ADHM Polytopes

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Abstract

We discuss the construction of ADHM data for Yang-Mills instantons with the symmetries of the regular polytopes in four dimensions. We show that the case of the pentatope can be studied using a simple modification of the approach previously developed for platonic data. For the remaining polytopes, we describe a framework in which the building blocks of the ADHM data correspond to the edges in the extended Dynkin diagram that arises via the McKay correspondence. These building blocks are then assembled into ADHM data through the identification of pairs of commuting representations of the associated binary polyhedral group. We illustrate our procedure by the construction of ADHM data associated with the pentatope, the hyperoctahedron and the 24-cell, with instanton charges 4, 7 and 23, respectively. Furthermore, we show that within our framework these are the lowest possible charges with these symmetries. Plots of topological charge densities are presented that confirm the polytope structure and the relation to JNR instanton data is clarified.

1 Introduction

There are many interesting connections between instantons, Skyrmions and monopoles (for a review see [1]), including the existence of solutions with platonic symmetries at the same specific values of the topological charges. Instantons were first related to Skyrmions by Atiyah and Manton [2, 3] who identified instanton holonomies with Skyrme fields, in a way that has subsequently found a natural interpretation in the context of holography [4, 5]. This connection is particularly useful because all instantons can be obtained using purely algebraic means via the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [6]. Combining the existence of known platonic Skyrmions with the Atiyah-Manton description motivated the search for platonic instantons. The first examples of instantons with tetrahedral and cubic symmetries were presented in [7] and a detailed understanding of the action of platonic symmetries within the ADHM formulation can be found in [8], including the explicit derivation of ADHM data for an instanton with icosahedral symmetry. The applicability of this approach has been demonstrated by the calculation of some fairly complicated ADHM data associated with symmetric polyhedra, including the truncated icosahedron [9]. Platonic ADHM data has recently found another application in generating explicit examples of platonic hyperbolic monopoles [10], by making use of an observation of Atiyah [11] that identifies circle invariant instantons as hyperbolic monopoles.

The ADHM construction yields the full $8N$-dimensional moduli space of $SU(2)$ Yang-Mills instantons of charge N. For $N \geq 3$ a subset of dimension $5N + 7$ can be obtained using the Jackiw-Nohl-Rebbi (JNR) ansatz [12], where the data consists of $N + 1$ distinct points in $\mathbb{R}^4$ together with a positive weight for each point. If all the weights are taken to be equal and the points placed at the vertices of a polyhedron in $\mathbb{R}^3 \subset \mathbb{R}^4$ then the instanton inherits the symmetries of the polyhedron and the topological charge density is localised along the edges (and particularly the vertices) of the polyhedron. This JNR approach to symmetric instantons therefore provides an upper bound on the minimal charge required to obtain an instanton associated with a given platonic solid, that is, $N \leq V - 1$, where V is the number of vertices of the platonic solid. For the platonic solids with triangular faces (the tetrahedron, octahedron and icosahedron) the minimal instanton charge is equal to $V - 1$ (that is, $N = 3, 5, 11$ respectively) and the instanton is of the JNR type. However, for instantons associated with the cube and the dodecahedron the minimal charges are 4 and 7 respectively [7, 8], which is less than $V - 1$, as these instantons are obtained using the ADHM construction and are not of the JNR type. Making use of connections between instantons, Skyrmions and monopoles, leads to an understanding of this minimal instanton number as $N = 1 + \frac{1}{2}F$, where F is the number of faces of the polyhedron, and the key ingredient is the fact that a degree N rational map between Riemann spheres has $2N - 2$ ramification points [13].

All the work mentioned above on symmetric instantons in $\mathbb{R}^4$ concerns platonic symmetries, that is, finite subgroups of $SO(3)$. This involves breaking the symmetries of $\mathbb{R}^4$ by selecting a distinguished direction, leaving an $SO(3)$ rotational symmetry in the remaining 3-dimensional space. Motivated by the interesting features found for platonic instantons and their subsequent utility, this leads on to a rather natural question concerning the existence

of symmetric instantons in $\mathbb{R}^4$, where all directions are treated on an equal footing, so that the full rotational symmetry group $SO(4)$ acts. In this case the role of the five platonic solids in three dimensions is played by the six regular polytopes in four dimensions, with the symmetry groups of the pentatope, hyperoctahedron/tesseract, 24-cell and 120-cell/600-cell, being the relevant finite subgroups of $SO(4)$. In this paper we address this issue by describing the action of these symmetry groups on the ADHM data of the instanton and providing a framework for the explicit construction of this symmetric data.

In analogy to the platonic situation described above, the JNR ansatz can be applied to construct instantons with the symmetries of the regular polytopes, by placing equal weight points at the vertices of the polytope. There is therefore again an upper bound, $N \leq V - 1$, on the minimal charge required to obtain an instanton associated with a given polytope, where V is the number of vertices of the polytope. In the platonic case, a prediction for the minimal charge can be obtained from the properties of rational maps between Riemann spheres, but there is no similar understanding for polytopes, so it is unknown whether or not one should expect the minimal charge instantons to saturate this bound and be of the JNR type. It appears that the only way to answer this question is to invoke the ADHM construction and explicitly calculate the symmetric ADHM data.

The double cover of $SO(4)$ is $SU(2) \times SU(2)$ and, as we shall see, the pentatope differs from the other polytopes in that the left and right $SU(2)$ actions are not independent, but rather there is an action of a twisted diagonal subgroup. This leads to a simplification that allows the techniques developed for finding platonic ADHM data to be modified in a simple way to apply to ADHM data with the symmetries of the pentatope. We describe this modification and apply an analysis to construct the ADHM data of a charge 4 instanton with the symmetries of the pentatope. Furthermore, we prove that there are no instantons of lower charge with this symmetry.

For the remaining polytopes, there are independent left and right actions of finite subgroups of $SU(2)$ corresponding to the binary versions of the dihedral group D_2, the tetrahedral group T and the icosahedral group Y, for the hyperoctahedron, 24-cell and 120-cell respectively. The McKay correspondence [14] associates these three groups to the extended Dynkin diagrams of the affine Lie algebras $\widetilde{d_4}$, $\widetilde{e_6}$, and $\widetilde{e_8}$. This proves to be a useful tool in our work, as we note that the building blocks of symmetric ADHM data are classified by the edges in the Dynkin diagram. Using these building blocks we describe a framework in which they can be assembled into symmetric ADHM data through the identification of pairs of commuting representations of the associated binary polyhedral group. We illustrate our procedure by the construction of ADHM data associated with the hyperoctahedron and the 24-cell, with instanton charges 7 and 23, respectively. Furthermore, we show that within our framework these are the lowest possible charges with these symmetries and we present plots of topological charge densities that confirm the polytope structure.

In the three examples of symmetric ADHM data that we have explicitly constructed, the charge is equal to that given by the JNR upper bound. We clarify this issue by demonstrating the equivalence of our ADHM data to JNR data and make some further comments regarding our current understanding of this aspect.

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2 The regular polytopes and their symmetries

The platonic solids are regular polyhedra, where all of the faces are identical regular polygons. In four dimensions, the analogue of the platonic solids are the regular polytopes, which are constructed from identical cells that are platonic solids. There are several choices of nomenclature for the six regular polytopes, including the convention in which each is named after the number of 3-dimensional cells it contains:

- The pentatope, or 5-cell, is the 4-dimensional analogue of the tetrahedron and is self-dual. It consists of 5 vertices, 10 edges and 10 triangular faces forming 5 tetrahedra.

- The tesseract, or 8-cell, is the 4-dimensional analogue of the cube and is dual to the hyperoctahedron. It consists of 16 vertices, 32 edges and 24 square faces forming 8 cubes.

- The hyperoctahedron, or 16-cell, is the 4-dimensional analogue of the octahedron and is dual to the tesseract. It consists of 8 vertices, 24 edges and 32 triangular faces forming 16 tetrahedra.

- The octaplex, or 24-cell, is self-dual and is unique to four dimensions, having no analogue in any other dimension. It consists of 24 vertices, 96 edges and 96 triangular faces forming 24 octahedra.

- The dodecaplex, or 120-cell, is the 4-dimensional analogue of the dodecahedron and is dual to the tetraplex. It consists of 600 vertices, 1200 edges and 720 pentagonal faces forming 120 dodecahedra.

- The tetraplex, or 600-cell, is the 4-dimensional analogue of the icosahedron and is dual to the dodecaplex. It consists of 120 vertices, 720 edges and 1200 triangular faces forming 600 tetrahedra.

The regular polytopes that are dual to each other share the same symmetry group, so the only symmetry groups that we need to consider are that of the 5-cell, the 16-cell, the 24-cell and the 600-cell. As we shall see, the explicit implementation of our general framework is stretched to its limit with the 24-cell, so we shall only briefly mention the application to the 120-cell and 600-cell in this paper. In the rest of this section we shall review the above symmetry groups, following [15].

If $\mathbb{R}^4$ is identified with the quaternions, then the action of any rotation, $\mathfrak{g} \in SO(4)$, on $x \in \mathbb{R}^4$, can be expressed as left and right multiplication by unit quaternions,

$$\mathfrak{g} \circ x = g_L \, x \, g_R^{-1}, \tag{2.1}$$

for some unit quaternions g_L and g_R, which may be identified with elements of $SU(2)$. The action of (g_L, g_R) is identical to the action of $(-g_L, -g_R)$, so there are two elements in $SU(2) \times SU(2)$ which correspond to the same element in $SO(4)$, reflecting the fact that $SU(2) \times SU(2)$ is the double cover of $SO(4)$. The symmetry groups of the 5-, 16-, 24- and 600-cell are all naturally expressed as subgroups of $SU(2) \times SU(2)$, with the true symmetry group being the projection to $SO(4)$.

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2.1 The 5-cell

The symmetry group of the 5-cell is realised in a different way to that of the other polytopes, so we shall consider it first.

The binary icosahedral group $\mathbb{Y}$, is generated by the unit quaternions

$$g_1 = \tfrac{1}{2}(1 + i + j + k), \quad \text{and} \quad g_2 = \tfrac{1}{2}(\tau + \tau^{-1}i + j), \tag{2.2}$$

where $\tau = \tfrac{1}{2}\left(\sqrt{5} + 1\right)$. These generators satisfy the relations

$$g_1^{\alpha} = g_2^{\beta} = (g_1 g_2)^{\gamma} = -1, \quad \text{with } \alpha = 3,\ \beta = 5,\ \gamma = 2. \tag{2.3}$$

The vertices of the 5-cell can be taken to be the five unit quaternions,

$$1, \tfrac{1}{4}\left(-1 \pm i \pm j \pm k\right), \tag{2.4}$$

where an odd number of plus signs is taken for each vertex. These vertices are permuted under the action of $\mathbb{Y}$, through a twisted diagonal embedding into $SU(2) \times SU(2)$. Explicitly,

$$x \mapsto g^{\sharp} x g^{-1}, \tag{2.5}$$

where $g \in \mathbb{Y}$, and $g^{\sharp}$ is the dual of g, obtained by making the replacement $\sqrt{5} \mapsto -\sqrt{5}$ in the generators. The double cover of the symmetry group of the 5-cell is therefore a subgroup of $SU(2)$ that is embedded in $SU(2) \times SU(2)$ via $g \mapsto (g^{\sharp}, g)$.

2.2 The 16-cell

The 8 vertices of the 16-cell may be taken to be at the intersection points of the four Cartesian axes with the unit four-sphere. As quaternions these vertices form the binary dihedral group $\mathbb{D}_2$, also known as the quaternion group,

$$\mathbb{D}_2 = \{\pm 1, \pm i, \pm j, \pm k\}. \tag{2.6}$$

$\mathbb{D}_2$ is a group under quaternionic multiplication and the left and right action of $\mathbb{D}_2$ permutes the vertices of the 16-cell

$$x \mapsto g_L\, x\, g_R^{-1}, \qquad g_L, g_R \in \mathbb{D}_2. \tag{2.7}$$

The double cover of the rotational symmetry group of the 16-cell is $\mathbb{D}_2 \times \mathbb{D}_2 \subset SU(2) \times SU(2)$, where $\mathbb{D}_2$ is generated by the two elements

$$g_1 = i, \quad \text{and} \quad g_2 = j. \tag{2.8}$$

The quaternion group generators satisfy

$$g_1^{\alpha} = g_2^{\beta} = (g_1 g_2)^{\gamma} = -1, \quad \text{with } \alpha = \beta = \gamma = 2. \tag{2.9}$$

The 16-cell is dual to the 8-cell, which shares the same symmetry group.

2.3 The 24-cell

The symmetry of the 24-cell has a similar structure to that of the 16-cell, with the binary dihedral group $\mathbb{D}_2$ replaced by the binary tetrahedral group $\mathbb{T}$. The 24 vertices may be taken to be

$$\mathbb{T} = \{\pm 1, \pm i, \pm j, \pm k, \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)\}. \tag{2.10}$$

which forms the group $\mathbb{T}$ under multiplication. The double cover of the rotational symmetry group of the 24-cell is $\mathbb{T} \times \mathbb{T}$, with rotations acting via left and right multiplication. $\mathbb{T}$ is generated by

$$g_1 = \tfrac{1}{2}(1 + i + j + k), \quad \text{and} \quad g_2 = \tfrac{1}{2}(1 + i + j - k), \tag{2.11}$$

which satisfy

$$g_1^\alpha = g_2^\beta = (g_1 g_2)^\gamma = -1, \quad \text{with } \alpha = \beta = 3, \ \gamma = 2. \tag{2.12}$$

2.4 The 600-cell

In a similar fashion to the 16-cell and 24-cell, the 120 vertices of the 600-cell form the binary icosahedral group

$$\mathbb{Y} = \{\pm 1, \pm i, \pm j, \pm k, \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k), \tfrac{1}{2}(\pm i \pm \tau j \pm \tau^{-1} k)\}. \tag{2.13}$$

The double cover of the rotational symmetry group of the 600-cell is therefore $\mathbb{Y} \times \mathbb{Y}$, with rotations acting via left and right multiplication. The generators of $\mathbb{Y}$ have already been presented in (2.2). The 600-cell is dual to the 120-cell, which shares the same symmetry group.

2.5 Group representations and the McKay correspondence

An n-dimensional representation of a group is a map, ρ, from the group to $\mathrm{GL}(n)$, such that the matrices, $\rho(g)$, preserve the group relations. For the binary polyhedral groups of interest in this paper, the representation matrices must satisfy the group relations

$$\rho(g_1)^\alpha = \rho(g_2)^\beta = \big(\rho(g_1)\rho(g_2)\big)^\gamma. \tag{2.14}$$

In our application to ADHM data we shall mainly be concerned with real representations, so it will be important to identify real irreducible representations, taking into account that representations that are reducible over $\mathbb{C}$ may be irreducible over $\mathbb{R}$.

All irreducible n-dimensional representations of the binary polyhedral group satisfy

$$\rho(g_1)^\alpha = \rho(g_2)^\beta = \big(\rho(g_1)\rho(g_2)\big)^\gamma = \varepsilon 1_n, \quad \text{where } \varepsilon = \pm 1. \tag{2.15}$$

If $\varepsilon = 1$ then the representation is also a representation of the polyhedral group, that is, of the finite subgroup of $SO(3)$, and we shall refer to this as a positive representation. If $\varepsilon = -1$ then the representation is not a representation of the polyhedral group but only of

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the binary polyhedral group, that is, of the finite subgroup of $SU(2)$, and we shall refer to this as a negative representation.

Our nomenclature for irreducible representations follows the notation that is common in chemistry, where representations are labelled by a letter which indicates their dimension. 1-dimensional representations are labelled by A, while 2-dimensional representations are labelled by E, 3-dimensional representations by F, and higher dimensions by going through the alphabet in sequence. Negative representations are indicated by a prime, for example, G' denotes a 4-dimensional negative representation. The fundamental quaternion representation, when viewed as a complex representation, is 2-dimensional and we shall denote it by E'. It is the fundamental representation obtained by the restriction of the 2-dimensional irreducible representation of $SU(2)$ to the binary polyhedral group

The McKay correspondence [14] provides a mapping between the binary polyhedral groups and the Dynkin diagrams of the affine simply-laced Lie algebras. For the binary polyhedral groups of interest in this paper, $\mathbb{D}_2, \mathbb{T}, \mathbb{Y}$, the associated affine Lie algebras are $\widetilde{d}_4$, $\widetilde{e}_6$, and $\widetilde{e}_8$, respectively. There is a one-to-one correspondence between the irreducible representations of the binary polyhedral group and the nodes in the extended Dynkin diagram. Furthermore, the nodes associated with the representations ρ_i and ρ_j are joined by an edge if and only if ρ_j is contained in the decomposition of $\rho_i \otimes E'$, where E' denotes the fundamental 2-dimensional representation, as described above. These features will prove to be useful in our computations.

3   Group actions on the ADHM data

In this section we shall discuss symmetric instantons and describe how the symmetry group acts on both the instanton and the underlying ADHM data. For an instanton to be symmetric, the gauge potential after the action of the symmetry must be gauge equivalent to the original gauge potential. As a consequence of this, the topological charge density of the instanton is invariant under the action of the symmetry group. For the moment, the symmetry can be any subgroup of $SO(4)$, such as the symmetry groups of the platonic solids, or of the regular polytopes. The gauge potential of an instanton, $a_i(x) \in su(2)$ for $i = 1, \ldots, 4$ is associated to its ADHM data, from which it can be constructed. In this section we shall see how the action of the symmetry group can be lifted to an action on the ADHM data.

If $\mathcal{G} \subset SO(4)$ is the symmetry group of an instanton then for each $\mathfrak{g} \in \mathcal{G}$ there must exist $\Omega_\mathfrak{g}(x) \in SU(2)$ such that

$$a_i(\mathfrak{g} \circ x) = \Omega_\mathfrak{g}(x) a_i(x) \Omega_\mathfrak{g}^{-1}(x) + \Omega_\mathfrak{g}(x) \partial_i(\Omega_\mathfrak{g}^{-1}(x)). \tag{3.1}$$

To find solutions with this symmetry we need to lift the action of $\mathcal{G}$ on a_i to an action on the underlying ADHM data. Recall that the ADHM data for a charge N instanton with gauge group $SU(2)$ is given by

$$\Delta(x) = \widehat{M} - Ux, \tag{3.2}$$

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where

$$\widehat{M} = \begin{pmatrix} L \\ M \end{pmatrix}, \quad \text{and} \quad U = \begin{pmatrix} 0 \\ 1_N \end{pmatrix}. \tag{3.3}$$

In this expression L is a length N quaternionic row vector and M is an $N \times N$ symmetric quaternionic matrix, which together satisfy the ADHM constraint that $\widehat{M}^\dagger \widehat{M}$ is a real non-singular matrix, where $\dagger$ denotes the quaternionic conjugate transpose. The spatial coordinate, x, is a quaternion in this construction, where $\mathbb{R}^4$ is identified with the quaternions as described earlier.

The gauge potential is obtained in terms of an $(N+1)$-component column vector, Ψ, of unit length $\Psi^\dagger \Psi = 1$, that solves the linear problem

$$\Psi^\dagger \Delta = 0. \tag{3.4}$$

The explicit formula for the gauge potential is

$$a_i = \Psi^\dagger \partial_i \Psi, \tag{3.5}$$

where a pure quaternion (that is, with no real component) is identified with an element of $su(2)$. Note that Ψ is unique only up to right multiplication by a unit quaternion, and this corresponds to a gauge transformation of a_i.

The topological charge density $\mathcal{N}$, whose integral over $\mathbb{R}^4$ gives the topological charge N, is given by

$$\mathcal{N} = -\frac{1}{32\pi^2} \varepsilon_{ijkl} \operatorname{Tr}\left(f_{ij} f_{kl}\right), \tag{3.6}$$

where f_{ij} is the gauge field $f_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j]$. This has the following useful expression in terms of the ADHM data

$$f_{ij} = -\Psi^\dagger U (\Delta^\dagger \Delta)^{-1} (e_i \bar{e}_j - e_j \bar{e}_i) U^\dagger \Psi, \tag{3.7}$$

where $e_i = \{i, j, k, 1\}$.

If the gauge potential is symmetric under the action of $\mathcal{G}$ then the ADHM data, $\Delta(x)$, must also transform in a way that yields a gauge equivalent gauge potential. The required transformation of the ADHM data takes the form

$$\Delta \to \begin{pmatrix} p & 0 \\ 0 & P \end{pmatrix} \Delta R^{-1}, \tag{3.8}$$

where p is a unit quaternion, P is an $N \times N$ quaternionic matrix such that $P^\dagger P = 1_N$, and R is an invertible $N \times N$ quaternionic matrix. For each element of the symmetry group, $\mathfrak{g} \in \mathcal{G}$, the transformed ADHM data is $\Delta(\mathfrak{g} \circ x)$. For a symmetric instanton, this must be equivalent to $\Delta(x)$, so there exists $p_\mathfrak{g}$, $P_\mathfrak{g}$ and $R_\mathfrak{g}$ such that

$$\Delta(\mathfrak{g} \circ x) = \begin{pmatrix} p_\mathfrak{g} & 0 \\ 0 & P_\mathfrak{g} \end{pmatrix} \Delta(x) R_\mathfrak{g}^{-1}. \tag{3.9}$$

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Recall that every rotation in $\mathbb{R}^4$ can be represented by left and right multiplication by unit quaternions,

$$\mathfrak{g} \circ x = g_L \, x \, g_R^{-1}. \tag{3.10}$$

Since the ADHM data is quaternionic, this is a natural way to represent the action of $\mathfrak{g}$. By comparing the terms in (3.8) that are linear in x, we see that $P_{\mathfrak{g}}$ and $R_{\mathfrak{g}}$ must factor into

$$P_{\mathfrak{g}} = Q_{\mathfrak{g}} \, g_L, \quad \text{and} \quad R_{\mathfrak{g}} = Q_{\mathfrak{g}} \, g_R, \tag{3.11}$$

for some real orthogonal matrix, $Q_{\mathfrak{g}}$. The left quaternion, g_L, may also be factored out of $p_{\mathfrak{g}}$, so that for symmetric ADHM data there must exist a quaternion $q_{\mathfrak{g}}$, and a real orthogonal matrix, $Q_{\mathfrak{g}}$, such that

$$\Delta(g_L \, x \, g_R^{-1}) = \begin{pmatrix} q_{\mathfrak{g}} & 0 \\ 0 & Q_{\mathfrak{g}} \end{pmatrix} g_L \, \Delta(x) \, g_R^{-1} \, Q_{\mathfrak{g}}^{-1}. \tag{3.12}$$

In terms of the blocks in the ADHM data, L and M, this condition is

$$Q_{\mathfrak{g}} \, g_L \, M = M \, g_R \, Q_{\mathfrak{g}} \quad \text{and} \quad q_{\mathfrak{g}} \, g_L \, L = L \, g_R \, Q_{\mathfrak{g}}. \tag{3.13}$$

To recap, if we have a symmetric instanton, then its ADHM data must be invariant under the action of each symmetry, $\mathfrak{g} \in \mathcal{G}$. This can be represented by the action of an element in the double cover of $\mathcal{G}$, which is a subgroup of $SU(2) \times SU(2)$, and acts by left and right quaternion multiplication. The transformed ADHM data must give a gauge equivalent gauge potential, and so there must exist $q_{\mathfrak{g}}$ and $Q_{\mathfrak{g}}$, as above, that relate it back to the original ADHM data. The construction of symmetric ADHM data involves using representation theory to determine possible choices for $q_{\mathfrak{g}}$ and $Q_{\mathfrak{g}}$, which then produces a simplified form for L and M, to which the ADHM constraint can be applied.

4 The ADHM 5-cell

As discussed above, the action of the 5-cell symmetry group is different from the other polytopes because the left and right actions do not act independently. This allows the existing machinery developed to study platonic instantons to be modified in a simple way to apply to this situation, as follows. We recall that the instantons considered in [8] are symmetric under the icosahedral group $Y \subset SO(3)$. The binary icosahedral group $\mathbb{Y} \subset SU(2)$, acts by quaternionic multiplication, as given by (2.1), with the diagonal embedding of $SU(2)$ into $SU(2) \times SU(2)$ given by $(g_L, g_R) = (g, g)$ for $g \in \mathbb{Y}$.

As described in Section 2.1, for the 5-cell this action is twisted, with the left quaternion replaced by the dual, $g^{\sharp}$, to give the twisted diagonal embedding $(g_L, g_R) = (g^{\sharp}, g)$ with $g \in \mathbb{Y}$. The representation theory remains largely the same, though the twisting turns out to allow a symmetric instanton with a lower charge than in the untwisted case, as we now show.
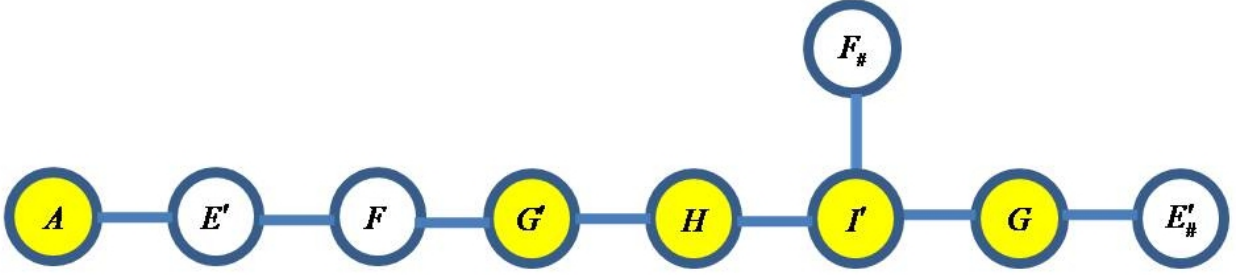
Figure 1: The Dynkin diagram of $\widetilde{e}_8$ providing a graphical illustration of the irreducible representations of $\mathbb{Y}$ via the McKay correspondence. The filled nodes denote self-dual representations.

Given the above discussion, in (3.13) we take $g_R = g \in \mathbb{Y}$ with $g_L = g^\sharp$ and $Q_{\mathfrak{g}} = Q(g)$. The matrices $Q(g)$ form a real N-dimensional representation of $\mathbb{Y}$, and with a slight abuse of notation we shall use Q to denote both the abstract representation and the explicit matrices. For the symmetry of the 5-cell it turns out to be more convenient not to factor out the left quaternion from $p_{\mathfrak{g}}$, so we write $p_{\mathfrak{g}} = q_{\mathfrak{g}} g^\sharp = p(g)$. The quaternions $p(g)$ form a representation of $\mathbb{Y}$ that may be viewed as a complex 2-dimensional representation, which we also denote by p. The condition for ADHM data L, M to have the symmetry of the 5-cell therefore becomes

$$Q(g_i)\, g_i^\sharp\, M = M\, g_i\, Q(g_i) \quad \text{and} \quad p(g_i)\, L = L\, g_i\, Q(g_i) \tag{4.1}$$

where $i = 1, 2$ and g_1, g_2 are the generators of $\mathbb{Y}$ given in (2.2).

The representation theory of $\mathbb{Y}$ is captured by the Dynkin diagram of $\widetilde{e}_8$, presented in Figure 1. There are nine irreducible representations of $\mathbb{Y}$, with one for every dimension from one up to six obtained as the restriction of the corresponding irreducible representation of $SU(2)$. Using the notation discussed in Section 2.5, we denote these six representations by A, E', F, G', H, I'. As for the three remaining representations, $E'_\sharp$ is the dual 2-dimensional representation obtained from the representation E' by making the replacement $\sqrt{5} \to -\sqrt{5}$ in the character table. Similarly, there is a 3-dimensional representation, $F_\sharp$, that is dual to F. The final representation is the 4-dimensional representation $G = E' \otimes E'_\sharp$. The representations $E, E_\sharp$ and $F, F_\sharp$ are dual pairs and we shall use the term self-dual for all the other irreducible representations, which we indicate by filled nodes in the Dynkin diagram.

As Q is a real N-dimensional representation then it must have a decomposition into the irreducible real representations $A, F, F_\sharp, G, H$, as the remaining irreducible representations are complex. Furthermore, as the 5-cell has 5 vertices then the JNR bound for the minimal charge is $N \leq 4$, so with this restriction the 5-dimensional representation H is already ruled out. We can neglect the trivial representation, so the only possibilities for Q that remain to be investigated are $F, F_\sharp$ and G.

From (4.1) we see that M is an invariant map

$$M: \ Q \otimes E' \mapsto Q \otimes E'_\sharp \tag{4.2}$$

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and L is an invariant map

$$L: \ Q \otimes E' \mapsto p. \tag{4.3}$$

By the McKay correspondence, the invariance of L requires that in the Dynkin diagram of $\widetilde{e}_8$ the node associated to the representation Q must be joined by an edge to the node associated with the 2-dimensional representation p. This eliminates the possibility that Q is equal to $F_\sharp$, since this node is not joined to the node of any 2-dimensional representation.

As $Q \otimes E'$ is equal to the nodes joined to Q, then taking the dual of this relation we see that $Q \otimes E'_\sharp$ is equal to the dual of the nodes joined to $Q_\sharp$, where $Q_\sharp = Q$ if Q is self-dual. The invariance of M therefore requires that there is a node common to the nodes joined to Q and the dual of the nodes joined to $Q_\sharp$. This rules out the possibility that $Q = F$, since the nodes joined to F are E' and G', whereas the dual of the only node joined to $F_\sharp$ is I'.

The only remaining possibility is $Q = G$, and this does yield an invariant map. In this case the nodes joined to Q are I' and $E'_\sharp$ and the dual of the nodes joined to $Q_\sharp = Q$ are I' and E'. The node I' is common to both sets and therefore there is an associated invariant map M. As G is joined to the node $E'_\sharp$ then there is an invariant map L with $p = E'_\sharp$.

With this choice, and using the canonical basis for G in which

$$G(g_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad G(g_2) = \frac{1}{4} \begin{pmatrix} -1 & \sqrt{5} & -\sqrt{5} & \sqrt{5} \\ -\sqrt{5} & -3 & -1 & 1 \\ \sqrt{5} & -1 & 1 & 3 \\ \sqrt{5} & -1 & -3 & -1 \end{pmatrix}, \tag{4.4}$$

equations (4.1) become

$$G(g_i)\, g_i^\sharp\, M = M\, g_i\, G(g_i) \quad \text{and} \quad g_i^\sharp\, L = L\, g_i\, G(g_i). \tag{4.5}$$

Solving these linear equations for L and M yields

$$L = l_0(1, i, j, k), \quad \text{and} \quad M = b_0 \begin{pmatrix} -3 & i & j & k \\ i & 1 & -\sqrt{5}k & -\sqrt{5}j \\ j & -\sqrt{5}k & 1 & -\sqrt{5}i \\ k & -\sqrt{5}j & -\sqrt{5}i & 1 \end{pmatrix}, \tag{4.6}$$

where l_0 and b_0 are arbitrary real parameters.

Imposing the ADHM constraint on this data reduces to the requirement that $l_0^2 = 4b_0^2$ and without loss of generality we can choose $b_0 = \lambda = -\frac{1}{2}l_0$ to give the ADHM data.

$$\widehat{M} = \lambda \begin{pmatrix} -2 & -2i & -2j & -2k \\ -3 & i & j & k \\ i & 1 & -\sqrt{5}k & -\sqrt{5}j \\ j & -\sqrt{5}k & 1 & -\sqrt{5}i \\ k & -\sqrt{5}j & -\sqrt{5}i & 1 \end{pmatrix}, \tag{4.7}$$

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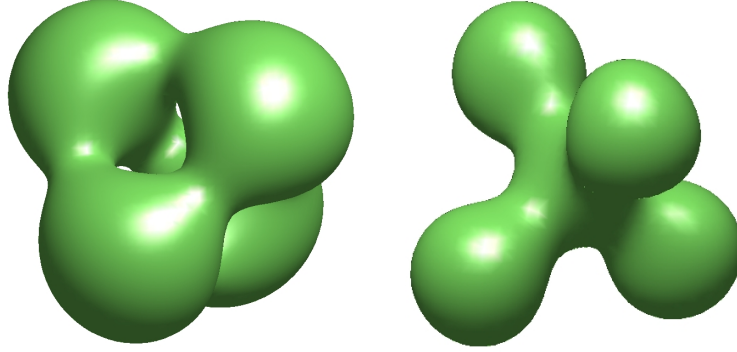
Figure 2: Surfaces of constant topological charge density for the charge 4 instanton with the symmetries of the 5-cell. The left image is the charge density in the hyperplane $x_4 = 0$, where the vertices of the 5-cell form a tetrahedron. The right image is the charge density integrated along the x_4-direction.

where λ is a real parameter that determines the scale of the instanton.

The vertices of the 5-cell that lie in the $x_4 = 0$ hyperplane form a tetrahedron. The topological charge density (3.6) of the 5-cell instanton in this hyperplane is plotted as an isosurface in the left image in Figure 2, using the formula (3.7). The tetrahedron is clearly visible in this image. The right image in Figure 2 captures more of the information about all the vertices by displaying an isosurface of the charge density integrated along the x_4-direction.

We have shown that this charge 4 ADHM data is invariant under the action of the symmetry group of the 5-cell, and moreover that there is no instanton of lower charge with this symmetry. As this charge is equal to the JNR bound then this ADHM data must be equivalent to JNR data in which the five points are placed at the vertices of a 5-cell with equal weights. We prove this equivalence explicitly in Section 8.

5 Constructing ADHM polytopes

Consider ADHM data that is invariant under the symmetries of one of the polytopes other than the 5-cell. The crucial difference is that now the left and right actions can be applied independently. First consider the right action of the generators of the binary polyhedral group, so that $g_L = 1$ and $g_R \in \{g_1, g_2\}$. Then there must exist matrices $Q_R(g_i)$ and quaternions $q_R(g_i)$, for $i = 1, 2$, which satisfy

$$Q_R(g_i)\, M = M g_i\, Q_R(g_i) \quad \text{and} \quad q_R(g_i)L = L\, g_i\, Q_R(g_i). \tag{5.1}$$

In our framework, these matrices in the right action form a real N-dimensional representation of the binary polyhedral group,

$$(Q_R(g_1))^\alpha = (Q_R(g_2))^\beta = \big(Q_R(g_1)Q_R(g_2)\big)^\gamma. \tag{5.2}$$

In keeping with the earlier notation, we shall use Q_R to denote both the abstract representation and the associated explicit matrices, given a basis for the representation that will be specified later. As we are free to choose a basis for Q_R, we can decompose it into the direct sum of irreducible representations. If we order the irreducible representations so that the positive representations form the upper blocks of Q_R and the negative representations form the lower blocks of Q_R then

$$(Q_R(g_1))^\alpha = (Q_R(g_2))^\beta = (Q_R(g_1)Q_R(g_2))^\gamma = \begin{pmatrix} 1_m & 0 \\ 0 & -1_n \end{pmatrix}, \tag{5.3}$$

where $m + n = N$. We write $Q_R = Q_R^+ \oplus Q_R^-$ as

$$Q_R(g) = \begin{pmatrix} Q_R^+(g) & 0 \\ 0 & Q_R^-(g) \end{pmatrix}, \tag{5.4}$$

where Q_R^+ is an m-dimensional positive representation and Q_R^- is an n-dimensional negative representation of the binary polyhedral group.

From (5.1) we see that M is an invariant map from the N-dimensional representation Q_R tensored with the quaternion representation E', back to the representation Q_R

$$M : Q_R \otimes E' \mapsto Q_R. \tag{5.5}$$

As E' is a negative representation then $Q_R^+ \otimes E'$ can have no component in common with Q_R^+ and $Q_R^- \otimes E'$ can have no component in common with Q_R^-. Thus M must map the component of a negative representation in Q_R^- to the component of a positive representation in Q_R^+, or vice-versa. As M is a symmetric matrix, then in our given basis it must take the off-diagonal form

$$M = \begin{pmatrix} 0 & B \\ B^\mathsf{T} & 0 \end{pmatrix}, \tag{5.6}$$

where B is an $m \times n$ quaternionic matrix.

Consider the irreducible (over $\mathbb{R}$) decomposition $Q_R = Q_{R,1}^+ \oplus \cdots \oplus Q_{R,s}^+ \oplus Q_{R,1}^- \oplus \cdots \oplus Q_{R,t}^-$ with the implied block structure

$$Q_R = \begin{pmatrix} Q_{R,1}^+ & & & & & & \\ & \ddots & & & & & \\ & & Q_{R,s}^+ & & & & \\ & & & Q_{R,1}^- & & & \\ & & & & \ddots & & \\ & & & & & Q_{R,t}^- \end{pmatrix}. \tag{5.7}$$

The ADHM data can also be decomposed into this block form, in particular

$$
M = \begin{pmatrix} & & & B_{11} & \cdots & B_{1t} \\ & 0 & & \vdots & \ddots & \vdots \\ & & & B_{s1} & \cdots & B_{st} \\ B_{11}^{\mathsf{T}} & \cdots & B_{s1}^{\mathsf{T}} & & & \\ \vdots & \ddots & \vdots & & 0 & \\ B_{1t}^{\mathsf{T}} & \cdots & B_{st}^{\mathsf{T}} & & & \end{pmatrix}, \tag{5.8}
$$

where the block B_{uv} satisfies

$$
Q_{R,u}^+(g_i)\, B_{uv} = B_{uv}\, g_i\, Q_{R,v}^-(g_i), \tag{5.9}
$$

with $u = 1, \ldots s$, and $v = 1, \ldots t$, which are not summed over in the above expressions. From this equation we see that B_{uv} is an invariant map

$$
B_{uv} : Q_{R,v}^- \otimes E' \mapsto Q_{R,u}^+, \tag{5.10}
$$

and therefore exists if and only if the representation $Q_{R,u}^+$ is contained in the irreducible decomposition of $Q_{R,v}^- \otimes E'$. However, we observe that this is precisely the condition that the nodes associated with the representations $Q_{R,u}^+$ and $Q_{R,v}^-$ are joined by an edge in the Dynkin diagram obtained using the McKay correspondence. In this way the building blocks of symmetric ADHM data are labelled by the edges in the Dynkin diagram. To complete this description we must add an extra label to the nodes that correspond to complex representations, since Q_R is a real representation. Explicitly, we introduce the notation $\rho_j[\rho_k]$ to denote that ρ_j is a complex irreducible representation and ρ_k is a real representation that is irreducible over $\mathbb{R}$ but is reducible over $\mathbb{C}$ and contains ρ_j in its decomposition into irreducible components. Each edge in the Dynkin diagram now corresponds to an invariant block between real representations, where we associate the real representation ρ_k with the node $\rho_j[\rho_k]$ and note that two edges may now be associated with the same invariant block, since two different complex representations may appear in the decomposition of the same real representation.

For each edge in the Dynkin diagram the associated invariant map given by the matrix B_{uv}, of size $\dim(Q_{R,u}^+) \times \dim(Q_{R,v}^-)$, can be obtained explicitly by solving the linear equation (5.9). This matrix will contain free parameters, for example it is clear from (5.9) that there is the freedom to multiply B_{uv} on the left by an arbitrary quaternion. In what follows it will be convenient to treat multiple copies of the same irreducible representation as a single representation, with the invariant map constructed from the single invariant block by the obvious tensor product.

If we now consider the left action, $g_R = 1$ and $g_L \in \{g_1, g_2\}$, then there must exist matrices $Q_L(g_i)$ and quaternions $q_L(g_i)$ which satisfy

$$
Q_L(g_i)\, g_i\, M = M\, Q_L(g_i) \quad \text{and} \quad q_L(g_i)\, g_i\, L = L\, Q_L(g_i). \tag{5.11}
$$

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The representation of the left action can be put in the same block form as the right action,

$$Q_L(g) = \begin{pmatrix} Q_L^+(g) & 0 \\ 0 & Q_L^-(g) \end{pmatrix}, \tag{5.12}$$

where Q_L^+ is an m-dimensional positive representation and Q_L^- is a n-dimensional negative representation. Furthermore, the representation of the left action shares the same block structure as the right action,

$$Q_L = \begin{pmatrix} Q_{L,1}^+ & & & & & \\ & \ddots & & & & \\ & & Q_{L,s}^+ & & & \\ & & & Q_{L,1}^- & & \\ & & & & \ddots & \\ & & & & & Q_{L,t}^- \end{pmatrix}, \tag{5.13}$$

where each pair of blocks in the left and right actions, $(Q_{L,u}^+, Q_{R,u}^+)$, or $(Q_{L,v}^-, Q_{R,v}^-)$, corresponds to a pair of left and right representations of the same dimension, but not necessarily the same representations. In particular, unlike the right action, these blocks in the left action are generally not in a basis where they are manifestly the direct sum of irreducible representations, since we no longer have the freedom to arbitrarily choose the basis, having already fixed Q_R in canonical form as a direct sum of irreducible representations. This is an important point and in particular it means that when we refer to a representation $Q_{L,u}^+$ or $Q_{L,v}^-$ we need to refer to a specific basis, and two representations which would usually be considered equivalent will need to be distinguished because the change of basis required to map one representation to the other is not compatible with preserving the canonical basis for the right representation.

So far we have considered the left and right actions independently, but the full set of rotations in $SO(4)$ are generated by acting with both a left and right action together. In terms of the action on the spatial coordinate $x \mapsto g_i x g_j^{-1}$, the order of the action is irrelevant, so either the left or right action can be applied first. This implies that the matrices in the left and right representations must commute,

$$Q_R(g_i) \, Q_L(g_j) = Q_L(g_j) \, Q_R(g_i). \tag{5.14}$$

At first sight it may appear that anti-commuting is also a possibility, but this can be ruled out by considering the left or right action of $(g_1)^{2\alpha} = 1$, under which $(Q_{L,R}(g_i))^{2\alpha} = 1_N$ and so must commute.

The procedure for constructing symmetric ADHM data will therefore involve calculating the invariant building blocks associated with the edges in the Dynkin diagram and assembling these into invariant data by identifying pairs of commuting representations. For an appropriate range of charges, such invariant data exists and has only a few free parameters,

which can then be constrained by the ADHM condition to determine whether or not an associated symmetric instanton exists.

The final issue we need to address in this section is that the upper row vector in the ADHM data, L, must also be invariant under the above left and right actions, as specified by the second equations in both (5.1) and (5.11). The first of these equations implies that L is an invariant map

$$L : Q_R \otimes E' \mapsto q_R. \tag{5.15}$$

We can write L in block form with the same block structure as Q_R and Q_L,

$$L = (L^+ \ L^-). \tag{5.16}$$

With this block structure, $L^\pm$ are invariant maps

$$L^\pm : Q_R^\pm \otimes E' \mapsto q_R. \tag{5.17}$$

As q_R is a 1-dimensional quaternionic representation then it is either positive, in which case L^+ must vanish, or negative, in which case L^- must vanish.

A similar consideration of the left action shows that q_L must be a representation of the same sign as q_R to be able to leave the remaining non-zero block in L invariant. By a similar argument to the one given above for the real representations Q_L and Q_R, the quaternion representations q_L and q_R must commute. However, there are no commuting negative representations for q_R and q_L, so they must both be positive representations.

In summary, L has the block form (5.16) with $L^+ = 0$ and L^- an invariant map

$$L^- : Q_R^- \otimes E' \mapsto q_R, \tag{5.18}$$

where q_R is a positive representation, and similarly for the left action. Putting everything together, the complete ADHM data takes the block form

$$\widehat{M} = \begin{pmatrix} & 0 & & L_1^- & \cdots & L_t^- \\ & & & B_{11} & \cdots & B_{1t} \\ & 0 & & \vdots & \ddots & \vdots \\ & & & B_{s1} & \cdots & B_{st} \\ B_{11}^\mathsf{T} & \cdots & B_{s1}^\mathsf{T} & & & \\ \vdots & \ddots & \vdots & & 0 & \\ B_{1t}^\mathsf{T} & \cdots & B_{st}^\mathsf{T} & & & \end{pmatrix}. \tag{5.19}$$

In this section we have introduced a framework for the construction of ADHM data with the symmetries of the regular polytopes. In the following two sections we shall apply this framework to two examples, namely the 16-cell, and then to the more complicated 24-cell.
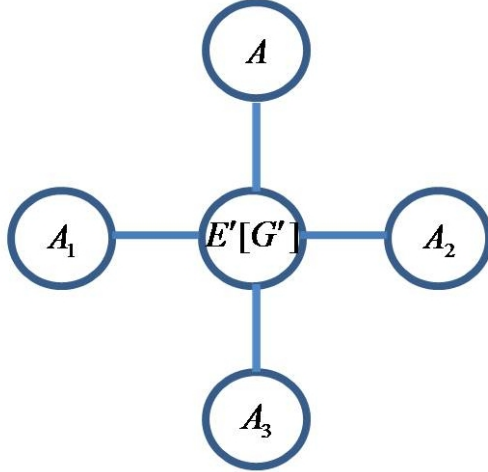
Figure 3: The Dynkin diagram of $\widetilde{d}_4$ providing a graphical illustration of the irreducible representations of $\mathbb{D}_2$ via the McKay correspondence.

6   The ADHM 16-cell

As discussed earlier, the 16-cell has 8 vertices and hence the JNR bound implies that the minimal charge for ADHM data with the symmetries of the 16-cell can be no greater than 7. In this section we enumerate and investigate all possibilities for symmetric ADHM data with charge $N \leq 7$, by a systematic consideration of all the possible representations for Q_L, Q_R, q_R and q_L, and the imposition of the ADHM constraint on the associated invariant data.

We have seen in Section 2.2 that the symmetry group of the 16-cell is generated by the left and right actions of the binary polyhedral group $\mathbb{D}_2$, generated by $g_1 = i$ and $g_2 = j$ which satisfy

$$g_1^2 = g_2^2 = (g_1 g_2)^2 = -1. \tag{6.1}$$

The representation theory of $\mathbb{D}_2$ is captured by the Dynkin diagram of $\widetilde{d}_4$, presented in Figure 3. This shows that there are four real 1-dimensional positive representations, A, A_1, A_2, A_3 and a real 4-dimensional negative representation G', that is irreducible over $\mathbb{R}$ but reducible over $\mathbb{C}$ with the decomposition $G' = E' \oplus E'$; recall our notation $E'[G']$ to signify this.

In a canonical basis we have

$$A(g_1) = 1, \qquad A(g_2) = 1, \tag{6.2}$$
$$A_1(g_1) = -1, \quad A_1(g_2) = -1, \tag{6.3}$$
$$A_2(g_1) = -1, \quad A_2(g_2) = 1, \tag{6.4}$$
$$A_3(g_1) = 1, \qquad A_3(g_2) = -1, \tag{6.5}$$

and

$$G'(g_1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad G'(g_2) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{6.6}$$

There are four edges in the Dynkin diagram, corresponding to the four invariant maps between $G' \otimes E'$ and any of the 1-dimensional representations. These four maps are 1×4 matrices that are easily obtained by solving (5.9) and are the building blocks of the ADHM data. Explicitly, the most general invariant maps between $G' \otimes E'$ and A, A_1, A_2, A_3 are given by

$$(1, -i, -j, -k), \quad (1, i, j, -k), \quad (1, i, -j, k), \quad (1, -i, j, k), \tag{6.7}$$

respectively, where there is the freedom to multiply on the left by an arbitrary quaternion.

In terms of the notation introduced in our earlier framework, the most general possibility is that $Q_R^+ = a_0 A \oplus a_1 A_1 \oplus a_2 A_2 \oplus a_3 A_3$, where a_0, a_1, a_2, a_3 are non-negative integers (at least one of which is non-zero) and

$$a_0 A = \underbrace{A \oplus \ldots \oplus A}_{a_0 \text{ times}}. \tag{6.8}$$

Furthermore, $Q_R^- = c_0 G'$, for some positive integer c_0.

As we are only concerned with $N = \dim(Q_R^+) + \dim(Q_R^-) = a_0 + a_1 + a_2 + a_3 + 4c_0 \leq 7$ then immediately we see that $c_0 = 1$, so that $Q_R^- = G'$ and $a_0 + a_1 + a_2 + a_3 \leq 3$.

Turning to the left action, Q_L^+ must also be the sum of some combination of A, A_1, A_2 and A_3, and Q_L^- must be a copy of G', but both blocks will generally be in a different basis to those of Q_R. To find the left action, we must find representations in a basis which commute with the right action. The matrices that commute with Q_R^+ are of the form $R_0 \oplus R_1 \oplus R_2 \oplus R_3$, where the R_i are arbitrary square matrices of dimension a_i. We can perform an arbitrary basis transformation on each of these blocks without affecting the right action, and so can also write the left action in its irreducible form as the direct sum of some copies of A, A_1, A_2 and A_3, although not necessarily grouped together as in the right action.

The matrices that commute with $G'(g_1)$ and $G'(g_2)$ are of the form:

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}. \tag{6.9}$$

For a matrix in this form to square to -1_4, it must satisfy $a = 0$, $b^2 + c^2 + d^2 = 1$. If we parameterise the two generators in the left action as

$$Q_L^-(g_1) = \begin{pmatrix} 0 & b & c & d \\ -b & 0 & -d & c \\ -c & d & 0 & -b \\ -d & -c & b & 0 \end{pmatrix}, \quad Q_L^-(g_2) = \begin{pmatrix} 0 & e & f & g \\ -e & 0 & -g & f \\ -f & g & 0 & -e \\ -g & -f & e & 0 \end{pmatrix}, \tag{6.10}$$

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then the conditions for them to satisfy the group relations are

$$b^2 + c^2 + d^2 = e^2 + f^2 + g^2 = 1,$$
$$be + cf + dg = 0. \tag{6.11}$$

This is the condition that (b, c, d) and (e, f, g) are orthogonal unit vectors in $\mathbb{R}^3$. These can be rotated to be $(1, 0, 0)$ and $(0, 1, 0)$ by transformation matrices of the form (6.9), which commute with the right action and so leave it invariant. The representation Q_L^- can therefore always be put in a basis where it has the following form

$$Q_L^-(g_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Q_L^-(g_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{6.12}$$

Note that this is the representation $Q_R^- = G'$ transformed by the matrix,

$$P = \operatorname{diag}(1, -1, -1, -1). \tag{6.13}$$

Finally, q_R and q_L must each be one of the representations, $2A$, $2A_1$, $2A_2$, or $2A_3$, where there are always two copies of the same 1-dimensional representation in order to be a 1-dimensional quaternionic representation.

As we have $5 \leq N \leq 7$, this presents us with a finite, and reasonably small, number of possibilities. All have been investigated to find the most general left and right invariant maps, which are then tested to see if any also satisfy the ADHM constraint. The result of this analysis is that there are no solutions with $N = 5$ or $N = 6$, hence the JNR bound is attained.

ADHM data is obtained for $N = 7$ by taking $Q_R^+ = Q_L^+ = A \oplus A_2 \oplus A_3$ and $q_R = q_L = 2A_1$. The symmetric data has the block structure

$$\widehat{M} = \begin{pmatrix} 0 & 0 & 0 & L_1^- \\ 0 & 0 & 0 & B_{11} \\ 0 & 0 & 0 & B_{21} \\ 0 & 0 & 0 & B_{31} \\ B_{11}^{\mathsf{T}} & B_{21}^{\mathsf{T}} & B_{31}^{\mathsf{T}} & 0 \end{pmatrix}, \tag{6.14}$$

where

$$B_{11} = b_1(1, -i, -j, -k), \tag{6.15}$$
$$B_{21} = b_2(1, i, -j, k), \tag{6.16}$$
$$B_{31} = b_3(1, -i, j, k), \tag{6.17}$$
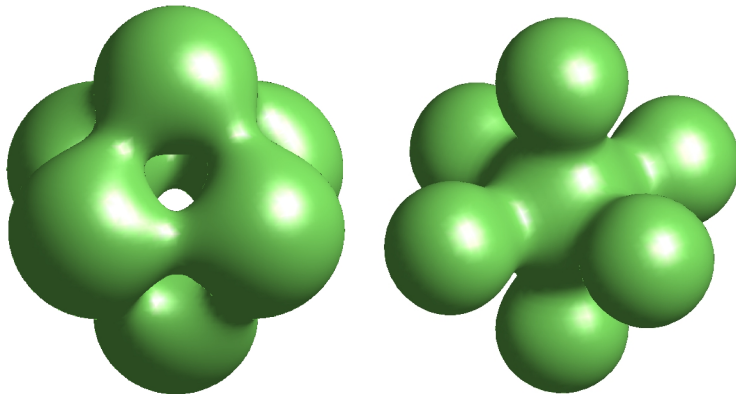$$L_1^- = l_0(1, i, j, -k), \tag{6.18}$$

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Figure 4: Surfaces of constant topological charge density for the charge 7 instanton with the symmetries of the 16-cell. The left image is the charge density in the hyperplane $x_4 = 0$, where the vertices of the 16-cell form an octahedron. The right image is the charge density integrated along the x_4-direction.

with arbitrary real parameters b_1, b_2, b_3, l_0. The right invariant building blocks (6.7) are manifest in the invariant maps B_{11}, B_{21}, B_{31} from $G' \otimes E'$ to A, A_2, A_3, with left invariance reducing the arbitrary quaternions to arbitrary real parameters. As $q_R = 2A_1$ then L_1^- is a right invariant map from from $G' \otimes E'$ to $2A_1$ and hence is formed from the remaining invariant building block in (6.7), where again left invariance reduces the arbitrary quaternion to an arbitrary real parameter.

The ADHM constraint applied to the data (6.14) reduces to the equations

$$l_0^2 = b_1^2 = b_2^2 = b_3^3. \tag{6.19}$$

Without loss of generality, we can take $l_0 = b_1 = b_2 = b_3 = \lambda$, with alternative choices of sign giving equivalent ADHM data. The remaining real parameter λ is the arbitrary instanton scale. Finally, we have the ADHM data of a charge 7 instanton with the symmetries of the 16-cell,

$$\widehat{M} = \lambda \begin{pmatrix} 0 & 0 & 0 & 1 & i & j & -k \\ 0 & 0 & 0 & 1 & -i & -j & -k \\ 0 & 0 & 0 & 1 & i & -j & k \\ 0 & 0 & 0 & 1 & -i & j & k \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -i & i & -i & 0 & 0 & 0 & 0 \\ -j & -j & j & 0 & 0 & 0 & 0 \\ -k & k & k & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.20}$$

The vertices of the 16-cell that lie in the $x_4 = 0$ hyperplane form an octahedron. The topological charge density (3.6) of the 16-cell instanton in this hyperplane is plotted as an isosurface in the left image in Figure 4, using the formula (3.7). The octahedron is clearly visible in this image. The right image in Figure 4 captures more of the information about

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all the vertices by displaying an isosurface of the charge density integrated along the x_4-direction.

In the ADHM data presented above, the representation A_1 was distinguished from the other three 1-dimensional representations. However, any one of the 1-dimensional representations can be chosen as the distinguished representation and this yields equivalent ADHM data. In detail, all choices $Q_R^+ = Q_L^+ = A_i \oplus A_j \oplus A_k$ for $i, j, k \in \{0, 1, 2, 3\}$, with $i \neq j \neq k$ are acceptable, where we have used the notation $A_0 \equiv A$. With this choice then q_R and q_L may both be taken to be equal to two copies of the 1-dimensional representation that is missing from Q_R^+.

As we have found a unique (up to scale) charge 7 instanton with the symmetries of the 16-cell then it must be equivalent to the JNR instanton mentioned earlier. This is shown explicitly in Section 8.

7   The ADHM 24-cell

Our treatment of the 24-cell is similar to the 16-cell in the previous section, upon replacing the binary dihedral group $\mathbb{D}_2$ by the binary tetrahedral group $\mathbb{T}$. The main difference is that all but one of the real irreducible representations of the binary tetrahedral group have dimension greater than one, which makes finding appropriate commuting representations more complicated. Furthermore, the JNR bound in this case is $N \leq 23$, so we may need to search up to charge 23.

We will first present the real irreducible representations of $\mathbb{T}$ in some canonical basis, which can be taken as the basis for the representations in the right action, and this generates the ADHM building blocks associated with each edge in the Dynkin diagram. We then find the form that the representations in the left action must take in order to commute with the representations in the right action. Finally, we enumerate all the possible combinations of these representations up to charge 23 and test these to find ADHM data with the symmetries of the 24-cell.

7.1   Representations of the right action of $\mathbb{T}$

The representation theory of $\mathbb{T}$ is captured by the Dynkin diagram of $\widetilde{e}_6$, presented in Figure 5. There are real irreducible, over $\mathbb{R}$, positive representations A, E, F, but over $\mathbb{C}$ the 2-dimensional representation is reducible as $E = A_1 \oplus A_2$. In terms of the generators (2.11)
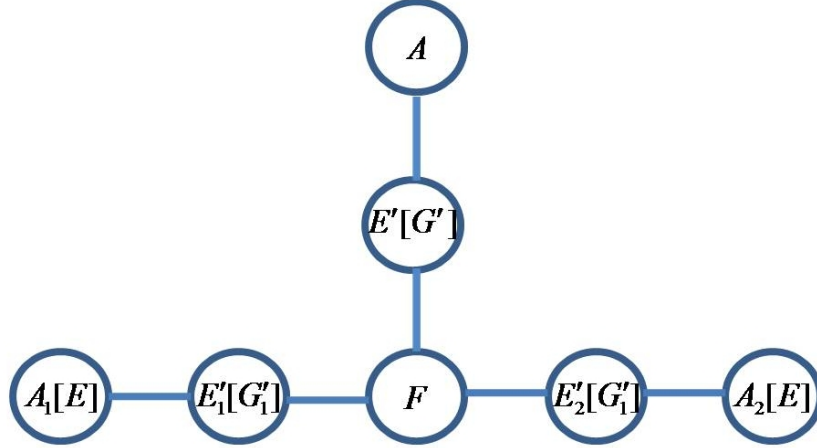
Figure 5: The Dynkin diagram of $\widetilde{e}_6$ providing a graphical illustration of the irreducible representations of $\mathbb{T}$ via the McKay correspondence.

a canonical basis is

$$A(g_1) = 1, \quad A(g_2) = 1, \tag{7.1}$$

$$E(g_1) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad E(g_2) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \tag{7.2}$$

$$F(g_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F(g_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}. \tag{7.3}$$

There are three complex negative representations, E', E'_1, E'_2 which combine to form two real irreducible, over $\mathbb{R}$, negative representations $G' = E' \oplus E'$ and $G'_1 = E'_1 \oplus E'_2$. A canonical basis for the generators is given by

$$G'(g_1) = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \quad G'(g_2) = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}, \tag{7.4}$$

$$G'_1(g_1) = \frac{1}{4}\begin{pmatrix} -1+\sqrt{3} & -1-\sqrt{3} & 1+\sqrt{3} & -1+\sqrt{3} \\ 1+\sqrt{3} & -1+\sqrt{3} & 1-\sqrt{3} & 1+\sqrt{3} \\ -1+\sqrt{3} & -1-\sqrt{3} & -1-\sqrt{3} & 1-\sqrt{3} \\ 1+\sqrt{3} & -1+\sqrt{3} & -1+\sqrt{3} & -1-\sqrt{3} \end{pmatrix},$$

$$G'_1(g_2) = \frac{1}{4}\begin{pmatrix} -1+\sqrt{3} & 1+\sqrt{3} & 1-\sqrt{3} & -1-\sqrt{3} \\ -1-\sqrt{3} & -1+\sqrt{3} & 1+\sqrt{3} & 1-\sqrt{3} \\ -1-\sqrt{3} & -1+\sqrt{3} & -1-\sqrt{3} & -1+\sqrt{3} \\ 1-\sqrt{3} & -1-\sqrt{3} & 1-\sqrt{3} & -1-\sqrt{3} \end{pmatrix}. \tag{7.5}$$

From the edges in the Dynkin diagram we see that there are four building blocks associated with the invariant mappings from $G' \otimes E'$ to A, from $G' \otimes E'$ to F, from $G'_1 \otimes E'$ to F, and from $G'_1 \otimes E'$ to E, where the last two blocks are both associated with two edges in the Dynkin diagram since G'_1 contains both E'_1 and E'_2 in its decomposition and E contains both A_1 and A_2 in its decomposition.

7.2   Representations of the left action of $\mathbb{T}$

To find appropriate representations of the left action, we need to find a basis in which the representations commute with those given for the right action. Note that the right representation may include multiple copies of the same irreducible representation, such as $E \oplus E$. The corresponding left block may then be a 4×4 block rather than two separate 2×2 blocks, because the off-diagonal blocks may be non-zero here.

We will systematically go through the blocks in the representation of the right action and find the possible commuting representations of the left action. We will only consider the most granular blocks, so for example, if the right representation contains the 4×4 block $E \oplus E$, we would not consider the representation where the left block is also $E \oplus E$, since these both split into two 2×2 blocks. However, we will consider the representation where the left block is $E \otimes E$ since this is not composed of smaller blocks, and the whole 4×4 block must be considered together.

We recall that when we refer to a representation we mean the explicit matrices in Section 7.1. Likewise, when we use the tensor product and direct sum, we are referring to the concrete Kronecker product and direct sum of the matrices respectively.

Let us start with the trivial cases. If a block in the right action is simply the identity matrix then any positive representation of the appropriate size may be used as the block in left action. We can take these to be in the canonical basis since we can perform any basis transformation without affecting the form of the right block. Likewise, for any block in the right action which is a positive non-trivial representation, the block in the left action may be taken to be the identity matrix.

Consider the right representation E. The only matrices which commute with both $E(g_1)$ and $E(g_2)$ are of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{7.6}$$

These must be rotation matrices and the only non-trivial rotation matrices which form a representation of $\mathbb{T}$ are $E(g_1)$ and $E(g_2)$. These two matrices are similar and the transformation between them is $P = \mathrm{diag}(1, -1)$, which does not commute with the right representation. So there are two possibilities for the left block: the original representation, E, and a twisted representation, E^t, where

$$E^t(g_1) = E(g_2), \quad E^t(g_2) = E(g_1). \tag{7.7}$$

When the right representation is $2E \equiv E \oplus E \equiv 1_2 \otimes E$, the commuting matrices are of

the form

$$\begin{pmatrix} a & -b & c & -d \\ b & a & d & c \\ e & -f & g & -h \\ f & e & h & g \end{pmatrix}. \tag{7.8}$$

Here $E \otimes E$, $E \otimes E^t$ and $E \otimes 2A$ are possible representations for the left block. The twisted product, $E \otimes E^t$, is related to $E \otimes E$ via the transformation matrix $P = \mathrm{diag}(1, -1, 1, -1)$. However, this does not leave the right action invariant and so $E \otimes E^t$ must be considered separately. Applying a twist to the first E in the product can be undone since the transformation will apply only to the identity part of the right representation, $2E = 1_2 \otimes E$, and therefore leave it invariant. There is no need to consider $E \oplus E$ as a left representation, because both representations are then composed of smaller blocks that we have already considered.

It is not clear that these are all possible left representations for the right block $2E$. There may be other 4×4 matrices which are of the form in equation (7.8) and form a representation but that are not related to $E \otimes E$ or $E \otimes E^t$ by a transformation which leaves the right action, $2E$, invariant. The condition for matrices of this form to be a representation is nonlinear and we have not been able to systemically rule out other possibilities. From now we will simply list possibilities for the left representations without claiming that these are exhaustive.

When the right representation is $3E$, the left representation must be 6-dimensional and have an analogous form to (7.8) but generalised to a 6×6 matrix. Three such representations are $F \otimes E$, $F \otimes E^t$ and $F \otimes 2A$. We are free to choose the basis for F since a transformation on the first term in the tensor product leaves the right block, $1_3 \otimes E$, invariant. We will therefore take F to be in the canonical basis above.

When the right representation is $4E$, let us start by considering the left representations in the form $\widetilde{G} \otimes E$, where $\widetilde{G}$ is some 4-dimensional representation. These will commute with the right representation for any choice of $\widetilde{G}$. We are free to choose a basis for $\widetilde{G}$ without affecting the right representation, and so can always take it to be composed of irreducible blocks. There is no irreducible 4-dimensional positive representation, so in the appropriate basis $\widetilde{G} \otimes E$ must be the direct sum of smaller blocks considered previously. Similarly, there is no need to consider left representations of the form $\widetilde{G} \otimes E^t$ or $\widetilde{G} \otimes 2A$.

We can also consider left representations in the form $\widetilde{E} \otimes (E \oplus E)$, where $\widetilde{E}$ is some 2-dimensional representation where we are free to choose the basis. The only choice for $\widetilde{E}$ that does not decompose into smaller blocks is $\widetilde{E} = E$, so that the left representation is $E \otimes (E \oplus E)$. By a similar argument, other possible left representations are of the form $E \otimes (\widetilde{E}_1 \oplus \widetilde{E}_2)$, where $\widetilde{E}_1, \widetilde{E}_2 = E, E^t$, or $2A$. Note that the ordering of the terms in the direct sum does not matter since these can be permuted without affecting the right representation.

There is no need to consider the left block in the form $\widetilde{E}_1 \otimes \widetilde{E}_2 \otimes \widetilde{E}_3 \otimes \widetilde{E}_4$ since this decomposes into blocks considered previously.

When the right representation is $5E$, there are no obvious possible 10-dimensional representations for the left representation which do not decompose into blocks we have already considered.

Following the same pattern, when the right representation is $6E$, the following left representations are possible and inequivalent: $F \otimes (\widetilde{E}_1 \oplus \widetilde{E}_2)$ and $E \otimes (\widetilde{E}_1 \oplus \widetilde{E}_2 \oplus \widetilde{E}_3)$, where $E_1, E_2, E_3 = E, E^t$, or $2A$, and permutations of the direct sum are again equivalent. As before, if the left action is in the form $\widetilde{I} \otimes \widetilde{E}_1$ for some 6-dimensional representation $\widetilde{I}$, then it can be written as the sum of blocks considered previously, after the appropriate basis transformation.

This pattern also extends to the right representation being $7E$, $8E$ or $9E$, and the results are shown in Table 1.

There are additional possibilities for the left representation when the right representation is $8E$. The matrices in G' and G'_1 are all in the form (7.8) and so commute with $2E$. We can also consider the twisted representation, G'^t, obtained by applying the transformation matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix} \tag{7.9}$$

that swaps $G'(g_1)$ and $G'(g_2)$. As this transformation commutes with $2E$ in the canonical basis, there is no need to consider G'^t separately. Similarly, there is a twisted representation $G'_1{}^t$, however, the transformation between G'_1 and $G'_1{}^t$ does not commute with $2E$, so these must be considered separately. The representations $G' \otimes G'$, $G' \otimes G'_1$, $G' \otimes G'_1{}^t$, $G'_1 \otimes G'_1$, $G'_1 \otimes G'_1{}^t$ and $G'_1 \otimes G'$ are therefore also possible representations for the left representation when the right representation is $8E$. Note that these representations are positive as they are the tensor product of two negative representations.

There is no need to consider $10E$ or higher, since Q_R^- must be at least 4-dimensional, and the highest charge that we need to consider is charge 23.

The F representation commutes only with the identity. If the right representation is F there is therefore no non-trivial left representation.

When the right representation is $2F$, the only possible left representation is $E \otimes 1_3$.

When the right representation is $3F$, the only possible left representation is $F \otimes 1_3$.

For any higher dimensional right representation, nF, with $n > 3$, the left representation must be in the form $\rho_n \otimes 1_3$, where ρ_n is an n-dimensional representation. However, we are free to choose the basis of ρ_n and so can decompose it into irreducible representations, where each block has been considered previously.

The G'_1 representation commutes with matrices of the form

$$\begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}. \tag{7.10}$$

Neither G' nor G'_1 can be put in this form since they are irreducible. For higher multiples of G'_1 in the right representation, the left representation must always occur with blocks of this

form. For example, when the right representation is $2G'_1$, the left representation must be in the form

$$
\begin{pmatrix}
a & -b & 0 & 0 & c & -d & 0 & 0 \\
b & a & 0 & 0 & d & c & 0 & 0 \\
0 & 0 & a & -b & 0 & 0 & c & -d \\
0 & 0 & b & a & 0 & 0 & d & c \\
e & -f & 0 & 0 & g & -h & 0 & 0 \\
f & e & 0 & 0 & h & g & 0 & 0 \\
0 & 0 & e & -f & 0 & 0 & g & -h \\
0 & 0 & f & e & 0 & 0 & h & g
\end{pmatrix}
\equiv
\begin{pmatrix}
a & -b & c & -d \\
b & a & d & c \\
e & -f & g & -h \\
f & e & h & g
\end{pmatrix}
\widetilde{\otimes}\, 1_2,
\tag{7.11}
$$

where we have defined $\widetilde{\otimes}$ as the Kronecker product acting on each 2×2 block. We therefore see that $G'\widetilde{\otimes}\,1_2$, $G'_1\widetilde{\otimes}\,1_2$ and $G'^{\,t}_1\widetilde{\otimes}\,1_2$ are possible left representations. The left representation $G'^{t}\widetilde{\otimes}\,1_2$ is equivalent to $G'\widetilde{\otimes}\,1_2$ since the transformation matrix between these is $P\widetilde{\otimes}\,1_2$, with P as in (7.9) and so commutes with the right action.

There are no additional possibilities when the right representation is $3G'_1$ or $5G'_1$. There is no need to consider $6G'_1$ or higher as we would exceed charge 23.

When the right representation is $4G'_1$, both $G'\otimes\widetilde{G}$ and $G'_1\otimes\widetilde{G}$ are suitable left representations, where $\widetilde{G}=1_4$, $E\oplus E$ or $E^t\oplus E^t$.

The following left representations are also possible when the right block is $4G'_1$: $(E\otimes G')\widetilde{\otimes}\,1_2$, $(E\otimes G'_1)\widetilde{\otimes}\,1_2$, $(E\otimes G'^{\,t}_1)\widetilde{\otimes}\,1_2$, $(G'\otimes E)\widetilde{\otimes}\,1_2$, $(G'\otimes E^t)\widetilde{\otimes}\,1_2$, $(G'_1\otimes E)\widetilde{\otimes}\,1_2$, and $(G'_1\otimes E^t)\widetilde{\otimes}\,1_2$. The left representations in this form, where the first term is twisted, are related to the untwisted representations by transformations which do not affect the right action. Once again, the transformation between $(E\otimes G')\widetilde{\otimes}\,1_2$ and $(E\otimes G'^{t})\widetilde{\otimes}\,1_2$ commutes with the right representation and so these do not need to be considered separately.

The final right representation to consider is G', which commutes with matrices of the form

$$
\begin{pmatrix}
a & -b & c & d \\
b & a & d & -c \\
-c & -d & a & -b \\
-d & c & b & a
\end{pmatrix}.
\tag{7.12}
$$

The left representation can be $PG'P^{\mathsf{T}}$ or $PG'^{t}P^{\mathsf{T}}$, where $P=\mathrm{diag}(-1,1,1,1)$. We can see from the discussion above that G'_1 can never commute with G' in any basis.

When the right representation is $2G'$, the left representation may be $E\otimes(PG'P^{\mathsf{T}})$ or $E\otimes(PG'^{t}P^{\mathsf{T}})$.

Similarly, when the right representation is $3G'$, the left representation may be $F\otimes PG'P^{\mathsf{T}}$ or $F\otimes PG'^{t}P^{\mathsf{T}}$.

When the right representation is $4G'$, both $G'\otimes\widetilde{G}$ and $G'_1\otimes\widetilde{G}$ are suitable left representations, where $\widetilde{G}$ is a positive representation in the form (7.12), $\widetilde{G}=4A$, $E\oplus E$, or $E^t\oplus E^t$.

Any left representation of the form $\widetilde{G}\otimes(PG'^{t}P^{\mathsf{T}})$, can be decomposed into blocks which we have been considered previously by transforming $\widetilde{G}$ to a basis where it is the direct sum

of irreducible representations.

When the right representation is $5G'$, all possibilities are composed of blocks that we have previously considered.

To recap, we have now presented all possible blocks in the right representation which can appear up to charge 23. For each block in the right representation, in the canonical basis, we have found possibilities for the corresponding block in the left representation, many of which are actually the same representation but in a different basis. However, we cannot transform between these bases without affecting the right block and so we must consider these as inequivalent representations of the left action. A list of these possible representations is given in Table 1. Unfortunately we have no method of systematically finding commuting representations and so we cannot rule out the possibility that there are other inequivalent left representations that we have not been able to find by inspection.

7.3 A charge 23 solution

Using computer algebra we have performed an automated and systematic test of all tractable combinations of the representations from the previous section to search for ADHM data up to charge 23. This has resulted in a unique solution with charge 23, in which the right and left representations are

$$Q_R = E \oplus 3F \oplus G' \oplus 2G_1', \tag{7.13}$$

and

$$Q_L = E \oplus (F \otimes 1_3) \oplus (PG'P^\mathsf{T}) \oplus (G_1' \mathbin{\widetilde{\otimes}} 1_2), \tag{7.14}$$

where $P = \operatorname{diag}(-1, 1, 1, 1)$ and $\widetilde{\otimes}$ is the Kronecker product on 2×2 blocks as in (7.11). In the block notation of our framework,

$$Q_{R,1}^+ = E, \; Q_{R,2}^+ = 3F, \; Q_{R,1}^- = G', \; Q_{R,2}^- = 2G_1', \tag{7.15}$$

$$Q_{L,1}^+ = E, \; Q_{L,2}^+ = F \otimes 1_3, \; Q_{L,1}^- = PG'P^\mathsf{T}, \; Q_{L,2}^- = G_1' \mathbin{\widetilde{\otimes}} 1_2. \tag{7.16}$$

The associated invariant blocks, which are again constructed from the building blocks corresponding to the edges in the Dynkin diagram, are

$$
\begin{aligned}
B_{12} = b_1 &\begin{pmatrix} -i & j & k & 1 & k & -1 & i & j \\ -j & -i & -1 & k & 1 & k & -j & i \end{pmatrix} \\
+ b_2 &\begin{pmatrix} j & i & 1 & -k & -1 & -k & j & -i \\ -i & j & k & 1 & k & -1 & i & j \end{pmatrix},
\end{aligned}
\tag{7.17}
$$

Right representation	Left Representation
1_n	A, E, or F (for $n = 1, 2, 3$ respectively)
E	$2A$, E, or E^t
$2E$	$E \otimes \widetilde{E}_1$, where $\widetilde{E}_1 = 2A$, E, or E^t.
$3E$	$F \otimes \widetilde{E}_1$
$4E$	$E \otimes (\widetilde{E}_1 \oplus \widetilde{E}_2)$
$5E$	—
$6E$	$F \otimes (\widetilde{E}_1 \oplus \widetilde{E}_2)$ or $E \otimes (\widetilde{E}_1 \oplus \widetilde{E}_2 \oplus \widetilde{E}_3)$
$7E$	—
$8E$	$E \otimes (\widetilde{E}_1 \oplus \widetilde{E}_2 \oplus \widetilde{E}_3 \oplus \widetilde{E}_4)$, $G' \otimes \widetilde{G}'$ or $G'_1 \otimes \widetilde{G}'$, where $\widetilde{G}' = G', G'_1$ or $G'^{\,t}_1$.
$9E$	$F \otimes (\widetilde{E}_1 \oplus \widetilde{E}_2 \oplus \widetilde{E}_3)$
F	1_3
$2F$	$E \otimes 1_3$
$3F$	$F \otimes 1_3$
nF, $n > 3$	—
G'	$PG'P^{\mathsf{T}}$, or $PG'^t P^{\mathsf{T}}$, where $P = \mathrm{diag}(-1, 1, 1, 1)$.
$2G'$	$E \otimes (PG'P^{\mathsf{T}})$, or $E \otimes (PG'^t P^{\mathsf{T}})$
$3G'$	$F \otimes (PG'P^{\mathsf{T}})$, or $F \otimes (PG'^t P^{\mathsf{T}})$
$4G'$	$G' \otimes \widetilde{G}$, or $G'_1 \otimes \widetilde{G}$, where $\widetilde{G} = 4A$, $E \oplus E$, or $E^t \oplus E^t$.
$5G'$	—
G'_1	—
$2G'_1$	$G' \widetilde{\otimes} 1_2$, $G'_1 \widetilde{\otimes} 1_2$ or $G'^{\,t}_1 \widetilde{\otimes} 1_2$
$3G'_1$	—
$4G'_1$	$G' \otimes \widetilde{G}$, or $G'_1 \otimes \widetilde{G}$, where $\widetilde{G} = 1_4$, $E \oplus E$ or $E^t \oplus E^t$; or $(E \otimes G') \widetilde{\otimes} 1_2$, $(E \otimes G'_1) \widetilde{\otimes} 1_2$, $(E \otimes G'^{\,t}_1) \widetilde{\otimes} 1_2$, $(G' \otimes E) \widetilde{\otimes} 1_2$, $(G' \otimes E^t) \widetilde{\otimes} 1_2$, $(G'_1 \otimes E) \widetilde{\otimes} 1_2$, or $(G'_1 \otimes E^t) \widetilde{\otimes} 1_2$.
$5G'_1$	—

Table 1: A summary of the possible blocks that make up the representations of the right and left actions of $\mathbb{T}$ when acting on the ADHM data.

and

$$B_{21} = b_3 \begin{pmatrix} 1 & -i & j & -k \\ -k & -j & -i & -1 \\ j & -k & -1 & i \\ k & -j & -i & 1 \\ 1 & i & -j & -k \\ -i & 1 & -k & j \\ -j & -k & 1 & i \\ i & -1 & -k & j \\ 1 & i & j & k \end{pmatrix}, \tag{7.18}$$

where b_1, b_2 and b_3 are arbitrary real coefficients, together with B_{22}, which is presented in Figure 6. Note that there is no invariant block B_{11} as there is no edge in the Dynkin diagram connecting E to G'.

The quaternionic representations are $q_R = q_L = 2A$. As there is no edge in the Dynkin diagram connecting G'_1 to A then $L_2^- = 0$ and the only non-vanishing block in L is the invariant L_1^-, which exists because there is an edge in the Dynkin diagram connecting G' to A. Explicitly, this right and left invariant row vector is

$$L_1^- = l_1(1, -i, -j, -k), \tag{7.19}$$

where l_1 is an arbitrary real parameter.

The ADHM data assembled from these invariant blocks is then

$$\widehat{M} = \begin{pmatrix} 0 & 0 & L_1^- & 0 \\ 0 & 0 & 0 & B_{12} \\ 0 & 0 & B_{21} & B_{22} \\ 0 & B_{21}^{\mathsf{T}} & 0 & 0 \\ B_{12}^{\mathsf{T}} & B_{22'}^{\mathsf{T}} & 0 & 0 \end{pmatrix}. \tag{7.20}$$

Applying the ADHM condition yields the following constraints on the coefficients

$$3(b_4^2 + b_5^2) = 3b_3^2 = 2(b_1^2 + b_2^2) = l_1^2. \tag{7.21}$$

These can be solved with the parameterisation

$$b_1 = \frac{\lambda}{2\sqrt{2}} \cos\theta_1, \quad b_2 = \frac{\lambda}{2\sqrt{2}} \sin\theta_1, \quad b_3 = \frac{\lambda}{2\sqrt{3}},$$
$$b_4 = \frac{\lambda}{2\sqrt{3}} \cos\theta_2, \quad b_5 = \frac{\lambda}{2\sqrt{3}} \sin\theta_2, \quad l_1 = \frac{\lambda}{2}, \tag{7.22}$$

where any choice of the parameters θ_1 and θ_2 gives equivalent ADHM data. The overall scale is given by λ and

$$\widehat{M}^\dagger \widehat{M} = \lambda^2 \, 1_{23}. \tag{7.23}$$

$$
B_{22} = b_4 \begin{pmatrix}
\frac{1}{2}\!\left(-i+\sqrt{3}j\right) & \frac{1}{2}\!\left(-\sqrt{3}i-j\right) & \frac{1}{2}\!\left(-\sqrt{3}-k\right) & \frac{1}{2}\!\left(1-\sqrt{3}k\right) & \frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(-1-\sqrt{3}k\right) & \frac{1}{2}\!\left(i+\sqrt{3}j\right) & \frac{1}{2}\!\left(\sqrt{3}i-j\right) \\[2pt]
j & -i & 1 & k & 1 & -k & -j & -i \\[2pt]
\frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(1+\sqrt{3}k\right) & \frac{1}{2}\!\left(i+\sqrt{3}j\right) & \frac{1}{2}\!\left(-\sqrt{3}i+j\right) & \frac{1}{2}\!\left(i-\sqrt{3}j\right) & \frac{1}{2}\!\left(-\sqrt{3}i+j\right) & \frac{1}{2}\!\left(\sqrt{3}+k\right) & \frac{1}{2}\!\left(1-\sqrt{3}k\right) \\[2pt]
j & -i & k & -1 & -1 & k & -j & -i \\[2pt]
\frac{1}{2}\!\left(i+\sqrt{3}j\right) & \frac{1}{2}\!\left(-\sqrt{3}i+j\right) & \frac{1}{2}\!\left(-1+\sqrt{3}k\right) & \frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(1+\sqrt{3}k\right) & \frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(-i+\sqrt{3}j\right) & \frac{1}{2}\!\left(\sqrt{3}+k\right) \\[2pt]
\frac{1}{2}\!\left(1-\sqrt{3}k\right) & \frac{1}{2}\!\left(\sqrt{3}+k\right) & \frac{1}{2}\!\left(-\sqrt{3}i-j\right) & \frac{1}{2}\!\left(i-\sqrt{3}j\right) & \frac{1}{2}\!\left(\sqrt{3}i+j\right) & \frac{1}{2}\!\left(-i-\sqrt{3}j\right) & \frac{1}{2}\!\left(-1+\sqrt{3}k\right) & \frac{1}{2}\!\left(-1-\sqrt{3}k\right) \\[2pt]
\frac{1}{2}\!\left(-\sqrt{3}-k\right) & \frac{1}{2}\!\left(-1+\sqrt{3}k\right) & \frac{1}{2}\!\left(i+\sqrt{3}j\right) & \frac{1}{2}\!\left(-\sqrt{3}i+j\right) & \frac{1}{2}\!\left(-\sqrt{3}i-j\right) & \frac{1}{2}\!\left(-1-\sqrt{3}k\right) & \frac{1}{2}\!\left(-\sqrt{3}+k\right) & \frac{1}{2}\!\left(-1-\sqrt{3}k\right) \\[2pt]
-i & j & -1 & -k & 1 & j & i & j
\end{pmatrix}
$$

$$
+\, b_5 \begin{pmatrix}
\frac{1}{2}\!\left(\sqrt{3}i+j\right) & \frac{1}{2}\!\left(-i+\sqrt{3}j\right) & \frac{1}{2}\!\left(-\sqrt{3}-k\right) & \frac{1}{2}\!\left(1+\sqrt{3}k\right) & \frac{1}{2}\!\left(-\sqrt{3}-k\right) & \frac{1}{2}\!\left(1+\sqrt{3}k\right) & \frac{1}{2}\!\left(-\sqrt{3}i+j\right) & \frac{1}{2}\!\left(i+\sqrt{3}j\right) \\[2pt]
i & j & 1 & k & 1 & -k & i & -j \\[2pt]
\frac{1}{2}\!\left(\sqrt{3}i-j\right) & \frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(i-\sqrt{3}j\right) & \frac{1}{2}\!\left(\sqrt{3}i+j\right) & \frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(\sqrt{3}i-j\right) & \frac{1}{2}\!\left(-1+\sqrt{3}k\right) & \frac{1}{2}\!\left(\sqrt{3}-k\right) \\[2pt]
i & j & \sqrt{3}i+j & -k & k & -1 & i & -j \\[2pt]
\frac{1}{2}\!\left(-\sqrt{3}-k\right) & \frac{1}{2}\!\left(i+\sqrt{3}j\right) & \frac{1}{2}\!\left(-i-\sqrt{3}j\right) & \frac{1}{2}\!\left(\sqrt{3}i+j\right) & \frac{1}{2}\!\left(-\sqrt{3}i+j\right) & \frac{1}{2}\!\left(-\sqrt{3}i-j\right) & \frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(-1-\sqrt{3}k\right) \\[2pt]
\frac{1}{2}\!\left(1-\sqrt{3}k\right) & \frac{1}{2}\!\left(-\sqrt{3}-k\right) & \frac{1}{2}\!\left(i+\sqrt{3}j\right) & \frac{1}{2}\!\left(i-\sqrt{3}j\right) & \frac{1}{2}\!\left(\sqrt{3}i+j\right) & \frac{1}{2}\!\left(i+\sqrt{3}j\right) & \frac{1}{2}\!\left(1+\sqrt{3}k\right) & \frac{1}{2}\!\left(-\sqrt{3}-k\right) \\[2pt]
\frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(i-\sqrt{3}j\right) & \frac{1}{2}\!\left(-\sqrt{3}i+j\right) & \frac{1}{2}\!\left(\sqrt{3}i+j\right) & \frac{1}{2}\!\left(i-\sqrt{3}j\right) & \frac{1}{2}\!\left(\sqrt{3}-k\right) & \frac{1}{2}\!\left(1-\sqrt{3}k\right) \\[2pt]
-j & -i & -1 & -k & -k & -1 & -j & i
\end{pmatrix}.
$$

Figure 6: The invariant map from $G'_1 \otimes E'$ to $3F$. The coefficients b_4 and b_5 are real and arbitrary.
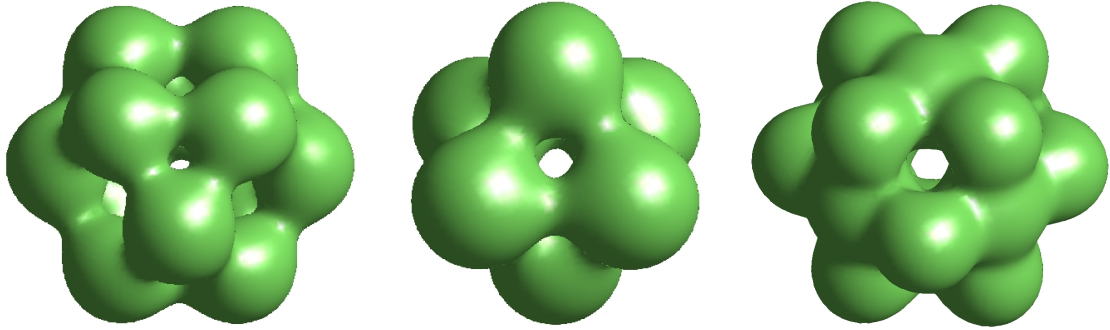
Figure 7: Surfaces of constant topological charge density for the charge 23 instanton with the symmetries of the 24-cell. The left image is the charge density in the hyperplane $x_4 = 0$, where the vertices of the 24-cell form a cuboctahedron. The centre image is the charge density in the hyperplane $x_4 = 1$, where the vertices of the 24-cell form an octahedron. The right image is the charge density integrated along the x_4-direction.

The vertices of the 24-cell can be divided into three hyperplanes, so that in the first of these hyperplanes the vertices form a cuboctahedron (a cube with each corner cut off to give an equilateral triangle face). The vertices in the two remaining hyperplanes form octahedrons. Figure 7 displays surfaces of constant topological charge density obtained from the above ADHM data. The first image is in the $x_4 = 0$ hyperplane, where the cuboctahedral structure is clear. The second image is in the $x_4 = 1$ hyperplane where the octahedral structure is clear. Finally, the third image is obtained by integrating the topological charge density along the x_4-direction and reveals a merged version of the two structures.

The ADHM data that we have found with the symmetries of the 24-cell has a charge equal to that given by the JNR bound and is therefore expected to be equivalent to a JNR instanton. We shall address this issue in the following section.

8   Equivalence to JNR data

The three examples of ADHM data that we have computed for the 5-cell, 16-cell and 24-cell all have a charge equal to the JNR bound. The ADHM data should therefore be equivalent to JNR data in which points with equal weights are placed at the vertices of these polytopes. In this section we shall explicitly demonstrate this equivalence.

The ADHM data corresponding to general JNR data has been presented in [16], but in a different format to the canonical form of ADHM data given by (3.3). For charge N JNR

data with equal weights and points in $\mathbb{R}^4$ given by $y_0, \ldots, y_N$, the ADHM data is [16]

$$\Delta(x) = \begin{pmatrix} y_0 & \cdots & y_0 \\ y_1 & & \\ & \ddots & \\ & & y_N \end{pmatrix} - \begin{pmatrix} 1 & \cdots & 1 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} x. \tag{8.1}$$

To convert this ADHM data to the canonical form (3.3) we need matrices $S \in \mathrm{O}(N+1)$ and $C \in \mathrm{GL}(N, \mathbb{R})$ such that

$$\begin{pmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = S \begin{pmatrix} 1 & \cdots & 1 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} C. \tag{8.2}$$

For general charge N, the following matrices will perform this transformation

$$C_{ij} = \begin{cases} 0 & \text{if } i > j \\ \dfrac{j}{\sqrt{j(j+1)}} & \text{if } i = j \\ -\dfrac{1}{\sqrt{j(j+1)}} & \text{if } i < j \end{cases} \qquad \text{where} \quad i, j = 1, \ldots, N, \tag{8.3}$$

and

$$S = \begin{pmatrix} -\frac{1}{\sqrt{N+1}} & \frac{1}{\sqrt{N+1}} & \cdots & \frac{1}{\sqrt{N+1}} \\ C_{11} & & & \\ -C_{12} & & \left(C^{\mathsf{T}}\right)_{ij} & \\ \vdots & & & \\ -C_{1N} & & & \end{pmatrix}. \tag{8.4}$$

This is a generalisation of the transformation presented in [17] for $N = 1, 2$.

In the case of the 5-cell with $N = 4$, the points are taken to be the five vertices

$$y_0 = \frac{1}{4}\left(1 - \sqrt{5}\,(i + j + k)\right), \quad y_1 = \frac{1}{4}\left(1 - \sqrt{5}\,(i - j - k)\right),$$
$$y_2 = \frac{1}{4}\left(1 - \sqrt{5}\,(-i + j - k)\right), \quad y_3 = \frac{1}{4}\left(1 - \sqrt{5}\,(-i - j + k)\right), \qquad y_4 = -1. \tag{8.5}$$

The ADHM 5-cell data $\widehat{M}$ presented earlier in (4.7) is equivalent to this data, when $\lambda = \frac{1}{4}$, since

$$\widehat{M} = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} S \begin{pmatrix} y_0 & \cdots & y_0 \\ y_1 & & \\ & \ddots & \\ & & y_4 \end{pmatrix} C\, Q^{-1}, \tag{8.6}$$

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where S and C are given above, and

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}. \tag{8.7}$$

Of course, the JNR data can be scaled to provide equivalence for any value of the scale λ.

To construct charge 7 JNR data with the symmetries of the 16-cell, we can take the 8 points to be the vertices of the 16-cell

$$y_0 = 1, \ y_1 = -1, \ y_2 = i, \ y_3 = -i, \tag{8.8}$$
$$y_4 = j, \ y_5 = -j, \ y_6 = k, \ y_7 = -k.$$

The ADHM data constructed previously, (6.20) with $\lambda = \frac{1}{2}$, is equivalent to this JNR data using the same transformations as above with

$$Q = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{15}} & \frac{2}{\sqrt{21}} & \frac{1}{\sqrt{7}} \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{\sqrt{5}} & -\frac{2}{\sqrt{15}} & -\frac{2}{\sqrt{21}} & -\frac{1}{\sqrt{7}} \\ 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{15}} & -\frac{2}{\sqrt{21}} & -\frac{1}{\sqrt{7}} \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{\sqrt{5}} & -\frac{\sqrt{3}}{\sqrt{5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}. \tag{8.9}$$

Due to the large dimension of the charge 23 ADHM data, it is difficult to find the transformation matrix between the solution in (7.20) and the JNR generated ADHM data. However, by examining the eigenvalues of the matrices Q_L and Q_R which leave the JNR generated ADHM data invariant under the left and right action of $\mathbb{T}$, we are able to confirm that they are the same representations as appear in the earlier ADHM data (7.20).

9 Discussion and conclusion

In this paper we have understood how the ADHM data of a charge N symmetric instanton transforms under the action of a finite subgroup of $SO(4)$. Given the description of the ADHM data in terms of quaternions, the natural way to represent the action of such a symmetry group is via the lift to the double cover, which is a subgroup of $SU(2) \times SU(2)$ and acts via right and left multiplication by unit quaternions. For the symmetry group of the 5-cell, the double cover is isomorphic to a subgroup of $SU(2)$, and the left and right actions are not independent. For this action, with elements $(g^\sharp, g)$, where $g^\sharp$ is dependent on g, the ADHM data transforms under a single N-dimensional real representation, Q, and

a single 1-dimensional quaternionic representation, q. These may always be taken to be in the canonical basis where Q is the direct sum of irreducible representations. It is then straightforward to enumerate all combinations of irreducible representations and search for any ADHM data that is invariant. This procedure allowed us to construct the ADHM data of a charge 4 instanton with the symmetries of the 5-cell, and show that this is lowest charge instanton with these symmetries.

The double cover of the symmetry groups of the remaining polytopes take the form $\mathbb{G} \times \mathbb{G}$, where $\mathbb{G}$ is one of the binary polyhedral groups $\mathbb{D}_2, \mathbb{T}$ or $\mathbb{Y}$, and the left and right actions of these groups are independent. This means that there are two independent representations of $\mathbb{G}$, Q_R and Q_L, and we only have the freedom to choose a basis in which either Q_R or Q_L is explicitly the direct sum of irreducible representations. However, Q_R and Q_L must commute, so the possible form of the representation Q_L is restricted when Q_R is in the canonical basis. In the case of the 16-cell, this has allowed us to uniquely determine all possibilities for Q_L given a choice of Q_R. For the 24-cell, we have only been able to determine the nonlinear constraints on the form of the representations in Q_L, and find the obvious examples by inspection.

With all possible combinations of Q_R and Q_L known for the 16-cell, and a large number known for the 24-cell, we have tested each combination to determine if there is invariant data that also satisfies the ADHM constraint. For the 16-cell we have found a solution at charge 7 and for the 24-cell we have found a solution at charge 23, both of which have been shown to be equivalent to JNR data.

In previous work on instantons with platonic symmetries, the minimal charge instantons associated with the cube and dodecahedron are not of the JNR type, and perhaps this is related to the fact that they are not deltahedra. In our search for instantons with polytope symmetries, the three minimal charge examples we have constructed are all of the JNR type, and perhaps the explanation again lies in the fact that the 5-cell, 16-cell and 24-cell all have triangular faces. This suggests that the 8-cell and the 120-cell may be more promising candidates to find minimal charge instantons that are not of the JNR type. However, the JNR bound for the 120-cell is $N \leq 599$, which is clearly beyond the limits of our approach. For the 8-cell, the JNR bound is $N \leq 15$ and this is also at the limits of our capabilities because there are four 1-dimensional representations and this rapidly generates a large number of possibilities as the charge increases, and in particular produces invariant data with too many parameters to make the ADHM constraint tractable.

As the 8-cell is dual to the 16-cell then they share the same symmetry group, so it might be tempting to conclude from our analysis that there is no instanton associated with the 8-cell with charge less than 8. However, there are a number of caveats to this conclusion, as we now discuss.

In the case of platonic symmetry, a polyhedral group acts as spatial rotations and there is an action on the gauge potential that covers this, but potentially the image of this representation may only be a quotient of the polyhedral group, rather than the full polyhedral group itself. If this is the case, then in passing to the binary polyhedral group, as is natural for the quaternionic ADHM description, there will be a double cover of this quotient group,

but this may not be equal to some quotient of the binary polyhedral group [8]. Precisely this situation occurs for the minimal charge instanton associated with the cube, and as the 8-cell is the 4-dimensional analogue of the cube then perhaps something similar might occur, taking the 8-cell outside our framework.

It is also possible that lower charge solutions exist, but outside of our framework, for the following reasons. In the 24-cell, there may be representations in the left action that we have not identified and yield a lower charge solution. To rule out this possibility would require the general solution of a set of nonlinear constraints to find the most general form of representations that commute with any given right representation, and it is not clear how to proceed with this. Our framework was therefore restricted to identifying obvious low-dimensional commuting representations and using these to form larger representations by forming tensor products. Some evidence to support the validity of this approach is the fact that we were able to obtain the charge 23 solution through this mechanism, which has a fairly complicated structure for both the left and right representations.

We have also assumed that both Q_R and Q_L form representations of the appropriate binary polyhedral group. It is possible that there are symmetric instantons with ADHM data that is invariant under some matrices Q_R and Q_L which are not strictly representations. For example, consider the right action of $g_i^2 = -1$ in the double cover of the 16-cell symmetry group. Then there must exist matrices, $Q_R(g_i)$, such that

$$(Q_R(g_i))^2 M = -M(Q_R(g_i))^2. \tag{9.1}$$

If Q_R is composed of irreducible representations then we saw previously that in the appropriate basis $(Q_R(g_i))^2 = \mathrm{diag}(1_m, -1_n)$. However, if N is even then the following is also a possibility,

$$(Q_R(g_i))^2 = \begin{pmatrix} 0 & 1_{N/2} \\ -1_{N/2} & 0 \end{pmatrix}, \tag{9.2}$$

where M takes the form

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{9.3}$$

with A and B symmetric matrices. The matrices $Q_R(g_i)$ do not form a representation of $\mathbb{D}_2$, for example $g_1^4 = 1$, yet $(Q_R(g_1))^4 = -1_N$. However, $Q_R(g_i)$ still obey the group action when applied to M since the sign is projected out. We have not been able to construct an argument why this cannot occur, though one may indeed exist.

Another possibility is that Q_R and Q_L are representations of opposite sign. We took Q_R to be composed of positive representations in the upper block and negative representations in the lower block, so that $(Q_R(g_i))^\alpha = \mathrm{diag}(1_m, -1_n)$. We also took a similar block structure for Q_L, but it is possible that Q_L consists of negative representations in the upper block and positive representations in the lower block so that $(Q_L(g_i))^\alpha = \mathrm{diag}(-1_m, 1_n)$. Again, this difference of sign is irrelevant in the action on the ADHM data. We have performed a similar analysis as in Section 7.2 with the left representations having the opposite sign, but we were not able to find any invariant data of this form. Again, as for the representations of

the same sign, our search was not exhaustive, but there may be some simple argument that rules out this possible structure.

Finally, it is possible that the matrices Q_R and Q_L only satisfy the group presentation up to a sign,

$$(Q_{R,L}(g_1))^\alpha = \pm(Q_{R,L}(g_2))^\beta = \pm\left(Q_{R,L}(g_1)Q_{R,L}(g_2)\right)^\gamma. \tag{9.4}$$

For the 24-cell symmetry group, where $\alpha = \beta = 3$, we can always choose the sign of $Q_{R,L}(g_i)$ such that the signs in this expression match. However, for the 16-cell, where $\alpha = \beta = 2$, it may be possible to have symmetric ADHM data which is invariant under some matrices $Q_{R,L}$ where the signs do not match. This would not be equivalent to a true representation. The core problem that generates all these possibilities outside of our framework is that the transformation of the ADHM data is unaffected by the sign of $Q_{R,L}$ and so they need only satisfy the group operation up to a sign,

$$Q_{R,L}(g)Q_{R,L}(h) = \pm Q_{R,L}(gh). \tag{9.5}$$

Our treatment in terms of representation theory is therefore only applicable when the signs agree with the group operation. We have been unable to find meaningful examples of suitable matrices when the signs do not agree.

Acknowledgements

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