A Search for Understanding

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ABSTRACT

We discuss an important breakthrough for us in our search for a technical meaning of 'understanding' in mathematics education. In this paper we describe the background to this discovery, the catalyst for the breakthrough and a concise explanation of our new definition. In discussing the consequences of this definition, both in terms of the theoretical implications for the internal characteristics and external manifestations of understanding with some initial practical consequences for a teacher's attempts to model a learner's understanding, we begin to describe our ongoing search for a more comprehensive theory.

Much of our work during the past five years has been devoted to developing a theory of learning which accounts for our own experiences as learners, teachers and researchers. The theory we have developed – in which we identify learning experiences as *natural*, *conflicting* or *alien*, and consider the ways in which learners might respond to them (Duffin and Simpson, 1993) – has been used to analyse a number of the learning incidents we have encountered (Duffin and Simpson, 1995). In the course of such analyses we found that the word 'understanding' often entered our discussions and eventually we felt compelled to seek a definition of the word which would fit within the basic framework of our theory and might enable us to make more sense of the incidents we encountered. The quest for such a definition has occupied us for more than three years and we will use this paper to explore how we came to our current position.

The paper is *not* focussed on understanding, its focus is on the journey we undertook to make sense of a particular term (which happens to be 'understanding) in the context of our on-going development of a personal theory of learning. As such, the paper contains almost as much about what we later came to see as mistaken (or to replace) as it does about the terminus of our journey: our current definition of understanding.

SETTING OUT ON OUR JOURNEY

Our search had three main phases, during each of which something has emerged that has been of significance to our thinking about the meaning of 'understanding'. Though subject to later modification as the work has progressed, each stage can be seen to have an influence on our present position.

Unusually for us, we began our work from a purely theoretical perspective. From the realisation that we were using a term like 'understanding' in an essentially undefined

way to explain our interpretations of learners' responses, we sensed a need to provide a technical meaning for the word within the broader theory we had developed up to that point. So, rather than beginning from critical incidents – our usual practice – with the aim of providing a theoretical framework which might explain them, we began from our existing framework with a question: 'what, within our theory, might understanding consist of?'

Our theory postulates that learners encounter three kinds of experience, which we have come to call natural, conflicting and alien, to which they respond by modifying their internal mental structures in different ways. We define a natural experience as one that fits the learner's current mental structures and as one to which they respond by strengthening the structure. A conflict is an experience which jars with the existing structures, either by showing that the way of working associated with the structure cannot cope in the way expected or by showing that more than one – previously dissociated – mental structure can be brought to bear on the same situation. We conjecture that learners respond to such conflicts by weakening or destroying the structure, by limiting the domain of experiences with which the structure is expected to cope or by constructing links between previously separate structures. In contrast, an alien experience is one that has no fit with the existing mental structures at all: it neither fits nor causes conflict. In response, a learner may ignore or avoid the experience or may absorb it as a new, separate structure that then becomes liable to modification through later natural, conflicting and alien experiences.

It was in the context of this theory, then, that we asked our question about a technical meaning for 'understanding': technical in the sense that our aim is not to replace the general use of the word, but to ground our personal meaning of it within our developing theory. Our first answer to this question came almost as soon as we asked

it and we developed our first definition of understanding, which was to stay with us for some considerable time before we felt the need to modify it. We defined understanding as:

> the awareness of connections between internal mental structures.

At this point, we began to focus on how such understanding may manifest itself and whether this new-found definition might provide an explanation for those manifestations. We initially examined our feelings, using our own internal perceptions to answer the question 'what do I feel when I understand?' and compared those perceptions with those of colleagues with whom we talked. We found considerable agreement about the affective aspects of understanding, including:

> When I understand
>
> - I feel comfortable
> - I feel confident
> - I feel able to forget the detail, confident that I can reconstruct it whenever I need it
> - I feel that the thing belongs to me
> - I can explain it to others

At the same time, we compared these feelings and our definition with our initial examination of the literature on understanding. From this we discerned two seemingly discrete approaches amongst researchers in the field, though – as we shall see later – our perceptions of what these researchers were saying have been modified in the light of our progress along this journey and, with some, we have moved from a sense of

profound disagreement to a realisation that we can absorb what they are saying within our new framework.

At the time, however, it appeared to us that the two approaches split those who believed understanding to be associated with an ability to do, or with a set of skills, from others who demanded that understanding has cognitive qualities besides the ability to perform tasks. We found ourselves surprised that the former group contained Gagné (1970) who, while he mentioned a relation to the 'internal state of the individual', wrote about understanding as "the attainment of an organised set of intellectual skills". In many senses (and in line with our theory) we found ourselves rejecting or ignoring those whose perceptions seemed alien to our own and were happier in aligning ourselves with those who expected more, like Van Engen (1953), who described understanding as "a process of organising and integrating knowledge according to a set of criteria" (an idea that resonates with more recent work, such a Hiebert and Carpenter, 1992), and others detailed in the review article of Byers (1980).

Some authors' views seemed to have a very close match with our own, particularly in emphasising the role of internal connections in the definition of 'understanding' and in describing manifestations of understanding which fit with our own. Thus Skemp (1976) makes the role of connections clear in his description of partial understanding which "would occur when the student forms some connections, but not enough to make use of the new knowledge in an assortment of different situations" and Poincaré (1908) talks of understanding as synthesis, which he linked with creativity and describes his feelings of his understanding of a proof as "if I have the feeling, the intuition … so as to perceive at a glance the reasoning as a whole, I need no longer fear lest I forget one of the elements."

The level of consonance with these other authors led us to feel some confidence in the relevance of our work so far, but at this point we found ourselves diverted from our quest until the second phase overtook us when two incidents occurred in quick succession, drawing us back to the practical roots of our work.

THE TWO INCIDENTS

The first incident concerned a student on a numeracy course who had been asked to do a subtraction (532 – 286) in the way she had been taught at school. Our attention was drawn to a snippet of conversation between her and another student who asked 'Why did you put a two there?' to which the reply was 'because one would not have been enough'. It transpired she was attempting to use the method of equal additions, adding a 1 to the unit 2 (making it 12) and another to the 8 which she read as 18. When she continued the procedure by adding a 1 to the 3 her common sense told her that this was insufficient and that she needed 23 at the top to enable the 18 to be subtracted from it; hence the initial comment which characterised the incident. Consternation arose for her when she then went to the hundreds column, where she added a 2 to the 2 to make 22, because at this stage she recognised that there was an insuperable problem because there was no way to proceed that would solve the problem she now had. She failed to complete the subtraction, saying that she wished she had been allowed to do the subtraction in her head.

The second incident concerned a colleague who was listening to us discussing the problems of the student in the first incident. He suddenly announced that he had been trying to remember the trigonometrical formulae for multiple angles, but had been unable to recall them. He was, however, very proud of himself when he was able to reconstruct them all just by recalling the formula $\cos^2 + \sin^2 = 1$.

MOVING OFF AGAIN

Our attention was re-engaged by these incidents which we then tried to analyse in the light of our definition. In the incidents, one learner tried and failed to carry out a task while the other, who confessed to having forgotten what he wanted to recall, was able to recreate the processes successfully by calling on connections he had available.

It was this recognition that, while both were about learners who had forgotten something, only one was successful in recapturing what had been forgotten, which was the catalyst for further investigation. It was clear to us that the success came from awareness of appropriate connections on the part of the second learner, precisely in accordance with our definition. The failure appeared to come because the other learner was merely trying to recall the steps of a procedure without using any significant connections that we might interpret as an understanding of the procedure.

The incident led the words *reconstructing* and *reproducing* to become central to refocusing on our search. At the time they depicted for us the actions of those who do or do not understand. It seemed that the learner who did not understand could only attempt (and fail) to reproduce a forgotten procedure, while the one who did understand was able, despite lack of recall, to reconstruct what was wanted by using his awareness of the appropriate connections. Moreover, the second incident further confirmed one of the characteristics we had identified: that if you understand something, you do not need to remember all the details. It also drew our attention back to the link Poincaré made between understanding and creativity, as our own investigation into creativity (Duffin and Simpson, 1991) had made links for us between that and reconstructing.

All these thoughts set us off again on our search, this time beginning to wonder about the ways in which a teacher might be able to recognise understanding in their students. We came to call such indications the *external manifestations* of the internal

characteristics we had earlier identified, though, again, these came to have a very different position as our work developed.

We identified a number of these manifestations and discovered more in the literature (notably Nickerson, 1985) which both extended our own discoveries and gave us a sense of being on a fruitful path. Amongst these external manifestations we included:

- Being able to explain
- Being able to recognise in other contexts
- Being able to derive consequences

We also developed a rough scale on which we could place these manifestations based on the reliability with which a teacher could use them to ascertain a learner's understanding. We felt that seeing a pupil do something (such as correctly perform a subtraction) was a low level indication of the pupil's understanding, since the teacher has no direct way of knowing whether the pupil has an understanding of what they have done rather than reproducing an algorithm from memory. On the other hand, the ability to derive consequences (which we had adopted from Nickerson) seemed to be less likely to have come from reproduction.

This work, we felt, had put us in a position to write something which we felt may have been of interest to others. We now had both a theoretical perspective on understanding and a layer of practical consequences for the teacher. However, all our efforts to write this at the time were in vain. While some of this failure may have been due to other circumstances, we feel that it may have been some deep-rooted doubt in what we had done that prevented us writing.

Our recognition of this difficulty led us to try a tactic to get us back on track. We decided to return to the literature for inspiration. We decided to re-read Sierpinska (1990), which turned our understanding of understanding upside down!

A BREAKTHROUGH

In reading Sierpinska's paper initially we had been puzzled by her question "is understanding an act, an emotional experience, an intellectual process or a way of knowing?" In our existing definition we saw understanding as a *state* and became puzzled when Sierpinska rejected this and worked with the notion of understanding as an act. Re-reading this paper did not clarify our confusion, but led us to look for clarification in her more substantial book on the subject (Sierpinska, 1994).

At the same time we had also begun to realise that there was a problem with our existing definition: there seemed to be a hole between the notion of internal characteristics and external manifestations. From our current perspective, we can see this hole being caused by the tension between seeing internal characteristics as static (feeling comfortable etc.) and external manifestations as dynamic (explaining, deriving consequences etc.) while for us, understanding was the state of 'awareness of connections between internal mental structures'. The realisation of this difficulty and the search for clarification of Sierpinska's meaning of an 'act of understanding' left us ripe for a breakthrough.

The breakthrough began when we realised that we could find a meaning within our theoretical framework which could describe an *act* of understanding and, moreover, such a meaning might have a vital role in dissipating the tension between the internal and external (albeit with a complete change of meanings as well). We interpreted an *act* of understanding as the *use* of understanding to solve a problem[1]. It suddenly

became clear to us that it is only through interpreting the physical manifestations of a learner's *use* of their understanding that the teacher can make any kind of judgement about the learner's existing understanding.

While it has since become redundant for us, this notion led us to develop an analogy which took us a stage further in our thinking. We saw an analogy between having and using understanding and the naïve view of potential and kinetic energy. In imagining a ball with potential energy, we note that we cannot directly see that energy. However, when the ball is falling, *while we still cannot see the energy*, we can infer from our observation of the ball moving and our theoretical perspective on the meaning of energy that the ball both has some kinetic energy and began with some potential energy. Similarly, we suggest that when a learner has some understanding we have no way of directly seeing it. However, when they enact that understanding, while we still cannot directly see it, we can begin to infer from our observations of the physical actions the learner performs, and our theoretical perspective on the meaning of understanding, something about the form of understanding the learner previously had[2].

The analogy, partial as it is, led us to think about the question of how the ball gets the energy by being lifted off the ground: that is, what might we now mean by 'building understanding'. Thus, we came to a three component definition of understanding.

THREE COMPONENTS OF UNDERSTANDING

We named the three components of understanding *building, having* and *enacting*. By the first of these we mean the formation of the connections between internal mental structures that we conjecture constitute the understanding which an individual has ready to be used to solve problems. With this meaning, it becomes clear that the

mechanisms for the formation (and destruction) of these connections are already encapsulated within our theory of responses to natural, conflicting and alien experiences.

The second component, which we term *having*, is the state of connections at any particular time. As teachers, it is this totality of connections (each having the potential to be used by the learner) that we are most interested in when we talk about a learner's understanding. It is important, at this point, to recognise that the emphasis on 'connections' gives the theory a different way of addressing the issue of whether understanding is 'all or nothing'. Byers (1980) states both that "teachers often speak of understanding in absolute, 'yes-or-no' terms" and that Poincaré uses understanding in an 'all or nothing' sense. In contrast, a number of authors (such as Skemp) develop theories which imply a continuum of understanding, some list different *kinds* of understanding (such as the tetrahedral model Byers and Herscovics, 1977) while others use the notion of discrete levels (such as Van Hiele, 1986). Our foundation in the notion of connections, however, allows us to adopt Nickerson's 'depth and breadth' idea

> One way to think of complexity is in terms of the breadth and depth of a concept's connectedness. That is, a concept might be immediately connected to many other concepts (which reflects its breadth of connections) and deeply connected to some of them, in the sense that the concepts to which it is connected are themselves complex and thus connected to many others in turn.
>
> (Nickerson, 1985, p231)

This notion has an influence on our third component, *enacting* understanding. By this we mean the use of the connections available *in the moment* to solve a problem or

construct a response to a question[3]. The breadth of understanding which a learner *has* may be evidenced by the number of different possible starting points they have for solving a problem, while the depth of their understanding may be evidenced by the ways in which they can unpack each stage in their solution in ever more detail, with reference to more concepts. As we shall see, it is this third component that has most to offer us in terms of enabling a teacher to model a learner's understanding.

Note that these three components bring us back to our problem with Sierpinska's question about whether understanding is an act, a state or a process. Our definition suggests that there are aspects of all three: an ongoing process of the development of connections (building), a state of the available connections at a given time (having) and the act of using the connections in response to a problem (enacting).

INTERNAL AND EXTERNAL ASPECTS OF THE THREE COMPONENTS

Much of the development for our original definition of understanding came in deciding what internal characteristics and external manifestations could be discerned when someone had some understanding. In doing this we developed two questions which we asked of ourselves and of colleagues. These two questions can be modified to determine what we mean by the internal and external aspects of our new definition.

We can try to determine internal characteristics by asking ourselves and other learners the questions:

$$\text{what do I feel when I } \left\{ \begin{array}{l} \text{am building} \\ \text{have} \\ \text{am enacting} \end{array} \right\} \text{ my understanding?}$$

and we can try to determine the external manifestations by asking ourselves and other teachers the questions

what do I expect to be able to see in my students when I believe that

they { are building / have / are enacting } their understanding?

The responses to these questions have led us to build a matrix of these possibilities (table 1).

INSERT TABLE 1 ABOUT HERE

This table has a number of important points which distinguish our current position from our previous development of the definition. Most notably, all the characteristics we had before (whether they were internal or external) we now see as internal, though they may be associated with different components of understanding.

It is important to note that, while we have separated out the components of understanding, we are not suggesting that they are discrete elements in the learning process. We do not suggest that learning is as simple as having a period of building, a quiescent state of having and finally a period of activity when understanding is enacted. Indeed, in solving a substantial problem a learner may use some recalled facts, enact some of their understanding, get stuck, find and resolve conflicts by building new connections, enact the understanding inherent in those new connections, bring in more recalled ideas and so on. All this may take place in a very short space of time during any particular learning incident.

Inevitably, there are limitation with this matrix and, we believe, some of these limitations are inherent in any discussion of the internal workings of a learner. Because we have no direct access to the mind of anyone other than our own selves,

we have no way of ensuring that any internal characteristics are immutably associated with understanding in all people. Through the act of sharing our perceptions of our own internal characteristics, we can begin to sense commonalities and, as we have already indicated, many of the descriptions here have resonated with the descriptions of others or have originated from others and have resonated with us to the extent that we have taken them on board as part of the definition and its consequences.

Similarly, we believe that there are some problems inherent in any discussion of the external aspects of understanding. Indeed, the crux of the problem we had with the tension between internal and external in our previous definition came, we feel, because we permitted too much to be taken as read in the interpretation of the actions a learner performs in enacting their understanding. Our current ideas try as much as possible to distinguish the act of interpretation from the observation of the physical act of the learner and addresses the issue of that interpretation separately. Before we do so, however, let us consider each component in turn.

Internal and External Aspects of Building Understanding

We have defined building understanding as the formation (or destruction) of connections as a result of responding to natural, conflicting and alien experiences. The problem of the inaccessibility of internal characteristics may be particularly acute with this component. We suggest that it is difficult for a learner to be simultaneously focussed on responding to the experiences they are having and on the mental reorganisations which that response engenders. The ability to do this is related to what we call the intellectual maturity of the learner and is at the heart of the discipline of noticing that Mason (1994) has developed.

What is accessible to the teacher, as a manifestation of the internal reorganisations which may be going on, are the reflex reactions to some of the responses to the experiences. For example, if a learner sees a conflict in which two previously dissociated structures are brought to bear and which leads them the see, through the new connection, a new way of solving the problem, the teacher may see the learner's eyes light up or some other physical manifestation associated with the 'aha' moment (described so vividly by Gardner, 1982).

Internal and External Aspects of Having Understanding

In a passive state, in which the learner is not having to reorganise their internal mental structures, or solve the problems put before them by enacting their understanding, the internal aspects of *having* understanding may be more accessible. We feel at ease with a topic if we feel we have sufficient connections to be able to cope with the kinds of situation we expect to encounter, even to the extent of feeling able to forget the details of an algorithm (for example), sure that we can reconstruct them later if needed.

As our earlier analogy with energy suggests, there can be no active external manifestations of the component of *having* understanding alone[4]. Manifestations can only come when the learner is doing something and, if separated out, the *having* of understanding is a passive state. This has very important implications for a teacher trying to model a learner's understanding: while it is the totality of what the learner might be able to do when they use all their available connections which we as teachers wish to determine when we consider a learner's understanding, it is only through our interpretations of their actions that we can come to this model. We cannot directly access the *having* component of understanding.

Internal and External Aspects of Enacting Understanding

We defined enacting understanding as the use of available connections in trying to solve a problem or respond to a question. This definition suggests a number of possible internal characteristics. A learner may be conscious of using connections to explain something to someone else, see an aspect of a concept in a context they have not used it in before or derive some new consequence of the concept.

Most important for our discussion of the teacher's ability to model understanding is the external manifestations of these characteristics. We suggest that these internal characteristics manifest themselves in a number of physical ways – speaking, writing, drawing etc. – which we have put under the heading of 'doing'. It is interesting to note that, while we initially rejected the notion of understanding as being associated with doing, it now forms what we feel is a central part of our model of how people understand. However, by doing, we do not mean just the completion of routine tasks, but any action that may have resulted from using understanding which then becomes available to be interpreted by the teacher in their attempt to model the learner's understanding. It is also important to note that the enacting of understanding will not necessarily lead to an external manifestation – a learner may solve a problem in their head, without recourse to any external representation of that solution or solving act. Just because a learner does not appear to be doing anything, it does not mean they aren't enacting their understanding, it just means that, as teachers we have no extra information we can use to model the understanding of the learner.

MODELLING UNDERSTANDING

We suggest that one of the major roles for a teacher is developing a model of the understanding a learner *has* which will allow them to set tasks and provide contexts to

enable the learner to increase the depth and breadth of their understanding, create new understandings and become fluent in their use of existing understandings. However, this role is an exceedingly complex one because the teacher can work only from their interpretation of their observation of the pupils' actions. In discussing the difficulty of interpretation, we may try to list the aspects of the teacher pupil interaction that can affect that interpretation. In doing so, we developed what was to us a useful, if complex picture (figure 1)

INSERT FIGURE 1 ABOUT HERE

In this the teacher has the role of setting tasks for the learner on the basis of their current model of that learner's understanding and on the teacher's own understanding of mathematics. The learner uses their expectations of the teacher (built up in the same way that the teacher has built up a model of the learner's understanding) – filtered through their theory of what teaching is meant to be – to decide how to react to the task. They then react with their understanding of mathematics in the ways we shall discuss below. The teacher then interprets the actions they observe, through the filter of their theory of learning, to modify their model of the learner's understanding. Such a model is a vast simplification of the interaction process and misses out many of its important features (and does not even begin to address learner-learner interaction), but it highlights the complexity of just the interaction facet of the issue of modelling learners' understanding through interpretation of their actions.

Even this simple model highlights two major problems: not all actions come from the enactment of understanding and, even if a teacher can be reasonably sure that the action is the result of enacting understanding, an enactment represents a single usea of a single pathway (whilst the teacher wishes to model the totality of potential pathways).

The first of these problems has its foundation for us in one of our earliest ideas: the contrast between reproduction and reconstruction. In our two incidents described earlier, the first seems to show a learner who is attempting to reproduce an algorithm and, while she shows a sensible interpretation of the meaning of place value, without realising that the algorithm has different meanings for the two '1's in 'put 1 here, put 1 there', she is unable to solve the problem and does not appear to be able to reconstruct it by enacting her understanding[5]. The second incident seems to show a learner who could not reproduce a solution, but instead needed to use his connections between some trigonometrical concepts to reconstruct the formulae he wanted. For our consideration of the ways in which a teacher can model understanding, however, we could consider hypothetical incidents in which the first learner does correctly recall and apply the equal additions algorithm and in which the second learner can recall, without reconstructing, the multiple angle formulae. If we had seen both of these hypothetical incidents, we would have seen actions that resulted from reproduction and, we suggest, we would have no substantial information with which to improve our models of those learner's understandings.

Indeed, a learner will reproduce a response if they can do so, even if they are quite capable of reconstructing by enacting their understanding. Bartlett (1932) suggests that there is considerably more mental effort involved in reconstruction than in recall. Thus in modelling understanding, a teacher may be misled into thinking that a solution to a problem is one obtained by enacting understanding but which could, instead, have been reproduced from memory. Indeed, if we note the analogies between our theory of understanding and the connectionist approach to artificial neural networks we may wonder about results, detailed in Patterson (1996), in which neural networks can become so specialised to specific tasks that the pathways become

over-trained. This analogy suggests that when a learner is easily able to reproduce, a question which asks them to reconstruct (such as a request to explain) may be very difficult to cope with and the shift of attention from the performance of the procedure to the reconstruction of an explanation may be hard to do.

We might naïvely assume that at least we could conclude something from a learner's failure to solve a problem. But even here there are difficulties: when a learner fails to reproduce a solution, they may try to enact their understanding to construct one, but while they could *have* the understanding needed, they may not have the awareness of the connections they need *at that moment* to respond to that situation. A failure to *enact* may not be a sign of a lack of *having* understanding.

The second problem – that even if the pupil is enacting their understanding, it may well be the result of the use of only a small portion of the understanding they have – is a signal that we have an even more complex issue. Modelling understanding is a long term process in which the teacher repeatedly revises the model every time they interpret an action as one which has come from enacting understanding.

THE FIRST CONSEQUENCES FOR TEACHING

Despite these two layers of complexity, we suggest that this new definition of understanding leads to some practical suggestions for a teacher trying to model their pupils' understanding:

- Teachers need to be aware that modelling is a long term process and they need to build a model from interpreting a large number of actions resulting from questions or problems which allow a range of broader and deeper understandings to be enacted

- To try to avoid the problem of reproduction as much as possible, teachers might use questions which make reproduction easier to spot: requests to explain, setting questions in new contexts etc. may all help do this.
- To try to avoid situations in which understanding cannot be enacted because of a lack of awareness of the connections that are available, questions might be put in many different ways and at many different times.

These are the beginnings of the implications for our new definition of understanding within our developing theory of learning. It is important to realise that our theory develops by moving between the theoretical search for implications, the discovery of practical consequences for ourselves and others and the application to incidents from our own teaching and learning. As such, the theory of learning progresses as a journey, with the definition of understanding being yet another step.

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Footnotes:

[1] It is important to note that the breakthrough came from *misreading* Sierpinska: we do not now believe that she uses 'act of understanding' in this sense, but that for her it is the overcoming of an epistemological obstacle. This alternative meaning has come to form part of our definition in what we later describe as 'building understanding'

[2] The energy analogy recalls the horrors detailed by Feynman (1985) in the nonsensical ways in which textbooks use the word 'energy'. Feynman describes a textbook in which students are asked what makes a wind-up toy go, what makes an automobile go etc. and was horrified to find the answer 'energy makes it go' to each question

> "Now that doesn't *mean* anything. Suppose it's "Wakalixes." That's the general principle: "Wakalixes makes it go." There's no knowledge coming in … it's just a word"

Similarly, understanding is too often just another 'Wakalixes" word.

[3] We do not, by the use of 'enacting', mean to imply any connection between this part of our definition and the theory of 'enactivism' espoused by Varela et al (1991).

[4] In a classroom context, it may be that the demeanour of a learner can indicate something about their confidence in the subject matter, but we suggest that this may not necessarily manifest itself through some action.

[5] That is not to say that the student had no understanding of subtraction – her words 'I wish she'd let us do it in our heads' at least seems to imply that she has a method (and perhaps an understanding) for solving the problem.

Table:

component of understanding	**building**	**having**	**enacting**
internal characteristics	noticing?	feeling comfortable feeling able to forget (and reconstruct later)	solving problems explaining to others using in a new context deriving consequences
external manifestation	'eyes lighting up' 'Aha'	none	Doing

Table 1

Figure:

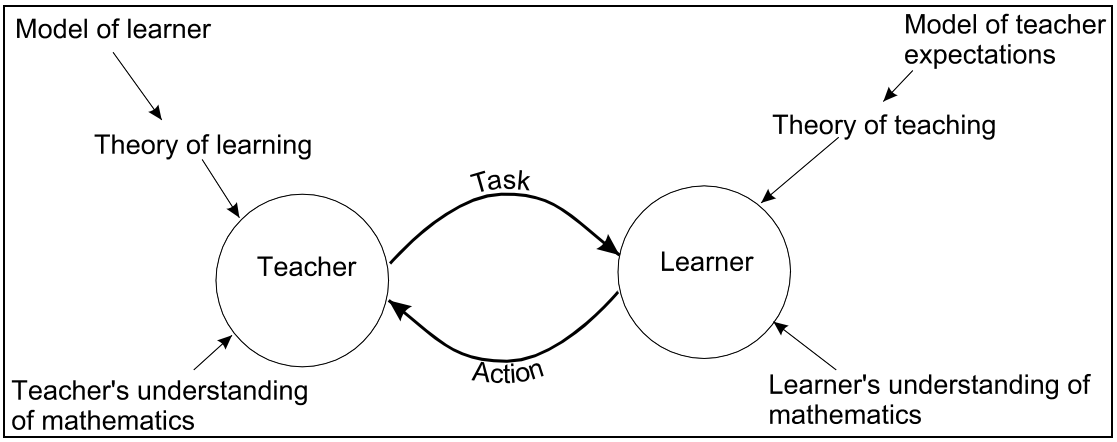

Figure 1