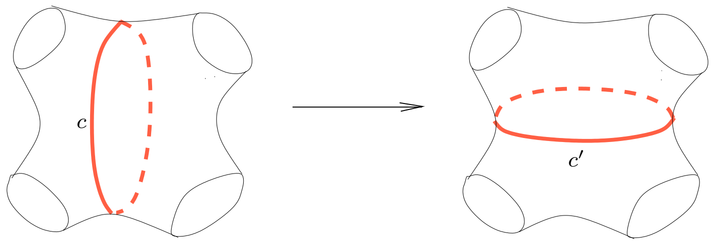


FIGURE 1.1. Flips of pants decomposition.

Definition 1.4 (*Double pants decomposition*)**.** A *double pants decomposition* $DP = (P_a, P_b)$ is a pair of pants decompositions P_a and P_b of the same surface such that the Lagrangian planes $\mathcal{L}_a = \mathcal{L}(P_a)$ and $\mathcal{L}_b = \mathcal{L}(P_b)$ spanned by these pants decompositions are in general position. P_a and P_b are called *parts* of DP.

See Fig. 1.2 for an example of a double pants decomposition.

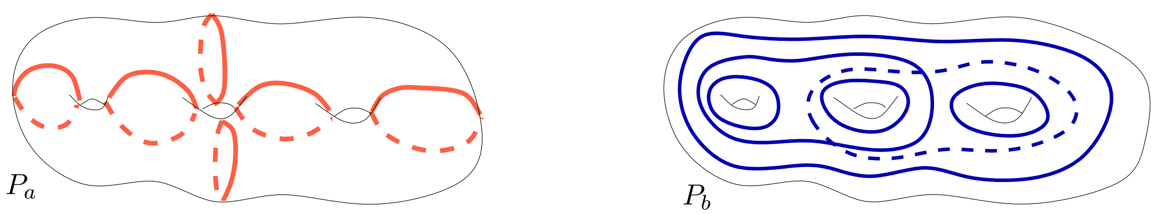


FIGURE 1.2. A double pants decomposition (P_a, P_b).

There are several natural transformations on the set of double pants decompositions:

- flips of P_a;
- flips of P_b;
- handle-twists (see Definition 1.5 below).

Definition 1.5 (*Handle-twists*)**.** Given a double pants decomposition $DP = (P_a, P_b)$ we define an additional transformation which may be performed if both parts P_a and P_b contain the same curve $a_i = b_i$ separating the same handle $\mathfrak{h}$, see Fig. 1.3(a). Let $a \in \mathfrak{h}$ and $b \in \mathfrak{h}$ be the only curves from P_a and P_b respectively. Then a *handle-twist* $t_a(b)$ (respectively, $t_b(a)$) is a Dehn twist along a (respectively, b) in any of two directions (see Fig. 1.3(b)).

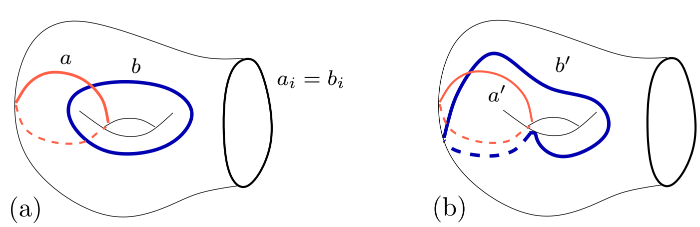


FIGURE 1.3. Handle-twists: (a) Double self-folded pair of pants; (b) The same pair of pants after a handle-twist $t_a(b)$

Notice that both flips and handle-twists are reversible transformations, so that flips and handle-twists generate a groupoid acting on the set of double pants decompositions.

Definition 1.6 (*Double curve*)**.** A curve $c \in (P_a, P_b)$ is *double* if $c \in (P_a \cap P_b)$ and $c \notin \partial S$.

Definition 1.7 (*Standard decomposition*)**.** A double pants decomposition (P_a, P_b) of $S_{g,n}$ is *standard* if there exist g double curves $c_1, \dots, c_g \in (P_a, P_b)$ such that c_i cuts out of S a handle $\mathfrak{h}_i$.

A standard double pants decomposition (P_a, P_b) is *strictly standard* if (P_a, P_b) contains $2g - 3 + n$ double curves (i.e. $c \in \{P_a \cup P_b\} \setminus \{P_a \cap P_b\}$ if and only if c is contained inside some handle).

See Fig. 1.4 for an example of a standard double pants decomposition (this decomposition may be turned into a strictly standard one in one flip).

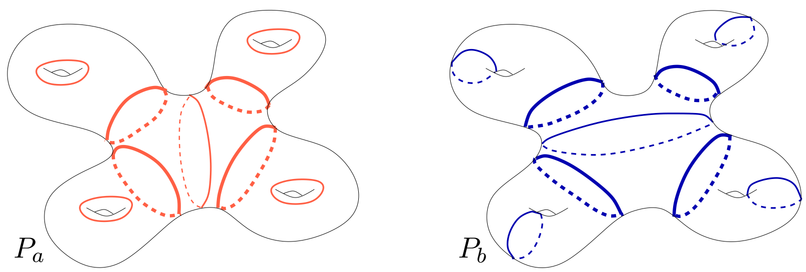


FIGURE 1.4. A standard double pants decomposition (P_a, P_b).

Definition 1.8 (*Admissible decomposition*)**.** A double pants decomposition (P_a, P_b) is *admissible* if it is possible to transform (P_a, P_b) to a standard pants decomposition by a sequence of flips.

For example, the decomposition shown in Fig. 1.2 is admissible.

The following theorem is the main result of [1].

Theorem 1.9 ([1])**.** *A groupoid generated by flips and handle-twists acts transitively on admissible double pants decompositions of $S = S_{g,n}$ (for any (g, n) such that $2g + n > 2$).*

Remark 1.10 (Admissible double pants decompositions and Heegaard splitting of $\mathbb{S}^3$). A set of admissible double pants decompositions have an invariant topological description in terms of Heegaard splittings of 3-manifolds. For each pants decomposition P of S one may construct a handlebody S_+ such that S is the boundary of S_+ and all curves of P are contractible inside S_+. A union of two pants decompositions of the same surface define two different handlebodies bounded by S. Attaching this handlebodies along S one obtains a Heegaard splitting of some 3-manifold $M^3(DP)$. It is shown in [1] that a pants decomposition DP is admissible if and only if $M^3(DP) = \mathbb{S}^3$, where $\mathbb{S}^3$ is a 3-sphere.

We will also use the following result proved in [1, Lemma 6.1].

Proposition 1.11 ([1])**.** *Let $S = S_{g,n}$, $2g + n > 2$, and $Mod(S)$ be its modular group. Let (P_a, P_b) be an admissible double pants decomposition without double curves. Then $\gamma \in Mod(S)$ fixes (P_a, P_b) if and only if $\gamma = id$.*

2. Preliminaries II: coordinates on Teichmüller space

Let $S = S_{g,n}$ be a hyperbolic surface of genus g with n boundary components. Each boundary component is assumed to be a geodesic of finite length.

A Teichmüller space $\mathcal{T} = \mathcal{T}_{g,n}$ is a parameter space of marked hyperbolic metrics on the surface $S_{g,n}$. For the marking on S we will usually use admissible double pants decompositions containing no double curves (this provides a correct marking since any elements $\gamma \neq e$ of the modular group $Mod(S_{g,n})$ acts non-trivially on the decomposition, see [1, Lemma 6.1]).

We will use Fenchel-Nielsen parameterization of the Teichmüller space. We shortly explain the parametrization below and refer to [14] for the details.

To build the parameterization one chooses a pants decomposition P of S. Each pair of pants is uniquely determined by the lengths of its boundary curves. To encode the concrete hyperbolic structure one need also to now how the adjacent pairs of pants a sewed together: one can choose an arbitrary way to attach them, and then rotate one piece along another by any real angle. More precisely, to determine the angle of the rotation one does the following:

1) for each pair of pants $p^k \in P$ one chooses three disjoint segments s^k_{ij}, $i, j \in \{1, 2, 3\}$ orthogonal to the boundary components b^k_i and b^k_j of p^k (so that p^k is decomposed into two right-angled hexagons);
2) then one fixes some way to attach the adjacent pairs of pants p^k and $p^{k'}$ so that the segments s^k_{ij} and $s^{k'}_{i'j'}$ intersect the curve $p^k \cap p^{k'}$ at the same points, this will produce some special gluing of pairs of pants, all other gluings (with other angles of rotation of p^k with respect to $p^{k'}$) will be compared with this special gluing;
3) for arbitrary gluing the angles of rotation are compared with the chosen special gluing, when the angle is changed by 2π one obtains the same hyperbolic structure on the surface, but the different point of the Teihmüller space.

So, the Fenchel-Nielsen coordinates on $\mathcal{T}$ build from the pants decomposition P consist of $3g - 3 + 2n$ length parameters $l(c_i)$ (lengths of all the curves $c_i \in P$ including the boundary curves of S) and $3g - 3 + n$ angle parameters $\alpha(c_j)$ (angles along all non-boundary curves $c_j \in P$, $c_j \notin \partial S$). We denote

$$FN(P) = \{l(c_i), \alpha(c_j) \mid c_i \in P; \quad c_j \in P, c_j \notin \partial S\}.$$

We will also assume that the Dehn twist along c_j changes $\alpha(c_j)$ by 2π.

The construction establishes the homeomorphism between $\mathcal{T}$ and $\mathbb{R}^{3g-3+2n}_{>0} \times \mathbb{R}^{3g-3+n}$ (where $\mathbb{R}_{>0}$ stays for positive real numbers).

Remark 2.1. After the Teichmüller space $\mathcal{T}$ is introduced using any given pants decomposition P_0 (or even using a marking of other type), one can choose any pants decomposition P to introduce the coordinates $FN(P)$ on the same space $\mathcal{T}$.

Our aim is to transform Fenchel-Nielsen coordinates to coordinates containing only length parameters.

Definition 2.2 (*Locally parametrizing decomposition*)**.** We say that a double pants decomposition DP is *locally parametrizing* at the point $\tau \in \mathcal{T}$ if the functions $l(DP) = \{l(c) \mid c \in DP\}$ provide a local homeomorphism from a neighborhood of τ to a neighborhood of some point in $\mathbb{R}^{6g-6+3n}$. By a *chart* $\mathfrak{C}(DP)$ we mean a pair $(X, l(DP))$ where X is the set of points $\tau \in \mathcal{T}$ such that DP is locally parametrizing at τ.

Our first aim is to prove that admissible double pants decompositions are locally parametrizing. As an intermediate technical step in the proof we will use *mixed* coordinates, containing some angle-parameters (but less than Fenchel-Nielsen coordinates).

Definition 2.3 (*Mixed coordinates*)**.** Let $DP = (P_a, P_b)$ be a double pants decomposition, possibly with some double curves. Let $FN(P_b)$ be some Fenchel-Nielsen coordinates build from P_b. Denote by $mix(DP, FN(P_b))$ the following set of functions:

$$mix(DP, FN(P_b)) = \{l(c), \alpha(c') \mid c \in DP, c' \in P_a \cap P_b\},$$

where $\alpha(c')$ is the corresponding angle coordinate in $FN(P_b)$.

3. Some properties of length functions

In this section we prove several facts from hyperbolic geometry. In particular, Lemmas 3.4 and 3.6 will be crucial for the construction of locally parametrizing double pants decompositions. Lemmas 3.1–3.3 are preparatory. We will denote the hyperbolic plane by $\mathbb{H}^2$.

Lemma 3.1. *Let $S = S_{0,4}$, let $c, d \in S$ be two closed curves $|d \cap c| = 2$. Let P be a pants decomposition of S, $c \in P$. Suppose that $d' \in S$ is a curve obtained from c by a flip of P. Then $d' = t_c^k(d)$ for some integer k, where t_c is a Dehn twist along c.*

The lemma follows immediately from [1, Lemma 1.16].

Lemma 3.2. *Let $p \in \mathbb{H}^2$ be a line separating points O and O'. Given the distances from p to O and O', the distance OO' is a monotonic function on the distance PP', where P and P' are the orthogonal projections of points O and O' to p.*

Proof. Suppose that the points P and O are fixed, and the point P' (together with O') glide away from P, see Fig. 3.1.b. Then the point $X = OO' \cap p$ glide away from P which implies that the distance OX grows monotonically when PP' increases. By the similar reason $O'X$ grows, and hence, OO' grows monotonically.

□

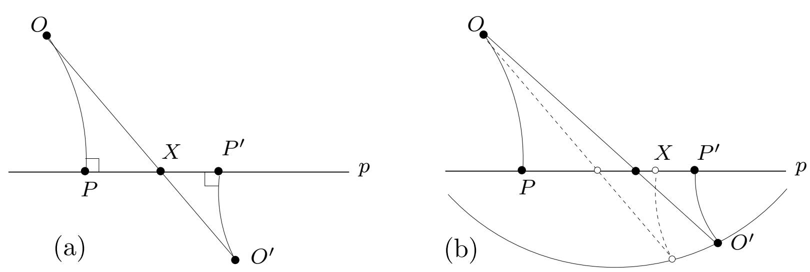


Figure 3.1. To the proof of Lemma 3.2

Lemma 3.3. *Let $S = S_{0,3}$ be a three-holed sphere with a boundary $\partial S = c_1 \cup c_2 \cup c_3$, and let s_{ij} be a segment orthogonal to c_i and c_j, for $i \neq j$, $i, j \in \{1, 2, 3\}$. Then the segments $s_{12}, s_{13}, s_{2,3}$ decompose S into two congruent right-angled hexagons.*

Proof. It is clear that the segments s_{ij} decompose S into two right-angled hexagons. Since a right-angled hexagon is determined (up to an isometry) by the lengths of three non-adjacent sides (the lengths of $s_{12}, s_{13}, s_{2,3}$), the hexagons are congruent.

□

If the curves $a, b \in S$ are orthogonal to each other we will write "$a \perp b$".

Lemma 3.4. *Let $S = S_{1,1}$ be a handle with a boundary curve c, let $a, b \subset S$ be two curves $|a \cap b| = 1$. Then the set of functions $\bar{x} = (l(a), l(b), l(c))$ is a local coordinate on $\mathcal{T} \setminus X$ where $X = \{\tau \in \mathcal{T} | a \perp b\}$. Moreover, $\bar{x}$ determines the point $\tau \in \mathcal{T}$ up to at most two possibilities.*

Proof. Shortly speaking, the coordinates $\bar{x} = (l(a), l(b), l(c))$ will be produced from Fenchel-Nielsen coordinates. More precisely, we fix Fenchel-Nielsen coordinates $FN(P) = (l(a), \alpha(a), l(c))$ arising from pants decomposition $P = \{a, c\}$. We fix some values of $l(a)$ and $l(c)$ and denote by α_0 the value of $\alpha(a)$ at the point where $l(a)$ and $l(c)$ have the chosen values and a is orthogonal to b. We will show that $l(b)$ is a monotonic function on the absolute value $|\alpha(a) - \alpha_0|$, which will imply all statements of the lemma. Below we explain this in more details.

First, we cut S along a and obtain a pair of pants S' with three boundary components c, a and a'. For each of the three pairs of boundary components of S' we draw a segment orthogonal to both of these two components. Denote these segments by $s_{c,a}$, $s_{c,a'}$, $s_{a,a'}$, see Fig. 3.2.a. The three segments decompose S' into two right-angled hexagons H_1 and H_2. Similarly, together with the curve a the three segments decompose the initial handle S into two hexagons.

Consider the covering of S by hyperbolic plane. We are interested in the tiling of the plane by the images of H_1 and H_2. Notice that the copies of H_1 and H_2 adjacent along the image of $s_{a,a'}$ (or $s_{c,a}$ or $s_{c,a'}$) have this side in common, while the gluing along the images of a and a' depends on the angle parameter $\alpha(a) \in FN(P)$. More precisely, when $\alpha(a) = \alpha_0$ the adjacent along a hexagons have a common side, otherwise the hexagons are shifted one along another as in Fig. 3.2.b. With growth of $\alpha(a)$ the hexagons in one row glide monotonically along the hexagons of the other row. We denote by p and p' the lines separating the rows.

Now, consider the curve $b \in S$, $|b \cap a| = 1$. First, suppose that $b \perp a$, i.e. the image $\hat{b}$ of b in the hyperbolic plane coincide with the image AA' of $s_{a,a'}$. Now, we increase $\alpha(a)$ and look at the image $\hat{b} \in \mathbb{H}^2$ of b: since b is a closed geodesic on S, $\hat{b}$ is a line forming the same angles with p and p'. This implies that $\hat{b}$ passes through the midpoint O of AA'. Hence, $AY = A'Y'$, where $Y = \hat{b} \cap p$ and $Y' = \hat{b} \cap p'$. Furthermore, the hexagon H_2' is shifted with respect to the hexagon H_2 to the distance $\rho = l(a)\frac{(\alpha(a)-\alpha_0)}{2\pi}$. Denote by T the vertex of H_2' projecting to the same point of S as A' (as in Fig. 3.2.b), then $TY = AY = A'Y'$. Hence, $AY = 1/2\rho = l(a)\frac{(\alpha(a)-\alpha_0)}{4\pi}$. The same formula holds for any positive value of $(\alpha(a) - \alpha_0)$ as well as for any negative one (in the latter case the point $Y \in l$ lies on the other side with respect to A).

This implies that the distance $YY' = l(b)$ grows monotonically with the growth of $|\alpha(a) - \alpha_0|$:

$$\cosh \frac{YY'}{2} = \cosh OY = \cosh OA \cosh AY = \cosh OA \cosh(l(a)\frac{\alpha(a) - \alpha_0}{4\pi})$$

Hence, $|\alpha(a) - \alpha_0|$ may be recovered from $l(b)$. So, given the lengths $(l(a), l(b), l(c))$ one may find the Fenchel-Nielsen coordinates $FN(P)$ up to two possibilities. In particular, in the neighborhood of a point $\tau \in \mathcal{T}$ where a is not orthogonal to b, the sign of $(\alpha(a) - \alpha_0)$ does not changes, which implies that the functions $(l(a), l(b), l(c))$ form a local coordinate in $\mathcal{T} \setminus X$, $X = \{\tau \in \mathcal{T} | a \perp b\}$.

□

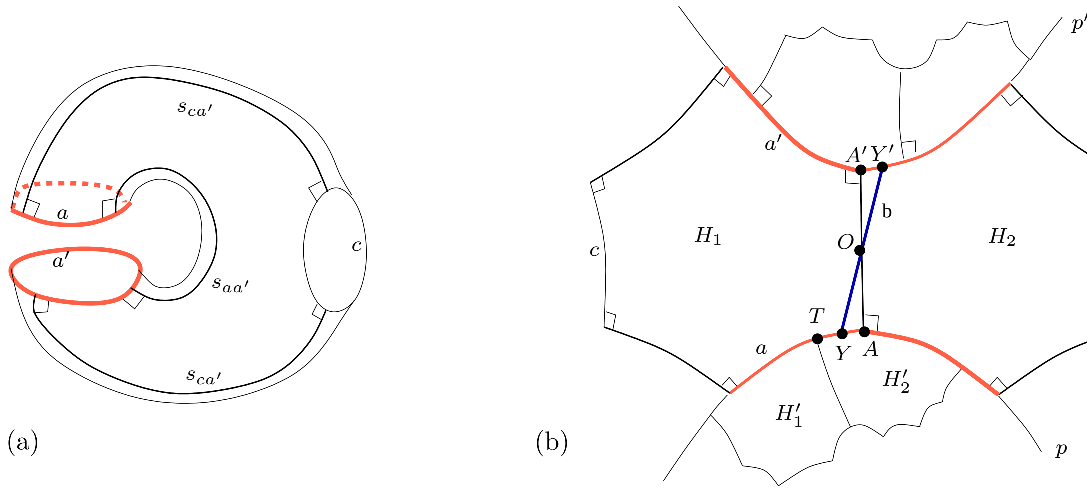


FIGURE 3.2. Length coordinates on a handle

Remark 3.5. Given Fenchel-Nielsen coordinates $(l(a), \alpha(a), l(c))$ on the handle, for each pair of lengths $l_0(a)$ and $l_0(c)$ there exists a unique angle $\alpha_0(a)$ such that a is orthogonal to b.

Lemma 3.6. *Let $S = S_{0,4}$ be a sphere with four holes, with boundary curves c_1, c_2, c_3, c_4. Let $a \in S$ be a closed geodesic and let $b \in S$ be a closed geodesic obtained from the curve a by a flip. Then*

(1) *the angle formed by a and b is of the same size for both intersections of a and b;*
(2) *the set of functions $\bar{x} = (l(a), l(b), l(c_1), l(c_2), l(c_3), l(c_4))$ is a local coordinate on $\mathcal{T} \backslash X$ where $X = \{\tau \in \mathcal{T} | a \perp b\}$;*
(3) *$\bar{x}$ determines the point $\tau \in \mathcal{T}$ up two at most two possibilities.*

Proof. The idea of the proof is the same as in the proof of Lemma 3.4: the coordinate $\bar{x}$ is obtained from Fenchel-Nielsen coordinates $FN(P)$ built from pants decomposition $P = \{a, c_1, c_2, c_3, c_4\}$. We show that given the values of $(l(a), l(c_1), l(c_2), l(c_3), l(c_4))$ the length $l(b)$ is a monotonic function on the absolute value $|\alpha(a) - \alpha_0|$, where α_0 is the value of $\alpha(a) \in FN(P)$ at the point of $\mathcal{T}$ such that a is orthogonal to b (and the values of $(l(a), l(c_1), l(c_2), l(c_3), l(c_4))$ are the chosen ones). Hence, $l(b)$ determines $\alpha(a)$ up to 2 possibilities. Moreover, in the neighborhood of a point $\tau \in \mathcal{T}$ where $|\alpha(a) - \alpha_0| \neq 0$, the sign of $(\alpha(a) - \alpha_0)$ is determined uniquely by the sign at τ.

In more details, the curve a decompose S into two pairs of pants, and each pair of pants is decomposed into two right-angled hexagons (respectively, by the segments $s_{ac_1}, s_{c_1c_2}, s_{c_2a}$ and $s_{a'c_3}, s_{c_3c_4}, s_{c_4a'}$ orthogonal to a pair of boundary components), Fig. 3.3.a. The images of four right-angled hexagons tile the covering hyperbolic plane: two hexagons adjacent by the image of the side a are shifted by the distance $\rho = l(a)\frac{\alpha(a)-\alpha_0}{2\pi}$ along the line containing the images of a, see Fig. 3.3.b.

Denote by O and O' the midpoints of images of s_{c_1,c_2} and s_{c_3,c_4}. Notice that the symmetry in the point O preserves the tiling of the hyperbolic plane by hexagons (compare with Lemma 3.3). The same holds for the symmetry in O'. Consider a line OO' and its intersection with the images of the curve a. It is easy to see that all angles made by OO' and images of a are equal. Furthermore, OO' intersects the images of s_{c_1,c_2} and s_{c_3,c_4} always in midpoints (to see that consider an image O'' of O with respect to the symmetry in O': it lies on OO' and in the midpoint of some image of $s_{c_1c_2}$, then consider the image of O' with respect to a symmetry in O'' and so on). This implies that the line OO' is the union of images of some closed geodesic $c \in S$, $|c \cap a| = 2$. Hence, c may be obtained from a by a flip. Notice that c intersects a in two points, forming two angles of the same size. The length $l(c) = 2 \cdot OO'$ increases as $|\alpha(a) - \alpha_0|$ increases (the distances from the points O and O' to the line p remain constant, but one point glide along p with respect to the other, so that we may apply Lemma 3.2).

Increasing the angle $\alpha(a)$, we increase the shift between the adjacent hexagons. Increasing $\alpha(a)$ by 2π we obtain the initial tiling of the plane by hexagons, but the line OO' in the new picture is moved, so that it is an image of another closed curve $c' \in S$ which may be obtained from a by a flip. Increasing (or decreasing) $\alpha(a)$ by $2\pi k$ we run through all curves on S which may be obtained by a flip from a (compare with Lemma 3.1). In particular, for some value of k we obtain the curve b. This implies statement (1). So, the length $l(b)$ increases with growth of $|\alpha(a) - \alpha_0|$. Hence $l(b)$ determines $\alpha(a)$ up to two possibilities, which implies that the set of functions $\bar{x}$ determines Fenchel-Nielsen coordinates $FN(P)$ up to two possibilities. This proves statement (3). If b is not orthogonal to a at $\tau \in \mathcal{T}$ then in the neighborhood of τ the function $l(b)$ (together with the chosen value of $\alpha(a)$ at τ) determines completely the function $\alpha(a)$, which implies that $\bar{x}$ is a set of local coordinates, and statement (2) is also proved.

□

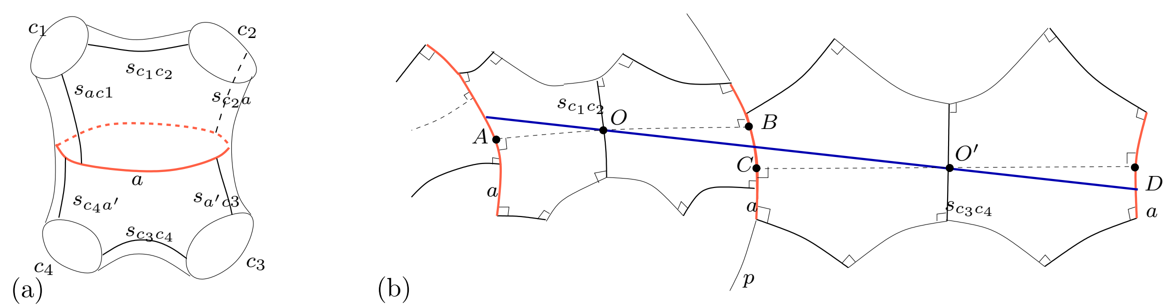


FIGURE 3.3. Length coordinates on a four-holed sphere

Remark 3.7. Given Fenchel-Nielsen coordinates on $S_{0,4}$, for each lengths $l_0(a)$ together with fixed lengths of the boundary components of $S_{0,4}$ there exists a unique angle $\alpha_0(a)$ such that a is orthogonal to b.

4. Locally parametrizing double pants decompositions

In this section we prove Theorem 4.11 which states that for an admissible double pants decomposition DP the functions $l(DP)$ provide a local parameter in neighborhoods of almost all points $\tau \in \mathcal{T}$.

The proof of the theorem is inductive. In Section 4.1, we build some examples of locally parametrizing double pants decompositions. These examples called *special decompositions* will be the base of the induction. In section 4.2, we show that any admissible double pants decomposition may be obtained from a special one by a sequence of flips. Finally, in Section 4.3 we show that flips preserve the parametrizing properties of double pants decompositions.

4.1. **Examples of locally parametrizing double pants decompositions.** In this section we present an example of a locally parametrizing double pants decomposition for each surface $S_{g,n}$. This will provide a base for the inductive proof of Theorem 4.11. The construction is obtained as a modification of Fenchel-Nielsen coordinates.

Definition 4.1 (*Special decomposition, conjugate curves*)**.** A double pants decomposition $DP = (P_a, P_b)$ is *special* with the *standard part* P_b if the following holds:

(1) DP contains no double curves;
(2) the part P_b is standard;
(3) DP may be obtained from a strictly standard double pants decomposition DP_0 via a sequence of $m = 3g - 3 + n$ flips $f_1, \dots, f_m$ of the P_a-part.

For a special decomposition $DP = (P_a, P_b)$ we will say that a curve $a_i \in P_a$ is *conjugate* to a curve $b_i \in P_b$ if either a_i is obtained by a flip f_i from b_i or a_i and b_i belong to the same handle in the standard decomposition P_b. In the former case (a_i, b_i) will called a *flip-conjugate pair*, in the latter case (a_i, b_i) will called a *handle-conjugate pair*.

See Fig. 4.1 for an example of a special decomposition. Notice, that any special double pants decomposition is admissible.

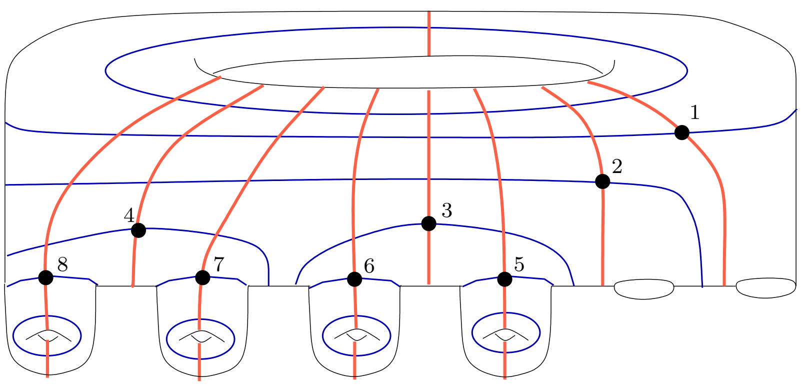


FIGURE 4.1. Example of a special double pants decomposition. The black nodes show the intersections of the conjugate curves. The number near the nodes show the sequence of flips taking the strictly standard decomposition to the special one.

Lemma 4.2. *For each standard pants decomposition P_b there exists a special double pants decomposition $DP = (P_a, P_b)$.*

Proof. To build the required decomposition we consider a strictly standard double pants decomposition $DP' = (P'_a, P_b)$ containing P_b and apply a flip of the P_a-part to each of the double curves. □

Notation 4.3. Let $DP = (P_a, P_b)$ be a special double pants decomposition. Denote by $Z(DP) \in \mathcal{T}$ the locus of points where a_i is orthogonal to b_i for at least one pair of conjugate curves $(a_i, b_i) \in DP$.

Remark 4.4. Let (a_i, b_i) be a pair of conjugate curves in a special double pants decomposition. Remarks 3.5 and 3.7 imply that the locus of points where a_i is orthogonal to b_i is homeomorphic to a hyperplane in $\mathcal{T} = \mathbb{R}^{3g-3+2n}_{>0} \times \mathbb{R}^{3g-3+n}$ (here Remarks 3.5 and 3.7 work for cases of handle-conjugate and flip-conjugate pairs respectively). Therefore, the set $Z(DP) \in \mathcal{T}$ is homeomorphic to a union of $3g-3+n$ hyperplanes in $\mathcal{T} = \mathbb{R}^{3g-3+2n}_{>0} \times \mathbb{R}^{3g-3+n}$.

Lemma 4.5. *Let $DP = (P_a, P_b)$ be a special double pants decomposition. Then*

(1) *$l(DP)$ is a local coordinate in $\mathcal{T} \setminus Z(DP)$;*
(2) *$l(DP)$ determine the point in $\mathcal{T}$ up to at most 2^{3g-3+n} choices.*

Proof. Suppose that P_b is a standard part of DP. Choose Fenchel-Nielsen coordinates $FN(P_b)$ based on the pants decomposition P_b. It is a global coordinate on $\mathcal{T}$. We will substitute angle coordinates of $FN(P_b)$ by length coordinates one by one.

Let $f_1, \dots, f_m$ be the sequence of flips described in the Definition 4.2, let $b_1, \dots, b_m$ be the curves of P_b such that f_i is a flip applied to b_i. Let $DP_i = f_i \circ \dots \circ f_1(DP_0)$, where DP_0 is the corresponding strictly standard double pants decomposition. Applying Lemma 3.4 sufficiently to all handle-conjugate pairs of curves $a_i, b_i \in DP$ we see that $mix(DP_0, FN(P_b))$ is a local coordinate away from $Z(DP_0)$ and defines the coordinate $FN(P_b)$ up to 2^g choices. Then, applying Lemma 3.6 to each pair of flip-conjugate curves successively (more precisely, to the subsurface $S_{0,4}$ obtained by a union of two pairs of pants adjacent to b_i in P_a-part of DP_i), we see that $mix(DP_i, FN(P_b))$ is a local coordinate away from $Z(DP_i)$ and defines $mix(DP_{i-1}, FN(P_b))$ up to 2 choices. This implies the lemma.

□

4.2. **Induction step: reduction to flips.**

Lemma 4.6. *Let DP be an admissible double pants decomposition. Then there exists a sequence of flips $f_1, \dots, f_k$ such that $DP_0 = f_k \circ \dots \circ f_1(DP)$ is a strictly standard double pants decomposition.*

Proof. Since DP is an admissible decomposition, there exists a sequence of flips taking DP to a standard double pants decomposition. It is known that flips act transitively on pants decompositions of $S_{0,k}$ (see [8]), which implies that any strictly standard double pants decomposition may be transformed to a strictly standard ones by flips.

□

Lemma 4.7. *Let DP be a double pants decomposition containing no double curves. Suppose that $DP' = f_k \circ \dots \circ f_1(DP)$, where f_i, $i = 1, \dots, k$, is a flip. If DP' contains no double curves then there exists a sequence of flips $g_1, \dots, g_r$ such that $DP' = g_r \circ \dots \circ g_1(DP)$ and no of the decompositions $g_i \circ \dots \circ g_1(DP)$, $i = 1, \dots, r$ contains double curves.*

Proof. Denote $DP = (P_a, P_b)$ and $DP' = (P'_a, P'_b)$ We will use the fact that flips of the P_a-part commute with flips of the P_b-part.

Let $C = \{c \mid c \in DP_i = f_i \circ \dots \circ f_1(DP), 0 \le i \le k\}$ be a set of all curves appearing during the transformation from DP to $DP' = f_k \circ \dots \circ f_1(DP)$.

First, for each of the curves $a_i \in P_a$ we apply a flip g_i so that $g_i(a_i) \notin C$: this is possible, since C is a finite set, while a set of flips for a given curve a_i in a given pants decomposition is either infinite or empty (in the later case, a_i lies in a handle bounded by some other curve a_j, so we can first destroy the handle applying a flip to a_j, and then apply a flip to a_i). Denote by P''_a the obtained P_a-part of the decomposition.

Second, we transform P_b to P'_b by the same sequence of flips as in $f_1, \dots, f_k$.

Third, there exists a sequence $f'_1, \dots, f'_l$ of flips taking P''_a to P'_a. Denote $\xi = f'_l \circ \dots \circ f'_1$. Denote $C' = \{c \mid c \in DP'' = f'_i \circ \dots \circ f'_1(DP), 0 \le i \le l\}$. For each of the curves $b_i \in P'_b$ we apply a flip g'_i so that $g'_i(b_i) \notin C'$.

Next, we transform P_b to P'_b by the same sequence of flips as in $f_1, \dots, f_k$.

Finally, we apply the inverse sequence ξ^{-1} to take the P_b-part back to the state P'_b.

Clearly, we can not obtain double curves at any stage of the transformation, so the lemma is proved.

□

Lemma 4.6 together with Lemma 4.7 imply the following lemma.

Lemma 4.8. *Let DP be an admissible double pants decomposition without double curves. Then there exists a special double pants decomposition DP' and a sequence of flips $f_1, \dots, f_k$ such that $DP_0 = f_k \circ \dots \circ f_1(DP)$ and no of the decompositions $f_i \circ \dots \circ f_1(DP)$, $i = 1, \dots, k$, contains double curves.*

4.3. **Induction step: flips.** In this section we show that flips take locally parametrizing double pants decompositions to locally parametrizing ones.

In the next lemma we show this property for almost all flips.

Lemma 4.9. *Let DP be a parametrizing double pants decomposition at $\tau \in \mathcal{T}$. Let f' and f'' be two different flips of the same curve $c \in DP$, such that neither $DP' = f'(DP)$ nor $DP'' = f''(DP)$ contain double curves. If DP' is not parametrizing at $\tau \in \mathcal{T}$ then DP'' is parametrizing at τ.*

Proof. Let $DP = (P_a, P_b)$, $c \in P_a$. Let $DP' = (P'_a, P_b)$, $DP'' = (P''_a, P_b)$. Denote by c' and c'' the curves of P'_a and P''_a obtained from c by flips f' and f'' respectively. In addition, denote by S_* a subsurface of S composed of two pairs of pants in P_a adjacent to the curve c.

Suppose that DP' is not a parametrizing double pants decomposition at $\tau \in \mathcal{T}$. By definition, this means that there exists a non-trivial deformation $\xi(\tau)$ of the hyperbolic structure, where ξ preserves all lengths of curves contained in (P'_a, P_b). This deformation may be described as a set of simultaneous small twists along the curves of P'_a (the rates of the twists need not coincide or to be constant).

Suppose that ξ contains no twist along c' (i.e. the twist along this curve is trivial, zero). Then the subsurface S_* is not changed, and the length of the curve c is preserved by ξ. Hence, ξ preserves the lengths of all curves in $(P_a, P_b) = DP$. By assumption, these lengths provide a local coordinate at τ, so the deformation ξ is trivial (does not change the point of Teichmüller space). The contradiction shows that ξ contains a non-trivial twist along c'.

On the other hand, consider another deformation η of the initial hyperbolic structure $\tau \in \mathcal{T}$, where η preserves all lengths of curves from (P_a, P_b) except the length of c. A locus of points of $\mathcal{T}$ obtained by η from τ is a 1-dimensional curve in a neighborhood of τ. This implies that $\eta = \xi$.

Suppose now that DP'' also is not parametrizing at τ. Similarly to the case of DP', this implies that there exists a deformation ψ preserving all lengths of curves from P''_a and containing a non-trivial twist along the curve $c'' \in P_a$. Similarly to ξ, the deformation ψ should coincide with η, so, $\xi = \psi$. However, these two transformations do not coincide in the subsurface S_*: one twists along c', another along $c'' \neq c'$. The contradiction shows that the double pants decomposition DP'' is parametrizing at τ.

□

Lemma 4.10. *Let DP be a locally parametrizing double pants decomposition at $\tau \in \mathcal{T}$. Let f_0 be a flip of DP such that the double pants decomposition $DP^{(0)} = f_0(DP)$ contains no double curves. Then $DP^{(0)}$ is a parametrizing double pants decomposition at $\tau \in \mathcal{T}$.*

Proof. Let $DP = (P_a, P_b)$ be a parametrizing double pants decomposition at $\tau \in \mathcal{T}$. Let $c \in DP$ be a curve flipped by f_0. Without loss of generality we may assume that $c \in P_a$. Denote $m = 3g - 3 + n$.

Consider an m-dimensional surface C_a through $\tau \in \mathcal{T}$ such that the lengths of all curves contained in $P_a \setminus P_b$ are constant in C_a. Let C_b be a similar surface for P_b. Denote by Π_a and Π_b the tangent planes to C_a and C_b in τ. Let $C_{\partial S}$ be an n-dimensional surface through τ such that all curves contained in ∂S have constant lengths in $C_{\partial S}$, let $\Pi_{\partial S}$ be the corresponding tangent plane. Since $DP = (P_a, P_b)$ is parametrizing at τ, the planes Π_a, Π_b and $\Pi_{\partial S}$ intersect each other in τ only (and span the whole tangent space at τ).

Let ψ_i, $i = 1, \dots, m$, be the curves in $\mathcal{T}$ on which all lengths of curves of DP are preserved except for the length of one curve $b_i \in P_b \setminus P_a$. Let $\bar{b}_1, \dots, \bar{b}_m$ be the tangent vectors to $\psi_1, \dots, \psi_m$ at τ. Clearly, the plane Π_a is spanned by the vectors $\bar{b}_1, \dots, \bar{b}_m$.

Now, consider a series of flips f_i of the curve $c \in P_a$ (including the flip f_0 described in the lemma): we will assume that the flip f_i takes c to the curves c_i of the same homology class; moreover, we assume that c_{i+1} may be obtained from c_i by a Dehn twist along c. For each of the flips f_i we denote $P_a^{(i)} = f_i(P_a)$. Denote by $\Pi_a^{(i)}$ the tangent planes at τ to the surfaces of the constant lengths of curves from $P_a^{(i)} \setminus P_b$.

If the double pants decomposition $DP^{(0)} = (P_a^{(0)}, P_b)$ is parametrizing at τ, then there is nothing to prove. So, suppose that $DP^{(0)}$ is not parametrizing at τ. By Lemma 4.9, this implies that all other double pants decompositions $DP^{(i)} = (P_a^{(i)}, P_b)$ are parametrizing at τ (with possible exclusion of at most one decomposition $DP^{(j)}$: at most one of these decompositions may contain a double curve c_i). Reasoning as above with Π_a, we show that the plane $\Pi_a^{(i)}$ is spanned by $b_1, \dots, b_m$. This implies that for $i \notin \{0, j\}$ all planes $\Pi_a^{(i)}$ coincide with Π_a.

Now, our aim is to show that $\Pi_a^{(0)} = \Pi_a$. Let t_c be a Dehn twist along c. The twist t_c takes c_i to c_{i+1}. On the other hand, t_c acts on $\mathcal{T}$ and takes Π_a^i to Π_a^{i+1}. Since $\Pi_a^i = \Pi_a$ for $i \notin \{0, j\}$, t_c preserves Π_a. Hence, $\Pi_a^i = \Pi_a$ for all $i \in \mathbb{Z}$.

Since $\Pi_a^{(0)} = \Pi_a$, the planes $\Pi_a^{(0)}$, Π_b and $\Pi_{\partial S}$ span the tangent space at τ, which implies that $DP^{(0)} = (P_a^{(0)}, P_b)$ is a parametrizing double pants decomposition at τ.

□

Theorem 4.11. *Let DP be an admissible double pants decomposition without double curves. Then DP together with the ordered set of lengths $l(DP) = \{l(c_i) | c_i \in DP\}$ is a local coordinate in $\mathcal{T} \setminus Z(DP')$ for some special double pants decomposition DP'.*

Proof. By Lemma 4.8 there exists a special double pants decomposition $DP' = (P'_a, P'_b)$, and a sequence ψ of flips taking DP to DP' and producing no double curves on its way. By Lemma 4.5 the lengths $l(DP')$ form a local coordinates in $\mathcal{T} \setminus Z(DP')$. By Lemma 4.10 each of the flips in the sequence ψ preserve the parametrizing property of double pants decomposition (i.e. the obtained decomposition provides a local parameter in $\mathcal{T} \setminus Z(DP')$). Hence, DP is parametrizing in $\mathcal{T} \setminus Z(DP')$.

□

5. Finite number of choices

In Section 4, we proved that the set of functions $l(DP)$ is a local parameter in almost all points of $\mathcal{T}$. In this section, we prove that the functions $l(DP)$ determine the point of $\mathcal{T}$ up to finitely many choices.

Consider an universal covering π of S by a hyperbolic plane $\mathbb{H}^2$, so that $S = \mathbb{H}^2/G$ where $G \in SL_2(\mathbb{R})$ is some finitely generated discrete group. Let $M_1, \ldots, M_s$ be a finite set of matrices generating G. Let $r_1(M_1, \ldots, M_s) = \cdots = r_n(M_1, \ldots, M_s) = E$ be the defining relations, where r_i is a word in the alphabet $A = \{M_1, M_1^{-1}, \ldots, M_s, M_s^{-1}\}$.

For each closed geodesic $c \subset S$ each connected component of the preimage $\pi^{-1}c$ is a line (denote it by $L_i(c)$, where integer index stays to emphasize that there are countably many of these preimages). The group G contains a hyperbolic transformation $\gamma(c)$ shifting $\mathbb{H}^2$ along $L_i(c)$ for the distance equal to $l(c)$. So, we have

$$\mathrm{tr}(\gamma(c)) = 2\cosh(l(c)/2). \tag{5.1}$$

Notice that $\gamma(c) = w(M_1, \ldots, M_s)$ for some word w in the same alphabet A. So, the Formula 5.1 may be considered as a finite set of polynomials in matrix elements of $M_1, \ldots, M_s$ with coefficients

$$\hat{l}(c) = 2\cosh(l(c)/2).$$

Theorem 5.1. *Let DP be an admissible double pants decomposition containing no double curves. Then $l(DP)$ determines a point of $\mathcal{T}$ up to finitely many choices.*

Proof. For each of the curves $c_i \in DP$ we consider one of its preimages on $\mathbb{H}^2$ together with the hyperbolic transformation $\gamma(c_i)$. Taking in account Formula 5.1, we obtain a system of polynomial equations in elements of M_i: the system consists of the equations arising from the following three sources:

(1) $M_i \in SL_2(\mathbb{R})$;
(2) $r_j(M_1, \ldots, M_s) = E$, where r_k is one of the defining relations;
(3) $\mathrm{tr}(\gamma(c)) = \hat{l}(c)$.

The matrix equations of the second type are considered as four scalar equations in matrix elements. Notice, that the equations of all three types are polynomial (here we use the fact that $M_i \in SL_2(\mathbb{R})$, and hence, all elements of M_i^{-1} are also elements of M_i). So, the three types compose a system of finitely many polynomial equations in matrix elements of M_i with integer coefficients and constant terms in $\mathbb{Z} \cup \{\hat{l}(c_1)), \ldots, \hat{l}(c_m)\}$. Suppose in addition that the values of $(\hat{l}(c_1)), \ldots, \hat{l}(c_m))$ correspond to at least one hyperbolic structure $\tau \in \mathcal{T}$ on S. Then the system of equations is solvable. On the other hand, Theorem 4.11 implies that the system is non-generate. Thus, there are finitely many solutions of this system.

In other words, for each set of values $l(DP)$ we can write a unique set of values $\hat{l}(DP) = \{\hat{l}(c_1)), \ldots, \hat{l}(c_m)\}$; for this set $\hat{l}(DP)$ there are finitely many possible values of matrix elements of M_i. So, for each value of $l(DP)$ there are finitely many distinct points in $\mathcal{T}$.

□

Corollary 5.2. Let DP be an admissible double pants decomposition of S without double curves. Let $c \in S$ be a closed curve. Then $\hat{l}(c)$ is an algebraic function of $\hat{l}(DP)$.

Proof. By Theorem 5.1 the value of $l(DP)$ determines the point of $\mathcal{T}$ up to finitely many choices. Each of these choices correspond to a unique (modulo conjugation) discrete subgroup $G \in SL_2(\mathbb{R})$ acting on $\mathbb{H}^2$. Consider the group G for one of these possibilities.

Following the proof of Theorem 5.1 consider a preimage $L(c)$ of c in $\mathbb{H}^2$ and a hyperbolic transformation $\gamma(c)$ which shifts along $L(c)$ by the distance $l(c)$. Then $\gamma(c) = w(M_1, \dots, M_s)$ where w is a word in the alphabet $\{M_i, M_i^{-1} | i = 1, \dots, s\}$. So, $\hat{l}(c) = \operatorname{tr}\gamma(c)$ is a polynomial in the matrix elements of $M_1, \dots, M_S$. Since the elements of matrices M_i are the solution of a system of polynomial equations, these elements are algebraic functions of $\hat{l}(DP)$. This implies, that $\hat{l}(c)$ is an algebraic function of $\hat{l}(DP)$ either.

□

6. An atlas on the Teichmüller space

In Section 4.3, we proved that for each admissible double pants decomposition DP the function $l(DP)$ provides a local coordinate in neighborhoods of almost all points in $\mathcal{T}$ (more precisely, away from a set of measure 0 formed by a finite union of hypersurfaces). In this section, we show that the coordinate charts with coordinates $l(DP)$ compose an atlas on $\mathcal{T}$. Moreover, the transition functions between the adjacent chart change exactly one coordinate (and correspond to flips and handle twists of double pants decompositions).

Lemma 6.1. *Let S be a surface with a fixed hyperbolic structure. Let $DP = (P_a, P_b)$ be a special double pants decomposition with a standard part P_b. Let $a_i, b_i \in DP$ be a pair of conjugate curves in DP. Let $b_j \in P_b$ be a curve such that $b_j \cap a_i \neq \emptyset$ and let t_{b_j} be a Dehn twist along b_j. If a_i is orthogonal to b_i then $t_{b_j}^k(a_i)$ is not orthogonal to b_i for all $k \in \mathbb{Z} \setminus 0$.*

Proof. First, notice that if $i = j$ than there is nothing to prove (the statement follows than from Lemma 3.4 in case of handle-conjugate curves and from Lemma 3.6 in case of flip-conjugate curves). From now on we assume $i \neq j$.

Notice that by construction of special decompositions, the condition $b_j \cap a_i \neq 0$ implies that (a_i, b_i) can not be a pair of handle-conjugate curves. So, (a_i, b_i) is a pair of flip-conjugate curves, and the curve b_i is homologically trivial. Suppose b_i is orthogonal to a_i as well as to $t_{b_j}^k(a_i)$, where $k \neq 0$. Since b_i is homologically trivial, b_i cuts S into two connected components S_1 and S_2. Let S_1 be the component containing the curve b_j. Denote $s = a_i \cap S_2$ and $s' = t_{b_j}^k(a_i) \cap S_2$. In view of Lemma 3.6 all ends of s and s' are orthogonal to b_i.

Since $b_j \in S_1$, and $b_j \cap b_i = \emptyset$, the topology of the decomposition of S_2 is not changed by t_{b_j} (however, geometrically $s \neq s'$). This implies that there exists an isotopy γ_x of s to s' (where $x \in [0, 1]$, $\gamma_0 = s, \gamma_1 = s'$) such that the ends of the segment $\gamma_x(s)$ belong to b_i. Notice that s can not intersect s', otherwise the segments s, s' and a part of b_j bound a hyperbolic triangle with two right angles $b_j s$ and $b_j s'$, which is impossible. On the other hand, if $s \cap s' = \emptyset$ then two parts of b_j, s and s' bound a hyperbolic quadrilateral with four right angles, which is also impossible. The contradiction shows the lemma.

□

Lemma 6.2. *For each point $\tau \in \mathcal{T}$ there exists a double pants decomposition DP_τ such that $l(DP_\tau)$ is a local coordinate in a neighborhood of τ.*

Proof. Consider an arbitrary special double pants decomposition $DP = (P_a, P_b)$ with a standard part P_b. By Theorem 4.11, $l(DP)$ is a local coordinate in $\mathcal{T} \setminus Z(DP)$. So, if $\tau \notin Z(DP)$ then there is nothing to prove. Suppose that $\tau \in Z(DP)$, i.e. there exists an orthogonal conjugate pair of curves $a_i, b_i \in DP$, (a pair of conjugate curves such that a_i is orthogonal to b_i in τ). We will apply to DP a twist t_{b_j} in some of the curves $b_j \in DP$ in order to reduce the number of orthogonal conjugate pairs.

To see that it is always possible, suppose that $a_i, b_i \in DP$ is an orthogonal conjugate pair. In this case there exists an integer k such that the special decomposition $t_{b_i}^k(DP)$ contains less orthogonal conjugate pairs than DP has (the pair a_i, b_i of this twisted decomposition is not orthogonal for each $k \neq 0$, Lemma 6.1 implies that for all but finitely many values of k the k-th degree of the twist will not produce new orthogonalities for other conjugate pairs). □

Lemma 6.2 shows that the charts with coordinates $l(DP)$ cover the space $\mathcal{T}$. Now, we consider the transition functions between the charts. In view of Theorem 1.9, it is natural to choose these transition functions as ones induced by flips and handle-twists of admissible double pants decompositions.

The case of flip is considered in Lemma 4.10: it is shown that as long as a flip f produces no double curves, f preserves the locus of points where the set of functions $l(DP)$ is a local coordinate. We have also shown in Lemma 4.8 that if DP and DP' are two double pants decompositions containing no double curves and DP can be turned into DP' by a sequence of flips, than one can choose this sequence of flips so that no double curves are produced on the way.

It is impossible to treat handle-twists directly in the same way: by definition no handle-twist can be applied to a double pants decomposition containing no double curves. To overcome this obstacle, we introduce the notion of a *quasi-handle-twist*.

Definition 6.3 (*Quasi-handle-twist*). Let DP be a double pants decomposition without double curves. Let $c \in DP$ be a curve such that there exists a flip $f(c)$ producing a handle $\mathfrak{h}$ in the decomposition $f(DP)$ (so that $f(c)$ is a double curve which cuts out the handle). Let $a \in DP \cap f(DP)$ be a curve contained in the handle $\mathfrak{h}$. By a *quasi-handle-twist* t_a of DP we mean a Dehn twist along a.

Remark 6.4. The quasi-handle-twist t_a may be written as $t_a = f^{-1} \circ \hat{t}_a \circ f$, where f is a flip as in Definition 6.3 and $\hat{t}_a$ is a handle twist in the handle $\mathfrak{h}$.

Remark 6.5. Since t_a is a Dehn twist, t_a acts on the Teichmüller space $\mathcal{T}$. We denote by $t_a(\tau)$ the point of $\mathcal{T}$ obtained from τ by the Dehn twist t_a.

Now, we will prove the counterparts to the Lemmas 4.10 and 4.8 for the case of quasi-handle-twists.

The next Lemma follows immediately from Definition 6.3 and Remarks 6.4 and 6.5.

Lemma 6.6. *Let DP be an admissible double pants decomposition without double curves. Let $\tau \in \mathcal{T}$ be a point such that $l(DP)$ is a local coordinate in τ. Let t be a quasi-handle-twist along the curve $c \in DP$. Then $l(t(DP))$ is a local coordinate in $\tau' = t(\tau)$.*

Lemma 6.7. *Let DP and DP' be two admissible double pants decompositions containing no double curves. Then there exists a sequence of flips and quasi-handle-twists which takes DP to DP' and produces no double curves on its way.*

Proof. By Theorem 1.9 there exists a sequence ψ of flips and handle-twists taking DP to DP'. In view of Lemma 4.7, each subsequence containing no handle-twist may be realized without producing double curves. It is sufficient to prove the lemma for the case when ψ contains one handle-twist only (and then apply inductional reasoning). Suppose that this unique handle-twist $\hat{t}_c$ is a twist in a curve $c \in DP^{st}$ where DP^{st} is a standard double pants decomposition flip-equivalent to DP (it is shown in [1, Lemma 4.1] that handle-twists in standard decompositions are sufficient for obtaining the transitivity theorem).

Let $DP^{st} = (P_a^{st}, P_b^{st})$ be the two parts, suppose that $c \in P_a^{st}$. Let $DP^{sp} = (P_a^{sp}, P_b^{sp})$ be a special decomposition with the standard part $P_b^{sp} = P_b^{st}$. By Lemma 4.7, there exists a sequence of flips taking DP to DP^{sp} without producing double curves. Then we apply a quasi-handle-twist t_c in c, so that we obtain another special decomposition DP_*^{sp}. In view of Remark 6.4, DP_*^{sp} is flip-equivalent to DP'. The sequence of flips taking DP_*^{sp} to DP' without producing double curves does exist in view of Lemma 4.7.

□

Summarizing results of Lemmas 6.2, 6.7 and Corollary 5.2 we obtain the following theorem.

Theorem 6.8. *(1) The charts $\mathfrak{C}(DP)$ with coordinates $l(DP)$, where DP is an admissible double pants decomposition without double curves, provide an atlas on Teichmüller space $\mathcal{T}$.*

(2) The elementary transition functions of these charts are induced by flips and quasi-handle-twists of double pants decompositions, each elementary transition function changes only one coordinate. This unique non-trivial transition function is algebraic.

(3) The compositions of elementary transition functions act transitively on the charts.

7. Deligne-Mumford compactification of moduli space

In Section 6, we showed that the Teichmüller space is covered by coordinate charts arising from admissible double pants decompositions. Since local coordinates on Teichmüller space are also local coordinates on the moduli space, the charts with coordinate $l(DP)$ also compose an atlas on the moduli space. In this section, we show that this atlas works also for most strata in Deligne-Mumford compactification of the moduli space.

Consider some Fenchel-Nielsen coordinates $FN(P)$ on the Teichmüller space

$$\mathcal{T} = \{l(c_i) > 0,\ \alpha(c_j) \in \mathbb{R} \mid c_i, c_j \in P, c_j \notin \partial S\}.$$

Given a pants decomposition P denote

$$\mathcal{T}_P = \{l(c_i) \geq 0,\ \alpha(c_j) \in \mathbb{R} \mid c_i, c_j \in P, c_j \notin \partial S\}.$$

The *augmented Teichmüller space* $\overline{\mathcal{T}}$ is the following closure of $\mathcal{T}$:

$$\overline{\mathcal{T}} = \cup_P \mathcal{T}_P$$

where the union is taken by all pants decompositions of the surface. The points of $\overline{\mathcal{T}} \setminus \mathcal{T}$ correspond to *nodal surfaces*, i.e. to the surfaces with *nodal singularities*: a nodal singularity arises when a non-trivial closed curve c in S is degenerated to a point (i.e. $l(c) \to 0$). A nodal surface is not a surface: a neighborhood of a nodal point is not homeomorphic to a disk. We denote by N the set of all nodal points on the nodal surface. It is known that $\overline{\mathcal{T}}/Mod = \overline{\mathcal{M}}$, where Mod is the modular group and $\overline{\mathcal{M}}$ is the Deligne-Mumford compactification of the modular space $\mathcal{M} = \mathcal{T}/Mod$.

The space $\overline{\mathcal{T}}$ inherits topology from $\cup_P T_p = \cup_P(\mathbb{R}^{3g-3+2n}_{\geq 0} \times \mathbb{R}^{3g-3+n})$.

Given an admissible double pants decomposition DP without double curves, we say that the *boundary of the chart* $\mathfrak{C}(DP)$ is the locus of points $\tau' \in \overline{\mathcal{T}}$ where $l(c) = 0$ for at least one $c \in DP$.

Theorem 7.1. *For each point $\tau' \in \overline{\mathcal{T}}$ there exists an admissible double pants decomposition DP containing no double curves and such that τ' belongs to the boundary of the chart $\mathfrak{C}(DP)$ with coordinates $l(DP)$.*

Proof. If $\tau' \in \mathcal{T}$, then there is nothing to prove in view of Theorem 6.8. Suppose that $\tau' \in (\overline{\mathcal{T}} \setminus \mathcal{T})$. Then there exists a set of mutually disjoint curves C on S such that the surface S' corresponding to τ' is obtained by contracting all curves $c_i \in C$. It is sufficient to show that there exists an admissible double pants decomposition DP containing no double curves and such that $C \in DP$.

Consider any pants decomposition P_a containing the set C. We will build the required decomposition $DP = (P_a, P_b)$ in the following four steps: first, we transform P_a by a sequence of flips to a standard decomposition P_a'; second, we build a standard double pants decomposition (P_a', P_b'); next, we transform P_a' back to P_a by flips; finally, we apply (if necessary) several flips to P_b' to avoid double curves.

□

Factorizing by the modular group Mod we obtain the charts on the Deligne-Mumford compactification of the modular space (with the natural notion of the *boundary of the chart* on $\overline{\mathcal{M}}$ defined as the boundary of the same chart on $\overline{\mathcal{T}}$ factorized by Mod). Applying the same reasoning as in Theorem 7.1 we obtain the following corollary.

Corollary 7.2. For each point $\tau' \in \overline{\mathcal{M}}$ there exists an admissible double pants decomposition DP containing no double curves and such that τ' belongs to the boundary of the chart $\mathfrak{C}(DP)$ with coordinates $l(DP)$.

Remark 7.3. For many of the points $\tau' \in \partial\overline{\mathcal{T}}$ the coordinates $l(DP)$ provide also a chart in a neighborhood $O'(\tau') = O(\tau') \cap \partial\overline{\mathcal{T}}$ (where $O(\tau')$ is some neighborhood of τ' in $\overline{\mathcal{T}}$. It would be natural to try to cover $\partial\overline{\mathcal{T}}$ (resp. the whole boundary of $\overline{\mathcal{M}}$) by these charts. However, in general it turns to be impossible (see Remark 7.14).

Below, we define a large subset of "good" points in the boundary and show that all points of this subset are covered by the charts $\mathfrak{C}(DP)$.

The boundary $\partial\overline{\mathcal{T}}$ is stratified: given a set C of mutually non-intersecting curves in S, a *stratum* $\mathcal{S}_C$ is a locus $\{l(c_i) = 0 \mid c_i \in C\}$. All nodal surfaces (of genus g with n boundary parts) with k nodal singularities compose a union $\mathcal{S}_{2k}$ of codimension $2k$ strata.

By a pants decomposition of a surface S with punctures we mean a decomposition into generalized pairs of pants (where a generalized pair of pants is either a sphere with three holes, or a sphere with two holes and a puncture, or a sphere with a hole and two punctures, or a sphere with three punctures).

By a *pants decomposition* (respectively, *double pants decomposition*) of a nodal surface S we mean a set of curves P composing a pants decomposition (respectively double pants decomposition) in all *components* of S (connected components of $S \setminus N$). The nodal points are not considered as curves of the pants decomposition.

A (double) pants decomposition of S is *standard* if the decompositions of all components are standard. Similarly, a double pants decomposition is *special* if decompositions of all components are special.

Let $DP = (P_a, P_b)$ be a double pants decomposition of S containing no double curves. Let $c \in DP$. Denote by S' the nodal surface obtained from S by collapsing c to a nodal singularity. Consider a set DP' of curves on S' obtained as a union of images of curves of DP which do not intersect c. Notice that DP' is not necessarily a double pants decomposition of S'; however, if $c \in P_b$ intersects only one other curve of DP then DP' is.

Lemma 7.4 ((Collar Lemma, [9])). *Let $c \in S$ be a simple closed geodesic on hyperbolic surface S of lengths $l = l(c)$. Define w by the relation*

$$\sinh l \sinh w = 1.$$

Then S contains a collar $Col(c)$ of width w defined by $Col(c) = \{x \in S \mid \rho_S(x, c) < w/2\}$, where $\rho_S(A, B)$ is the distance in S from the set A to the set B.

It follows immediately from the Collar Lemma that if $a, b \in S$ are closed geodesics $b \cap a \neq \emptyset$ then contracting a so that $l(a) \to 0$ implies $l(b) \to \infty$.

The Collar Lemma implies that the local coordinates $l(DP)$ degenerate while the curves c_i are collapsing: if $a_i \cap c_i \neq \emptyset$ then $l(a_i) \to \infty$ while $c_i \to 0$ (the curve a_i intersecting c_i do exists since $c_i \notin P_a$ and P_a is a maximal set of disjoint curves in S). For the case $C \subset DP$ we define the new set of functions $\tilde{l}(DP, C)$ as follows:

$$\tilde{l}(DP, C) = \{l(c_i), \frac{1}{l(c_j)} \mid c_i \in C, c_j \in DP \setminus C\}.$$

Clearly, $\tilde{l}(DP, C)$ is a local coordinate in all points of $\mathcal{T}$, where $l(DP)$ is a local coordinate. Moreover, this set of functions remains correctly defined while the curves of the set C are collapsed.

Definition 7.5 (*Inversion*). An *inversion* of a k-th function of $\tilde{l}(DP, C)$ is an exchange of $l(c_k)$ or $\frac{1}{l(c_k)}$ (where $c_k \in DP$) by $\frac{1}{l(c_k)}$ or $l(c_k)$ respectively.

It is clear the transformation from a set of functions $\tilde{l}(DP, C)$ to any other set of functions $\tilde{l}(DP', C')$ may be obtained as a composition of inversions and transformations induced by flips and quasi-handle-twists of double pants decompositions (here DP and DP' are admissible double pants decompositions containing no double curves, C and C' are sets of disjoint curves).

Definition 7.6 (*Strong and weak curves*). Let $C = \{c_1, \dots, c_k\}$ be a set of mutually disjoint curves on S. Each curve $c \in C$ appears two times in the boundary of $S \setminus C$. We say that c is a *strong curve* of C if two copies of c appear in two different connected components of $S \setminus C$. Otherwise, we say that c is *weak*.

We denote by $C_{strong} \subset C$ the subset of all strong curves.

We denote by $S^1, \dots, S^l$ the connected components of $S \setminus C$. By $\hat{S}^i$ we denote the connected component of $S \setminus C_{strong}$ corresponding to the component S^i of $S \setminus C$ ($\hat{S}^i$ is obtained from S^i by gluing along the pairs of boundary components arising from the weak curves).

Definition 7.7 (*Good set of curves*). We say that a set $C = \{c_1, \dots, c_k\}$ of mutually disjoint curves on S is *good* if each connected component $\hat{S}^i$ of $S \setminus C_{strong}$ is either a surface of positive genus or has at least two boundary components contained in ∂S.

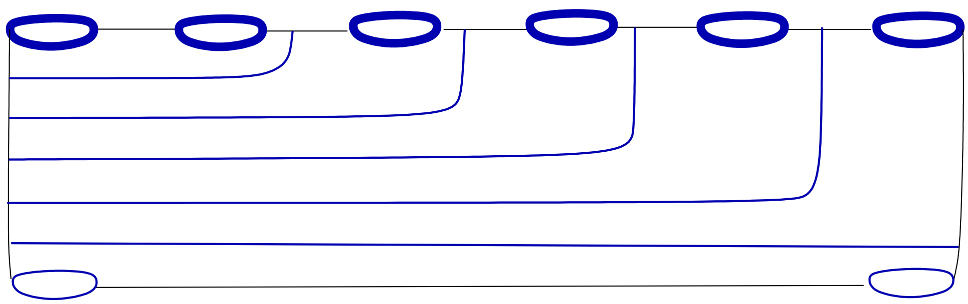
Let $\mathcal{S}_{good}$ be a union of all strata $\mathcal{S}_C$ where C is a good set.

Remark 7.8. It is easy to see that $\mathcal{S}_2 \subset \mathcal{S}_{good}$, where $\mathcal{S}_2$ is a union of all codimension 2 strata.

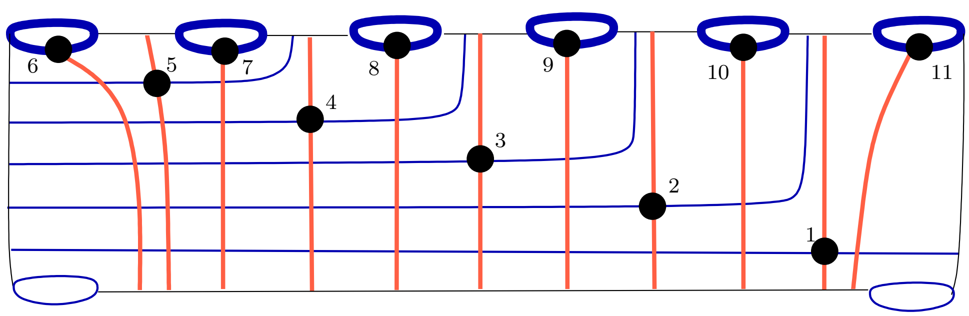
Lemma 7.9. *Let $C = \{c_1, \dots, c_k\}$ be a good set of curves. Then there exists a special double pants decomposition DP of S such that $C \subset DP$ and each curve $c_i \in C$ is intersected by a unique curve of $DP \setminus c$. Moreover, collapsing any curve $c_i \in C$ to a nodal singularity leads to a special double pants decomposition of the obtained nodal surface.*

Proof. We build a special double pants decomposition $DP = (P_a, P_b)$ with a standard part P_b such that P_b contains all strong curves of C and P_a contains all weak curves of C. We construct the decomposition DP separately for each connected component $\hat{S}^i$ of $S \setminus C_{strong}$.

If $\hat{S}^i$ is a sphere with holes, then we build the decomposition DP as shown in Fig. 7.1: since C is a good set of curves, at least two boundary components of $\hat{S}^i$ do not belong to C (the two bottom boundary components in the figure). In Fig. 7.1.a we show the part P_a of DP, in Fig. 7.1.b we show the whole decomposition $DP = (P_a, P_b)$, notice that each curve of C is intersected by a unique curve of P_b.



(a)



(b)

FIGURE 7.1. Special double pants decomposition containing C: case $\hat{S}^i = S_{0,r}$. The curves of C are bold, each intersects a unique other curve of DP. The figure shows only the front part of the surface, the decomposition of the back part is the same. The black nodes show the intersections of the conjugate curves.

Now, suppose that S^i contains at least one handle.

First, we build a standard pants decomposition P containing the set C. To do this for the component $\hat{S}^i$, we build a standard decomposition with a linear structure as in Fig. 7.2.a: first come all handles than come all holes. Moreover, for each strong curve $c_j \in \hat{S}^i$ the curve c_j is contained inside one of the handles (more precisely, first we build the curves $\tilde{c}_j \in S^i$ which together with both copies of c_j bounds a pair of pants in S^i, then in $\hat{S}_i$ the curve $\tilde{c}_j$ cuts out a handle $\mathfrak{h}_j$ containing c_j).

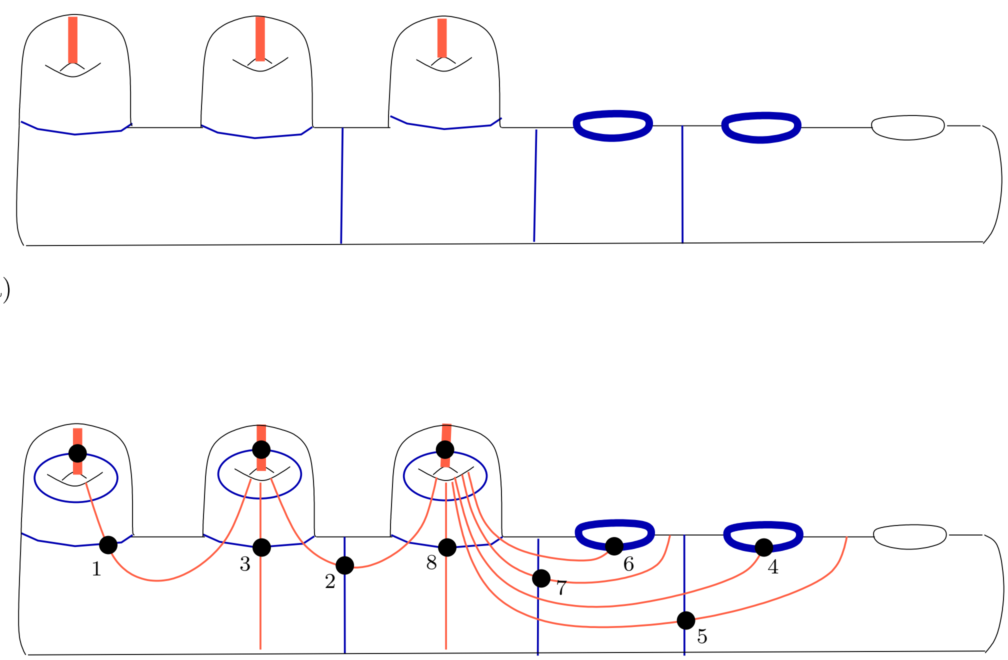
Next, we build the standard part P_b of the special decomposition $DP = (P_a, P_b)$: we take the standard decomposition P and for each handle $\mathfrak{h}_j$ of P we substitute the curve $c_j \in P \cap C$

by any other curve $c'_j \in \mathfrak{h}_j$ such that $|c_j \cap c'_j| = 1$ (in the handles containing no curves of C we do nothing).

Now, we build the part P_a of the special decomposition $DP = (P_a, P_b)$. We build the restriction of P_a to $\hat{S}^i$ as it is shown in Fig. 7.2.b: namely, each of the weak curves $c_j \in C$ is intersected only by a unique curve of P_a lying in the same handle as c_j; each of the strong curves is intersected only by a curve passing through the handle $\mathfrak{h}_0^i$.

The obtained decomposition DP is special: it may be transformed to a standard decomposition by a sequence of flips as shown in Fig. 7.1 and Fig. 7.2 (we show the order of flips by numbering the intersection points of conjugate curves). It is easy to see that collapsing any curve $c_i \in C$ to a point we get a special double pants decompositions DP' of the obtained nodal surface: the sequence of flips taking DP' to a standard decomposition almost coincide with the corresponding sequence for DP (the only difference is that in case of strong curve c_i one needs to omit the flip in the curve conjugated to c_i).

□



(a)

(b)

FIGURE 7.2. Special double pants decomposition containing C (the curves of C are bold, each intersects a unique other curve of DP). The figure shows only the front part of the surface, the decomposition of the back part is the same. The black nodes show the intersections of the conjugate curves.

Let DP be a special double pants decomposition of S, let $C \in DP$ be a good set of curves. Denote by $\overline{Z}(DP, C)$ the locus in $\overline{\mathcal{T}}$ where at least one of the conjugate pairs of $DP \setminus C$ is an orthogonal pair.

Remark 7.10. Let (a_i, b_i) be a conjugate pair of DP and let $b_i \in C$. It is easy to see that while b_i is collapsed, the angle formed up by a_i and b_i tends to the right angle (if lengths of other curves of P_b remain fixed). This implies that $\mathcal{S}_C$ belongs to the closure of $Z(DP)$ in $\overline{\mathcal{T}}$. Therefore, we can not hope that the set of functions $\tilde{l}(DP)$ will provide a local coordinate in the whole neighborhood of a given point $\tau' \in \mathcal{S}_C$.

Instead, we will show that for any point $\tau' \in \mathcal{S}_C$ there exists a suitable special double pants decomposition DP such that $\tilde{l}(DP)$ is a local coordinate in the neighborhood of τ' in $\mathcal{S}_C$ as well as a local coordinate in almost all points of the neighborhood of τ' in $\overline{\mathcal{T}}$ (more precisely, $\tilde{l}(DP)$ is a local coordinate in $O(\tau') \setminus Z(DP)$ where $O(\tau')$ is a neighborhood of τ' in $\overline{\mathcal{T}}$.

This motivates the following definition:

Definition 7.11 (*Almost chart*)**.** Let C be a good set of curves, let $\mathcal{S}_C \subset \overline{\mathcal{T}}$ be the corresponding stratum and let $\tau' \in \mathcal{S}_C$ be a point. An *almost chart* centered at τ' is a pair $(O(\tau'), f)$ where $O(\tau') \subset \overline{\mathcal{T}}$ is a neighborhood of τ' and $f = (f_1, \dots, f_k)$ is a set of k functions, $k = \dim \mathcal{T} = 6g - 6 + 3n$ satisfying the following conditions:

1) the functions f are defined and continuous in $O(\tau')$;
2) f is a local coordinate in a neighborhood $O'(\tau') = O(\tau') \cap S_C$;
3) there exists a finite set X of codimension 1 surfaces in $\overline{\mathcal{T}}$ such that f is a local coordinate in a neighborhood of each point $\tau \in O(\tau') \cap (\overline{\mathcal{T}} \setminus X)$.

Lemma 7.12. *Let S be a marked hyperbolic surface considered as a point of $\mathcal{T} = \mathcal{T}(S)$. Let $S_{good} \subset \overline{\mathcal{T}}$ be a union of the good strata. Let S' be a nodal surface with nodal singularities, such that the marked hyperbolic structure τ' of S' belongs to $\mathcal{S}_{good}$.*

Then there exists an admissible double pants decomposition DP of S which degenerates to an admissible double pants decomposition DP' of S' such that $\tilde{l}(DP', C)$ provides an almost chart centered in τ'.

Proof. Since S' belongs to S_{good}, the nodal surface S' is obtained from S by collapsing the curves contained in some good set C.

By Lemma 7.9 there exists a special double pants decomposition $DP = (P_a, P_b)$ with standard part P_b, such that

(1) $c_i \in P_b$ for all strong curves c_i of C;
(2) $c_i \in P_a$ for all weak curves c_i of C;
(3) for each $c_i \in C$ the decomposition (P_a, P_b) contains a unique curve d_i intersecting c_i.

By Lemma 7.9 by collapsing a curve $c_i \in C$ one obtains a special double pants decomposition of the obtained nodal surface, and, after collapsing all curves $c_i \in C$, we obtain a special double pants decomposition DP' of S'. Clearly, the set of functions $\tilde{l}(DP, C)$ is defined and continuous in a neighborhood O' of τ'.

Using Lemma 6.1 (as in the proof of Lemma 6.2) we may apply to DP several twists (along the curves of P_b) so that the resulting special decomposition $DP_* = t_{c_m}^{k_m} \circ \dots \circ t_{c_1}^{k_1}(DP)$ satisfies $\tau' \notin \overline{Z}(DP'_*)$.

Suppose that some of the twists t_{c_j} changes a curve $c \in C$. Then $c \in P_a$, so c is a weak curve of C. The curve c_j then is the curve conjugated to c in DP. Clearly, we may substitute a degree of the twist t_{c_j} by a degree of the twist t_c so that in the resulting double pants decomposition the images of curves c and c_j are not orthogonal to each other. So, after several substitutions we transform DP_* to a special decomposition DP_{**} such that $\tau' \notin \overline{Z}(DP'_{**})$ and $C \in DP_{**}$. This implies the conditions 2) and 3) of Definition 7.11. The condition 1) of the same definition holds for $\tilde{l}(DP_{**}, C)$ in some neighborhood $O'(\tau') \subset \overline{\mathcal{T}}$ trivially. Hence, the pair $(O'(\tau'), \tilde{l}(DP_{**}, C))$ provides an almost chart centered at τ'.

□

Now, consider the moduli space $\mathcal{M} = \mathcal{T}/Mod$. A local chart in a neighborhood of $\tau \in \mathcal{T}$ projects to a local chart in a neighborhood of $\pi(\tau) \in \mathcal{M}$ (where π is a factorization by Mod) unless τ is a hyperbolic structure with non-trivial automorphism group, or, equivalently, unless $\pi(\tau)$ is an orbifold point of $\mathcal{M}$. Composing this with Lemma 7.12 and Theorem 6.8 we obtain the following theorem:

Theorem 7.13. *Let S be a nodal surface, let $\mathcal{M}(S)$ be its moduli space and let $\overline{\mathcal{M}}(S)$ be the Deligne-Mumford compactification of $\mathcal{M}$. Let $\mathcal{S}^{\mathcal{M}}_{good} = \mathcal{S}_{good}/Mod$ be the union of good strata in $\mathcal{M}$. Let O be a locus of orbifold points of $\mathcal{M}$, let $\overline{O}$ be the closure of O in $\overline{\mathcal{M}}$. Then*

1. *the charts with coordinates $\tilde{l}(DP, C)$ provide an atlas on $\mathcal{M} \setminus O$ and on $\mathcal{S}^{\mathcal{M}}_{good} \setminus \overline{O}$, (here C is a good set and DP is an admissible double pants decomposition without double curves);*
2. *each point $\tau' \in S^{\mathcal{M}}_{good} \setminus \overline{O}$ is covered by some almost chart $(O'(\tau'), \tilde{l}(DP, C))$;*
3. *the elementary transition functions of these charts (almost charts) are inversions and transformations induced by flips and quasi-handle-twists of double pants decompositions; each elementary transition function change only one coordinate; this unique non-trivial transition function is algebraic;*
4. *the compositions of elementary transition functions act transitively on the union of charts and almost charts.*

Remark 7.14. We do not claim that the Definition 7.7 of the good strata exhaust all the points of $\partial\overline{\mathcal{T}}$ (resp. $\partial\overline{\mathcal{M}}$) covered by the almost charts of our atlas. However, some restrictions for the "good" points covered by the atlas are indispensable. For example, if $S = S_{3,0}$ and C is a set of three curves cutting a pair of pants out of C (see Fig. 7.3) then it is possible to prove that in each admissible double pants decomposition DP such that $C \in DP$ the set of curves $\{c_i \in DP \setminus C,\ c_i \cap C \neq \emptyset\}$ contains more than three curves. Hence, after retracting the curves of C, any decomposition DP contains less curves (of finite non-zero length) than required. This implies that the points $\tau' \in \mathcal{S}_C$ can not be covered by any chart of our atlas.

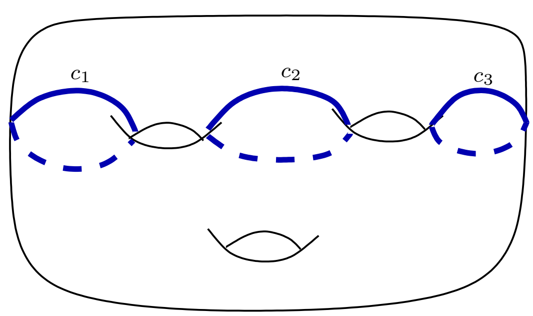


FIGURE 7.3. Example of the stratum not covered by the atlas: $S = S_{3,0}$, $C = \{c_1, c_2, c_3\}$.

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