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Bernstein estimator for unbounded copula densities

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Summary: Copulas are widely used for modeling the dependence structure of multivariate data. Many methods for estimating the copula density functions are investigated. In this paper, we study the asymptotic properties of the Bernstein estimator for unbounded copula density functions. We show that the estimator converges to infinity at the corner and we establish its relative convergence when the copula density is unbounded. Also, we provide the uniform strong consistency of the estimator on every compact in the interior region. We investigate the finite sample performance of the estimator via an extensive simulation study and we compare the Bernstein copula density estimator with other nonparametric methods. Finally, we consider an empirical application where the asymmetric dependence between international equity markets (US, Canada, UK, and France) is examined.

1 Introduction

The copula function has the advantage to model completely the dependence among variables. In fact, any continuous joint distribution function can be controlled by the marginal distributions, which give the information on each component, and a unique copula that captures the dependence between components. This gives rise to a flexible two step modelling approach where in the first step one models the marginal distributions and in the second one characterizes the dependence using a copula function; see Nelsen (2006) for textbook details. In finance, for example, copulas are a powerful tool for modelling dependence between risky assets, and are applicable in multi-asset pricing, credit portfolio modelling, risk management, etc. The aim of the present paper is to investigate the properties of the nonparametric Bernstein estimator of the copula density. Although many common families of copula densities are unbounded (e.g. Clayton, Gumbel, Gaussian and Student), the properties of the Bernstein copula density estimator have been studied only under the boundedness condition of the copula density at the corners. Hence, in this paper we examine the asymptotic properties of the Bernstein estimator for unbounded density copula functions.

Several approaches have been proposed to estimate the copula functions. The first method is a parametric approach that imposes a specific model for the density copula that is known up to some parameters.

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These parameters can be estimated using the maximum likelihood or inference function for margins methods. Theses approaches are widely used in practice because of heir simplicity; see Joe (1997) and Joe (2005) for more details. The second possibility is a semiparametric approach that assumes a parametric model for the density copula and a nonparametric model for the marginal distributions; for more details, see Genest and Rivest (1995) and Shih and Louis (1995). Liebscher (2005) proposes to estimate the density function based on parametric copulas and on the standard kernel estimator for the marginal densities, which solves the curse of dimensionality problem but not the boundary problem. Bouezmarni and Rombouts (2008) estimate the multivariate density function using parametric copula and asymmetric kernels for the marginal densities, which allows them to address the boundary and the curse of dimensionality problems simultaneously. In a recent paper, Kim, Silvapulle, and Silvapulle (2007) compare semiparametric and parametric methods for estimating copulas. The third way of estimating copulas is based on a nonparametric approach. The advantage of this approach is its flexibility to adapt to any kind of dependence structure. An important contribution is Deheuvels (1979) who suggests the multivariate empirical distribution to estimate the copula function. (Gijbels and Mielniczuk 1990) estimate a bivariate copula density using smoothing kernel methods. They also suggest the reflection method in order to solve the well known boundary bias problem of the kernel methods. (Chen and Huang 2007) propose a bivariate copula estimator based on the local linear estimator, which is consistent everywhere in the support of the copula function, and Rödel (1987) uses the orthogonal series method.

Motivated by Weierstrass theorem, Bernstein polynomials are considered by Lorentz (1953) who proves that any continuous function on the interval [0,1] can be approximated by Bernstein polynomials. For density functions, estimation using the Bernstein polynomials is suggested by Vitale (1975) and, with a slight modification, by Grawronski and Stadtmüller (1981). Tenbusch (1994) investigates the Bernstein estimator for bivariate density functions and Bouezmarni and Rolin (2007) prove the consistency of the Bernstein estimator for unbounded probability density functions. Kakizawa (2004) and (Kakizawa 2006) consider the Bernstein polynomial to estimate density and spectral density functions, respectively. Tenbusch (1997) and (Brown and Chen 1999) propose estimators of the regression functions based on the Bernstein polynomial. In the Bayesian context, Bernstein polynomials are explored by Petrone (1999a), Petrone (1999b), Petrone and Wasserman (2002), and Ghosal (2001). The Bernstein estimator for bounded copula densities was first studied by Sancetta and Satchell (2004) for independent and identically distributed (i.i.d.) data and by Bouezmarni, Rombouts, and Taamouti (2010) for time series data.

In this paper, we focus our attention on the behavior of the Bernstein copula estimator at the boundary regions. In finance, for example, having a good estimator of the copula density at the boundary region is essential for obtaining a valid risk evaluation (risk management). To show the performance of the Bernstein estimator for a copula density that is not necessarily bounded at the corners, we study the consistency of the estimator at the boundary and the interior region. Without assuming any unnecessary assumption like the existence of the first derivative, we are able to prove that the Bernstein copula density estimator converges to infinity at the corner when the copula density is unbounded and we establish the relative strong convergence at the boundary region. Also, we provide its uniform strong consistency on each compact in the interior region. To show the last results, the boundedness of the copula density at the corners is not required.

Further, we ran a simulation study to investigate the finite sample properties of the Bernstein estimator for the copula density. The results show that this estimator has a good performance compared to many other well known estimators like Local linear estimator, Mirror-reflection estimator, Beta kernel estimator and the Transformation estimator using multiplicative Epanechnikov kernel and Gaussian transformation. Since the Bernstein copula density estimator depends on a ”smoothing” parameter, we also investigate the least square cross validation method to select the optimal smoothing parameter. Finally, we consider an empirical

application where the asymmetric dependence between international equity markets (US, Canada, UK, and France) is re-examined. We find that the Bernstein copula density estimator is a good estimator at the extremes. The results show that this estimator is able to capture the well known asymmetric dependence phenomena that is observed in the international equity markets.

This paper is organized as follows. The Bernstein copula density estimator is introduced in Section 2. Section 3 provides the asymptotic properties of the Bernstein copula density estimator at the corners and the interior region. In Section 4, we provide simulation results that show the performance of the Bernstein estimator compared to other existing nonparametric estimators of the copula density. In Section 5, we investigate the least square cross validation method to select the optimal smoothing parameter. Section 6 presents an empirical illustration using financial data and Section 7 concludes.

2 Bernstein copula estimator

Let $X \equiv (X_1, \ldots, X_d)^\top$ be a random vector in $\mathbb{R}^d$ with distribution function F and density function f from which an $i.i.d$ sample of length n, say $\left\{X_i \equiv (X_{i1}, \ldots, X_{id})^\top,\ i = 1, \ldots, n\right\}$, is observed. According to Sklar (1959), the distribution function of X can be expressed via a copula:

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)), \tag{2.1}$$

where F_j, for $j = 1, \ldots, d$, is the marginal distribution function of the random variable X_j, and C is a copula function that captures the dependence structure in the vector X. If we differentiate (2.1) with respect to $(x_1, \ldots, x_d)$, we obtain the density function of X that can be expressed as follows:

$$f(x_1, \ldots, x_d) = \left(\prod_{j=1}^{d} f_j(x_j)\right) \times c\left(F_1(x_1), \ldots, F_d(x_d)\right),$$

where f_j, for $j = 1, \ldots, d$, is the marginal density of the random variable X_j and c is the copula density. Hence, the estimation of the joint density function can be done by estimating the univariate marginal densities, the univariate marginal distributions and the copula density function.

Sancetta and Satchell (2004) proposed the Bernstein copula function which is defined as follows:

$$\tilde{C}(\mathbf{u}) = \sum_{v_1=0}^{k} \cdots \sum_{v_d=0}^{k} C\left(\frac{v_1}{k}, \ldots, \frac{v_d}{k}\right) \prod_{j=1}^{d} p_{v_j,k}(u_j), \text{ for } \mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d, \tag{2.2}$$

where k is an integer that plays the role of a smoothing parameter and $p_{v_j,k}(u_j)$, for $j = 1, \ldots, d$, is the binomial distribution function:

$$p_{v_j,k}(u_j) = \binom{k}{v_j} u_j^{v_j}\ (1 - u_j)^{k - v_j}.$$

Sancetta and Satchell (2004) showed that the Bernstein approximation, $\tilde{C}$, is itself a copula function. In practice the Bernstein copula can not be used because it depends on C which is unknown. To answer that, (Sancetta and Satchell 2004) proposed the Bernstein estimator of copula function which is defined as follows:

$$\hat{C}(\mathbf{u}) = \sum_{v_1=0}^{k} \cdots \sum_{v_d=0}^{k} C_n\left(\frac{v_1}{k}, \ldots, \frac{v_d}{k}\right) \prod_{j=1}^{d} p_{v_j,k}(u_j), \text{ for } \mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d, \tag{2.3}$$

where $k \equiv k_n$ is an integer that depends on the sample size n and C_n is the empirical copula function of the vector $X = (X_1, \ldots, X_d)$ given by :

$$C_n(\mathbf{u}) = F_n\left(F_{n1}^{-1}(u_1), \ldots, F_{nd}^{-1}(u_d)\right)$$

with F_n (resp. F_{n1},..., F_{nd}) the empirical distribution function of X (resp. of X_1,..., X_d). In a very recent paper, Janssen, Swanepoel, and Ververbeke (2012) investigated the asymptotic properties of the Bernstein estimator of copula. They established the almost sure consistency, the asymptotic normality of the estimator and they provided the asymptotic bias and variance of $\hat{C}$.

In this paper, our interest lies in the estimation of the unbounded copula density function using Bernstein polynomials. Indeed, if we differentiate (2.3) with respect to $\mathbf{u}$ we obtain the following Bernstein copula density estimator :

$$\hat{c}(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n} K_{k,S_i}(\mathbf{u}), \tag{2.4}$$

where $S_i = (F_{n1}(X_{i1}), \ldots, F_{nd}(X_{id}))$, $i = 1, \ldots, n$, are the pseudo-observations,

$$K_{k,S_i}(\mathbf{u}) = k^d \sum_{\upsilon_1=0}^{k-1} \cdots \sum_{\upsilon_d=0}^{k-1} A_{S_i,\upsilon_1,\ldots,\upsilon_d} \prod_{j=1}^{d} p_{\upsilon_j,k-1}(u_j),$$

and

$$A_{S_i,\upsilon_1,\ldots,\upsilon_d} = \mathbf{1}_{\{S_i \in B_{\upsilon_1,\ldots,\upsilon_d}\}}, \quad \text{with} \quad B_{\upsilon_1,\ldots,\upsilon_d} = \left]\frac{\upsilon_1}{k}, \frac{\upsilon_1+1}{k}\right] \times \cdots \times \left]\frac{\upsilon_d}{k}, \frac{\upsilon_d+1}{k}\right].$$

Hereafter, we will denote $p_{\upsilon_j,k-1}(u_j)$ by $p_{\upsilon_j}(u_j)$, $A_{S_i,\upsilon_1,\ldots,\upsilon_d}$ by $A_{S_i,\upsilon}$, $B_{\upsilon_1,\ldots,\upsilon_d}$ by B_υ and the sums $\sum_{\upsilon_1=0}^{k-1} \cdots \sum_{\upsilon_d=0}^{k-1}$ by $\sum_\upsilon$.

3 Main results

In this section, we study the asymptotic properties of the Bernstein estimator for unbounded copula densities. Recall that for i.i.d data and when the copula density has a finite first derivative everywhere on its support, Sancetta and Satchell (2004) derived upper bounds for the bias and variance of the Bernstein copula density estimator and showed the pointwise convergence of this estimator. Moreover, (Bouezmarni, Rombouts, and Taamouti 2010) provided asymptotic properties of the Bernstein copula density estimator in the presence of time series data. They derived the asymptotic bias, asymptotic variance and showed the uniform strong convergence of the estimator when the underlying density is continuous on its support. Also, they established the asymptotic normality of the Bernstein copula density estimator. However, although many common copula density families are unbounded at the corners (e.g. Clayton, Gumbel, Gaussian and Student copulas), the derivation of the previous results required the boundedness of the copula density.

In this section, we show that the Bernstein estimator is still a consistent estimator for unbounded copula densities. For the univariate random variables, Bouezmarni and Rolin (2007) studied the properties of the Bernstein estimator for unbounded probability density function but they investigated the case where the random variables are observed and defined on $[0, 1]$. But here the marginal distributions are

replaced by their empirical version, hence the Bernstein estimator is based on the pseudo-observations $S_i = (F_{n1}(X_{i1}), \ldots, F_{nd}(X_{id}))$. Also, we study the properties of the Bernstein estimator for unbounded copula densities which are more common in practice.

The following proposition establishes the uniform strong consistency of the Bernstein copula density estimator on any compact set I in the interior region, without imposing boundedness condition of the copula density at the corners.

Proposition 3.1 *Let $c(.)$ be a continuous copula density function on $(0,1)^d$. Let I be a compact set in $(0,1)^d$, and $\hat{c}(.)$ the Bernstein copula density estimator of c. If*

$$k \to \infty \quad \textit{and} \quad k^d n^{-1/2} (\log\log(n))^{1/2} \to 0,$$

then

$$\sup_{\mathbf{u} \in I} |\hat{c}(\mathbf{u}) - c(\mathbf{u})| \xrightarrow{a.s} 0, \quad \textit{as} \quad n \to \infty.$$

Proof: For simplicity of exposition we consider the case where $d = 2$. The Bernstein density copula estimator can be rewritten as follows:

$$\hat{c}(u_1, u_2) = k^2 \sum_{v} C_n(B_v) p_{v_1}(u_1) p_{v_2}(u_2)$$

with

$$C_n(B_v) = C_n\left(\frac{v_1+1}{k}, \frac{v_2+1}{k}\right) - C_n\left(\frac{v_1}{k}, \frac{v_2+1}{k}\right) - C_n\left(\frac{v_1+1}{k}, \frac{v_2}{k}\right) + C_n\left(\frac{v_1}{k}, \frac{v_2}{k}\right).$$

Now, observe that:

$$\begin{aligned}\sup_{(u_1,u_2)\in I} |\hat{c}(u_1,u_2) - c(u_1,u_2)| &\leq \sup_{(u_1,u_2)\in I} |\hat{c}(u_1,u_2) - \tilde{c}(u_1,u_2)| + \sup_{(u_1,u_2)\in I} |\tilde{c}(u_1,u_2) - c(u_1,u_2)| \\ &= I_{k,n} + I_k,\end{aligned}$$

where

$$\tilde{c}(u_1, u_2) = k^2 \sum_{v} C(B_v) p_{v_1}(u_1) p_{v_2}(u_2)$$

with

$$C(B_v) = C\left(\frac{v_1+1}{k}, \frac{v_2+1}{k}\right) - C\left(\frac{v_1}{k}, \frac{v_2+1}{k}\right) - C\left(\frac{v_1+1}{k}, \frac{v_2}{k}\right) + C\left(\frac{v_1}{k}, \frac{v_2}{k}\right).$$

We can show that I_k converges to zero if the smoothing parameter k tends to infinity.
For $(u_1, u_2) \in I$, we have

$$\begin{aligned}|\tilde{c}(u_1,u_2) - c(u_1,u_2)| &= \left| k^2 \sum_{v} \int_{\frac{v_1}{k}}^{\frac{v_1+1}{k}} \int_{\frac{v_2}{k}}^{\frac{v_2+1}{k}} \{c(t_1,t_2) - c(u_1,u_2)\}\, dt_1 dt_2 p_{v_1}(u_1) p_{v_2}(u_2) \right| \\ &= \left| k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} \{c(t_1,t_2) - c(u_1,u_2)\} dt_1 dt_2 \right) \right| \\ &\leq k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} |c(t_1,t_2) - c(u_1,u_2)|\, dt_1 dt_2 \right),\end{aligned}$$

where ξ_1 and ξ_2 are two independent Binomial random variables with corresponding parameters $(k-1, u_1)$ and $(k-1, u_2)$, respectively.
Let δ be a positive real number such that $\delta < \min(u_1, 1-u_1, u_2, 1-u_2)$. Then we have

$$\begin{aligned}|\tilde{c}(u_1,u_2) - c(u_1,u_2)| &\leq k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} |c(t_1,t_2) - c(u_1,u_2)| \, \mathbf{1}_{A_\delta} \, dt_1 dt_2 \right) \\ &\quad + k^2 \mathbb{E}_{(\xi_1,\xi_2)} \left(\int_{\frac{\xi_1}{k}}^{\frac{\xi_1+1}{k}} \int_{\frac{\xi_2}{k}}^{\frac{\xi_2+1}{k}} |c(t_1,t_2) - c(u_1,u_2)| \, \mathbf{1}_{A_\delta^c} \, dt_1 dt_2 \right) \\ &= I_{k,1}^* + I_{k,2}^*,\end{aligned}$$

where $A_\delta \equiv \left\{ |\frac{\xi_1}{k} - u_1| \leq \delta \text{ and } |\frac{\xi_2}{k} - u_2| \leq \delta \right\}$ and A_δ^c is the complementary event of A_δ. Observe that A_δ^c contains 4 events, that is $A_\delta^C = A_1 \cup A_2 \cup A_3 \cup A_4$ where $A_1 = \left\{ \frac{\xi_1}{k} - u_1 > \delta \right\}$, $A_2 = \left\{ \frac{\xi_1}{k} - u_1 < -\delta \right\}$, $A_3 = \left\{ \left| \frac{\xi_1}{k} - u_1 \right| \leq \delta \text{ and } \frac{\xi_2}{k} - u_2 > \delta \right\}$ and $A_4 = \left\{ \left| \frac{\xi_1}{k} - u_1 \right| \leq \delta \text{ and } \frac{\xi_2}{k} - u_2 < -\delta \right\}$.

Thus, using Lemma 2.1 of Bouezmarni and Rolin (2007), we obtain

$$I_{k,2}^* \leq 4 \left(k^2 + \sup_{(u_1,u_2)\in I} |c(u_1,u_2)| \right) \exp(-2(k-1)\delta^2).$$

Also, because of the uniform continuity of the copula density on I, it is straightforward to show that $I_{k,1}^* = o(1)$. Hence, $I_k \to 0$, when $k \to \infty$.

Next, we show that $\sup_I |\hat{c}(u_1,u_2) - \tilde{c}(u_1,u_2)|$ converges to zero. Using Lemma 1 in Janssen, Swanepoel, and Ververbeke (2012), we have

$$\begin{aligned} I_{k,n} &= \sup_{(u_1,u_2)\in I} \left| k^2 \sum_{v} \{C_n(B_v) - C(B_v)\} \, p_{v_1}(u_1) p_{v_2}(u_2) \right| \\ &\leq 4k^2 \sup_{(t_1,t_2)\in[0,1]^2} |C_n(t_1,t_2) - C(t_1,t_2)| \\ &= O\left(k^2 n^{-1/2} (\log\log(n))^{1/2} \right) \quad \text{a.s., } n \to \infty. \end{aligned}$$

Hence, we conclude the proof of Proposition 3.1. □

The next proposition shows that the Bernstein copula density estimator converges to infinity when the density is unbounded at the corners. It also provides the relative convergence of the estimator at the corners. Without loss of generality, the following results are derived when the density is unbounded at $(0,0)$.

Proposition 3.2 *Let $c(.)$ be a copula density function that is unbounded at $(0,0)$. Let $\hat{c}(.)$ be the Bernstein copula density estimator of c. Then, under the conditions of Proposition 3.1, we have*

$$\hat{c}(0,0) \xrightarrow{a.s} \infty, \quad \textit{as} \quad n \to \infty.$$

Further, we have

$$\frac{|\hat{c}(u_1,u_2) - c(u_1,u_2)|}{c(u_1,u_2)} \xrightarrow{a.s} 0, \quad \textit{as} \quad n \to \infty, \quad \textit{and} \quad (u_1,u_2) \to (0,0).$$

Proof: We can first show that $\hat{c}(0,0)$ converges to infinity. From the proof of Proposition 3.1, we have $|\hat{c}(0,0) - \tilde{c}(0,0)| \xrightarrow{a.s.} 0$. Thus, it remains to show that $\tilde{c}(0,0)$ converges to infinity when the smoothing parameter k tends to infinity. The copula density, c, is unbounded at $(0,0)$, then for $L > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $c(u_1, u_2) > L$, for $u_1 < \delta_1$ and $u_2 < \delta_2$. For n sufficiently large and for k_n tending to infinity, we have $\min(\delta_1, \delta_2) > \frac{1}{k}$. Hence,

$$\tilde{c}(0,0) = k^2 \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} c(u_1, u_2)\, du_1 du_2 > L, \text{ for } n \text{ sufficiently large.}$$

We can show the relative convergence of the Bernstein copula estimator in the boundary region of unbounded copula density functions by using similar arguments to those in the proof of Proposition 3.1. This concludes the proof of Proposition 3.2. □

4 Monte Carlo simulations

In this section, we run Monte Carlo simulations to evaluate the performance of the Bernstein estimator of copula density in the interior region and at the corners. We compare the finite sample properties of the Bernstein copula density estimator [hereafter **BR**] with those of:

(1) Local linear estimator with multiplicative Epanechnikov kernel [hereafter **LL**];

(2) Mirror-reflection estimator with multiplicative Epanechnikov kernel [hereafter **MR**];

(3) Beta kernel estimator [hereafter **BT**];

(4) Transformation estimator using multiplicative Epanechnikov kernel and Gaussian transformation [hereafter **TR**].

We choose these estimators because they are known to have a good behavior at the borders. For more details about the LL estimator and the MR, BT and TR estimators, the reader can consult Chen and Huang (2007) and Charpentier, Fermanian, and Scaillet (2006), respectively.

To study the performance of these estimators in different contexts that one can encounter in practice, we consider several data generating processes (DGPs). We simulate our bivariate data $\left\{(X_{i1}, X_{i2})^\top\right\}_{i=1}^n$ using a uniform distribution $Unif[0,1]^2$ and under one of the following copula densities: **(1)** Normal copula [hereafter $c_{(n)}$]; **(2)** Student copula [hereafter $c_{(t)}$]; **(3)** Clayton copula [hereafter $c_{(cl)}$]; **(4)** Gumbel copula [hereafter $c_{(g)}$]; and **(5)** Frank copula [hereafter $c_{(f)}$]**.** These copulas densities are extremely useful in practice. Except Frank copula, all these copula densities are unbounded at $(0,0)$ or/and at $(1,1)$. We consider two scenarios corresponding to two level of dependency as measured by the Kendall rank correlation coefficient: **(a)** $\tau = 0.25$ for weak dependence and **(b)** $\tau = 0.75$ for strong dependence.

To keep the computing time reasonable, we consider small and moderate sample sizes: $n = 50$ and $n = 150$ and we perform $N = 1000$ simulations. We evaluate the performance of each estimator using two measures: the Averaged Mean Squared Error (AMSE) and the Averaged Median Absolute Relative Error (AMAE).

$$\text{AMSE} = \frac{1}{I} \sum_{i=1}^{I} \text{MSE}(\mathbf{u}_i),$$

where $\mathrm{MSE}(\mathbf{u}_i) = \frac{1}{N}\sum_{j=1}^{N}\left(\hat{c}_j(\mathbf{u}_i) - c(\mathbf{u}_i)\right)^2$, $\mathbf{u}_i = (u_{1i}, u_{2i}) \in \mathcal{S}$, a subset of $[0,1]\times[0,1]$ of size I, and $\hat{c}_j(.), j = 1, \ldots, N$, is the estimator of the copula density corresponding to the j-th replication.

$$AMAE = \frac{1}{I}\sum_{i=1}^{I} MAE(\mathbf{u}_i),$$

where $MAE(\mathbf{u})$ is the empirical median of the sequence $\left\{\left|\frac{\hat{c}_j(\mathbf{u})-c(\mathbf{u})}{c(\mathbf{u})}\right|\right\}_{j=1}^{N}$. Clearly, this criterion is less sensitive to extreme deviations than the classical AMSE which may be the result of only few atypically large deviations. Also, observe that $\mathrm{AMSE} = \frac{1}{N}\sum_j \mathrm{ASE}(\hat{c}_j)$, where $ASE(\hat{c}) = \frac{1}{I}\sum_i(\hat{c}(\boldsymbol{u}_i) - c(\boldsymbol{u}_i))^2$ is the averaged squared error of $\hat{c}$. To assess the uncertainty in the AMSE measure, we also report the standard deviation of $\{\mathrm{ASE}(\hat{c}_j)\}_{j=1}^{N}$, i.e. $\mathrm{SASE} = \sqrt{\frac{1}{N}\sum_j\left(\mathrm{ASE}(\hat{c}_j) - \mathrm{AMSE}\right)^2}$.

We calculate the AMSE and AMAE using $h = 0.001, 0.04, \ldots, 0.97$ for BT, LL, TR and MR estimators and using $k = 3, 6, \ldots, n$ (n is the sample size) for BR estimator. Since the optimal smoothing parameter (the one that minimize AMSE) may depend strongly on the local behavior of our target function (the copula density) and since we are interested in studying the copula estimation not only in the interior but also at the boundary, we do the above calculation in four different regions:

(1) $\mathcal{S} = \{(u_{1i}, u_{2i}) = (0.01, 0.01), (0.01, 0.03), \ldots, (0.99, 0.99)\}$;

(2) $\mathcal{S}_0 = S \cap \left\{(u_{1i}, u_{2i}) : \sqrt{u_{1i}^2 + u_{2i}^2} < 0.56\right\}$, i.e. the 25% extreme left points;

(3) $\mathcal{S}_1 = S \cap \left\{(u_{1i}, u_{2i}) : \sqrt{u_{1i}^2 + u_{2i}^2} > 0.98\right\}$, i.e. the 25% extreme right points; and

(4) $\mathcal{S}_i = \mathcal{S} \setminus \{\mathcal{S}_0, \mathcal{S}_1\}$, i.e. all points in $\mathcal{S}$ except the extreme ones.

Table 4.1 reports the AMSE for the BR estimator using the optimal smoothing parameter $k_{opt} = \arg\min_k \mathrm{AMSE}(k)$ on $\mathcal{S}$, $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_i$. As expected, we see that the AMSE decreases with the sample size n and increases with the Kendall's rank correlation τ. In other words, we find that the strength of the dependence between X_1 and X_2 makes the estimation of copula density more difficult. Except for the Clayton copula density and for $\tau = 0.75$, we obtain relatively small integrated mean squared errors. We also find that the performance of the BR estimator depends on the target region ($\mathcal{S}$, $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_i$). in the "interior" region ($\mathcal{S}_i$), the estimator behaves clearly better than at the borders. Further, it can be seen from Table 4.1 that k_{opt}, the optimal smoothing parameter, increases with n and τ, as predicted by the theory. The fact that k_{opt} increases with n reflect the fact that for Bernstein copula density estimator to be (uniformly) consistent, $k \to \infty$, as $n \to \infty$, see Proposition 1. On the other hand, when τ increases, the dependence in the data increases and the estimation becomes more complicated. In such a case, in order to reduce the bias, one need a larger value of k , see e.g. (Sancetta and Satchell 2004). Interestingly, we see that k_{opt} also depends on the target region. One should use a larger value of k near the extreme points where the values of the copula function become very large. Thus, in practice an adaptive smoothing parameter should be used in order to get a better approximation. One such an adaptation could be the method of "shrinking" of the smoothing parameter at the borders.

In Tables 4.2 and 4.3, we compare the performance of BR estimator with the other estimating methods cited above for $n = 50$ and $n = 150$, respectively. To facilitate such a comparison, we provide the relative efficiency of each estimator ($E = LL, MR, BT, TR$) with respect to Bernstein copula estimator. Thus, we

			$\mathcal{S}_0$		$\mathcal{S}_1$		$\mathcal{S}_i$		$\mathcal{S}$	
n	τ	Cop.	k_{opt}	AMSE	k_{opt}	AMSE	k_{opt}	AMSE	k_{opt}	AMSE
50	0,25	$c_{(cl)}$	9	0.597	3	0.071	6	0.056	6	0.216
		$c_{(f)}$	3	0.098	3	0.121	6	0.050	3	0.080
		$c_{(g)}$	3	0.084	6	0.534	3	0.043	6	0.185
		$c_{(n)}$	3	0.133	6	0.163	6	0.047	6	0.097
		$c_{(t)}$	9	0.656	9	0.725	6	0.072	9	0.385
	0,75	$c_{(cl)}$	50	33.348	9	0.453	27	0.201	50	9.402
		$c_{(f)}$	15	1.052	15	0.893	36	0.247	21	0.635
		$c_{(g)}$	21	1.353	50	11.249	27	0.190	45	3.431
		$c_{(n)}$	30	2.277	33	2.251	24	0.158	30	1.197
		$c_{(t)}$	50	8.567	50	8.455	30	0.239	50	4.364
150	0,25	$c_{(cl)}$	21	0.365	9	0.057	9	0.027	12	0.139
		$c_{(f)}$	6	0.054	6	0.066	6	0.025	6	0.042
		$c_{(g)}$	9	0.063	21	0.343	9	0.025	9	0.128
		$c_{(n)}$	6	0.082	6	0.091	6	0.024	6	0.054
		$c_{(t)}$	24	0.430	24	0.470	9	0.040	21	0.256
	0,75	$c_{(cl)}$	150	28.236	15	0.240	48	0.108	150	7.542
		$c_{(f)}$	30	0.584	27	0.492	65	0.133	36	0.351
		$c_{(g)}$	39	0.762	95	8.669	45	0.105	95	2.480
		$c_{(n)}$	65	1.343	65	1.292	39	0.084	54	0.698
		$c_{(t)}$	150	6.812	150	6.789	54	0.133	150	3.415

Table 4.1 Averaged Mean Squared Error (AMSE) of the BR estimator on the unit square ($\mathcal{S}$), near $(0,0)$ ($\mathcal{S}_0$), near $(1,1)$ ($\mathcal{S}_1$) and in ($\mathcal{S}_i$), under different families of copula and using optimal smoothing parameter.

provide $\mathrm{RES} = \frac{\mathrm{AMSE}(E)}{\mathrm{AMSE}(BR)}$ and $REA = \frac{AMAE(E)}{AMAE(BR)}$. We also report the ratio of the averaged variance to AMSE $\left(\frac{\mathrm{AVAR}}{\mathrm{AMSE}}\right)$ and the standard deviation of the averaged squared error (SASE); see the definition given above. Note that the values reported in Tables 4.2 and 4.3 are expressed as percentages. From these, we see that for weakly dependent data ($\tau = 0.25$), the BR and LL estimators are quite comparable and give the best results in terms of AMSE and AMAE. The BR estimator performs better than the rest of the estimators for three of the five copulas that we studied here (Clayton, Gumbel and Student copulas). Looking at the ratio $\frac{\mathrm{AVAR}}{\mathrm{AMSE}}$ and at the standard deviation of the averaged squared error (SASE), we see clearly that the BR estimator has the advantage to be less variable than the LL estimator. We also see that the Transformation estimator (TR) gives the worse result with a very high variance and a large AMSE. However, the opposite happens for strongly dependent data ($\tau = 0.75$). In the latter case the TR method becomes the best since it behaves better for three of the five copulas considered here (Clayton, Gumbel and Student copulas). The performance measure AMAE indicates that the BR estimator still behaves reasonably well. The only case where the BR estimator is beaten by the TR estimator is when we consider the Clayton copula with $\tau = 0.75$.

5 Smoothing parameter selection

Here, we investigate the performance of the Bernstein copula density estimator when an automatic data-driven smoothing parameter is used. We use the least-squared cross-validation (LSCV) method which

		$\tau = 0.25$				$\tau = 0.75$			
Copula	Method	RES	REA	$\frac{\text{AVAR}}{\text{AMSE}}$	SASE	RES	REA	$\frac{\text{AVAR}}{\text{AMSE}}$	SASE
$c_{(cl)}$	BR	100.0	100.0	29.6	5.5	100.0	100.0	20.2	244.8
	LL	101.3	112.3	32.7	6.1	103.2	87.4	14.2	167.5
	BT	104.5	110.6	29.9	5.6	103.6	2087.0	16.1	206.8
	MR	114.8	120.1	20.0	5.6	116.8	87.4	8.1	74.1
	TR	168.7	113.8	87.5	19.8	***89.4***	***72.9***	24.3	326.3
$c_{(f)}$	BR	100.0	100.0	40.1	3.9	100.0	100.0	52.7	17.3
	LL	***95.2***	100.0	91.0	6.1	112.3	107.3	63.3	34.4
	BT	108.3	109.9	46.1	3.8	94.8	113.5	53.3	15.0
	MR	110.1	111.8	44.2	4.9	***92.8***	106.6	47.0	18.0
	TR	399.8	149.5	88.1	16.5	161.3	110.4	48.1	38.8
$c_{(g)}$	BR	100.0	100.0	33.1	4.7	100.0	100.0	24.5	79.9
	LL	100.0	108.4	36.1	5.5	107.0	119.2	27.8	75.0
	BT	105.2	108.1	27.4	4.6	111.1	101.4	41.5	96.4
	MR	114.1	124.4	22.4	4.8	116.1	120.1	13.6	31.0
	TR	196.2	133.1	88.0	18.9	***92.5***	***89.9***	28.4	99.5
$c_{(n)}$	BR	100.0	100.0	62.6	4.3	100.0	100.0	45.5	31.0
	LL	***89.9***	100.0	73.6	5.3	116.8	121.3	43.8	46.2
	BT	106.8	116.8	36.7	3.7	108.4	105.9	24.4	23.5
	MR	114.0	125.4	32.5	4.5	124.4	122.8	24.7	19.7
	TR	338.5	133.2	89.8	17.3	107.0	99.2	46.5	41.3
$c_{(t)}$	BR	100.0	100.0	29.7	7.5	100.0	100.0	31.9	107.6
	LL	111.7	112.5	33.1	10.4	108.2	120.4	31.7	88.1
	BT	103.0	103.5	28.5	7.4	107.0	105.4	34.3	107.5
	MR	115.8	124.3	22.2	7.1	122.3	123.8	16.5	44.6
	TR	127.9	104.0	80.2	22.0	***91.1***	100.9	32.0	117.9

Table 4.2 The ratios RES (%), REA (%) and AVAR / AMSE (%), and the SASE (%) of the estimators BR, LL, MR, BT, and TR on the set $\mathcal{S}$ and for $n = 50$.

selects a smoothing parameter $\hat{k}$ that minimizes the following function:

$$\text{LSCV}(k) = \int_0^1 \int_0^1 \hat{c}^2(u_1, u_2) du_1 du_2 - 2n^{-1} \sum_{i=1}^{n} \hat{c}^{(-i)}(S_{1,i}, S_{2,i}),$$

where $\hat{c}^{(-i)}(\cdot, \cdot)$, for $i = 1, \ldots, n$, is the Bernstein copula density estimator calculated using a smoothing parameter k and all the data except the observation $(S_{1,i}, S_{2,i})$. Observe that:

$$\mathbb{E}\left(\text{LSCV}(k)\right) = \mathbb{E}\left(\int_0^1 \int_0^1 (\hat{c}(u_1, u_2) - c(u_1, u_2))^2 du_1 du_2\right) - \int_0^1 \int_0^1 c^2(u_1, u_2) du_1 du_2.$$

Indeed,

$$\begin{aligned}
&\mathbb{E}\left[\hat{c}^{(-i)}(S_{1,i}, S_{2,i})\right] \\
&\quad = \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} \left(k^2 \sum_{\upsilon_1=0}^{k-1} \sum_{\upsilon_2=0}^{k-1} \mathbb{E}\left[A_{S_j, \upsilon_1, \upsilon_2}\, p_{\upsilon_1}(S_{1,i}) p_{\upsilon_2}(S_{2,i})\right] \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} \left(k^2 \sum_{\upsilon_1=0}^{k-1} \sum_{\upsilon_2=0}^{k-1} \mathbb{E}\left[A_{S_j,\upsilon_1,\upsilon_2}\right] \mathbb{E}\left[p_{\upsilon_1}(S_{1,i}) p_{\upsilon_2}(S_{2,i})\right] \right) \\
&= \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} \left(k^2 \sum_{\upsilon_1=0}^{k-1} \sum_{\upsilon_2=0}^{k-1} \mathbb{E}\left[A_{S_j,\upsilon_1,\upsilon_2}\right] \int_0^1 \int_0^1 p_{\upsilon_1}(u_1) p_{\upsilon_2}(u_2) c(u_1,u_2) du_1 du_2 \right) \\
&= \int_0^1 \int_0^1 \mathbb{E}\left(\frac{1}{n-1} \sum_{j=1, j\neq i}^{n} k^2 \sum_{\upsilon_1=0}^{k-1} \sum_{\upsilon_2=0}^{k-1} A_{S_j,\upsilon_1,\upsilon_2}\, p_{\upsilon_1}(u_1) p_{\upsilon_2}(u_2) \right) c(u_1,u_2) du_1 du_2 \\
&= \int_0^1 \int_0^1 \mathbb{E}\left(\hat{c}^{(-i)}(u_1,u_2) \right) c(u_1,u_2) du_1 du_2 \\
&= \int_0^1 \int_0^1 \mathbb{E}\left(\hat{c}(u_1,u_2) \right) c(u_1,u_2) du_1 du_2.
\end{aligned}
$$

		$\tau = 0.25$				$\tau = 0.75$			
Copula	Method	RES	REA	$\frac{\text{AVAR}}{\text{AMSE}}$	SASE	RES	REA	$\frac{\text{AVAR}}{\text{AMSE}}$	SASE
$c_{(cl)}$	BR	100,0	100.0	41.2	3.8	100.0	100.0	15.6	213.2
	LL	104.8	103.9	39.5	4.0	109.3	80.0	9.5	147.0
	BT	114.0	113.4	35.0	3.7	114.7	2865.0	6.0	128.4
	MR	135.8	126.7	29.5	3.6	126.6	80.0	17.2	127.1
	TR	108.4	115.6	69.7	6.3	***79.0***	***60.5***	31.9	317.2
$c_{(f)}$	BR	100.0	100.0	48.0	1.9	100.0	100.0	58.0	8.5
	LL	***65.8***	***86.6***	75.5	1.7	***98.2***	105.3	62.3	13.2
	BT	116.7	112.1	55.9	2.0	110.2	105.8	27.0	6.5
	MR	127.0	116.6	43.3	2.5	***89.8***	105.9	56.8	8.8
	TR	306.7	147.7	70.8	5.6	166.2	110.8	51.4	18.5
$c_{(g)}$	BR	100.0	100.0	38.2	3.0	100.0	100.0	24.1	66.3
	LL	104.5	105.7	36.0	3.2	107.9	123.2	30.7	61.4
	BT	111.8	111.4	27.0	2.9	108.0	108.5	19.6	58.6
	MR	126.6	130.7	29.3	2.9	129.4	125.8	17.8	31.0
	TR	117.7	130.1	70.5	5.7	***84.1***	***93.4***	39.4	91.7
$c_{(n)}$	BR	100.0	100.0	56.1	2.1	100.0	100.0	53.0	20.3
	LL	***81.7***	***92.0***	47.9	1.8	121.4	116.4	58.7	28.0
	BT	122.3	122.3	41.7	2.2	118.8	108.6	63.4	22.0
	MR	140.5	134.6	35.4	2.6	149.8	123.7	28.0	12.6
	TR	243.9	134.1	74.4	5.9	104.6	***94.4***	53.6	23.9
$c_{(t)}$	BR	100.0	100.0	43.5	5.8	100.0	100.0	13.3	65.4
	LL	111.7	108.4	38.8	7.1	100.0	119.3	25.0	63.5
	BT	110.5	110.4	37.8	5.5	104.0	102.2	15.0	64.4
	MR	131.7	132.3	27.9	4.4	127.3	124.3	38.1	63.0
	TR	***88.7***	110.7	58.3	7.6	***77.2***	100.9	40.2	104.8

Table 4.3 The ratios RES (%), REA (%) and AVAR / AMSE (%), and the SASE (%) of the estimators BR, LL, MR, BT, and TR on the set $\mathcal{S}$ and for $n = 150$.

Hence,

$$\mathbb{E}(\mathrm{LSCV}(k)) = \mathbb{E}\left(\int_0^1\int_0^1 \hat{c}^2\,(u_1,u_2)\,du_1u_2 - 2\int_0^1\int_0^1 \hat{c}(u_1,u_2)c(u_1,u_2)\,du_1u_2\right).$$

Thus, the smoothing parameter $\hat{k}$ minimizes an unbiased estimator of the expected integrated squared error. One can also show that

$$\int_0^1\int_0^1 \hat{c}^2(u_1,u_2)du_1du_2 = n^{-2}\sum_i\sum_j \tilde{B}(\lfloor S_{1,i}k\rfloor,\lfloor S_{1,j}k\rfloor)\tilde{B}(\lfloor S_{2,i}k\rfloor,\lfloor S_{2,j}k\rfloor),$$

where $\lfloor\cdot\rfloor$ denotes the floor function and

$$\tilde{B}(a,b) = \frac{B(a+b+1,2k-a-b-1,)}{B(a+1,k-a)B(b+1,k-b)},$$

where $B(a,b)$ is the usual Beta function, i.e. $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$. The above formula facilitates the calculation of the LSCV function, which reduces the simulation run time.

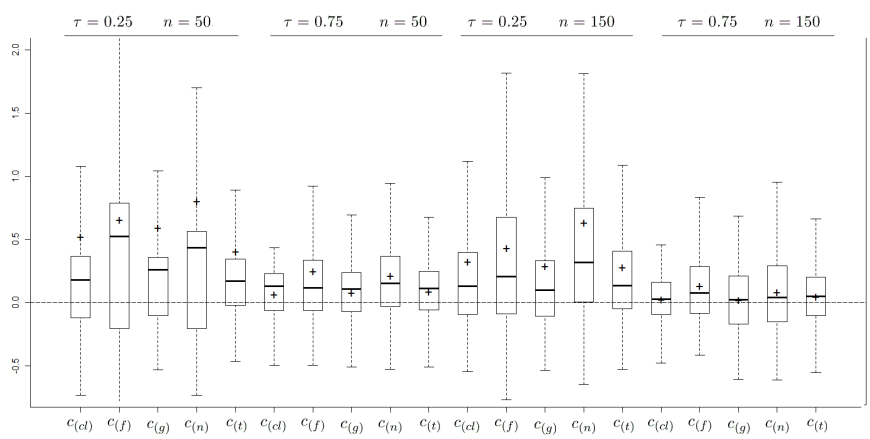


Figure 5.1 The ratio $(\mathrm{ASE}(\hat{k}) - \mathrm{AMSE}(k_{opt}))/\,\mathrm{AMSE}(k_{opt})$ of the BR estimator under weak and strong dependence and different families of the copula, $n = 150$

We repeat the simulation study described in Section 4 using the data-driven smoothing parameter $\hat{k}$ instead of the optimal smoothing parameter k_{opt}. Figure 5.1 shows a box-plot of 1000 observations of the ratio $\frac{ASE(\hat{c}_{j,\hat{k}_j})-\mathrm{AMSE}(k_{opt})}{\mathrm{AMSE}(k_{opt})}$. The latter should fluctuate around zero if the data-driven smoothing parameter $\hat{k}$ and the optimal smoothing parameter k_{opt} lead approximately to the same averaged squared error. Although our results show that this is not the case for all considered scenarios, the averaged squared error obtained using $\hat{k}$ remains reasonably small and typically does not exceed $2 * \mathrm{AMSE}(k_{opt})$. The results

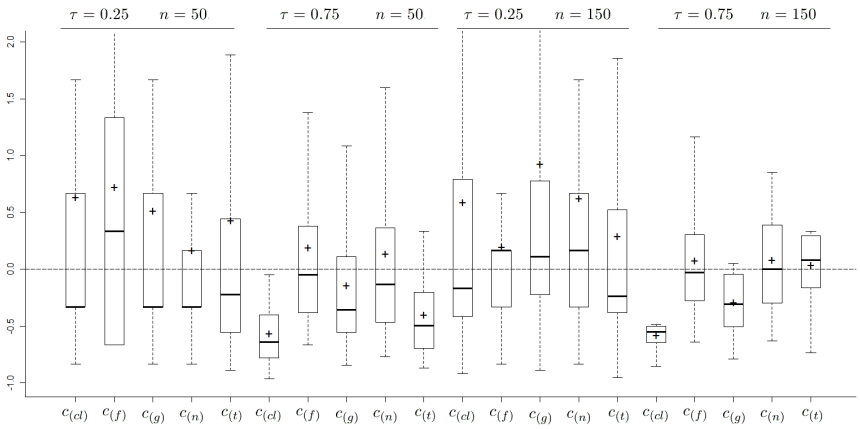


Figure 5.2 The ratio $(\hat{k} - k_{opt})/k_{opt}$ of the BR estimator under weak and strong dependence and different families of the copula, $n = 150$

for $n = 50$ are not very satisfactory, probably because the sample size is too small. However, these results improve when the sample size n increases, which seems to indicate the consistency of the smoothing parameter selection method.

For weakly dependent data ($\tau = 0.25$), we see that the averaged squared error changes a lot across the simulations, especially for Normal and Frank copulas. Moreover, for the latter two cases, the smoothing parameter $\hat{k}$ seems to lead to some bias approximations. Surprisingly, when the dependence between X_1 and X_2 is strong ($\tau = 0.75$), we find much better results in terms of ASE, even if the boundary problems are more severe in this case. An explanation can be obtained by comparing $\hat{k}$ to k_{opt}. Figure 5.2 shows a box-plot of 1000 observations of the ratio $\frac{\hat{k}_j - k_{opt}}{k_{opt}}$, where $\hat{k}_j$ is the smoothing parameter that corresponds to the j-th replication selected using the least-squared cross-validation (LSCV) method. This figure clearly shows that the LSCV method tend to choose a large smoothing parameter k when τ is small, which leads to an over-smoothing. However, the opposite happens when τ is large, but the under-smoothing is much less severe except for the Clayton copula density. Consequently, we recommend to correct the LSCV smoothing parameter by taking into account the "degree of unboundedness" at corners of the copula density, in particular for small size data. This can be done, for example, by adapting the method of "Shrinkage" proposed by Omelka, Gijbels, and Veraverbek (2009). This will be investigated and studied, theoretically and by simulations, in a future work.

6 Empirical illustration

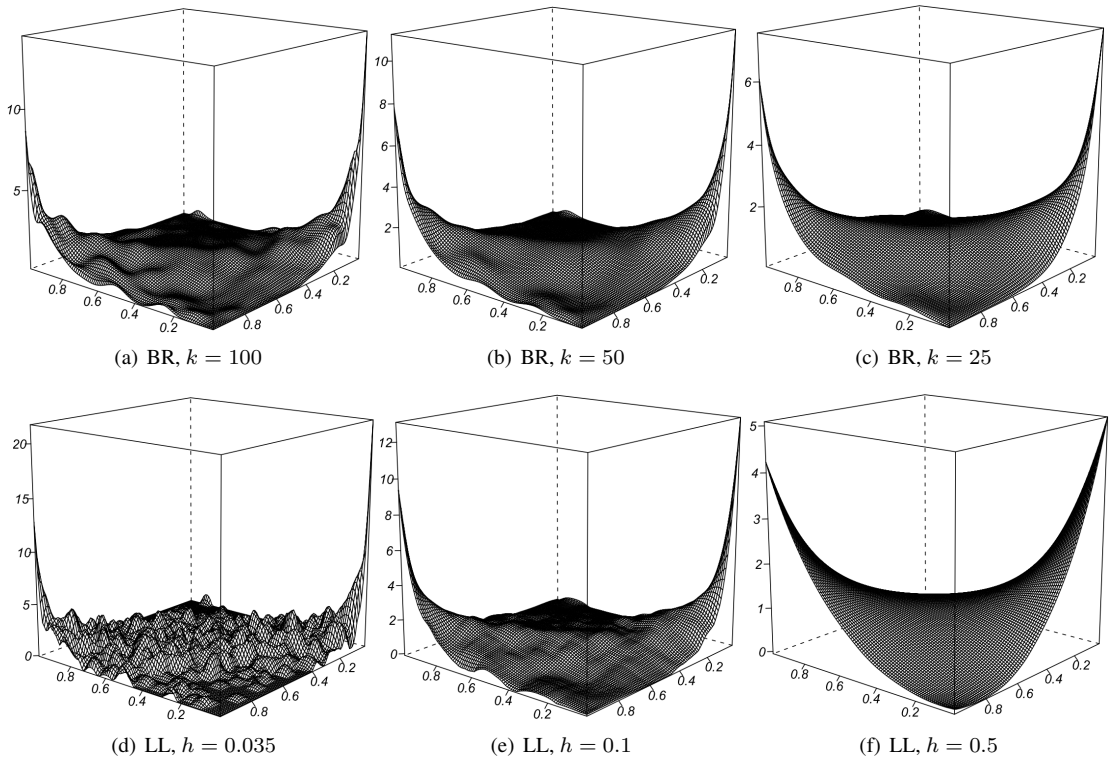
In this section, we re-examine the asymmetric dependence between international equity markets using two nonparametric estimators of copula densities. Recent research suggests an increase in the correlation between international equity markets during volatile periods. This increase is especially observed during market downturns. Ang and Bekaert (2002) use a two-regime switching model and find evidence of one

state with low returns and high correlation and volatilities, and a second state with high returns and low correlation and volatilities. Longin and Solnik (2001) use extreme value theory and develop a new concept named exceedance correlation, and find a high correlation between large negative returns and zero correlation between large positive returns.

Rather than to use correlation coefficients, here we use copula densities which can be viewed as a natural way to model the dependence between equity market returns. We focus on four equity markets (US, Canada, UK, and France) and we use weekly observations that span 19 years.

6.1 Data description

Our data consists of weekly observations on MSCI Equity Indices series for the US, Canada, the UK and France. The sample runs from October 16th 1984 to December 21th 2004 for a total of 1054 observations. The returns are computed using the standard continuous compounding formula. All returns are derived on a weekly basis from daily prices expressed in US dollars. Summary statistics (not reported) for the US, Canada, the UK and France equity returns indicate that the unconditional distributions of these returns exhibit high kurtosis and negative skewness. The sample kurtosis is greater than the normal distribution value of three. The values of Jarque-Bera test statistic show that the equity returns are not normally distributed.

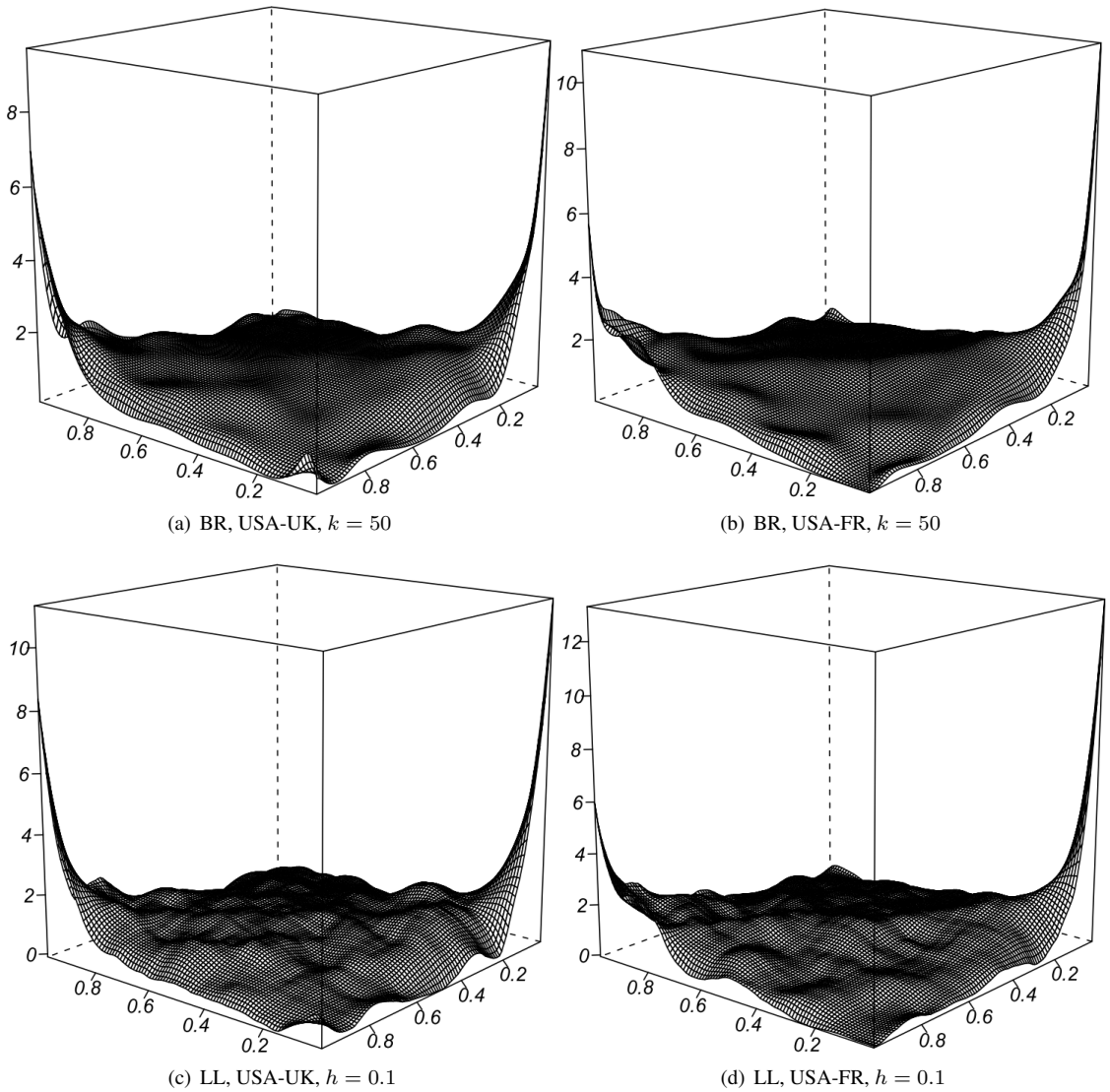


(a) BR, $k = 100$ (b) BR, $k = 50$ (c) BR, $k = 25$

(d) LL, $h = 0.035$ (e) LL, $h = 0.1$ (f) LL, $h = 0.5$

Figure 6.1 Empirical Bernstein (BR) and Local linear (LL) estimators of the copula density for the pair US-Canada, using weekly equity returns and different bandwidth parameters.

6.2 Results



(a) BR, USA-UK, $k = 50$ (b) BR, USA-FR, $k = 50$

(c) LL, USA-UK, $h = 0.1$ (d) LL, USA-FR, $h = 0.1$

Figure 6.2 empirical Bernstein (BR) and Local linear (LL) estimators of the copula density for the pairs US-UK and US-France, using weekly equity returns for.

To estimate the dependence between US, Canada, UK and France equity markets, we use the two best estimators of the copula density that we have selected on the basis of simulation results of Section 4. These estimators are the Bernstein copula estimator (BR) and the Local linear estimator with multiplicative Epanechnikov kernel (LL). To assess the sensitivity of our estimation results, we consider various values

for the smoothing parameter k. These values are $k = 25, 50, 100$ for the BR estimator and $h = 0.035$, 0.1 and 0.5 for the LL estimator. The smoothing parameter of the BR estimator plays the inverse role of the bandwidth of the LL estimator, that is a large value of BR's smoothing parameter reduces the bias but increases the variance.

The empirical results for the copula density estimation for the pairs US-Canada, US-UK and US-France are presented in Figures 6.1 and 6.2. From these, we see that using a small smoothing parameter it over-smooths the BR estimator, whereas a large smoothing parameter under-smooths the estimator. The opposite happens with LL estimator: we over-smooth the estimation of the copula density when we choose a large value of the bandwidth and we under-smooth the estimator when a small bandwidth is chosen. Intermediate values like $k = 50$ for the BR estimator and $k = 0.1$ for the LL estimator, produce more reasonable results. As expected, we find that the dependence between US and Canada, UK, France equity markets is asymmetric. That is, the international equity market returns are more dependent during the bear market than during the bull market. The latter result is confirmed by comparing the values that takes the estimator of the copula density at the extremes (0,0) and (1,1); with (0,0) corresponds to the bear market and (1,1) to the bull market. We find that at (0,0) the estimator of the copula density takes a larger value than the one it takes at (1,1). The result is quite stable when we use BR estimator, and it is more striking in the US-France case. Since a large bandwidth tends to increase the bias in the LL estimator and to decrease its variance, we find that using a large bandwidth for the LL estimator decreases the asymmetry in the estimated dependence [see Figure 6.1]. Thus, the dependence between US and Canada, UK, France equity markets look more symmetric. This is due to the high value of the bandwidth, so to the high-biased LL estimator.

To sum up, it seems that the Bernstein copula estimator is a good estimator at the extremes. The empirical results show that this estimator is able to capture the asymmetric dependence phenomena which is observed in the international equity markets.

7 Conclusion

In this paper, we examined the asymptotic properties of the Bernstein estimator (BR) for copula density functions which is not necessarily bounded at the corners. We showed that the BR estimator converges to infinity at the corners. We established its relative convergence when the copula is unbounded and its uniform strong consistency on every compact in the interior region. Furthermore, we studied the finite simple performance of the estimator via an extensive simulation study and we compared with other well known nonparametric estimators. Finally, we considered an empirical application where we re-examined the asymmetric dependence between international equity markets US, Canada, UK, and France. We compared the empirical results using the Bernstein copula density estimator and the Local linear estimator with multiplicative Epanechnikov kernel. We found that the Bernstein copula density estimator is a good estimator at the extremes. Our results showed that this estimator is able to capture the well known asymmetric dependence phenomena observed in the international equity markets.

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