Massive gravity, the elasticity of space-time, and perturbations in the dark sector

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We consider a class of phenomenological modified gravity models where the terms added to the standard Einstein-Hilbert Lagrangian are just a function of the metric only. For linearized perturbations around an isotropic space-time, this class of models is entirely specified by a rank-4 tensor that encodes possibly time-dependent masses for the gravitons. This tensor has the same symmetries as an elasticity tensor, suggesting an interpretation of massive gravity as an effective rigidity of space-time. If we choose a form for this tensor that is compatible with the symmetries of Friedmann-Robertson-Walker and enforce full reparametrization invariance, then the only theory possible is a cosmological constant. However, in the case where the theory is only time translation invariant, the ghost-free massive gravity theory is equivalent to the elastic dark energy scenario with the extra Lorentz violating vector giving rise to 2 transverse and 1 longitudinal degrees of freedom, whereas when one demands spatial translation invariance one is left with a theory where the entropy perturbation is not gauge invariant.

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I. INTRODUCTION

The realization that the Universe appears to be accelerating has fueled the search for alternative theories of gravity [1] as a possible explanation for what has become called *the dark sector*. In this paper we will focus on what is the simplest subset of such theories, in which the dark sector Lagrangian is only a function of the metric (and no extra derivatives thereof) following the approach discussed in [2]. The action for this type of theory is given by

$$S = \int d^4x\sqrt{-g}[R + 16\pi G\mathcal{L}_{\rm m} - 2\mathcal{L}_{\rm d}(g_{\mu\nu})]. \qquad (1.1)$$

If $T_{\mu\nu}$ is the energy-momentum tensor for the matter sector Lagrangian $\mathcal{L}_{\rm m}$ and $U_{\mu\nu}$ is that associated with the dark sector Lagrangian $\mathcal{L}_{\rm d}$, then the Einstein equation is $G_{\mu\nu} = 8\pi GT_{\mu\nu} + U_{\mu\nu}$. Typically, we will be interested in space-times that are isotropic where $U_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu}$ can be specified in terms of a density, ρ, and pressure, $P = w\rho$. There are two classes of theories that can be described by (1.1): elastic dark energy and theories of massive gravitons. Typically, theories of massive gravitons have been considered as a fundamental theory around Minkowski space-time, but it is also possible to think about masses for the gravitons as being induced by some unknown effective physics encoded by $\mathcal{L}_{\rm d}$. This phenomenological approach to cosmological perturbation theory [2,3] is at the heart of the present paper.

The study of massive gravity theories (see, for example, [4,5]) has a long history. This started with the linearized theories of Feirz and Pauli [6], progressing to the studies of Boulware and Deser [7]. It has received a new lease of life in recent times with the proposal of nonlinear de Rahm-Gabadadze-Tolley massive gravity theory [8–11] and its connections to the Vainshtein screening mechanism [12–15]. Massive gravity theories are built upon the pretext that the resulting theory should be ‘‘ghost-free’’ [10,16–22], and they have begun to be studied in cosmological backgrounds [23–37]. Such theories are usually presumed to be Lorentz invariant that leads to the Pauli-Fierz tuning. Giving up Lorentz invariance is another way to remove ghosts from the theory, as pointed out by [38,39] and further studied in [4,40–43].

Elastic dark energy (EDE) is an idea that has been developed from relativistic elasticity theory [44–50]. The basic concept is that the stress-energy component responsible for the dark energy has rigidity that stabililizes perturbations that would, if modeled as those of a perfect fluid, give rise to exponential growth in the density contrast. The framework was adapted for cosmological purposes in [51,52] in order to provide a phenomenological model for the domain wall as an explanation for accelerated expansion that has $P/\rho = w_{\rm dw} = -2/3$. However, in principle the equation of state parameter, w, is allowed to take ‘‘any’’ value as long as the rigidity modulus, μ, is sufficiently large. Indeed, the theory is well defined in

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the limit $w \to -1$ where the elastic medium becomes a "cosmological constant" and $w \to 0$ and $\mu \to 0$, which corresponds to cold dark matter. The standard assumption, which we will use in this paper, is that the elasticity tensor is isotropic, but one can also construct anisotropic models [53,54].

The aim of this paper is to point out the connections among linearized massive gravity theories, EDE, and the framework for linearized perturbations in the generalized phenomenological models for the dark sector discussed in [2,55]. The reason for this connection is that, at linearized order, one can represent all possible Lagrangians by a generalized function that is quadratic in the fields. In the specific case we are concerned with here, this is just a quadratic function of the metric that is parametrized by a rank-4 tensor. This has the same symmetries as an elasticity tensor, suggesting an interpretation of massive gravitons as creating an effective rigidity of space-time. This tensor can split in a way that is compatible with the symmetries of the Friedmann-Robertson-Walker space-time, which is more general than the usual Pauli-Fierz case. We will make a survey of the possible mechanisms by which ghost modes can be removed, both Lorentz invariant and violating. The ghosts are associated with a breaking of reparametrization invariance, and we show that its reimposition leads to three interesting subcases, one of which is compatible with a Lorentz invariance that is a cosmological constant and the other two which violate either time or spatial translation invariance. The one that violates spatial translation invariance happens to be the EDE model that we see is a Lorentz-violating ghost-free massive gravity theory.

II. THE GENERAL "METRIC ONLY" THEORY

The action that will give linearized field equations for the perturbed field variables is given by

$$S_{\{2\}} = \int \mathrm{d}^4x\sqrt{-g}[\diamondsuit^2 R + 16\pi G \diamondsuit^2 \mathcal{L}_{\mathrm{m}} - 2\mathcal{L}_{\{2\}}]. \quad (2.1)$$

We use $\diamondsuit^2 Q$ to denote the second measure-weighted variation of the quantity Q, defined as $\diamondsuit^2 Q \equiv \frac{1}{\sqrt{-g}}\delta^2(\sqrt{-g}\mathrm{Q})$. $\mathcal{L}_{\{2\}}$ is the Lagrangian for dark sector perturbations, given by

$$\mathcal{L}_{\{2\}} = \frac{1}{8}\mathcal{W}^{\mu\nu\alpha\beta}\delta g_{\mu\nu}\delta g_{\alpha\beta}, \quad (2.2)$$

where $\delta g_{\mu\nu}$ is the metric fluctuation. This clearly looks like a mass term for the metric perturbations. The tensor

$$\mathcal{W}^{\mu\nu\alpha\beta} = \mathcal{W}^{(\mu\nu)(\alpha\beta)} = \mathcal{W}^{\alpha\beta\mu\nu} \quad (2.3)$$

is the *mass matrix* determining how the components of $\delta g_{\mu\nu}$ mix to provide the mass; all linearized massive gravities are encoded by choices of $\mathcal{W}$. Hence, the complete linearized theory we study is

$$S_{\{2\}} = \int \mathrm{d}^4x\sqrt{-g}\Big[\diamondsuit^2 R + 16\pi G\diamondsuit^2\mathcal{L}_{\mathrm{m}} - \frac{1}{4}\mathcal{W}^{\mu\nu\alpha\beta}\delta g_{\mu\nu}\delta g_{\alpha\beta}\Big]. \quad (2.4)$$

This theory contains metric fluctuations that have a kinetic term and a mass term. We will see that the spatial components of $\mathcal{W}$ can be interpreted as an elasticity tensor.

To isolate the degrees of freedom in the theory, $\delta g_{\mu\nu}$ can be decomposed as

$$\delta g_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}. \quad (2.5)$$

In the parlance of [45,50,56,57] $h_{\mu\nu}$ is the Eulerian metric perturbation and ξ_μ is a vector field representing possible coordinate transformations. In standard general relativity, the action is independent of ξ_μ, but in more general theories this can become a physical field. This formulation is equivalent to what is sometimes called the Stuckelberg trick [5,19,58]. Inserting (2.5) into the action (2.4) and integrating by parts reveals that

$$S_{\{2\}} = \int \mathrm{d}^4x\sqrt{-g}[(\delta_{\mathrm{E}}G^{\mu\nu} - 8\pi G\delta_{\mathrm{E}}T^{\mu\nu} - \delta_{\mathrm{E}}U^{\mu\nu})h_{\mu\nu} + 2\xi_{(\mu}\delta_{\mathrm{E}}(\nabla_{\nu)}U^{\mu\nu})], \quad (2.6)$$

where the variational operator "δ_{E}" denotes that the quantity is evaluated with the metric perturbation variable $h_{\mu\nu}$ (rather than $\delta g_{\mu\nu}$), and where we have defined the perturbed dark energy-momentum tensor

$$\delta_{\mathrm{E}}U^{\mu\nu} = -\frac{1}{2}(\mathcal{W}^{\mu\nu\alpha\beta} + U^{\mu\nu}g^{\alpha\beta})\delta g_{\alpha\beta} - \xi^\alpha\nabla_\alpha U^{\mu\nu} + 2U^{\alpha(\mu}\nabla_\alpha\xi^{\nu)}. \quad (2.7)$$

It is now a simple matter to obtain the functional derivatives of the action with respect to the perturbed metric $h_{\mu\nu}$ and ξ^μ fields,

$$\frac{\hat{\delta}}{\hat{\delta}h_{\mu\nu}}S_{\{2\}} = \delta_{\mathrm{E}}G^{\mu\nu} - 8\pi G\delta_{\mathrm{E}}T^{\mu\nu} - \delta_{\mathrm{E}}U^{\mu\nu} = 0, \quad (2.8a)$$

$$\frac{\hat{\delta}}{\hat{\delta}\xi_\mu}S_{\{2\}} = \delta_{\mathrm{E}}(\nabla_\nu U^{\mu\nu}) = 0. \quad (2.8b)$$

The variational principle was used to demand that these expressions vanish, yielding the perturbed gravitational field equations and perturbed conservation equation, respectively. Using (2.7) to evaluate (2.8b) yields

$$L^{\mu\nu\alpha\beta}\nabla_\mu\nabla_\alpha\xi_\beta + (\nabla_\mu\mathcal{W}^{\mu\nu\alpha\beta})\nabla_\alpha\xi_\beta + (\nabla_\mu\nabla_\alpha U^{\mu\nu})\xi^\alpha = -\frac{1}{2}((\nabla_\mu\mathcal{W}^{\mu\nu\alpha\beta})h_{\alpha\beta} + P^{\mu\nu\alpha\beta}\nabla_\mu h_{\alpha\beta}), \quad (2.9)$$

where we defined the *effective metric* $L^{\mu\nu\alpha\beta}$ and *derivative-coupling* $P^{\mu\nu\alpha\beta}$ terms,

$$L^{\mu\nu\alpha\beta} \equiv \mathcal{W}^{\mu\nu\alpha\beta} + U^{\mu\nu} g^{\alpha\beta} - 2U^{\alpha(\mu} g^{\nu)\beta},$$
$$P^{\mu\nu\alpha\beta} \equiv \mathcal{W}^{\mu\nu\alpha\beta} + U^{\alpha\beta} g^{\mu\nu} - 2g^{\nu\beta} U^{\alpha\mu}. \tag{2.10}$$

We impose spatial isotropy upon the background with the $(3+1)$ decomposition, and in doing so we will obtain the most general linearized massive gravity Lagrangian compatible with the spatial isotropy of the background. We foliate the four-dimensional (4D) space-time by three-dimensional (3D) sheets with a timelike unit vector, u_μ, being everywhere orthogonal to the sheets. The 4D space-time has metric $g_{\mu\nu}$, and the 3D sheets have metric $\gamma_{\mu\nu}$. The $(3+1)$ decomposition of the 4D metric is $g_{\mu\nu} = \gamma_{\mu\nu} - u_\mu u_\nu$, where $u^\mu u_\mu = -1$, $\gamma^{\mu\nu} u_\mu = 0$. This structure provides an extrinsic curvature tensor $K_{\mu\nu} = K_{(\mu\nu)}$ on the 3D sheets, given by $K_{\mu\nu} = \nabla_\mu u_\nu$ and satisfying $u^\mu K_{\mu\nu} = 0$ (the extrinsic curvature tensor is given by $K_{\mu\nu} = \frac{1}{3} K\gamma_{\mu\nu}$). We define "time" and "space" differentiation as the derivative operator projected along the time and space directions,

$$\dot{\psi} \equiv u^\mu \nabla_\mu \psi, \qquad \bar{\nabla}\psi \equiv \gamma^\nu{}_\mu \nabla_\nu \psi. \tag{2.11}$$

Using this technology we decompose the gradient of a scalar into two orthogonal terms,

$$\nabla_\mu \psi = -\dot{\psi} u_\mu + \bar{\nabla}\psi. \tag{2.12}$$

This enables us to find the values of two useful "kinetic scalars,"

$$\nabla^\mu \psi \nabla_\mu \psi = -\dot{\psi}^2 + \bar{\nabla}^\mu \psi \bar{\nabla}_\mu \psi, \tag{2.13a}$$
$$\Box\psi \equiv \nabla^\mu \nabla_\mu \psi = -\ddot{\psi} + \bar{\nabla}^\mu \bar{\nabla}_\mu \psi. \tag{2.13b}$$

The last term of each expression simply selects the spatial derivatives of the scalar field. Another useful application of the $(3+1)$ decomposition is to find all the freedom in a tensor that is compatible with spatial isotropy of the background space-time.

We use the $(3+1)$ decomposition to isolate the components of the perturbed metrics by writing

$$\delta g_{\mu\nu} = 2\Phi u_\mu u_\nu + 2N_{(\mu} u_{\nu)} + \bar{H}_{\alpha\beta} \gamma^\alpha{}_\mu \gamma^\beta{}_\nu, \tag{2.14a}$$
$$h_{\mu\nu} = 2\phi u_\mu u_\nu + 2n_{(\mu} u_{\nu)} + \bar{h}_{\alpha\beta} \gamma^\alpha{}_\mu \gamma^\beta{}_\nu, \tag{2.14b}$$

and we isolate the timelike and spacelike components of the vector field via

$$\xi_\mu = -\chi u_\mu + \omega_\mu, \tag{2.14c}$$

where $N^\mu u_\mu = n^\mu u_\mu = 0$, $u^\mu \bar{H}_{\mu\nu} = u^\mu \bar{h}_{\mu\nu} = 0$ and $u^\mu \omega_\mu = 0$.

In [2] we showed that the general decomposition of the mass-matrix $\mathcal{W}$ compatible with spatial isotropy is

$$\begin{aligned}\mathcal{W}^{\mu\nu\alpha\beta} = {} & A_{\mathcal{W}} u^\mu u^\nu u^\alpha u^\beta + B_{\mathcal{W}} (u^\mu u^\nu \gamma^{\alpha\beta} + u^\alpha u^\beta \gamma^{\mu\nu}) \\ & + 2C_{\mathcal{W}} (\gamma^{\alpha(\mu} u^{\nu)} u^\beta + \gamma^{\beta(\mu} u^{\nu)} u^\alpha) \\ & + D_{\mathcal{W}} \gamma^{\mu\nu} \gamma^{\alpha\beta} + 2E_{\mathcal{W}} \gamma^{\mu(\alpha} \gamma^{\beta)\nu},\end{aligned} \tag{2.15}$$

where there are only five free functions that depend only on time. Using the mass matrix (2.15) and (2.14a) in the Lagrangian (2.2) yields

$$\begin{aligned}8\mathcal{L}_{\{2\}} = {} & 4A_{\mathcal{W}} \Phi^2 + 4B_{\mathcal{W}} \Phi \bar{H} + 2C_{\mathcal{W}} N_\alpha N^\alpha \\ & + D_{\mathcal{W}} \bar{H}^2 + 2E_{\mathcal{W}} \bar{H}^{\alpha\beta} \bar{H}_{\alpha\beta}.\end{aligned} \tag{2.16}$$

This can be written in terms of "graviton masses" (see, e.g., [4]) where the free functions $\{A_{\mathcal{W}}, \ldots, E_{\mathcal{W}}\}$ are given by

$$\begin{aligned}2\mathcal{L}_{\{2\}} = {} & m_0^2 (\delta g_{00})^2 + 2m_1^2 (\delta g_{0i})^2 - m_2^2 (\delta g_{ij})^2 \\ & + m_3^2 (\delta g_{ii})^2 - 2m_4^2 \delta g_{00} \delta g_{ii}\end{aligned} \tag{2.17}$$

with $A_{\mathcal{W}} = m_0^2$, $B_{\mathcal{W}} = -2m_4^2$, $C_{\mathcal{W}} = 4m_1^2$, $D_{\mathcal{W}} = 4m_3^2$, and $E_{\mathcal{W}} = -2m_2^2$. One should keep in mind, therefore, that when we talk about the $\{A_{\mathcal{W}}, \ldots, E_{\mathcal{W}}\}$, we are actually talking about the graviton masses m_i^2, albeit ones that depend on time. The values of these masses for a "Goldstone" theory are given in [59] and those that are induced by perturbations in scalar fields in [2].

Using (2.14) to evaluate (2.5) yields

$$\Phi = \phi + \dot{\chi}, \qquad N_\alpha = n_\alpha - \dot{\varpi}_\alpha - \bar{\nabla}_\alpha \chi,$$
$$\bar{H}_{\alpha\beta} = \bar{h}_{\alpha\beta} + 2\bar{\nabla}_{(\alpha} \omega_{\beta)} - \frac{2}{3} K \gamma_{\alpha\beta} \chi - \frac{2}{3} K \omega_{(\alpha} u_{\beta)}, \tag{2.18}$$

where $\dot{\varpi}_\alpha \equiv (\dot{\omega}_\alpha - \frac{1}{3} K \omega_\alpha)$. Substituting (2.14a) into the action (2.4) one finds the absence of $\dot{\phi}^2$ and $\dot{n}_\alpha^2$ terms in the kinetic part of the Einstein-Hilbert Lagrangian; this can also be seen in results given by [4,39,42] and in the ADM formulation [60]. ϕ and n_α are now Lagrange multipliers whose equations of motion are constraint equations, allowing them to be eliminated. Using redefinitions of the coefficients, we can effectively set $\phi = 0$ and $n_\alpha = 0$, which is equivalent to choosing the synchronous gauge. We will make this choice in what follows.

The two independent components of the equation of motion (2.9), after inserting the $(3+1)$ decomposition, are given by

$$[A_{\mathcal{W}}+\rho]\ddot{\chi}+[\dot{A}_{\mathcal{W}}+\mathcal{H}(4A_{\mathcal{W}}+\rho-3P)]\dot{\chi}+[P+C_{\mathcal{W}}]\nabla^2\chi-\Big[\mathcal{H}(3\dot{P}+2\dot{\rho}-\dot{A}_{\mathcal{W}}+3\dot{B}_{\mathcal{W}})$$
$$-(2A_{\mathcal{W}}+5\rho+3P-3B_{\mathcal{W}}-9D_{\mathcal{W}}-6E_{\mathcal{W}})\mathcal{H}^2+(2\rho+3B_{\mathcal{W}}-A_{\mathcal{W}})\frac{\ddot{a}}{a}+\ddot{\rho}\Big]\chi$$
$$-[\dot{B}_{\mathcal{W}}+\mathcal{H}(3B_{\mathcal{W}}+3D_{\mathcal{W}}+2E_{\mathcal{W}}-2P)]\partial_i\omega^i+[B_{\mathcal{W}}+C_{\mathcal{W}}]\partial_i\dot{\omega}^i$$
$$=\frac{1}{2}[\dot{B}_{\mathcal{W}}+\mathcal{H}(3B_{\mathcal{W}}+3D_{\mathcal{W}}+2E_{\mathcal{W}}-2P)]h+\frac{1}{2}[B_{\mathcal{W}}-P]\dot{h}, \tag{2.19a}$$
$$[\rho-C_{\mathcal{W}}]\ddot{\omega}^i-[\dot{C}_{\mathcal{W}}+\mathcal{H}(4C_{\mathcal{W}}-\rho+3P)]\dot{\omega}^i-[E_{\mathcal{W}}-P]\nabla^2\omega^i-[E_{\mathcal{W}}+D_{\mathcal{W}}]\partial^i\partial_k\omega^k$$
$$+[\dot{C}_{\mathcal{W}}-\mathcal{H}(3D_{\mathcal{W}}-B_{\mathcal{W}}+2E_{\mathcal{W}}-4C_{\mathcal{W}}-2P)]\partial^i\chi+[B_{\mathcal{W}}+C_{\mathcal{W}}]\partial^i\dot{\chi}$$
$$=-[P-E_{\mathcal{W}}]\partial_j h^{ij}+\frac{1}{2}[P+D_{\mathcal{W}}]\partial^i h. \tag{2.19b}$$

ρ and P are the density and pressure coming from the dark fluid (i.e., the components of the background dark energy momentum tensor $U_{\mu\nu}$). The benefit of using the (3 + 1) decomposition has become apparent: we are able to identify the degrees of freedom. There are the tensor degrees of freedom h_+ and $h_\times$ that are present in standard general relativity, a vector degree of freedom ω^i, that can be split into a longitudinal (scalar) and two transverse (vector) degrees of freedom, and a scalar degree of freedom χ. Therefore, *prima facie* there are 6 extra degrees of freedom. As we will discuss below either χ or ω^i can be a ghost, and therefore the coefficients must be chosen to suppress one or both of them. In the case where χ is the ghost then there are 5 degrees of freedom with those in ω^i being split into a longitudinal (scalar) and two transverse (vector) degrees of freedom. If ω^i is the ghost, then there are only 3 degrees of freedom.

III. MECHANISMS FOR THE ELIMINATION OF GHOSTS

From (2.6) we see that the kinetic terms of the ω_μ and χ fields enter the theory via

$$\frac{1}{2}S_{\{2\}}\supset\int d^4x\sqrt{-g}[(A_{\mathcal{W}}+\rho)\dot{\chi}^2+(C_{\mathcal{W}}+P)\bar{\nabla}_\mu\chi\bar{\nabla}^\mu\chi$$
$$+(C_{\mathcal{W}}-\rho)\dot{\omega}_\mu\dot{\omega}^\mu+(D_{\mathcal{W}}+P)(\bar{\nabla}_\mu\omega^\mu)^2$$
$$+2(E_{\mathcal{W}}-P)\bar{\nabla}_\mu\omega_\nu\bar{\nabla}^\mu\omega^\nu]. \tag{3.1}$$

Let us now focus on the standard scenario of perturbations around Minkowski space-time when both $\rho=P=0$. If $A_{\mathcal{W}}>0$, then $C_{\mathcal{W}}<0$ is required for χ to have a kinetic term with the "proper sign." But if this is the case, then ω_μ has a kinetic term with the "wrong sign" (the same is true if $A_{\mathcal{W}}<0$). Hence, one of χ, ω_μ must be a ghost. There are a few ways to get out of this.

First, one can make the coefficient of $\dot{\chi}^2$ vanish by setting $A_{\mathcal{W}}=0$, which removes χ as a propagating mode and where the equation of motion is a constraint that can be enforced in a Lorentz invariant theory. Second, one could set $C_{\mathcal{W}}=0$, since that would remove ω_μ as a propagating mode. Finally, one could set $\chi\equiv 0$ directly, which requires a breaking of Lorentz invariance (since we will be manually forcing one of the four components of a 4-vector to zero).

When $A_{\mathcal{W}}=0$, there is no $\dot{\chi}^2$ term in the Lagrangian and the equation of motion simply becomes a constraint equation specifying the value of χ from the other field variables. This can be back-substituted into the action so that the theory explicitly does not contain the χ field. From our presentation it is clear that in this case the ghost can be identified with the timelike degree of freedom χ. We have been able to deduce this since we used a (3 + 1) decomposition. Performing a transverse-longitudinal decomposition does not aid the identification of the ghost.

If we choose the five parameters in (2.15) to be given by $A_{\mathcal{W}}=X+Y$, $B_{\mathcal{W}}=-X$, $C_{\mathcal{W}}=-\frac{1}{2}Y$, $D_{\mathcal{W}}=X$, $E_{\mathcal{W}}=\frac{1}{2}Y$, then the theory is Lorentz invariant and the mass matrix is given by

$$\mathcal{W}^{\mu\nu\alpha\beta}=Xg^{\mu\nu}g^{\alpha\beta}+Yg^{\mu(\alpha}g^{\beta)\nu}, \tag{3.2}$$

where X, Y are two parameters that are dependent on background field variables only. The standard route for isolating the ghost in the Lorentz invariant theory [58] decomposes ξ^μ into its transverse and longitudinal modes as

$$\xi_\mu=\zeta_\mu+\nabla_\mu\kappa, \tag{3.3}$$

where $\nabla_\mu\zeta^\mu=0$ and κ is a scalar field. For Minkowski background space-time, inserting (2.5) and (3.3), and the mass-matrix (3.2) into the Lagrangian (2.2), while assuming that X, Y are constants, yields

$$\mathcal{L}_{\{2\}}=\frac{1}{8}(Xh^2+Yh^{\mu\nu}h_{\mu\nu})+\frac{1}{2}Yh^{\mu\nu}\partial_{(\mu}\zeta_{\nu)}$$
$$+\frac{1}{2}(Xh\Box\kappa+Yh^{\mu\nu}\partial_\mu\partial_\nu\kappa)+\frac{1}{2}Y\partial^\mu\zeta^\nu\partial_{(\mu}\zeta_{\nu)}$$
$$+Y\partial^{(\mu}\zeta^{\nu)}\partial_\mu\partial_\nu\kappa+\frac{1}{2}(X+Y)(\Box\kappa)^2. \tag{3.4}$$

This expression has made the ghost problem manifest in the Lorentz invariant language. The existence of the last term, $(X+Y)(\Box\kappa)^2$, means that ghosts are inevitable (see, e.g., [16,61,62]). The cure is to set $X=-Y$, which removes the problematic kinetic term and leaves the Pauli-Feriz mass term, $\mathcal{L}_{\{2\}}\supset\mathcal{L}_{\rm PF}=h^2-h^{\mu\nu}h_{\mu\nu}$. The parameter choice $X=-Y$ is called the *Pauli-Feirz tuning* and will render massive gravitons ghost-free at linearized order on Minkowski backgrounds. In this case $A_{\mathcal{W}}=0$, $B_{\mathcal{W}}=-X$, $C_{\mathcal{W}}=\frac{1}{2}X$, $D_{\mathcal{W}}=X$, $E_{\mathcal{W}}=-\frac{1}{2}X$, which is a special case of the more general situation discussed earlier.

Rather than retain the Lorentz invariance and be forced to use the Pauli-Feirz tuning to remove the ghost, it has been suggest that one can just fix the field $u^\mu\xi_\mu=0$, which implies that $\chi=0$, removing it as a physical degree of freedom. Of course, this is not really a solution to the problem of the ghost, since we have just set the field to zero. However, we will see in the next section that it is possible to impose a symmetry that is equivalent to this. The condition $u^\mu\xi_\mu=0$ imposes an interesting structure upon the fields in theory when we use the transverse-longitudinal split language. Using this and (3.3) implies that $\dot\kappa=-u^\mu\zeta_\mu$, which can be differentiated to yield $\ddot\kappa=-u^\mu\dot\zeta_\mu$. This shows us that the κ field (i.e, the longitudinal component of the ξ^μ field) does not propagate. Instead, the nth time derivative of κ is replaced by the $(n-1)$th time derivative of ζ_μ. Evaluating the "kinetic scalars" (2.13) for the scalar κ yields

$$\nabla^\mu\kappa\nabla_\mu\kappa=-u^\mu u^\nu\zeta_\mu\zeta_\nu+\bar\nabla^\mu\kappa\bar\nabla_\mu\kappa, \quad (3.5a)$$

$$\Box\kappa=u^\mu\dot\zeta_\mu+\bar\nabla^\mu\bar\nabla_\mu\kappa. \quad (3.5b)$$

Using (3.5b), the previously offensive term in (3.4) becomes

$$(X+Y)(\Box\kappa)^2=(X+Y)(u^\mu u^\nu\dot\zeta_\mu\dot\zeta_\nu+2u^\alpha\dot\zeta_\alpha\bar\nabla^\mu\bar\nabla_\mu\kappa+\bar\nabla^\mu\bar\nabla_\mu\kappa\bar\nabla^\alpha\bar\nabla_\alpha\kappa), \quad (3.6)$$

and we observe that the multiple derivatives of κ that are present are entirely spatial. The upshot is that there are no time derivatives of the scalar κ left, and *crucially* no second time derivatives of κ in $\Box\kappa$. The term $(X+Y)\times(\Box\kappa)^2$ in (3.4) is no longer problematic and does not require removal.

IV. IMPOSING REPARAMETRIZATION INVARIANCE

A key aspect of the theories under consideration here is the spontaneous violation of reparametrization invariance. It is interesting to see under what conditions it can be reimposed on the theory. Therefore, we consider how the vector field ξ^μ sources the perturbed gravitational field equations, and under what circumstances its components decouple from the field equations. From (2.1) the field equations for the metric are $\delta_{\rm E}G^{\mu\nu}=8\pi G\delta_{\rm E}T^{\mu\nu}+\delta_{\rm E}U^{\mu\nu}$, where $\delta_{\rm E}U^{\mu\nu}$ is the dark energy momentum tensor and contains contributions to the field equations from the dark sector. In [2] we showed that

$$\delta_{\rm E}U^{\mu\nu}=-\frac{1}{2}(\mathcal{W}^{\mu\nu\alpha\beta}+U^{\mu\nu}g^{\alpha\beta})\delta g_{\alpha\beta}-\xi^\alpha\nabla_\alpha U^{\mu\nu}+2U^{\alpha(\mu}\nabla_\alpha\xi^{\nu)}. \quad (4.1)$$

It is useful to note the terms in $\delta_{\rm E}U^{\mu\nu}$ that are present due to the background dark energy-momentum tensor $U^{\mu\nu}$. The components of $\delta_{\rm E}U^{\mu\nu}$ are written as perturbed fluid variables,

$$\delta_{\rm E}U^\mu{}_\nu=\delta\rho u^\mu u_\nu+2(\rho+P)v^{(\mu}u_{\nu)}+\delta P\gamma^\mu{}_\nu+P\Pi^\mu{}_\nu, \quad (4.2)$$

where one can obtain

$$\delta\rho=(\dot\rho+\mathcal{H}[2\rho-A_{\mathcal{W}}+3B_{\mathcal{W}}])\chi-(A_{\mathcal{W}}+\rho)\dot\chi+(\rho+B_{\mathcal{W}})\left(\frac{1}{2}h+\partial_i\omega^i\right), \quad (4.3a)$$

$$\delta P=-(\dot P+\mathcal{H}[2P-B_{\mathcal{W}}+3D_{\mathcal{W}}+2E_{\mathcal{W}}])\chi+(B_{\mathcal{W}}-P)\dot\chi-\frac{1}{3}(P+3D_{\mathcal{W}}+2E_{\mathcal{W}})\left(\frac{1}{2}h+\partial_i\omega^i\right), \quad (4.3b)$$

$$(\rho+P)v^i=(P+C_{\mathcal{W}})\partial^i\chi+(\rho-C_{\mathcal{W}})\dot\omega^i, \quad (4.3c)$$

$$P\Pi^i{}_j=2(P-E_{\mathcal{W}})\left(\frac{1}{2}h^i{}_j+\partial^{(i}\omega_{j)}-\frac{1}{3}\delta^i{}_j\left(\frac{1}{2}h+\partial_k\omega^k\right)\right). \quad (4.3d)$$

These effective fluid variables define how the components of the vector field ξ^μ sources the gravitational field equations. If one, or both, of the fields χ and ω^i does not appear in (4.3), then that field is not dynamical and hence can be completely ignored. It is clear that with particular choices of the free functions in the mass matrix it will be possible to achieve this. When one or both does not appear, it means that the theory is invariant under the symmetry associated with that field. Therefore, we can impose reparametrization invariance in three natural ways:

(i) The ξ^μ field decouples from the system when $\dot\rho+3\mathcal{H}(\rho+P)=0$, $\dot P+\mathcal{H}(P+3D_{\mathcal{W}}+2E_{\mathcal{W}})=0$, $A_{\mathcal{W}}=-\rho$, $B_{\mathcal{W}}=P$, $C_{\mathcal{W}}=-P$, $\rho=-B_{\mathcal{W}}$, $\rho=C_{\mathcal{W}}$, $P=E_{\mathcal{W}}$, $D_{\mathcal{W}}=-P$. Hence, in the "fully" reparametrization invariant case, where the theory is forced to be invariant under $x^\mu\longrightarrow x^\mu+\xi^\mu$, the only values of ρ, P that are allowed are those that are provided by a cosmological constant, $\rho = -P$, and all perturbed fluid variables vanish. Neither the χ nor the ω^i field propagate.

(ii) The $\chi = u_\mu \xi^\mu$ field decouples from the system when the parameters satisfy $A_{\mathcal{W}} = -\rho$, $B_{\mathcal{W}} = P$, $C_{\mathcal{W}} = -P$, $\dot{\rho} + \mathcal{H}(2\rho - A_{\mathcal{W}} + 3B_{\mathcal{W}}) = 0$, $\dot{P} + \mathcal{H}(2P - B_{\mathcal{W}} + 3D_{\mathcal{W}} + 2E_{\mathcal{W}}) = 0$ from which we can deduce that $\dot{\rho} + 3\mathcal{H}(\rho + P) = 0$ and $\dot{P} + \mathcal{H}(P + 3D_{\mathcal{W}} + 2E_{\mathcal{W}}) = 0$. These equations appear to leave two coefficients, $D_{\mathcal{W}}$ and $E_{\mathcal{W}}$, unspecified. If we now define two parameters β and μ via $D_{\mathcal{W}} = \beta - P - \frac{2}{3}\mu$ and $E_{\mathcal{W}} = \mu + P$, then we find that $\beta = (\rho + P)\frac{\mathrm{d}P}{\mathrm{d}\rho}$, which is the definition of the relativistic bulk modulus, and μ can then be interpreted as a rigidity modulus of an elastic medium. Hence, in the case where we impose time translation invariance, $t \to t + \chi$, but not spatial translation invariance, then we find that the resulting massive gravity theory must be EDE. The equations of motion (2.19) become

$$-3\mathcal{H}[\dot{P} + 3\beta\mathcal{H}]\chi = 0, \tag{4.4a}$$

$$[\rho + P]\ddot{\omega}^i + [\dot{P} + \mathcal{H}(P+\rho)]\dot{\omega}^i - [E_{\mathcal{W}} - P]\nabla^2\omega^i - [E_{\mathcal{W}} + D_{\mathcal{W}}]\partial^i\partial_k\omega^k = -[P - E_{\mathcal{W}}]\partial_j h^{ij} + \frac{1}{2}[P + D_{\mathcal{W}}]\partial^i h. \tag{4.4b}$$

Note that (4.4a) vanishes for arbitrary values of χ since $\beta = (\rho + P)\dot{P}/\dot{\rho}$, and that there is a propagating vector degree of freedom, ω^i. Equation (4.4b) is the equation of motion presented in [63]. In this case the mass term for the gravitons is

$$\mathcal{L}_{\{2\}} = \frac{1}{8}\rho\left[\left(w^2 - \frac{2}{3}\hat{\mu}\right)\bar{h}^2 + 2(w + \hat{\mu})\bar{h}_{\mu\nu}\bar{h}^{\mu\nu}\right], \tag{4.5}$$

where $w = P/\rho$, $\hat{\mu} \equiv \mu/\rho$. This case is equivalent to setting $A_{\mathcal{W}} = 0$ in the Minkowski space case. Since χ does not appear in (4.3), it is no longer a physical degree and there is no ghost.

(iii) ω^i decouples when $\rho + B_{\mathcal{W}} = 0$, $P + 3D_{\mathcal{W}} + 2E_{\mathcal{W}} = 0$, $\rho = C_{\mathcal{W}}$, $P = E_{\mathcal{W}}$ from which we can deduce that $B_{\mathcal{W}} = -C_{\mathcal{W}} = -\rho$ and $D_{\mathcal{W}} = -E_{\mathcal{W}} = -P$. Therefore, we see that in the case where we impose spatial translation invariance, $x^i \to x^i + \xi^i$, but not time translation invariance, then the perturbations have some of the characteristics of massive scalar field theory as explained in [2]. The equations of motion (2.19) become

$$\begin{aligned}
&[A_{\mathcal{W}} + \rho]\ddot{\chi} + [\dot{A}_{\mathcal{W}} + \mathcal{H}(4A_{\mathcal{W}} + \rho - 3P)]\dot{\chi} + [\rho + P]\nabla^2\chi \\
&\quad + [(\dot{A}_{\mathcal{W}} + 3(A_{\mathcal{W}} - \rho - 2P)\mathcal{H})\mathcal{H} + (A_{\mathcal{W}} + 4\rho + 3P)\dot{\mathcal{H}}]\chi = -\frac{1}{2}[\rho + P]\dot{h}, \qquad (4.6a)\\
&[\dot{\rho} + 3\mathcal{H}(\rho + P)]\partial^i\chi = 0. \qquad (4.6b)
\end{aligned}$$

Note that (4.6b) vanishes for arbitrary values of χ due to the background conservation equation. If we define the entropy

$$w\Gamma = \left(\frac{\delta P}{\delta\rho} - w\right)\delta \tag{4.7}$$

and set $A_{\mathcal{W}} = [w + \epsilon(1 + w)]\rho$, we find that

$$w\Gamma = \left(\frac{1}{1+\epsilon} - w\right)\left[\delta - 3\mathcal{H}\left(\frac{w(1+\epsilon)+1}{w(1+\epsilon)-1}\right)(1+w)\theta\right]. \tag{4.8}$$

This form for Γ is not gauge invariant, which is a consequence of breaking time-translation invariance. The mass term is given by

$$\mathcal{L}_{\{2\}} = -\frac{1}{8}w\rho[\bar{h}^2 - 2\bar{h}_{\mu\nu}\bar{h}^{\mu\nu}], \tag{4.9}$$

which we note does not satisfy the Pauli-Feirz tuning. This case is equivalent to setting $C_{\mathcal{W}} = 0$ in the Minkowski space case. Since ω^i does not appear in (4.3), it is no longer a physical degree and there is no ghost.

V. DISCUSSION

It is well established in the literature that there can be ghosts in *general* massive gravity theories, and various methods have been devised to remove them. If one imposes Lorentz invariance, then one is forced to use the Pauli-Feirz tuning to excise the ghost. If one is willing to give up Lorentz invariance, then certain parameter choices allow for ghost-free massive gravity theories. We have shown that the theory (2.2) with the (3 + 1) decomposition of the mass matrix (2.15) imposed with just time-translation, $SO(1, 0)$, invariance constitutes a linearized theory with healthy Lorentz-violating massive gravitons with 5 physical degrees of freedom. That theory *is* exactly the EDE model previously discussed in the literature [63]. In addition, if one imposes spatial translation $SO(0, 3)$ invariance, then there is another ghost-free massive gravity theory with 3 degrees of freedom.

In terms of the EDE parameters and graviton masses, $A_{\mathcal{W}} = m_0^2 = -\rho$, $B_{\mathcal{W}} = -2m_4^2 = P$, $C_{\mathcal{W}} = 4m_1^2 = -P$, $D_{\mathcal{W}} = 4m_3^2 = \beta - P - \frac{2}{3}\mu$, $E_{\mathcal{W}} = -2m_2^2 = \mu + P$. The masses can be conveniently parametrized by some overall mass scale $M^2 \equiv \rho \sim H_0^2$ (since we wrote $U_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu}$, ρ has units of mass squared),

$$\begin{aligned} m_0^2 &= -M^2, \\ m_1^2 &= -\frac{1}{4}wM^2, \\ m_2^2 &= -\frac{1}{2}(\hat{\mu} + w)M^2, \\ m_3^2 &= \frac{1}{4}\left(w^2 - \frac{2}{3}\hat{\mu}\right)M^2, \\ m_4^2 &= -\frac{1}{2}wM^2. \end{aligned} \tag{5.1}$$

We defined $\hat{\mu} \equiv \mu/\rho$ in analogy with $w = P/\rho$.

To connect to dark energy, we note that the fraction of the total energy density that is dark energy is linked to the mass scale M^2 via $\Omega_{\rm de} = M^2/(3H_0^2)$. Hence, we see that the "natural" scale for the masses in order for the modification of gravity to act as a source of cosmic acceleration is of the order the Hubble parameter and are all multiplied by order unity "corrections" defined by two parameters that encode the properties of the elastic medium: its equation of state parameter w and shear modulus $\hat{\mu}$. The longitudinal and transverse sound speeds of EDE are [63]

$$c_{\rm s}^2 = w + \frac{\frac{4}{3}\hat{\mu}}{1+w}, \qquad c_{\rm v}^2 = \frac{\hat{\mu}}{1+w}. \tag{5.2}$$

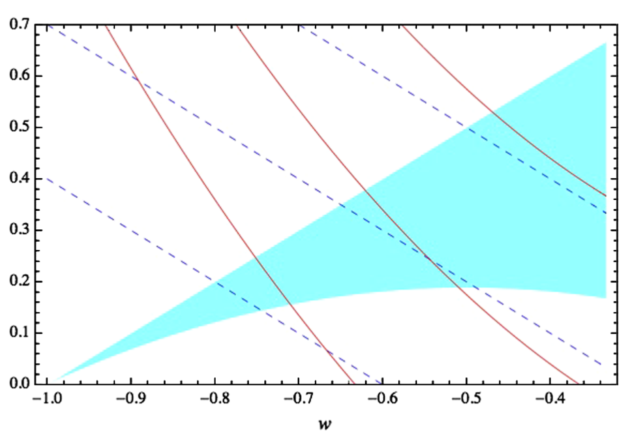

FIG. 1 (color online). The shaded region denotes the range of values of the equation of state w and shear modulus $\hat{\mu}$ that yield sound speeds less than unity, which is where the inequalities (5.3) are satisfied. The red (solid) lines denote lines of constant $m_2^2 = \frac{1}{2}M^2(0.6, 0.3, 0)$ from left to right and the blue (dashed) lines of constant $m_3^2 = \frac{1}{6}M^2(0.6, 0.2, -0.2)$, again from left to right.

Stability and subluminality require that $0 \le c_i^2 \le 1$, so that we have the following constraints on the possible values of $\hat{\mu}$:

$$-\frac{3}{4}w(1+w) \le \hat{\mu} \le \frac{3}{4}(1-w^2), \qquad 0 \le \hat{\mu} \le 1 + w. \tag{5.3}$$

In Fig. 1 we plot the allowed values of $(w, \hat{\mu})$ that satisfy (5.3) and some lines of constants m_2^2 and m_3^2. Observationally the values of w, M^2, and $\hat{\mu}$ (only $\hat{\mu}$ is the "new" parameter) can be constrained and then (5.1) used to obtain the graviton masses.

In the scenario with invariance under spatial translations [leading up to (4.6)], the masses are given in terms of the mass scale $M^2 \equiv \rho \sim H_0^2$ by

$$\begin{aligned} m_0^2 &= [w + \epsilon(1+w)]M^2, \\ m_1^2 &= \frac{1}{4}M^2, \\ m_2^2 &= -\frac{1}{2}M^2, \\ m_3^2 &= -\frac{1}{4}M^2, \\ m_4^2 &= \frac{1}{2}M^2. \end{aligned} \tag{5.4}$$

In this theory there is a residual shift symmetry in the time-translation direction that is similar to that discussed in [38,58,64].

Elastic dark energy and massive gravity share two common features. First, they are both constructed from rank-4 tensors (the elasticity tensor and mass matrix, respectively), and these tensors have identical symmetries in their indices. Second, they both have five propagating degrees of freedom. The extra degrees of freedom in elastic dark energy may have a different fundamental origin from those in massive gravity, but they enable an interesting interpretation and physical intuition to be extracted from what is otherwise an abstract theory of massive gravities. The interpretation gained is that *massive gravity is the manifestation of rigidity of space-time*.

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