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Perturbative correlation functions of null Wilson loops and local operators

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Abstract: We consider the correlation function of a null Wilson loop with four edges and a local operator in planar MSYM. By applying the insertion procedure, developed for correlation functions of local operators, we give an integral representation for the result at one and two loops. We compute explicitly the one loop result and show that the two loop result is finite.

Keywords: Duality in Gauge Field Theories, Scattering Amplitudes, Wilson, 't Hooft and Polyakov loops

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1 Introduction

Correlation functions of gauge invariant local operators are the natural observables of any conformal field theory. Over the last few years, there has been rapid progress in the understanding/computation of correlations functions of $\mathcal{N} = 4$ SYM, see for instance [1–3], and now explicit results, that would be impossible to obtain by standard Feynman diagram techniques, are available.

Given an $n-$point correlation function $\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle$ an interesting limit to consider is the one where consecutive (after choosing a specific ordering) distances became null $x_{i,i+1}^2 \to 0$, at equal rate. It was argued in [4] that in such a limit one obtains

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle}{\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_{\text{tree}}} = \langle W_{\text{adj}}^n[\mathcal{C}] \rangle \tag{1.1}$$

where $W_{\text{adj}}^n[\mathcal{C}]$ is a Wilson loop in the adjoint representation, over the null polygonal path $\mathcal{C}$, with cusps at x_i. This relation is quite general and does not require the theory to be planar. If we focus on a planar theory, as we will do in this paper, then $\langle W_{\text{adj}}^n[\mathcal{C}] \rangle = \langle W_{\text{fund}}^n[\mathcal{C}] \rangle^2$, the square of a Wilson loop in the standard fundamental representation.

One can also consider a generalization of the above limit, in which all distances but one became null. It was argued in [5], see also [6, 7], that in this limit one obtains

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \mathcal{O}(y) \rangle}{\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle} = \frac{\langle W_{\text{adj}}^n[\mathcal{C}] \mathcal{O}(y) \rangle}{\langle W_{\text{adj}}^n[\mathcal{C}]] \rangle} \,. \tag{1.2}$$

On the right hand side we obtain the correlation function of a null Wilson loop with a local operator. This is a very interesting class of objects, in particular, they interpolate between a Wilson loop and a correlation function, and they are finite, since UV divergences in the numerator and denominator cancel out. The planar limit of a Wilson loop with operator insertions was discussed in detail in [6], where it was shown that $\langle W^n_{\rm adj}[\mathcal{C}]\mathcal{O}(y)\rangle \to 2\langle W^n_{\rm fund}[\mathcal{C}]\rangle\langle W^n_{\rm fund}[\mathcal{C}]\mathcal{O}(y)\rangle$. Hence, in the planar limit

$$\lim_{x^2_{i,i+1}\to 0}\frac{\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\mathcal{O}(y)\rangle}{\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle} = 2\frac{\langle W^n_{\rm fund}[\mathcal{C}]\mathcal{O}(y)\rangle}{\langle W^n_{\rm fund}[\mathcal{C}]]\rangle} \tag{1.3}$$

In this paper we will focus on the simplest case, where the polygonal null Wilson loop has four edges, i.e. $n=4$. In this case conformal symmetry implies:

$$\frac{\langle W^4(x_1,x_2,x_3,x_4)\mathcal{O}(y)\rangle}{\langle W^4(x_1,x_2,x_3,x_4)\rangle} = \frac{|x_{13}x_{24}|^2}{\prod_{i=1}^4|y-x_i|^2}F(\zeta)\,, \tag{1.4}$$

where ζ is the cross-ratio that can be constructed out of the location of the local operator y and the location of the cusps x_i:

$$\zeta = \frac{|y-x_2|^2|y-x_4|^2x^2_{13}}{|y-x_1|^2|y-x_3|^2x^2_{24}}\ . \tag{1.5}$$

Hence $F(\zeta)$ is a function of a single variable ζ, in addition to the coupling constant $a=\frac{g^2N}{4\pi^2}$. From the definition of the cross-ratio and cyclic symmetry of the location of the cusps, we expect $F(\zeta)$ to have "crossing" symmetry:

$$F(\zeta) = F(1/\zeta) \tag{1.6}$$

For the case of $\mathcal{O}=\mathcal{O}_{\rm dil}$, the operator that couples to the dilaton (i.e. the $\mathcal{N}=4$ action), this function was computed in [5] at leading order both in the weak and strong coupling expansions[1]

$$F(\zeta) = -\frac{a}{4\pi^2}+\ldots, \qquad a\ll 1 \tag{1.7}$$

$$F(\zeta) = \frac{\zeta}{(1-\zeta)^3}(2(1-\zeta)+(\zeta+1)\log\zeta)\frac{\sqrt{a}}{2\pi^2}+\ldots, \qquad a\gg 1 \tag{1.8}$$

we can see that both expressions satisfy the crossing symmetry (1.6). The aim of the present paper is to compute $F(\zeta)$ to higher orders in perturbation theory.

A related quantity, namely the four-point correlation function of the stress-tensor multiplet, has been extensively studied in the past as well as more recently and has now been explicitly computed at the integrand level to 6 loops [1, 2, 8–11]. This multiplet, in particular, contains the chiral Lagrangian of $\mathcal{N}=4$ SYM. Computations of the correlator have made extensive use of the method of Langrangian insertions. This method relies on the

[1] In [5] the strong coupling result was found to be $F(\zeta)=c\frac{-\zeta}{3(1-\zeta)^3}(2(1-\zeta)+(\zeta+1)\log\zeta)\sqrt{\lambda}$, with $\lambda=4\pi^2a$. In the appendix we show that $c=-3/(4\pi^3)$ in order for (1.10) to be satisfied. As for the weak coupling result, one is to set $\hat{c}_{\rm dil}=1/2$ in [5] , and further multiply by 1/4, since [5] used a non-standard convention for traces in the fundamental representation.

observation that derivatives with respect to the coupling constant of any correlation function can be expressed in terms of a correlation function involving an additional insertion of the $\mathcal{N}=4$ SYM action. For instance,

$$a\frac{\partial}{\partial a}\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\rangle=\int d^4x_5\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\mathcal{L}_{\mathcal{N}=4}(x_5)\rangle\ . \tag{1.9}$$

This method is very powerful: by successive differentiation with respect to the coupling, it allows one to express the $\ell-$loop correction for the four-point correlation in terms of the integrated tree-level correlation function with ℓ additional insertions of the $\mathcal{N}=4$ SYM Lagrangian.

From the discussion above it is clear that a particular limit of those integrands will produce the integrands for $\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\mathcal{L}_{\mathcal{N}=4}(x_5)\rangle$ in the particular null limit we are interested in. This will give integrand expressions for loop corrections to $\langle W^4\mathcal{L}_{\mathcal{N}=4}(x_5)\rangle$.

In the next section we start by writing down those integral expressions. Then we compute the one-loop correction to $F(\zeta)$ (proportional to a^2) and show that the two-loop correction (proportional to a^3) is finite. This is to be expected, but it is far from obvious from the integral expressions, since each integral diverges as $\frac{1}{\epsilon^4}$ in dimensional-regularization.

Before proceeding, let us finish with a brief comment. The insertion procedure in particular implies an integral constraint on $F(\zeta)$, namely

$$x_{13}^2x_{24}^2\int d^4y\frac{F(\zeta)}{\prod_{i=1}^4|y-x_i|}=a\partial_a\log\langle W^4\rangle \tag{1.10}$$

One can check that this equation is indeed satisfied by the leading results at weak and at strong coupling, and we do so in the appendix.

2 Explicit results

2.1 General expressions and one-loop result

Following [1, 13, 14] we introduce

$$\begin{aligned}\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\rangle&=G_4=\sum_{\ell=0}^{\infty}a^\ell G_4^{(\ell)}(1,2,3,4)\\ \langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\mathcal{L}(x_5)\rangle&=1/4\int d^4\rho_5G_{5;1}=1/4\sum_{\ell=0}^{\infty}a^{\ell+1}\int d^4\rho_5G_{5;1}^{(\ell)}(1,2,3,4,5)\end{aligned}$$

here ρ is a Grassmann variable, $\mathcal{O}$ is the lowest component of the stress-tensor multiplet and $\mathcal{L}$ is the component proportional to ρ^4. We define the 'tHooft coupling constant $a=g^2N/(4\pi^2)$. The object we want to compute is then simply given by

$$\frac{\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\mathcal{L}(x_5)\rangle}{\langle\mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\rangle}=\frac{\int d^4\rho_5G_{5;1}}{4G_4} \tag{2.1}$$

Expressions for $G_4^{(\ell)}$ and $G_{5;1}^{(\ell)}$ (in terms of certain functions to be defined bellow), can be found in [1]. In general, those depend on the insertion points, together with certain auxiliary

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harmonic variables y_i. In the null limit considered in this paper, however, the dependence on the harmonic variables factors out, and goes away when taking the ratio (2.1). In the null limit we obtain

$$G_4^{(\ell)}(1,2,3,4) \;=\; \frac{1}{\ell!}\frac{2\,x_{13}^2 x_{24}^2}{(-4\,\pi^2)^\ell}\,G_4^{(0)} \int d^4x_5 \ldots d^4x_{4+\ell} f^{(\ell)}(x_1,\ldots,x_{4+\ell}) \tag{2.2}$$

$$\int d^4\rho_5 G_{5;1}^{(\ell)} \;=\; \frac{8}{\ell!}\frac{x_{13}^2 x_{24}^2}{(-4\,\pi^2)^{\ell+1}}\,G_4^{(0)} \int d^4x_6 \ldots d^4x_{5+\ell} f^{(\ell+1)}(x_1,\ldots,x_{5+\ell}) \tag{2.3}$$

which is consistent with the insertion formula

$$a\frac{\partial}{\partial a}G_4 \;=\; 1/4 \int d^4x_5 \int d^4\rho_5 G_{5;1}\;. \tag{2.4}$$

Finally we also need expressions for the f functions. These have a remarkably simple form [1]. At $1,2,3$ loops these are given by[2]

$$\begin{aligned}
f^{(1)}(x_1,\ldots,x_5) &= \frac{1}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}\,,\\
f^{(2)}(x_1,\ldots,x_6) &= \frac{\frac{1}{48}\sum_{\sigma\in S_6} x^2_{\sigma(1)\sigma(2)}x^2_{\sigma(3)\sigma(4)}x^2_{\sigma(5)\sigma(6)}}{(x_{15}^2x_{25}^2x_{35}^2x_{45}^2)(x_{16}^2x_{26}^2x_{36}^2x_{46}^2)x_{56}^2}\\
f^{(3)}(x_1,\ldots,x_7) &= \frac{\frac{1}{20}\sum_{\sigma\in S_7} x^4_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_4\sigma_5}x^2_{\sigma_5\sigma_6}x^2_{\sigma_6\sigma_7}x^2_{\sigma_7\sigma_3}}{(x_{15}^2x_{25}^2x_{35}^2x_{45}^2)(x_{16}^2x_{26}^2x_{36}^2x_{46}^2)(x_{17}^2x_{27}^2x_{37}^2x_{47}^2)(x_{56}^2x_{57}^2x_{67}^2)}\,.
\end{aligned} \tag{2.5}$$

These functions satisfy certain symmetries. Upon multiplication by the product of all external kinematic invariants ($x_{12}^2x_{13}^2x_{14}^2x_{23}^2x_{24}^2x_{34}^2$) and for generic (non-null-separated) points, these functions are completely symmetric under interchange of any two points and can be written as $\frac{P^{(\ell)}(x_1,\ldots,x_{4+\ell})}{\prod_{1\le i<j\le 4+\ell} x_{ij}^2}$, where $P^{(\ell)}$ is a homogeneous polynomial in x_{ij}^2 of uniform weight $-(\ell-1)$ at each point. These properties hold at all loops in perturbation theory [1]. When taking the null limit the functions $f^{(\ell)}$ will have fewer terms, but some symmetries will be lost.

Let us now consider the ratio (2.1) order by order in perturbation theory

$$\begin{aligned}
\frac{\int d^4\rho_5 G_{5;1}}{G_4} = \int d^4\rho_5 \Bigg\{ & a\left[\frac{G_{5;1}^{(0)}}{G_4^{(0)}}\right] \;+\; a^2\left[\frac{G_{5;1}^{(1)}}{G_4^{(0)}} - \frac{G_{5;1}^{(0)}}{G_4^{(0)}}\frac{G_4^{(1)}}{G_4^{(0)}}\right]\\
&+ a^3\left[\frac{G_{5;1}^{(2)}}{G_4^{(0)}} - \frac{G_{5;1}^{(1)}}{G_4^{(0)}}\frac{G_4^{(1)}}{G_4^{(0)}} - \frac{G_{5;1}^{(0)}}{G_4^{(0)}}\frac{G_4^{(2)}}{G_4^{(0)}} + \frac{G_{5;1}^{(0)}}{G_4^{(0)}}\left(\frac{G_4^{(1)}}{G_4^{(0)}}\right)^2\right] + \ldots\Bigg\}
\end{aligned} \tag{2.6}$$

Hence, at leading order in perturbation theory (proportional to a) we find

$$\begin{aligned}
\left(\frac{\langle W^4\mathcal{L}\rangle}{\langle W^4\rangle}\right)^{(0)} &= \frac{a}{8}\frac{\int d^4\rho_5 G_{5;1}^{(0)}}{G_4^{(0)}} = \frac{a\,x_{13}^2x_{24}^2}{(-4\,\pi^2)}\times f^{(1)}(x_1,\ldots,x_5)\\
&= \frac{a}{(-4\,\pi^2)}\frac{x_{13}^2x_{24}^2}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}\,,
\end{aligned} \tag{2.7}$$

[2] Note that the functions $f^{(\ell)}$ are multiplied by the overall factor $(x_{12}^2x_{13}^2x_{14}^2x_{23}^2x_{24}^2x_{34}^2)$ compared to the definition in [1].

which precisely agrees with the leading order result found in [5]. At next order we find

$$\left(\frac{\langle W^4\mathcal{L}\rangle}{\langle W^4\rangle}\right)^{(1)} = \frac{a^2}{(-4\,\pi^2)^2}\times x_{13}^2x_{24}^2\times\Bigg[\int d^4x_6 f^{(2)}(x_1,\dots,x_5,x_6) - 2x_{13}^2x_{24}^2 f^{(1)}(x_1,\dots,x_5)\int d^4x_6 f^{(1)}(x_1,\dots,x_4,x_6)\Bigg] \tag{2.8}$$

In the light-like limit the numerator of $f^{(2)}$ becomes simply

$$\frac{1}{48}\sum_{\sigma\in S_6} x^2_{\sigma(1)\sigma(2)}x^2_{\sigma(3)\sigma(4)}x^2_{\sigma(5)\sigma(6)} = x_{13}^2x_{24}^2x_{56}^2 + x_{15}^2x_{36}^2x_{24}^2 + x_{25}^2x_{46}^2x_{13}^2 + x_{35}^2x_{16}^2x_{24}^2 + x_{45}^2x_{26}^2x_{13}^2 \tag{2.9}$$

When integrating over x_6 in (2.8) we recognize two kinds of contributions

$$F(1,2,3,4) = -\frac{1}{4\pi^2}\int d^4x_6\frac{x_{13}^2x_{24}^2}{x_{16}^2x_{26}^2x_{36}^2x_{46}^2} \tag{2.10}$$

$$F(1,2,3,5) = -\frac{1}{4\pi^2}\int d^4x_6\frac{x_{13}^2x_{25}^2}{x_{16}^2x_{26}^2x_{36}^2x_{56}^2} \tag{2.11}$$

The first is the conformal massless box function, while the second is the two mass hard box function (since x_{51} and x_{53} are not null).[3] To be more precise, we have

$$\left(\frac{\langle W^4\mathcal{L}\rangle}{\langle W^4\rangle}\right)^{(1)} = \frac{a^2}{(-4\,\pi^2)}\times\frac{x_{13}^2x_{24}^2}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2} \times\Big(F(1,2,3,5)+F(4,1,2,5)+F(3,4,1,5)+F(2,3,4,5)-F(1,2,3,4)\Big)\,. \tag{2.12}$$

The first ($f^{(2)}$) term in (2.8) contributes a similar expression with all coefficients $+1$ whereas the second term in (2.8) subtracts a term proportional to $2F(1,2,3,4)$ thus swapping the sign of the last term.

The explicit expression for the box functions can be found for instance in [16, 17], where dimensional regularization is used. Even though each box function is divergent, the above combination is finite. Furthermore this combination is dual conformally invariant (see for example (2.23,2.22) of [15] for the divergences and conformal variation of the box functions in dimensional regularization). Plugging the analytic expressions for the box functions and expanding up to finite terms we obtain

$$\left(\frac{\langle W^4\mathcal{L}\rangle}{\langle W^4\rangle}\right)^{(1)} = \frac{a^2}{(-4\,\pi^2)}\times\frac{x_{13}^2x_{24}^2}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}\times\left(-\frac{1}{4}\right)\left(\log^2\zeta+\pi^2\right) \tag{2.13}$$

$$F(\zeta) = \frac{a^2}{(-4\pi^2)}\left(-\frac{1}{4}\right)\left(\log^2\zeta+\pi^2\right) \tag{2.14}$$

This result has homogeneous degree of transcendentality and the correct symmetry $F(\zeta) = F(1/\zeta)$.

[3]These integrals are of course infrared divergent and need regularisation. The combination of these integrals we consider below will be finite however and so we do not specify a regulator. In practise we will use dimensional regularisation (where the x's are interpreted as dual momenta).

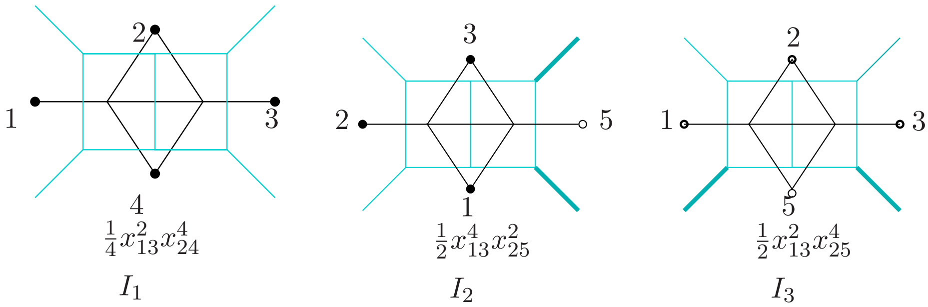

Figure 1. All contributing double box integrals at 2 loops with the corresponding numerator.

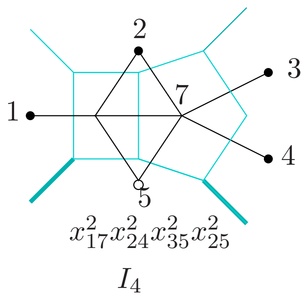

Figure 2. The 2 mass pentabox which contributes at 2 loops with the corresponding numerator.

2.2 Two-loop result

At $O(a^3)$ we have

$$\left(\frac{\langle W^4\mathcal{L}\rangle}{\langle W^4\rangle}\right)^{(2)} = \frac{1}{2}\frac{a^3x_{13}^2x_{24}^2}{(-4\pi^2)^3} \times \int d^4x_6d^4x_7 \times$$
$$\times \Big[f^{(3)}(x_1,\dots,x_6,x_7) - 4x_{13}^2x_{24}^2f^{(2)}(x_1,\dots,x_6)f^{(1)}(x_1,\dots,x_4,x_7)$$
$$- 2x_{13}^2x_{24}^2f^{(1)}(x_1,\dots,x_5)f^{(2)}(x_1,\dots,x_4,x_6,x_7)$$
$$+ 8(x_{13}^2x_{24}^2)^2f^{(1)}(x_1,\dots,x_5)f^{(1)}(x_1,\dots,x_4,x_6)f^{(1)}(x_1,\dots,x_4,x_7)\Big] \tag{2.15}$$

The integrals which arise from this are a 2-mass pentabox, 2-mass (2 types) and massless double boxes, and products of massless and 2 mass boxes. All of these are illustrated in the figures.

More specifically we have

$$\int d^4x_6d^4x_7\frac{f^{(3)}(x_1,\dots,x_6,x_7)}{f^{(1)}(x_1,\dots,x_5)} = \sum_{16\text{ perms}}\Big(I_1+I_2+I_3+I_4+I_6+I_7\Big)$$
$$\frac{x_{13}^2x_{24}^2x_{13}^2x_{24}^2}{f^{(1)}(x_1,\dots,x_5)}\int d^4x_6d^4x_7f^{(2)}(x_1,\dots,x_6)f^{(1)}(x_1,\dots,x_4,x_7) = \sum_{16\text{ perms}}\left(I_5+\frac{1}{2}I_6\right)$$

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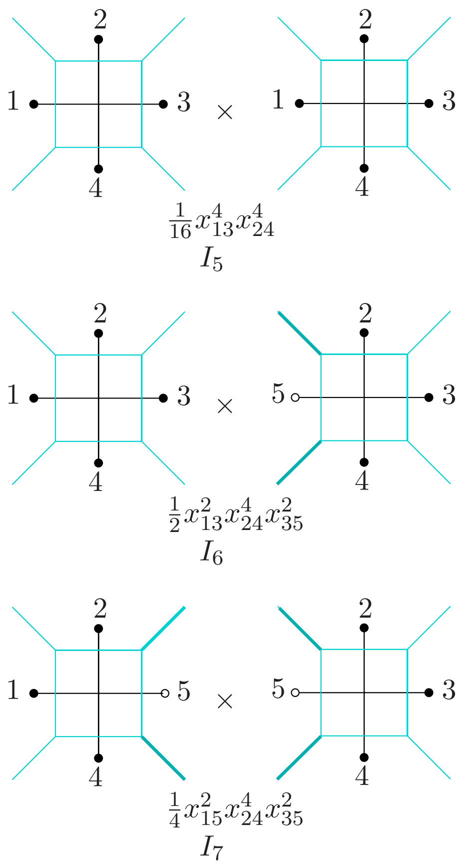

Figure 3. The 3 types of products of boxes which contribute at two loops with the corresponding numerator.

$$
\begin{aligned}
x_{13}^2 x_{24}^2 \int d^4x_6 d^4x_7 f^{(2)}(x_1,\ldots,x_4,x_6,x_7) &= \sum_{16 \text{ perms}} \Big(I_1 + I_5\Big) \\
(x_{13}^2 x_{24}^2)^2 \int d^4x_6 d^4x_7 f^{(1)}(x_1,\ldots,x_4,x_6) f^{(1)}(x_1,\ldots,x_4,x_7) &= \sum_{16 \text{ perms}} I_5
\end{aligned} \tag{2.16}
$$

where the sum over 16 permutations indicates that we must sum over 16 permutations generated by cycling the external points x_1, x_2, x_3, x_4, parity ($x_1 \leftrightarrow x_4,\ x_2 \leftrightarrow x_3$) together with swapping the internal coordinates x_6, x_7. These permutations will not always produce a different integrand (for example I_5 is completely symmetric under all such permutations). We have divided by the corresponding symmetry factor in the definition of the integral (see figures).

Putting this all together into (2.15) gives

$$
\left(\frac{\langle W^4 \mathcal{L}\rangle}{\langle W^4\rangle}\right)^{(2)} = \frac{1}{2}\frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \times \frac{a^3}{(-4\pi^2)^3} \times \sum_{16 \text{ perms}} \Big(-I_1 + I_2 + I_3 + I_4 + 2I_5 - I_6 + I_7\Big)\,. \tag{2.17}
$$

2.3 Finiteness of the two-loop result

We wish to check that the combination of (divergent) two-loop integrals (2.17) is finite. To do this at the level of the integrand we have to first understand where the divergences come from, and then see if they cancel. As is well-known there are two overlapping sources of infrared divergences for massless integrals, soft (when the internal momentum between two massless external legs vanishes) and collinear (when the integration momentum becomes collinear to a massless external leg) [18]. We make these divergences explicit in the current situation, by changing integration variables. Firstly we perform a transformation and Lorentz transformation to put $x_1 = 0$ and $x_2 = (b/2, b/2, 0, 0)$. Next transform to the following variables $(\phi, \epsilon, \hat{x}_6)$ and $(\phi', \epsilon', \hat{x}_7)$ where

$$x_6^\mu = \begin{pmatrix} \phi(1+\epsilon^2)/2 \\ \phi(1-\epsilon^2)/2 \\ \phi\epsilon\hat{x}_6 \end{pmatrix} \qquad x_7^\mu = \begin{pmatrix} \phi'(1+\epsilon'^2)/2 \\ \phi'(1-\epsilon'^2)/2 \\ \phi'\epsilon'\hat{x}_7 \end{pmatrix} . \tag{2.18}$$

Thus when $\epsilon \to 0$ we have x_{61} collinear with x_{12} whereas when $\phi \to 0$ we have $x_{71} \to 0$, so that the collinear and soft singularities occur when $\epsilon = 0$ and $\phi = 0$ respectively. Similarly for ϵ' and ϕ' with x_7. Making this change of variables, and focusing only on the potential divergences as $\epsilon, \epsilon', \phi, \phi' \to 0$, a generic two-loop integral takes the form

$$\text{finite} \times \int \frac{d\epsilon}{\epsilon}\frac{d\phi}{\phi}\frac{d\epsilon'}{\epsilon'}\frac{d\phi'}{\phi'}\frac{\text{numerator}(\epsilon,\epsilon',\phi.\phi')}{x_{67}^2}\Big(1+O(\epsilon,\epsilon',\phi,\phi')\Big) \tag{2.19}$$

where

$$x_{67}^2 = \phi^2\epsilon^2(1+\hat{x}_6^2) + \phi'^2\epsilon'^2(1+\hat{x}_7^2) + \phi\phi'\Big(\epsilon^2+\epsilon'^2+2\epsilon\epsilon'\hat{x}_6\cdot\hat{x}_7\Big) \tag{2.20}$$

and where the numerator term is generically 4th order in $\epsilon, \epsilon', \phi, \phi'$. So for example for the two loop ladder diagram we have

$$\text{numerator}_{\text{ladder}} \sim \Big(\epsilon^2\phi^2 + \epsilon'^2\phi'^2\Big)\Big(1+O(\epsilon,\epsilon',\phi,\phi')\Big) , \tag{2.21}$$

where in defining the integrand we always sum over permutations of the integration variables, giving the two terms here. We then see that the integral diverges as

$$\text{ladder} \;\sim\; \text{finite} \times \int d\epsilon d\phi \frac{d\epsilon'}{\epsilon'}\frac{d\phi'}{\phi'}\frac{\epsilon\phi}{x_{67}^2}\Big(1+O(\epsilon,\epsilon',\phi,\phi')\Big) \tag{2.22}$$

$$\sim\; \text{finite} \times \int d\epsilon d\phi \frac{d\epsilon'}{\epsilon'}\frac{d\phi'}{\phi'}\frac{1}{\epsilon\phi}\Big(1+O(\epsilon,\epsilon',\phi,\phi')\Big) , \tag{2.23}$$

where to get the second line we use that $x_{67}^2 = \phi^2\epsilon^2(1+\hat{x}_6^2) + O(\epsilon', \phi')$ an approximation we can make, since there are poles in ϵ' and ϕ'. We thus see the expected $\log^4$ divergence of the two-loop ladder.

Now, before addressing the case of interest, we consider another interesting two-loop integral, namely the logarithm of the amplitude at 2-loops. We know that this has a reduced infrared divergence, and it is interesting to see how this manifests itself at the level of the integrand. Again, performing the change of variables above we find that the numerator for the log of the amplitude takes the form

$$\text{numerator}_{\text{log of amplitude}} \sim \epsilon\phi\epsilon'\phi' \times \Big(1 + O(\epsilon, \epsilon', \phi, \phi')\Big) \ . \tag{2.24}$$

Notice that this vanishes in any collinear ($\epsilon \to 0$) or soft ($\phi \to 0$) limit in distinction with the ladder diagram alone above. This shows that it must have a reduced divergence compared to the ladder diagram. Indeed this simple fact (that the numerator vanishes when the loop integration becomes collinear with a massless external momentum) was used to great effect recently for determining high-loop four-point amplitudes [19, 20] and correlation functions [1]. Here we go slightly further and consider the exact degree of divergence in this case.

Implementing the change of integration variables, the log of the amplitude takes the form

$$\text{log of amplitude} \ \sim \ \text{finite} \times \int d\epsilon d\phi d\epsilon' d\phi' \frac{1}{x_{67}^2}\Big(1 + O(\epsilon, \epsilon', \phi, \phi')\Big) \ . \tag{2.25}$$

To see the degree of divergence in this integral, it is useful to change variable once more, and let $\epsilon' = \epsilon\alpha$, $\phi' = \phi\beta$. Then the potential divergences occur at $\epsilon, \phi, \alpha, \beta \to 0$ (also when $\alpha, \beta \to \infty$ but we have symmetrized the integration variables, allowing us to concentrate on the former case). In these variables, from (2.20)

$$x_{67}^2 = \epsilon^2\phi^2\Big(1 + \hat{x}_6^2 + \alpha^2\beta^2(1 + \hat{x}_7^2) + \beta\big(1 + \alpha^2 + 2\beta\hat{x}_6 \cdot \hat{x}_7\big)\Big) \tag{2.26}$$

$$d\epsilon\, d\phi\, d\epsilon'\, d\phi' \ \to \ d\epsilon\, d\phi\, d\alpha\, d\beta \times \epsilon\phi \ . \tag{2.27}$$

Then one can see there are no singularities when $\alpha, \beta \to 0$ (unlike in the two-loop ladder case), and so the log of the amplitude takes the form

$$\text{log of amplitude} \ \sim \ \text{finite} \times \int d\epsilon d\phi \frac{1}{\epsilon\phi}\Big(1 + O(\epsilon, \alpha, \phi, \beta)\Big) \,, \tag{2.28}$$

and we identify the $\log^2$ divergence.

Finally then, we consider the case of interest, the two-loop integral defined in (2.17). Making the change of variables we find that this time the numerator is of degree 6 in the $\epsilon, \epsilon', \phi.\phi'$ variables, in particular

$$\text{numerator} \sim \epsilon\epsilon'\phi\phi'\big(A\,\epsilon\phi + B\,\epsilon'\phi'\big) \times \Big(1 + O(\epsilon, \epsilon', \phi, \phi')\Big) \ . \tag{2.29}$$

for some finite A, B.

Let us then consider the degree of divergence of such an integral. Plugging into (2.19) and changing to the α, β variables the numerator $\sim \epsilon^3\phi^3\alpha\beta\big(a + b\alpha\beta\big)$ and there are then no poles at all as $\epsilon, \phi, \alpha, \beta \to 0$ and thus the integral is completely finite there.

While perhaps not a completely rigorous proof of finiteness, the above argument gives a strong indication that the above integral is finite.[4] Furthermore it provides an integrand-level criterion for obtaining finite integrals: they must have numerators of the form form (2.29). Indeed implementing this criterion on an arbitrary linear combination of the integrals $I_1, \ldots I_7$ gives the unique solution (2.17).

[4]Furthermore, numerical results are consistent with a cancelation of the two leading poles.

3 Conclusions

In this paper we considered the correlation function of a local operator (the $\mathcal{N}=4$ Lagrangian) with a four cusped null Wilson loop $\frac{\langle W^4 \mathcal{O}(y)\rangle}{\langle W^4\rangle}$ in perturbation theory. This correlation function is expected to be finite, and conformal symmetry implies that the non-trivial dependence is encoded in a function $F(\zeta)$ of a single cross-ratio and the coupling constant. By using previous results on correlation functions, we computed $F(\zeta)$ at one-loop in perturbation theory, obtaining

$$F(\zeta) = -\frac{a}{4\pi^2}\left(1 - \frac{1}{4}(\log^2\zeta + \pi^2)a + \dots\right) . \tag{3.1}$$

Our result is consistent with crossing-symmetry $F(\zeta) = F(1/\zeta)$ and furthermore has the expected degree of transcendentality. Furthermore, we have given an integral representation for the two-loop contribution to $F(\zeta)$. This is given in terms of seven integrals, including double boxes and pentaboxes, plus permutations. Even though each contribution diverges as $1/\epsilon^4$ in dimensional regularization, we argue that this particular combination is finite. This claim is also supported by a numerical analysis of the integrals. We hope to come back in the future with a more detailed analysis, and hopefully an analytic answer, of the two-loop result.

Finally, let us mention that the computation of $F(\zeta)$ should be simpler in certain limits. For instance, if the insertion point is null separated to one of the cusps (but not to the other) ζ vanishes (or becomes infinity), hence it should be possible to understand this limit in terms of the light-cone OPE for correlation functions. We hope to go back to this question in the future.

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A Integration over the insertion point

A.1 Tree level

In this appendix we will explicitly check that the normalization of the tree level result is consistent with (1.10). At tree-level we have obtained $F = -\frac{a}{4\pi^2}$, hence, we should consider

$$I = -\frac{s\ t\ a}{4\pi^2}\int d^4y \frac{1}{\prod_{i=1}^4 |y - x_i|} \tag{A.1}$$

where we have introduced $s = x_{13}^2$ and $t = x_{24}^2$. This is the usual massless scalar box function, and it has been computed in dimensional regularization, for instance, in [16]. At leading order in ϵ we obtain

$$I = -a\frac{1}{\epsilon^2} + \dots$$

We need to compare this with the divergent part of the right-hand-side of (1.10). This is

$$a\,\partial_a \log\langle W^4\rangle_{div} = -\ell\, a^\ell \frac{\Gamma^{(\ell)}_{cusp}}{\epsilon^2} \tag{A.2}$$

since $\Gamma_{cusp} = a + \ldots$, we obtain the desired result.

A.2 Strong coupling

Let us now consider (1.10) at strong coupling. The unintegrated left hand side is the result of an integral over the world-sheet variables (u, v) [5]

$$x_{13}^2 x_{24}^2 \frac{F(\zeta)}{\prod_{i=1}^4 |y - x_i|^2} \\ = 2c \int_{-\infty}^{\infty} \left(\frac{(\cosh u \sinh v)^{-1}}{1 + y^2 - 2y_1 \tanh u - 2y_2 \tanh v + 2y_0 \tanh u \tanh v} \right)^4 du dv \tag{A.3}$$

where the world-sheet corresponds ends on the regular polygon with four edges. If we integrate over the world-sheet coordinates (u, v) we obtain $F(\zeta)$ at strong coupling, quoted in the body of the text. On the other hand, we could also integrate over the location of the insertion point y. The integrals are quite elementary and we obtain

$$x_{13}^2 x_{24}^2 \int d^4y \frac{F(\zeta)}{\prod_{i=1}^4 |y - x_i|^2} = 2c \frac{\pi^2}{6} \int_{-\infty}^{\infty} du dv \tag{A.4}$$

If we set $c = -3/(4\pi^3)$, the right hand side coincides exactly with the action of the regular polygon with four edges found in [21].

References

[1] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Hidden symmetry of four-point correlation functions and amplitudes in N = 4 SYM*, *Nucl. Phys.* **B 862** (2012) 193 [arXiv:1108.3557] [INSPIRE].

[2] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Constructing the correlation function of four stress-tensor multiplets and the four-particle amplitude in N = 4 SYM*, *Nucl. Phys.* **B 862** (2012) 450 [arXiv:1201.5329] [INSPIRE].

[3] B. Eden, P. Heslop, G.P. Korchemsky, V.A. Smirnov and E. Sokatchev, *Five-loop Konishi in N = 4 SYM*, *Nucl. Phys.* **B 862** (2012) 123 [arXiv:1202.5733] [INSPIRE].

[4] L.F. Alday, B. Eden, G.P. Korchemsky, J. Maldacena and E. Sokatchev, *From correlation functions to Wilson loops*, *JHEP* **09** (2011) 123 [arXiv:1007.3243] [INSPIRE].

[5] L. Alday, E. Buchbinder and A. Tseytlin, *Correlation function of null polygonal Wilson loops with local operators*, *JHEP* **09** (2011) 034 [arXiv:1107.5702] [INSPIRE].

[6] O.T. Engelund and R. Roiban, *On correlation functions of Wilson loops, local and non-local operators*, *JHEP* **05** (2012) 158 [arXiv:1110.0758] [INSPIRE].

[7] T. Adamo, *Correlation functions, null polygonal Wilson loops and local operators*, *JHEP* **12** (2011) 006 [arXiv:1110.3925] [INSPIRE].

[8] B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, *Four point functions in $N = 4$ supersymmetric Yang-Mills theory at two loops*, Nucl. Phys. **B 557** (1999) 355 [hep-th/9811172] [INSPIRE].

[9] B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, *Simplifications of four point functions in $N = 4$ supersymmetric Yang-Mills theory at two loops*, Phys. Lett. **B 466** (1999) 20 [hep-th/9906051] [INSPIRE].

[10] F. Gonzalez-Rey, I. Park and K. Schalm, *A note on four point functions of conformal operators in $N = 4$ super Yang-Mills*, Phys. Lett. **B 448** (1999) 37 [hep-th/9811155] [INSPIRE].

[11] B. Eden, C. Schubert and E. Sokatchev, *Three loop four point correlator in $N = 4$ SYM*, Phys. Lett. **B 482** (2000) 309 [hep-th/0003096] [INSPIRE].

[12] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *Anomalous dimensions in $N = 4$ SYM theory at order g^4*, Nucl. Phys. **B 584** (2000) 216 [hep-th/0003203] [INSPIRE].

[13] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *The super-correlator/super-amplitude duality: part I*, Nucl. Phys. **B 869** (2013) 329 [arXiv:1103.3714] [INSPIRE].

[14] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *The super-correlator/super-amplitude duality: part II*, Nucl. Phys. **B 869** (2013) 378 [arXiv:1103.4353] [INSPIRE].

[15] A. Brandhuber, P. Heslop and G. Travaglini, *One-loop amplitudes in $N = 4$ super Yang-Mills and anomalous dual conformal symmetry*, JHEP **08** (2009) 095 [arXiv:0905.4377] [INSPIRE].

[16] G. Duplancic and B. Nizic, *Dimensionally regulated one loop box scalar integrals with massless internal lines*, Eur. Phys. J. **C 20** (2001) 357 [hep-ph/0006249] [INSPIRE].

[17] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, Nucl. Phys. **B 425** (1994) 217 [hep-ph/9403226] [INSPIRE].

[18] G.F. Sterman, *Partons, factorization and resummation, TASI 95*, hep-ph/9606312 [INSPIRE].

[19] J.L. Bourjaily, A. DiRe, A. Shaikh, M. Spradlin and A. Volovich, *The soft-collinear bootstrap: $N = 4$ Yang-Mills amplitudes at six and seven loops*, JHEP **03** (2012) 032 [arXiv:1112.6432] [INSPIRE].

[20] J. Golden and M. Spradlin, *Collinear and soft limits of multi-loop integrands in $N = 4$ Yang-Mills*, JHEP **05** (2012) 027 [arXiv:1203.1915] [INSPIRE].

[21] L.F. Alday and J.M. Maldacena, *Gluon scattering amplitudes at strong coupling*, JHEP **06** (2007) 064 [arXiv:0705.0303] [INSPIRE].