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NON-ABELIAN CONGRUENCES BETWEEN SPECIAL VALUES OF L-FUNCTIONS OF ELLIPTIC CURVES; THE CM CASE

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In this work we prove congruences between special values of L-functions of elliptic curves with CM that seem to play a central role in the analytic side of the non-commutative Iwasawa theory. These congruences are the analogue for elliptic curves with CM of those proved by Kato, Ritter and Weiss for the Tate motive. The proof is based on the fact that the critical values of elliptic curves with CM, or what amounts to the same, the critical values of Grössencharacters, can be expressed as values of Hilbert-Eisenstein series at CM points. We believe that our strategy can be generalized to provide congruences for a large class of L-values.

[a]

1. Introduction

In [8,15] a vast generalization of the Main Conjecture of the classical (abelian) Iwasawa theory to a non-abelian setting is proposed. As in the classical theory, the non-abelian Main Conjecture predicts a deep relation between an analytic object (a non-abelian p-adic L-function) and an algebraic object (a Selmer group or complex over a non-abelian p-adic Lie extension). However, the evidences for this non-abelian Main Conjecture are still very modest. One of the central difficulties of the theory seems to be the construction of non-abelian p-adic L-functions. Actually, the only known results in this direction are mainly restricted to the Tate motive over particular p-adic Lie extensions as, for example, in [16,22,23,27]. We should also mention here that for elliptic curves there are some evidences for the existence of such non-abelian p-adic L-functions offered in [4,10] and also some computational evidences offered in [11,14]. Finally, there is some recent progress, achieved in [5], for elliptic curves with complex multiplication defined over $\mathbb{Q}$ with repsect the p-adic Lie extension obtained by adjoining to $\mathbb{Q}$ the p-power torsion points of the elliptic curve.

The main aim of the present work is to address the issue of the existence of the non-abelian p-adic L-function for an elliptic curve with complex multiplication (but see also the remark later in the introduction) with respect specific p-adic Lie extension as for ex-

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ample, the so-called false Tate type extension or the Heisenberg type extensions. Actually, we prove congruences, under some assumptions, that are the analogue for elliptic curves with CM of those proved by Ritter and Weiss in [27] for the Tate motive. We remark that such type of congruences is the main input from the analytic theory in order to prove the existence of the non-abelian p-adic L-function as done, for example, in [23] or in [22] for the Tate motive. We start by making our setting concrete.

Let E be an elliptic curve defined over $\mathbb{Q}$ with CM by the ring of integers $\mathfrak{R}_0$ of a quadratic imaginary field K_0. We fix an isomorphism $\mathfrak{R}_0 \cong End(E)$ and we write Σ_0 for the implicit CM type of E. Let us write ψ_{K_0} for the Grössencharacter attached to E. That is, ψ_{K_0} is a Hecke character of K_0 of (ideal) type $(1,0)$ with respect to the CM type Σ_0 and satisfies $L(E,s) = L(\psi_{K_0}, s)$. We fix an odd prime p where the elliptic curve has good ordinary reduction. We fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and, using the selected CM type, we fix an embedding $K_0 \hookrightarrow \bar{\mathbb{Q}}$. The ordinary assumption implies that p splits in K_0, say to $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ where we write $\mathfrak{p}$ for the prime ideal that corresponds to the p-adic embedding $K_0 \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. We write N_E for the conductor of E and $\mathfrak{f}$ for the conductor of ψ_{K_0}.

We consider a finite totally real extension F (resp. $F' \supset F$) of $\mathbb{Q}$ which we assume unramified at the primes of $\mathbb{Q}$ that ramify in K_0 and at p. We write $\mathfrak{r}$ (resp. $\mathfrak{r}'$) for its ring of integers and we fix an integral ideal $\mathfrak{n}$ of $\mathfrak{r}$ that is relative prime to p and to N_E. Let K (resp K') be the CM-field FK_0 (resp. $F'K_0 = F'K$) and let $\mathfrak{R}$ (resp. $\mathfrak{R}'$) be its ring of integers. Let $F(p^\infty\mathfrak{n})$ be the ray class field of conductor $p^\infty\mathfrak{n}$ and let F' be ramified only at primes above p, hence $F \subset F' \subset F(p^\infty\mathfrak{n})$. Furthermore, assume F'/F to be cyclic of order p and that the primes of F' that ramify in F'/F are split in K'. That is if we write $\theta_{F'/F}$ for the relative different of F'/F then $\theta_{F'/F} = \mathfrak{P}\bar{\mathfrak{P}}$ in K'. We write Γ for the Galois group $Gal(F'/F) \cong Gal(K'/K)$. Note that in both F and F' all primes above p split in K and K' respectively. Finally we write τ for the nontrivial element (complex conjugation) of $Gal(K/F) \cong Gal(K'/F')$ and we set $g := [F : \mathbb{Q}]$.

We now consider the base changed elliptic curves E/F over F and E/F' over F'. We note that the above setting gives the following equalities between the L functions,

$$L(E/F,s) = L(\psi_K, s), \;\; L(E/F',s) = L(\psi_{K'}, s) \tag{1.1}$$

where $\psi_K := \psi_{K_0} \circ N_{K/K_0}$ and $\psi_{K'} := \psi_K \circ N_{K'/K} = \psi_{K_0} \circ N_{K'/K_0}$, that is the base-changed characters of ψ_{K_0} to K and K'.

We write G_F for the Galois group $Gal(F(p^\infty\mathfrak{n})/F)$ and $G_{F'} := Gal(F'(p^\infty\mathfrak{n})/F')$ for the analogue for F'. Note that the above setting (the ramification of F' over F) introduces a transfer map $ver : G_F \to G_{F'}$. Moreover we have an action of $\Gamma = Gal(F'/F)$ on $G_{F'}$ by conjugation. We consider the measures $\mu_{E/F}$ of G_F and $\mu_{E/F'}$ of $G_{F'}$ that interpolate the critical value at $s=1$ of the elliptic curve E/F and E/F' respectively twisted by finite order characters of conductor dividing $p^\infty\mathfrak{n}$. The precise interpolation property is a delicate issue in our setting that we will discuss in the next section. However we can state now the main theorem of our work. We write $\mathfrak{j}$ for the smallest ideal of $\mathfrak{r}$ which contains $\mathfrak{n}\mathfrak{f}\mathfrak{R} \cap F$ and such that its prime factors are inert or ramified in K. If we write $\mathfrak{J} := \mathfrak{j}\mathfrak{R}$ then we denote by $Cl_K(\mathfrak{J})$ the ray class group of the ray class field $K(\mathfrak{J})$. We define $Cl_K^-(\mathfrak{J})$ as the quotient of $Cl_K(\mathfrak{J})$ by the natural image of $(\mathfrak{r}/\mathfrak{j})^\times$. Similarly, we make the analogous definitions for K'.

Theorem 1.1. *We make the assumptions*

(1) The natural map $Cl_K^-(\mathfrak{J}) \to Cl_{K'}^-(\mathfrak{J})^\Gamma$ is an isomorphism,
(2) The natural map $Cl_F(1) \to Cl_{F'}(1)$ is an injection,
(3) The relative different $\theta_{F'/F}$ of F' over F is trivial in $Cl_{F'}^+$, the strict ideal class group of F'. That is, there is a totally positive $\xi \in F'$ such that $\theta_{F'/F} = (\xi)$.

Then,

$$\int_{G_F} \epsilon \circ ver \quad d\mu_{E/F} \equiv \int_{G_{F'}} \epsilon \quad d\mu_{E/F'} \mod p\mathbb{Z}_p \tag{1.2}$$

for all ϵ locally constant $\mathbb{Z}_p$-valued functions on $G_{F'}$ such that $\epsilon^\gamma = \epsilon$ for all $\gamma \in \Gamma$, where $\epsilon^\gamma(g) := \epsilon(\tilde{\gamma} g \tilde{\gamma}^{-1})$ for all $g \in G_{F'}$ and for some lift $\tilde{\gamma} \in Gal(F'(p^\infty \mathfrak{n})/F))$ of γ. More generally, if we relax the assumption (1) and assume only that $\imath : Cl_K^-(\mathfrak{J}) \hookrightarrow Cl_{K'}^-(\mathfrak{J})^\Gamma$ is injective, then equation (1) reads

$$\int_{G_F} \epsilon \circ ver \quad d\mu_{E/F} \equiv \int_{G_{F'}} \epsilon \quad d\mu_{E/F'} + \Delta(\epsilon) \mod p\mathbb{Z}_p, \tag{1.3}$$

where $\Delta(\epsilon)$ is an "error term" that depends on the cokernel of the map $\imath$.

Remarks:

(1) It can be shown, see for example [27], that the above congruences imply the following congruences between measures. If we write $f_{E/F}$ for the element in the Iwasawa algebra $\mathbb{Z}_p[[G_F]]$ that corresponds to the measure $\mu_{E/F}$ and similarly $f_{E/F'}$ for that in $\mathbb{Z}_p[[G'_F]]$, then we obtain the congruences

$$ver(f_{E/F}) \equiv f_{E/F'} \mod T, \tag{1.4}$$

where T is the trace ideal in $\mathbb{Z}_p[[G'_F]]^\Gamma$ generated by elements $\Sigma_{i=0}^{p-1} a^{\gamma^i}$ for $a \in \mathbb{Z}_p[[G'_F]]$. Note that $f_{E/F'}$ is in $\mathbb{Z}_p[[G'_F]]^\Gamma$ as it comes from base change from F. It is exactly this implication that motivates our work. The aim is to use this kind of congruences to establish the existence of non-commutative p-adic L-functions for our elliptic curve with respect to specific p-adic Lie groups, as for example Heisenberg type Lie groups, very much in the same spirit as done by Kato for the Tate motive $\mathbb{Z}_p(1)$ in [23] and by Kakde for false Tate curve extensions also for the Tate motive in [22].
(2) Our assumption that the elliptic curve is defined over $\mathbb{Q}$ is made mainly for simplicity reasons. Our considerations could be applied in a more general setting. One can consider as starting "object" a Hilbert-modular form over F with CM by K. The delicate issue however is the understanding of the various motivic periods that are associated to it. However the "philosophy" of our proof applies also in this setting.
(3) We believe that the term $\Delta(\epsilon)$ always vanishes but we cannot prove it yet.

(4) The assumption that ϵ is $\mathbb{Z}_p$-valued can be relaxed and consider any integrally-valued locally constant function. Then simply one obtains the congruences

$$\int_{G_F} \epsilon \circ ver \quad d\mu_{E/F} \equiv \int_{G_{F'}} \epsilon \quad d\mu_{E/F'} \mod p\mathbb{Z}_p[\epsilon], \tag{1.5}$$

where $\mathbb{Z}_p[\epsilon]$ is the ring of integers of the smallest extension of $\mathbb{Q}_p$ that contains the values of ϵ.

(5) The assumption that F'/F is ramified only at p could be modified and in general consider F' such that $F \subseteq F' \subseteq F(p^\infty \mathfrak{n})$ and all the primes of F that ramify in F' are split in K.

On the strategy of the proof: Let us finish the introduction by briefly explaining the main idea of the proof. As we will shortly explain we are going to construct the measures $\mu_{E/F}$ and $\mu_{E/F'}$ by using the so-called Katz measure for Hecke characters of CM fields. The reason for this should be intuitively clear from the above equation of L functions. These measures are constructed by using the fact (going back to Damerell's theorem) that the special values of the L-functions of Grössencharacters of a CM field K can be expressed as values of Hilbert-Eisenstein series on particular CM points. The modular meaning of these CM points is that they correspond to Hilbert-Blumenthal abelian varieties (HBAV) with CM by K of the same type as the character under consideration. In our relative setting we have that the Grössencharacter $\psi_{K'}$ is the base change of ψ_K. In particular, as we will explain in the next section, if we write (K, Σ) for the CM type of ψ_K, then the CM type of $\psi_{K'}$ is (K', Σ') where Σ' is the lift of Σ to K'. But now the key observation is that the HBAV with CM of type (K', Σ') are isogenous to $[K' : K]$-copies of HBAV with CM (K, Σ). In particular this says that the CM points that we need to evaluate our Eisenstein series over F' are in some sense coming from F through the natural diagonal embedding $\Delta : \mathbb{H}_F \hookrightarrow \mathbb{H}_{F'}$ of the Hilbert upper half planes. Note here the importance of Σ' being lifted from Σ. Hence we can pull-back the Hilbert-Eisenstein series that is used over F' to obtain a Hilbert-Eisenstein series over F, so that its values on the CM points of $\mathbb{H}_F$ under consideration are the same with those of the one over F' evaluated on the image of these CM points with respect to Δ. It is mainly this idea that we will use to prove the above congruences. We note here that a similar strategy was used by Kato [23] and Ritter and Weiss [27] for the cyclotomic character but in their works the L values appeared as the constant term of Hilbert-Eisenstein series (or as "values" at the cusp at infinity). We believe that this strategy is more general. We have applied similar considerations in [3] for other L-values that can be understood either as values at CM points or at infinity of Eisenstein series of the unitary group. Actually, what we are doing here could be rephrased in the unitary group setting, but we defer this discussion for our forthcoming work [3].

2. The Measures Attached to the Elliptic Curves E/F and E/F'

The statement of our main theorem involves measures on G_F (resp $G_{F'}$) with the property that integrating these measures against finite characters of G_F (resp $G_{F'}$) we obtain critical values of E/F (resp E/F')) twisted with these characters up to some modification. Now we proceed in explaining the construction of these measures and their interpolation properties. We point right away that there are various construction of these measures; the modular symbol construction, which we will not discuss at all, the construction of Katz, Hida and Tilouine, which we will use in the present work and, finally, in our specific setting, the construction of Colmez and Schneps which we also discuss shortly below. In order to explain the definition of the above-mentioned measures we need to introduce some more notation.

Archimedean and p-adic periods: Since the elliptic curve E is defined of $\mathbb{Q}$, we have that the class number of K_0 is one. In particular, we can fix a well-defined complex period for E as follows. We write Λ for the lattice of E, that is $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$. Then we define $\Omega_\infty(E) \in \mathbb{C}^\times$, uniquely up to elements in $\mathfrak{R}_0^\times$, as $\Lambda = \Omega_\infty(E)\sigma_0(\mathfrak{R}_0)$, where $\sigma_0 : K \hookrightarrow \mathbb{C}$ is the selected embedding. Moreover we define a p-adic period $\Omega_p(E) \in \mathcal{J}_\infty^\times$, where $\mathcal{J}_\infty$ denotes the ring of integers of the p-adic completion of the maximal abelian unramified extension of $\mathbb{Q}_p$. If we write Φ for the extension of Frobenious that operates on it, then it is well-known that this period is uniquely determined, up to elements in $\mathbb{Z}_p^\times$, by the property

$$\frac{\Omega_p(E)^\Phi}{\Omega_p(E)} = u \in \mathbb{Z}_p^\times, \tag{2.1}$$

where u is the p-adic unit determined by the equation (note that, as p is good ordinary for E, we have $a_p(E) := p + 1 - \sharp\tilde{E}_p(\mathbb{F}_p) \in \mathbb{Z}_p^\times$)

$$1 - a_p(E)X + pX^2 = (1 - uX)(1 - wX). \tag{2.2}$$

CM-types: We fix some CM-types for the CM fields K_0, K, K'. We have already fixed an embedding of $K_0 \hookrightarrow \mathbb{C}$, say σ_0 and defined the CM type of K_0 by $\Sigma_0 = \{\sigma_0\}$. We normalized things so that the character ψ_{K_0} is of infinite type $1\ \Sigma_0$. Now we fix a CM type Σ of K by taking the lift of Σ_0 to K. That is, we pick embeddings that restrict to σ_0 in K_0. We also define Σ' to be the lift of Σ in K'. We note two things for these CM-types. First the characters ψ_K and ψ'_K are of type 1Σ and $1\Sigma'$. Second the types just picked are also p-ordinary in the terminology of Katz, that simply amounts to picking the primes of K and K' that are above $\mathfrak{p}$. We denote these sets of primes as Σ_p and Σ'_p respectively. Of course we set also $\Sigma_{0,p} = \{\mathfrak{p}\}$. Finally, we note that all abelian varieties of dimension $[F : \mathbb{Q}]$ with CM by K (resp dimension $[F' : \mathbb{Q}]$ and CM by K') and type Σ (resp. Σ') are isogenous to the product of $[F : \mathbb{Q}]$ (resp. $[F' : \mathbb{Q}]$) copies of the elliptic curve E (see [26, Theorem 4.4. page 19]).

The ∞-types of the Grössencharacters: For the Grössencharacter ψ_{K_0} we have that $\psi_{K_0}\bar{\psi}_{K_0} = N_{K_0/\mathbb{Q}}$ and that $\psi_{K_0}(\bar{\mathfrak{q}}) = \overline{\psi_{K_0}(\mathfrak{q})}$. In particular,

$$L(\psi_{K_0}^{-1}, 0) = L(\bar{\psi}_{K_0} N_{K/\mathbb{Q}}^{-1}, 0) = L(\bar{\psi}_{K_0}, 1) = L(\psi_{K_0}, 1). \tag{2.3}$$

Moreover, if we consider twists by finite cyclotomic characters, that is characters of the form $\chi = \chi' \circ N_{K_0/\mathbb{Q}}$ for χ' a finite Dirichlet character of $\mathbb{Q}$, we have that $L(\psi_{K_0}\chi, 1) = L(\psi_{K_0}^{-1}\chi, 0)$. The same considerations apply of course for ψ_K and $\psi_{K'}$. So from now on we are going to consider characters of infinite type $-k\Sigma_0$, $-k\Sigma$ and $-k\Sigma'$ for the various CM-types and $k \geq 1$ and the critical values that we study are at $s = 0$. The above equation explains why these are the values that we are interested in.

We now recall the interpolation properties of a slight modification of a p-adic measure μ_δ^{KHT} for Hecke characters constructed by Katz [25] and later extended by Hida and Tilouine in [17]. Let $\mathfrak{C}$ be some integral ideal of K relative prime to p. Then, for a Hecke character χ of $G_K := Gal(K(\mathfrak{C}p^\infty)/K)$ of infinite type $-k\Sigma$, we have

$$\frac{\int_{G_K} \chi(g)\mu_\delta^{KHT}(g)}{\Omega_p^{k\Sigma}} = (\mathfrak{R}^\times : \mathfrak{r}^\times) Local(\Sigma, \chi, \delta) \frac{(-1)^{kg}\Gamma(k)^g}{\sqrt{D_F}\Omega_\infty^{k\Sigma}} \times$$

$$\prod_{\mathfrak{q}|\mathfrak{F}\mathfrak{I}} (1 - \check{\chi}(\bar{\mathfrak{q}})) \prod_{\mathfrak{q}|\mathfrak{F}} (1 - \chi(\bar{\mathfrak{q}})) \prod_{\mathfrak{p}\in\Sigma_p} (1 - \chi(\bar{\mathfrak{p}}))(1 - \check{\chi}(\bar{\mathfrak{p}}))L(0, \chi),$$

where the ideals $\mathfrak{F}, \mathfrak{I}$ are some factors of $\mathfrak{C}$ and will be defined in the next section and $\check{\chi}$ is a Hecke character defined by the equality $\chi(\mathfrak{q})\check{\chi}(\bar{\mathfrak{q}}) = N_{K/\mathbb{Q}}(\mathfrak{q})^{-1}$. Also, in the next section, we will explain in details the construction of the above measure, but for the time being we just want to indicate three points:

(1) The measure depends on a choice of totally imaginary element $\delta \in K$, with $Im(\delta^\sigma) > 0$ for $\sigma \in \Sigma$ and such that its valuation at $\mathfrak{p} \in \Sigma_p$ is equal with the valuation of the absolute different of K.
(2) The periods (archimedean and p-adic) that appear above depend only on the CM type Σ and not at all on the finite part of the Hecke character χ.
(3) The factor $Local(\chi, \Sigma, \delta)$ is similar to some epsilon factor of χ, but not equal. We will explain more on that shortly.

We have fixed above a Grössencharacter ψ_K (note that $k = 1$ for this character). We set, with notation as in the introduction, $\mathfrak{C} := \mathfrak{n}\mathfrak{f}\mathfrak{R}$ and we consider the measure of G_K defined for every finite character χ of G_K by

$$\int_{G_K} \chi(g)\mu_{\psi_K,\delta}^{KHT}(g) := \int_{G_K} \chi(g)\hat{\psi}_K^{-1}(g)\mu_\delta^{KHT}(g), \tag{2.4}$$

where $\hat{\psi}_K$ is the p-adic avatar of ψ_K constructed by Weil, see for example [21, Theorem 1.1 page 12]. We will show later that in this case we can set $\Omega_p^\Sigma = \Omega_p(E)^g$ and $\Omega_\infty^\Sigma = \Omega(E)^g$. Then, we define the measure $\mu_{E/F}$ discussed above by (recall that $G_F := Gal(F(\mathfrak{n}p^\infty)/F)$)

$$\int_{G_F} \chi(g)\mu_{E/F}(g) := \frac{\int_{G_K} \tilde{\chi}(g)\mu_{\psi_K,\delta}^{KHT}(g)}{\Omega_p(E)^g}, \tag{2.5}$$

where $\tilde{\chi}$ is the base change of χ from F to K. Then, from our remarks on the critical value $L(E/F, 1)$, we see that this measure interpolates twists of this critical value of E/F. The same considerations apply also for the datum $(K', F', \psi_{K'}, G_{F'}, G_{K'})$. We now observe that our main theorem above amounts to prove the following congruences, under of course the same assumptions as in the theorem above,

$$\frac{\int_{G_K} \epsilon \circ ver \quad d\mu_{\psi_K,\delta}^{KHT}}{\Omega_p(E)^g} \equiv \frac{\int_{G_{K'}} \epsilon \quad d\mu_{\psi_{K'},\delta'}^{KHT}}{\Omega_p(E)^{pg}} \mod p\mathbb{Z}_p \tag{2.6}$$

for all ϵ locally constant $\mathbb{Z}_p$-valued functions on $G_{K'}$ with $\epsilon^\gamma = \epsilon$, which belong to the cyclotomic part of it, i.e. when it is written as a sum of finite order characters it is of the form $\epsilon = \sum c_\chi \chi$ with $\chi^\tau = \chi$.

However, these congruences do not hold when the extension F'/F is ramified. In order to overcome this difficulty we will need to modify (twist) the measure of Katz-Hida-Tilouine over K'. The key question is whether the factor $Local(\chi, \Sigma, \delta)$ is the "right" one. We believe that this is not so when the extension F'/F is ramified (in the appendix we offer evidences for this) and actually with our modification we aim to overcome this problem. In short, we will define for the datum $(K', F', \psi_{K'}, G_{F'}, G_{K'})$ the measure $\mu_{E/F'}$ as

$$\int_{G_{F'}} \chi(g)\mu_{E/F'}(g) := \frac{\int_{G_{K'}} \tilde{\chi}(g)\mu^{KHT,tw}_{\psi_{K'},\delta,\xi}(g)}{\Omega_p(E)^{pg}} := \frac{\int_{G_{K'}} \tilde{\chi}(g)\hat{\psi}^{-1}_{K'}(g)\mu^{KHT,tw}_{\delta,\xi}(g)}{\Omega_p(E)^{pg}}, \tag{2.7}$$

where the measure $\mu^{KHT,tw}_{\delta,\xi}$, called in this work the twisted Katz-Hida-Tilouine measure, will be defined in section 4. Then we will show that

$$\frac{\int_{G_K} \epsilon \circ ver \quad d\mu^{KHT}_{\psi_K,\delta}}{\Omega_p(E)^g} \equiv \frac{\int_{G_{K'}} \epsilon \quad d\mu^{KHT,tw}_{\psi_{K'},\delta,\xi}}{\Omega_p(E)^{pg}} \mod p\mathbb{Z}_p \tag{2.8}$$

for all ϵ locally constant $\mathbb{Z}_p$-valued functions on $G_{K'}$ with $\epsilon^\gamma = \epsilon$ which belong to the cyclotomic part of it.

The measure of Colmez and Schneps: We close this section by making a few more observations. In the setting that we consider we can apply the construction of [9]. Indeed, in this work Colmez and Schneps construct a measure of $G_K := Gal(K(\mathfrak{C}p^\infty)/K)$ such that, for every Grössencharacter χ of K of infinite type $\chi((a)) = N_{K/K_0}(a))^{-k}$ for $a \equiv 1$ modulo the conductor of χ,

$$\int_{G_K} \chi(g)\mu^{CS}(g) = (-1)^{kg}\Gamma(k)^g \prod_{\mathfrak{p}\in\Sigma_p} e_{\mathfrak{p}}(\chi, \psi, dx_1) \prod_{\mathfrak{q}|\mathfrak{C}} (1-\chi(\mathfrak{q})) \prod_{\mathfrak{p}\in\Sigma_p} (1-\chi(\bar{\mathfrak{p}}))(1-\check{\chi}(\bar{\mathfrak{p}}))L(0,\chi).$$

Although Colmez and Schneps do not work the algebraicity of the measure we see here that their measure is normalized differently from that of Katz-Hida-Tilouine with respect to the local factors. Here one gets the epsilon factors of Deligne as local factors. It is exactly this construction that we explore in a common work with Filippo Nuccio [6] where we try to obtain a different proof of the congruences hoping also to relax some of the assumptions of the current work.

3. The (twisted-)Eisenstein Measure of Katz-Hida-Tilouine

We start by recalling some Eisenstein series appearing in the work of Katz [25] and Hida and Tilouine [17]. For reasons that will become clear later we need to introduce a slight modification of these series. We follow the notations of Hida and Tilouine and introduce the setting described in their paper. We consider a totally real field F with ring of integers $\mathfrak{r}$ and write θ for the different of $F/\mathbb{Q}$. We also fix an odd prime p. For an ideal $\mathfrak{a}$ of F we write $\mathfrak{a}^* = \mathfrak{a}^{-1}\theta^{-1}$. We fix a fractional ideal $\mathfrak{c}$ and take two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$. Let $\phi : \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f}')\} \times \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f}'')\} \to \mathbb{C}$ be a locally constant function such that $\phi(\epsilon^{-1}x, \epsilon y) = N(\epsilon)^k \phi(x, y)$, for all $\epsilon \in \mathfrak{r}^\times$, k some positive integer and $\mathfrak{f}'$ and $\mathfrak{f}''$ integral ideals relative prime to p. We put $\mathfrak{f} := \mathfrak{f}' \cap \mathfrak{f}''$.

On the ideals $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ we put the following restrictions. For the ideal $\mathfrak{a}$ we always assume that $(\mathfrak{a}, p\mathfrak{f}) = 1$, i.e. it is taken relative prime to $p\mathfrak{f}$. The ideal $\mathfrak{c}$ will always be taken of the form $\mathfrak{c}_1\mathfrak{c}_2$, where $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are fractional ideals that satisfy,

(1) $(\mathfrak{c}_1, p\mathfrak{f}) = 1$,
(2) $\mathfrak{c}_2 = \prod_{\mathfrak{p}_j | p} \mathfrak{p}^{a_{\mathfrak{p}_j}}$, with $a_{\mathfrak{p}_j} \geq 0$. Here the $\mathfrak{p}_j$'s denote the prime ideals of F above p.

This in turn implies that $\mathfrak{b}$ is of the form $\mathfrak{b}_1\mathfrak{b}_2$ with $(\mathfrak{b}_1, p\mathfrak{f}) = 1$ and $\mathfrak{b}_2 = \mathfrak{c}_2^{-1}$. We now define the spaces $T_1 := \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f})\}$ and $T_2 := \{(\mathfrak{c}^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p) \times (\mathfrak{r}/\mathfrak{f})\}$. Note that our assumptions imply that $\mathfrak{r}_p \subset \mathfrak{c}^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p$ and hence

$$T \subseteq T_1 \times T_2,$$

where $T := \{\mathfrak{r}_p \times \mathfrak{r}/\mathfrak{f}\} \times \{\mathfrak{r}_p \times \mathfrak{r}/\mathfrak{f}\}$. Using the canonical projection

$$T \to \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f}')\} \times \{\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f}'')\},$$

we may consider ϕ as a locally constant function on T. We extend ϕ by zero to a function of $T_1 \times T_2$. Actually, in our applications, we will be given functions on $T^\times := \{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\} \times \{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\}$, which we will extend trivially by zero to functions on $T_1 \times T_2$. We define the partial Fourier transform of the first variable of ϕ and write

$$P\phi : \{F_p/\theta_p^{-1} \times \mathfrak{f}^*/\theta^{-1}\} \times T_2 \to \mathbb{C} \tag{3.1}$$

as

$$P\phi(x, y) = p^{-\alpha[F:\mathbb{Q}]N(\mathfrak{f})^{-1}} \sum_{a \in X_\alpha} \phi(a, y) e_F(ax), \quad e_F(ax) := exp(2\pi i Tr_{F/\mathbb{Q}}(2ax)), \tag{3.2}$$

for ϕ factoring through $X_\alpha \times T_2$ with $X_\alpha := \mathfrak{r}_p/\alpha\mathfrak{r}_p \times (\mathfrak{r}/\mathfrak{f})$ and $\alpha \in \mathbb{N}$. We now attach an Eisenstein series to ϕ. This is realized over the complex number as a rule on triples $(\mathcal{L}, \lambda, \imath)$, where $\mathcal{L}$ a lattice in $\mathbb{C}^{[F:\mathbb{Q}]}$ with real multiplication, λ-polarized with λ an isomorphism $\bigwedge_{\mathfrak{r}}^2 \mathcal{L} \cong \theta^{-1}\mathfrak{c}^{-1}$ and $\imath$ a $p^\infty\mathfrak{f}^2$ level structure.

The partial Tate module: From the $p^\infty\mathfrak{f}^2$ structure after restriction we obtain a projection π'

$$\pi' : (\mathcal{L} \otimes_{\mathfrak{r}} \mathfrak{r}_p) \times \mathcal{L}/\mathfrak{f}\mathcal{L} \to (\mathfrak{c}^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p) \times (\mathfrak{r}/\mathfrak{f}\mathfrak{r}) = T_2. \tag{3.3}$$

The second component of this projection, that is $\mathcal{L}/\mathfrak{f}\mathcal{L} \to \mathfrak{r}/\mathfrak{f}\mathfrak{r}$ is obtained as in [17, page 206]. For the first, we follow Katz [25, page 246] we note that we have a short exact sequence of free $\mathfrak{r}_p$-modules

$$0 \to \theta^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p \to \mathcal{L} \otimes_{\mathfrak{r}} \mathfrak{r}_p \to ? \to 0. \tag{3.4}$$

From the given polarization we obtain an isomorphism

$$\bigwedge_{\mathfrak{r}_p}^2 (\mathcal{L} \otimes_{\mathfrak{r}} \mathfrak{r}_p) \cong \theta^{-1}\mathfrak{c}^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p. \tag{3.5}$$

We conclude that

$$? \cong \mathfrak{c}^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p, \tag{3.6}$$

which explains the first part of the projection π'. Following Hida and Tilouine (loc. cit.) we then define the partial Tate module $PV(\mathcal{L})$ as a submodule of $\mathcal{L} \otimes_{\mathfrak{r}} F_{p\mathfrak{f}}$ that contains $\mathcal{L} \otimes_{\mathfrak{r}} \mathfrak{r}_{p\mathfrak{f}}$ such that

$$PV(\mathcal{L})/\mathcal{L} \otimes_{\mathfrak{r}} \mathfrak{r}_{p\mathfrak{f}} \cong Im(F_p/\theta^{-1} \times \mathfrak{f}^*/\theta^{-1} \to p^{-\infty}\mathcal{L}/\mathcal{L} \times \mathfrak{f}^{-1}\mathcal{L}/\mathcal{L}). \tag{3.7}$$

Then, as explained in [17], one obtains the projections

$$\pi' : PV(\mathcal{L}) \twoheadrightarrow T_2 \quad and, \quad \pi : PV(\mathcal{L}) \twoheadrightarrow F_p/\theta_p^{-1} \times \mathfrak{f}^*/\theta^{-1}. \tag{3.8}$$

We set $\mathcal{L}(\mathfrak{f}p) := \mathfrak{f}^{-1}p^{-\infty}\mathcal{L} \cap PV(\mathcal{L})$ and, for a $w \in \mathcal{L}(\mathfrak{f}p)$ we define $P\phi(w) := P\phi(\pi(w), \pi'(w))$. For an integer $k \geq 1$ we define the $\mathfrak{c}$-polarized HMF $E_k(\phi, \mathfrak{c})$ by

$$E_k(\phi, \mathfrak{c})(\mathcal{L}, \lambda, \imath) := \frac{(-1)^{kg}\Gamma(k+s)^g}{\sqrt{(D_F)}} \sum_{w \in \mathcal{L}(\mathfrak{f}p)/\mathfrak{r}^\times} \frac{P\phi(w)}{N(w)^k |N(w)^{2s}|} \,|_{s=0} \,. \tag{3.9}$$

We now have the proposition:

Proposition 3.1. *There exists a $\mathfrak{c}$-HMF $E_k(\phi, \mathfrak{c})$ of level $p^\infty \mathfrak{f}^2$ and weight k such that, if $k \geq 2$ or $\phi(a, 0) = 0$ for all a, then its q-expansion at the cusp $(\mathfrak{a}, \mathfrak{b})$ (with our assumptions on $\mathfrak{a}$ and $\mathfrak{b}$) is given by*

$$\begin{aligned} E_k(\phi, \mathfrak{c})(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, \omega_{can}, i_{can}) = N(\mathfrak{a})\{2^{-g}L(1-k, \phi, \mathfrak{a}) \\ + \sum_{0 \ll \xi \in \mathfrak{a}\mathfrak{b}} \sum_{(a,b) \in (\mathfrak{a} \times \mathfrak{b})/\mathfrak{r}^\times, ab = \xi} \phi(a, b) sgn(N(a)) N(a)^{k-1} q^\xi\} \end{aligned} \tag{3.10}$$

where $L(s; \phi, \mathfrak{a}) = \sum_{x \in (\mathfrak{a}-0)/\mathfrak{r}^\times} \phi(x, 0) sgn(N(x))^k |N(x)|^{-s}$.

Proof. The proposition is in principle the one stated in [25, page 247, theorem (3.2.3)] and [17, page 208]. Indeed, the fact that the defined series is a Hilbert modular form of parallel weight k follows as in Katz. First, as Katz remarks, we can restrict ourselves to work over $\mathbb{C}$. Then (see also [25, page 236, equation (2.3.36)] for a moduli interpretation of the following equation),

$$E_k(\phi, \mathfrak{c})(a^{-1}\mathcal{L}, a\bar{a} < \cdot, \cdot >, a^{-1} \times \imath) = N(a)^k |N(a)^{2s}| E_k(\phi, \mathfrak{c})(\mathcal{L}, < \cdot, \cdot >, \imath), \tag{3.11}$$

since for $w' = a^{-1}w \in a^{-1}\mathcal{L}$ with $w \in \mathcal{L}$ we have $P\phi(w') = P\phi(w)$. Indeed the projections $\pi_{\mathcal{L}}$ and $\pi'_{\mathcal{L}}$ of $PV(\mathcal{L})$ and $\pi_{a^{-1}\mathcal{L}}$ and $\pi'_{a^{-1}\mathcal{L}}$ of $PV(a^{-1}\mathcal{L})$ are related by

$$\pi_{\mathcal{L}}(a \times y) = \pi_{a^{-1}\mathcal{L}}(y), \;\; y \in a^{-1}\mathcal{L}$$

and

$$\pi'_{\mathcal{L}}(a \times y) = \pi'_{a^{-1}\mathcal{L}}(y), \;\; y \in a^{-1}\mathcal{L}.$$

These last equalities follow by observing that the exact sequence

$$0 \to \theta^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p \xrightarrow{\imath} \mathcal{L} \otimes_{\mathfrak{r}} \mathfrak{r}_p \xrightarrow{\pi_{\mathcal{L}}} \mathfrak{c}^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p \to 0$$

induces the exact sequence

$$0 \to \theta^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p \overset{a^{-1}\times\imath}{\to} a^{-1}\mathcal{L} \otimes_{\mathfrak{r}} \mathfrak{r}_p \overset{\pi_{\mathcal{L}}(a\times\cdot)}{\to} \mathfrak{c}^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p \to 0,$$

from which the partial Tate module $PV(a^{-1}\mathcal{L})$ is constructed.

Since both sides of the equation 3.11 have analytic continuation, evaluating at $s = 0$, we see that the defined series have parallel weight k. The holomorphicity at $s = 0$ follows as in Katz by studying the q-expansion. Now the proof of the q-expansion written above is exactly as in Katz or in Hida and Tilouine (loc. cit.). However we have to comment on an assumption that is made on both of these works with respect the ideals $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. Namely, there it is assumed that $(\mathfrak{a}, p\mathfrak{f}) = 1$ (as in our case) but also that $(\mathfrak{c}, p\mathfrak{f}) = 1$. However this second assumption in both of these works is made not in order to establish the above proposition but to prove the functional equation of the Eisenstein series (see [25, page 253] and [17, page 225] where the assumption is crucially used). Indeed, one can follow the proof in [25, pages 248-252] or [17, pages 207-208] to see that the assumption on $\mathfrak{c}$ (and hence also on $\mathfrak{b}$, given the restrictions on $\mathfrak{a}$ and the equation $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$) does not play any role at all. Indeed, for the proof of these propositions, one uses crucially that $(\mathfrak{a}, p\mathfrak{f}) = 1$. That is, the cusp $(\mathfrak{a}, \mathfrak{b})$ must be unramified for the given numerical structure (see [12, page 259] for the definition of unramified cusps), since only in the first variable is taken the Fourier transform (in the second variable there is nothing happening). Of course, if one wants to prove the functional equation for the Eisenstein series, he has to interchange the roles of $\mathfrak{a}$ and $\mathfrak{b}$. In particular, in the case that $(\mathfrak{c}, p\mathfrak{f}) \neq 1$, the unramified cusp $(\mathfrak{a}, \mathfrak{b})$ will be associated to the ramified cusp $(\mathfrak{b}, \mathfrak{a})$. Concluding, we have that our modification still do give us the q-expansion stated above but not the functional equation stated in [25, page 254 theorem (3.3.13)] or [17, page 227 equation (5.2)]. □

Actually, in all of our applications we will have that the function ϕ will be supported in $\{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\} \times \{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\}$ (i.e. it will be zero outside this domain). Hence, because of the support assumption of the second variable, the sum that appear in [17, page 206] or [25, page 249])

$$\sum_{(a,b)\in\{(p^n\mathfrak{a}\mathfrak{f})^*\times\mathfrak{b}\}/\mathfrak{r}^\times} \frac{P\phi(a,b)}{N(a+bz)^k|N(a+bz)^{2s}|},$$

with $(p^n\mathfrak{a}\mathfrak{f})^* = p^{-n}\mathfrak{a}^{-1}\mathfrak{f}^{-1}\theta^{-1}$, will be simplified to

$$\sum_{(a,b)\in\{(p^n\mathfrak{a}\mathfrak{f})^*\times\mathfrak{b}_1\}/\mathfrak{r}^\times} \frac{P\phi(a,b)}{N(a+bz)^k|N(a+bz)^{2s}|},$$

where we recall $\mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2$ with $\mathfrak{b}_1$ and $\mathfrak{b}_2$ as defined above. In particular $(\mathfrak{b}_1, p\mathfrak{f}) = 1$ so one can redo the proof of Katz or Hida and Tilouine from this point on. Note that the q-expansion then will be of the form

$$E_k(\phi, \mathfrak{c})(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, \omega_{can}, i_{can}) =$$

$$N(\mathfrak{a})\{\sum_{0\ll\xi\in\mathfrak{a}\mathfrak{b}_1}\sum_{(a,b)\in(\mathfrak{a}\times\mathfrak{b}_1)/\mathfrak{r}^\times, ab=\xi}\phi(a,b)sgn(N(a))N(a)^{k-1}q^\xi\}.$$

Remark: The following remarks are in order:

(1) In the case that the locally constant function ϕ is supported on $\{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\} \times \{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\}$ then the Eisenstein series has constant term equal to zero at the cusp $(\mathfrak{a},\mathfrak{b})$.
(2) Note that the p-integrality of the q-expansion follows from the values of the function ϕ and from the fact that $(\mathfrak{a}, p) = 1$.

The Eisenstein Measure of Katz-Hida-Tilouine: Hida and Tilouine extended the work of Katz to obtain measures of the Galois group $Gal(K(\mathfrak{C}p^\infty)/K)$ for K a CM field and $\mathfrak{C}$ an integral ideal of K. We briefly describe the construction and the interpolation properties of these measures. We start with the decomposition $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_{\mathfrak{c}}\mathfrak{I}$ such that

$$\mathfrak{F} + \mathfrak{F}_c = \mathfrak{R},\ \ \mathfrak{F} + \mathfrak{F}^c = \mathfrak{R},\ \ \mathfrak{F}_c + \mathfrak{F}_c^c = \mathfrak{R},\ \ \mathfrak{F}_c \supset \mathfrak{F}^c \tag{3.12}$$

and $\mathfrak{I}$ consists of ideals that are inert or ramify in K/F. We set $\mathfrak{f}' := \mathfrak{F}\mathfrak{I} \cap F$ and $\mathfrak{f}'' := \mathfrak{F}_c\mathfrak{I}\cap F$, $\mathfrak{f} := \mathfrak{f}'\cap\mathfrak{f}'' = \mathfrak{f}'$, $\mathfrak{s} = \mathfrak{F}_c\cap F$ and $\mathfrak{j} := \mathfrak{I}\cap F$. As in Hida and Tilouine, we consider the homomorphism obtained from class field theory

$$i : \{(\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times \times \mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{s})^\times)/\overline{\mathfrak{r}^\times}\} \to Cl_K(\mathfrak{C}p^\infty). \tag{3.13}$$

We write $Cl_K^-(\mathfrak{I})$ for the quotient of $Cl_K(\mathfrak{I})$ by the natural image of $(\mathfrak{r}/\mathfrak{j})^\times$. If $\{\mathfrak{U}_j\}_j$ are representatives of $Cl_K^-(\mathfrak{I})$, which we pick relative prime to $p\mathfrak{C}\mathfrak{C}^c$, then we have that $Cl_K(\mathfrak{C}p^\infty) = \coprod_j Im(i)[\mathfrak{U}_j]^{-1}$ where $[\mathfrak{U}_j]$ the image of $\mathfrak{U}_j$ in $Cl_K(\mathfrak{C}p^\infty)$. We use the surjection $(\mathfrak{r}/\mathfrak{f})^\times \to (\mathfrak{r}/\mathfrak{s})^\times$ to obtain a projection

$$T' := \{(\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times \times \mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times)/\overline{\mathfrak{r}^\times}\} \twoheadrightarrow \{(\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times \times \mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{s})^\times)/\overline{\mathfrak{r}^\times}\}. \tag{3.14}$$

Given a continuous function ϕ of $Cl_K(\mathfrak{C}p^\infty) \cong Gal(K(\mathfrak{C}p^\infty)/K) =: G$ we define a function $\tilde{\phi}_j$ on $Im(i)$ by $\tilde{\phi}_j(x) := \phi(x[\mathfrak{U}_j^{-1}])$ and through the above projection we view $\tilde{\phi}_j$ as function on T'. Moreover we write $\mathbf{N}$ for the function

$$\mathbf{N} : \{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\} \times \{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\} \to \mathbb{Z}_p^\times \tag{3.15}$$

given by $\mathbf{N}(x, a, y, b) = \prod_{\sigma\in\Sigma_p} x_\sigma$. Then we define functions ϕ_j on $\{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\} \times \{\mathfrak{r}_p^\times \times (\mathfrak{r}/\mathfrak{f})^\times\}$ by $\phi_j(x, a, y, b) := \mathbf{N}(x)^{-1}\tilde{\phi}_j(x^{-1}, a^{-1}, y, b)$. We view ϕ_j and $\mathbf{N}$ as functions on $T_1 \times T_2$ by extending them trivially by zero as we described at the beginning of the section.

In order to define the measure of Katz, Hida and Tilouine we need to pick polarization of HBAV with complex multiplication by $\mathfrak{R}$ and CM type Σ. We pick an element $\delta \in K$ such that

(1) $\delta^c = -\delta$ and $Im(\delta^\sigma) > 0$ for all $\sigma \in \Sigma$,
(2) The polarization $< u, v >:= \frac{u^c v - u v^c}{2\delta}$ on $\mathfrak{R}$ induces the isomorphism $\mathfrak{R} \wedge_{\mathfrak{r}} \mathfrak{R} \cong \theta^{-1}\mathfrak{c}^{-1}$ for $\mathfrak{c}$ relative prime to p.

After the above choice of δ, we can attach (see [17] page 211 for details) to the fractional ideals $\mathfrak{U}_j$ of K a datum $(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), \imath(\mathfrak{U}_j))$ consisting of a HBAV $X(\mathfrak{U}_j)$ with CM of type (K, Σ), a polarization $\mathfrak{c}\mathfrak{U}_j^{-1}\mathfrak{U}_j^{-c}$ and a level structure $\imath(\mathfrak{U}_j)$ of type $p^\infty\mathfrak{f}^2$. We define the measure μ_δ^{KHT} as (see [25, pages 260-261])

$$\int_G \phi(g)\mu_\delta^{KHT}(g) := \sum_j \int_T \tilde{\phi}_j dE_j := \sum_j E_1(\phi_j, \mathfrak{c}_j)(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), \imath(\mathfrak{U}_j)), \tag{3.16}$$

where $\mathfrak{c}_j := \mathfrak{c}(\mathfrak{U}_j\mathfrak{U}_j^c)^{-1}$. We note here that, when ϕ is a character of infinite type $-k\Sigma$, then we have that

$$E_1(\phi_j, \mathfrak{c}_j)(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), \imath(\mathfrak{U}_j)) = \phi\left([\mathfrak{U}_j]^{-1}\right) E_k(\phi_{finite}, \mathfrak{c}_j)(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), \imath(\mathfrak{U}_j), \omega^{can}(\mathfrak{U}_j)), \tag{3.17}$$

where ϕ_{finite} is as in [25] page 277 and the above equation is explained in (5.5.7) of (loc. cit.). Using this equation we remark that for a character ϕ of infinite type $-k\Sigma$ we have

$$\int_G \phi(g)\mu_\delta^{KHT}(g) = \sum_j E_1(\phi_j, \mathfrak{c}_j)\left(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), i(\mathfrak{U}_j)\right) =$$

$$\sum_j \phi\left([\mathfrak{U}_j^{-1}]\right) E_1\left(\mathbf{N}(x)^{k-1}\phi_{finite}(x^{-1}, a^{-1}, y, b), \mathfrak{c}_j\right)\left(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), i(\mathfrak{U}_j)\right) =$$

$$\sum_j \phi\left([\mathfrak{U}_j^{-1}]\right) E_k\left(\phi_{finite}(x^{-1}, a^{-1}, y, b), \mathfrak{c}_j\right)\left(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), i(\mathfrak{U}_j), \omega^{can}(\mathfrak{U}_j)\right).$$

For our later applications (in section 7) it is important for us to consider the integrals $\int_G \phi(g)\chi(g)\mu_\delta^{KHT}(g)$ where ϕ is a locally constant function on G and χ a character of infinite type $-k\Sigma$. We may write $\chi(x, a, y, b) = \mathbf{N}(x)^{-k}\chi_{finite}(x, a, y, b)$ and hence

$$\widetilde{\phi\chi}_j(t) = \tilde{\phi}_j(t)\chi(t)\chi(\mathfrak{U}_j^{-1}) = \tilde{\phi}_j(x, a, y, b)\mathbf{N}(x)^{-k}\chi_{finite}(x, a, y, b)\chi(\mathfrak{U}_j^{-1}).$$

We write

$$(\phi\chi)_j(x, a, y, b) := \mathbf{N}(x)^{-1}\widetilde{\phi\chi}_j(x^{-1}, a^{-1}, y, b)$$

$$= \tilde{\phi}_j(x^{-1}, a^{-1}, y, b)\mathbf{N}(x)^{k-1}\chi_{finite}(x^{-1}, a^{-1}, y, b)\chi(\mathfrak{U}_j^{-1}).$$

Then we have,

$$\int_G \phi(g)\chi(g)\mu_\delta^{KHT}(g) = \sum_j E_1\left((\phi\chi)_j, \mathfrak{c}_j\right)\left(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), i(\mathfrak{U}_j)\right)$$

$$= \sum_j \chi(\mathfrak{U}_j^{-1})E_1\left(\tilde{\phi}_j(x^{-1}, a^{-1}, y, b)\mathbf{N}(x)^{k-1}\chi_{finite}(x^{-1}, a^{-1}, y, b), \mathfrak{c}_j\right)\left(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), i(\mathfrak{U}_j)\right)$$

$$= \sum_j \chi(\mathfrak{U}_j^{-1})E_k\left(\tilde{\phi}_j(x^{-1}, a^{-1}, y, b)\chi_{finite}(x^{-1}, a^{-1}, y, b), \mathfrak{c}_j\right)\left(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), i(\mathfrak{U}_j), \omega^{can}(\mathfrak{U}_j)\right).$$

In particular, setting $(\phi\chi)_{j,finite} := \tilde{\phi}_j(x^{-1}, a^{-1}, y, b^{-1})\chi_{finite}(x^{-1}, a^{-1}, y, b)\chi(\mathfrak{U}_j^{-1})$,

$$\int_G \phi(g)\chi(g)\mu_\delta^{KHT}(g) = \sum_j E_k\left((\phi\chi)_{j,finite}, \mathfrak{c}_j\right)\left(X(\mathfrak{U}_j), \lambda(\mathfrak{U}_j), i(\mathfrak{U}_j), \omega^{can}(\mathfrak{U}_j)\right).$$

We now state the interpolation properties of the measure. For a reminder of the definition of the archimedean and p-adic periods appearing below the reader can check [25, page 269] or the definition right after the statement of proposition 3.5 in this paper.

Theorem 3.2 (Interpolation Properties). *For a character χ of $G := Gal(K(\mathfrak{C}p^\infty)/K)$ of infinite type $-k\Sigma$ we have*

$$\frac{\int_G \chi(g)\mu_\delta^{KHT}(g)}{\Omega_p^{k\Sigma}} = (\mathfrak{R}^\times : \mathfrak{r}^\times) Local(\Sigma, \chi, \delta)\frac{(-1)^{kg}\Gamma(k)^g}{\sqrt{D_F}\Omega_\infty^{k\Sigma}} \times$$

$$\prod_{\mathfrak{q}|\mathfrak{F}\mathfrak{I}}(1 - \check{\chi}(\bar{\mathfrak{q}}))\prod_{\mathfrak{q}|\mathfrak{F}}(1 - \chi(\bar{\mathfrak{q}}))\prod_{\mathfrak{p}\in\Sigma_p}(1 - \chi(\bar{\mathfrak{p}}))(1 - \check{\chi}(\bar{\mathfrak{p}}))L(0, \chi). \tag{3.18}$$

Proof. This is in principle the measure constructed by Katz and Hida-Tilouine in [17,25]. The main difference of the above formula with the one in Theorem 4.1 of [17] is that we do also the partial Fourier transform for the primes that divide $\mathfrak{F}\mathfrak{I}$ (this is why in our definition we used ϕ and not ϕ^0 as Hida and Tilouine do (page 209). Note that the computations in their work are local, so what we do amounts simply moving some of the epsilon factors away from p to the other part of the functional equation (compare with theorem 4.2 in Hida and Tilouine). □

The reason for doing this slight modification is related with the values of the measures $\mu_{E/F}$ and $\mu_{E/F'}$ that we will define later. If we want these measures to take $\mathbb{Z}_p$ values then we have to make sure that we put the right epsilon factors (viewed as periods) also away from p. We now explain the local factor $Local(\chi, \Sigma, \delta)$ that shows up in the interpolation formula above. So we let χ be a Grössencharacter of a CM field K of infinite type (after fixing $incl(\infty) : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$)

$$\chi_\infty : K^\times \to \bar{\mathbb{Q}} \hookrightarrow \mathbb{C} \tag{3.19}$$

given by

$$\chi_\infty(a) = \prod_{\sigma\in\Sigma}\frac{1}{\sigma(a)^k}\left(\frac{\sigma(\bar{a})}{\sigma(a)}\right)^{d(\sigma)}. \tag{3.20}$$

We write $c : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ for the corresponding adelic character and we decompose it to $c = \prod_{\sigma\in\Sigma} c_\sigma \prod_v c_v$. The infinite type of the character can be read from the parts at infinite $c_\sigma : \mathbb{C}^\times \to \mathbb{C}^\times$. These are given by

$$c_\sigma(re^{i\theta}) = c_\sigma(z) = \frac{z^{k+d(\sigma)}}{\bar{z}^{d(\sigma)}} = r^k e^{i\theta(k+2d(\sigma))}. \tag{3.21}$$

Let as pick $\mathfrak{q}$, a prime ideal of K which we also take relative prime to 2. Then we define

$$Local(\chi,\delta)_{\mathfrak{q}} := \frac{\hat{F}_{\mathfrak{q},1}\left(\frac{-1}{2\delta a}\right)}{c_{\mathfrak{q}}(a)}, \tag{3.22}$$

where $a \in K$ is such that $ord_{\mathfrak{q}}(a) = ord_{\mathfrak{q}}(cond(\chi))$. Here

$$\hat{F}_{\mathfrak{q},1}(x) := \frac{1}{N(\mathfrak{q})^{ord_{\mathfrak{q}} cond(\chi)}} \sum_{u \in (\mathfrak{R}/\mathfrak{q})^{\times}} c_{\mathfrak{q}}(y) exp(-2\pi i\, Tr_{\mathfrak{q}}(ux)), \tag{3.23}$$

where $Tr_{\mathfrak{q}}(y) := Tr_{K_{\mathfrak{q}}/\mathbb{Q}_q}(y) \mod \mathbb{Z}_q$ and $q := \mathfrak{q} \cap \mathbb{Q}$. Then in the formula we have

$$Local(\chi,\Sigma,\delta) := \prod_{\mathfrak{q}|\mathfrak{F}\mathfrak{I}} Local(\chi,\delta)_{\mathfrak{q}} \prod_{\mathfrak{p}\in\Sigma_p} Local(\chi,\delta)_{\mathfrak{p}}. \tag{3.24}$$

The discrepancy of the ϵ-factors: Our next goal is to understand the relation of the local factor $Local(\Sigma,\chi,\delta)$ appearing in the interpolation properties of the Hida-Katz-Tilouine measure and the standard epsilon factors of Tate-Deligne. We start by normalizing properly the epsilon factors. We follow Tate's article [29] for the definition and properties of the epsilon factors of Deligne. We denote Deligne's factor with $e_{\mathfrak{p}}(\chi,\psi,dx)$ as is defined in Tate's article [29] where as $\psi(\cdot)$ we pick the additive character of $K_{\mathfrak{p}}$ given by $exp(-2\pi i Tr_{\mathfrak{p}}(\cdot))$ (as above in the Gauss sum appearing in Katz's work) and dx we pick the Haar measure that gives measure 1 to the units of $\mathfrak{R}_{\mathfrak{p}}$. From the formula (3.6.11) in Tate (there is a typo there!) we have that

$$e_{\mathfrak{p}}(\chi^{-1},\psi,dx) = c_{\mathfrak{p}}^{-1}(\alpha) N(\theta_K(\mathfrak{p})) \sum_{u\in(\mathfrak{R}/\mathfrak{p})^{\times}} c_{\mathfrak{p}}(y) exp(-2\pi i\, Tr_{\mathfrak{p}}(\frac{u}{\alpha})), \tag{3.25}$$

where α is an element with $ord_{\mathfrak{p}}(\alpha) = n(\chi) + n(\psi)$, with $n(\chi)$ (resp. $n(\psi)$) the exponent of the conductor of χ (resp. ψ) and $\theta_K(\mathfrak{p})$ is the different of $K_{\mathfrak{p}}/\mathbb{Q}_p$ and hence we have also the equality $N(\theta_K(\mathfrak{p})) = N(\mathfrak{p})^{n(\psi)}$ from the very definition of ψ and the different. In particular we conclude that

$$e_{\mathfrak{p}}(\chi^{-1},\psi,dx) = N(\mathfrak{p})^{ord_{\mathfrak{p}} cond(\chi)} c_{\mathfrak{p}}^{-1}(\delta) N(\theta_K(\mathfrak{p})) Local(\chi,\Sigma,\delta)_{\mathfrak{p}}. \tag{3.26}$$

We conclude

Lemma 3.3. *The relation between Katz and Deligne's epsilon factors is given by*

$$e_{\mathfrak{p}}(\chi^{-1},\psi,dx) = N(\mathfrak{p})^{ord_{\mathfrak{p}} cond(\chi)} c_{\mathfrak{p}}^{-1}(\delta) N(\theta_K(\mathfrak{p})) Local(\chi,\Sigma,\delta)_{\mathfrak{p}} \tag{3.27}$$

Now we take in the lemma above χ equal to $\chi\psi_K^{-1}$ for χ a finite character of K. Then, for $\pi_{\mathfrak{p}}$ a prime element of $\mathfrak{p}$, we have that

$$e_{\mathfrak{p}}(\chi^{-1}\psi_K,\psi,dx) = e_{\mathfrak{p}}(\chi^{-1},\psi,dx)\psi_K(\pi_{\mathfrak{p}}^{n(\chi)+n(\psi)}) \tag{3.28}$$

In particular that implies

$$Local(\chi\psi_K^{-1},\Sigma,\delta)_{\mathfrak{p}} = N(\mathfrak{p})^{-n(\chi)} c_{\mathfrak{p}}(\delta) N(\theta_K(\mathfrak{p})^{-1}) e_{\mathfrak{p}}(\chi^{-1}\psi_K,\psi,dx) =$$

$$= N(\mathfrak{p})^{-n(\chi)} c_{\mathfrak{p}}(\delta) N(\theta_K(\mathfrak{p})^{-1}) e_{\mathfrak{p}}(\chi^{-1},\psi,dx)\psi_K(\pi_{\mathfrak{p}}^{n(\chi)+n(\psi)}) =$$

$$= c_{\mathfrak{p}}(\delta) e_{\mathfrak{p}}(\chi^{-1}, \psi, dx) \frac{\psi_K(\pi_{\mathfrak{p}}^{n(\chi)})}{N(\mathfrak{p})^{n(\chi)}} \frac{\psi_K(\pi_{\mathfrak{p}}^{n(\psi)})}{N(\mathfrak{p})^{n(\psi)}}, \tag{3.29}$$

where $c_{\mathfrak{p}}(\delta)$ is the value of the adelic counterpart of $\chi\psi_K^{-1}$ at δ, with $\delta \in K \subset K_{\mathfrak{p}}$ as introduced above (used to define polarization of the HBAV's). But as ψ_K is unramified at $\mathfrak{p}$ we have that $c_{\mathfrak{p}}(\delta) = \psi_K(\pi_{\mathfrak{p}}^{-n(\psi)})\chi_{\mathfrak{p}}(\delta)$. So we conclude that

$$Local(\chi\psi_K^{-1}, \Sigma, \delta)_{\mathfrak{p}} = \chi_{\mathfrak{p}}(\delta) e_{\mathfrak{p}}(\chi^{-1}, \psi, dx) \left(\frac{\psi_K(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{n(\chi)} \frac{1}{N(\mathfrak{p})^{n(\psi)}}. \tag{3.30}$$

Remarks on the values of the measure of Katz-Hida-Tilouine and the periods: In order to determine where the measures $\mu_{E/F}$ and $\mu_{E/F'}$ defined in section 2 above take their values, we need first to explain where the measures $\mu^{KHT}_{\psi_K,\delta}$ and $\mu^{KHT}_{\psi_{K'},\delta'}$ of Hida-Katz-Tilouine take their values. The key point is to understand how the interpolation formulas of these measures are related to the period conjectures of Deligne that were proved by Blasius [1] in our setting. As mentioned above in Theorem 3.2, the interpolation properties of the Katz-Hida-Tilouine measure for a character χ of $G := Gal(K(\mathfrak{m}p^\infty)/K)$ of infinite type $k\Sigma$ are

$$\frac{\int_G \chi(g)\mu_\delta^{KHT}(g)}{\Omega_p^{k\Sigma}} = (\mathfrak{R}^\times : \mathfrak{r}^\times) Local(\Sigma, \chi, \delta) \frac{(-1)^{kg}\Gamma(k)^g}{\sqrt{D_F}\Omega_\infty^{k\Sigma}} \times$$

$$\prod_{\mathfrak{q}|\mathfrak{F}\mathfrak{I}} (1 - \check{\chi}(\bar{\mathfrak{q}})) \prod_{\mathfrak{q}|\mathfrak{F}} (1 - \chi(\bar{\mathfrak{q}})) \prod_{\mathfrak{p}\in\Sigma_p} (1 - \chi(\bar{\mathfrak{p}}))(1 - \check{\chi}(\bar{\mathfrak{p}})) L(0, \chi) \tag{3.31}$$

and we have fixed a Grössencharacter ψ_K associated to E/F, unramified above p and considered the measure of G defined for every locally constant function χ of G by

$$\int_G \chi(g)\mu^{KHT}_{\psi_K\delta}(g) := \int_G \chi(g)\hat{\psi}_K^{-1}(g)\mu_\delta^{KHT}(g), \tag{3.32}$$

where $\hat{\psi}_K$ is the p-adic avatar of ψ_K constructed by Weil. Then we consider the question in which field the algebraic elements $\frac{\int_G \chi(g)\hat{\psi}_K(g)\mu_\delta^{KHT}(g)}{\Omega_p^{k\Sigma}}$ belong, which is equivalent to addressing the question where the values

$$Local(\Sigma, \chi\psi_K^{-1}, \delta) \frac{L(0, \chi\psi_K^{-1})}{\sqrt{|D_F|}\Omega_\infty^{\Sigma}} \tag{3.33}$$

exactly belong. As we will see later we can replace $Local(\Sigma, \chi\psi_K^{-1}, \delta)$ with $Local(\Sigma, \chi, \delta)$ as the two differ by an element in $K^\times$. Now we note that the element Ω_∞ defined by Katz depends only on the infinite type of ψ_K. However we will assume that Ω_∞ is selected in such a way that $\sqrt{|D_F|}\Omega_\infty^{\Sigma}$ is equal to Deligne's period $c^+(\psi_K^{-1})$. We note that this is not always possible in Katz's construction as one is restricted to pick abelian varieties with CM by K that arise from fractional ideals of K. However in our setting, as everything will be "coming" from an elliptic curve $E/\mathbb{Q}$, we are allowed this assumption and actually we will prove later that we are allowed to take $\Omega_p^{\Sigma} = \Omega(E)_p^g$ and $\Omega_\infty^{\Sigma} = \Omega(E)^g$, where we recall $g = [F : \mathbb{Q}]$. So we may assume that $\frac{L(0,\psi_K^{-1})}{\sqrt{|D_F|\Omega^g(E)}} \in K$. As we have mentioned above, Blasius has proved in [1] Deligne's conjecture for Hecke characters of CM fields, in particular we know that

$$\frac{L(0,\chi\psi_K^{-1})}{c^+(\chi\psi_K^{-1})} \in K(\chi), \tag{3.34}$$

where $c^+(\chi\psi_K)$ is Delinge's period for the Hecke character $\chi\psi_K^{-1}$. In general one has that $c^+(\chi\psi_K^{-1}) \neq c^+(\chi)c^+(\psi_K^{-1})$. Indeed it is shown in [28] (page 107 formula 3.3.1) that

$$\frac{c^+(\chi\psi_K^{-1})}{c^+(\psi_K^{-1})} = c(\Sigma,\chi) \mod K(\chi)^\times. \tag{3.35}$$

Here $c(\Sigma,\chi) \in (K(\chi)\otimes\bar{\mathbb{Q}})^\times$ is a period associated to the finite character χ and depending on the CM-type of the Grössencharacter ψ_K. Actually it can be determined, up to elements in $K(\chi)^\times$, from the following reciprocity law. If we write $F := K^+$ for the maximal totally real subfield of K then one can associate to the CM type Σ the so-called half-transfer map of Tate (see [28] page 106)

$$Ver_\Sigma : Gal(\bar{\mathbb{Q}}/F) \to Gal(\bar{\mathbb{Q}}/K). \tag{3.36}$$

Then one has that

$$(1\otimes\tau)c(\Sigma,\chi) = (\chi\circ Ver_\Sigma)(\tau)c(\Sigma,\chi), \quad \tau\in Gal(\bar{\mathbb{Q}}/F). \tag{3.37}$$

So for our considerations we need to consider the question if $Local(\chi,\Sigma,\delta)$ is equal to $c(\Sigma,\chi)$ up to elements in $K(\chi)^\times$. This is in general **not** the case (for a similar discussion see also [5, page 399]). Indeed, as it is explained by Blasius in [2, page 66], if we denote by E the reflex field of (K,Σ), this is a CM field itself, then the extension $E_\Sigma := E(c(\Sigma,\chi),\ \chi)$, where we adjoin to E the values $c(\Sigma,\chi)$ for finite order characters χ over K, is the field extension of E generated by values of arithmetic Hilbert modular functions on CM points of $\mathbb{H}^{[F:\mathbb{Q}]}$ of type (K,Σ), i.e. correspond to Hilbert-Blumenthal abelian varietes of dimension $[F:\mathbb{Q}]$ with CM of type (K,Σ). This extension of E is not included in $E\mathbb{Q}^{ab}$. However we will see later that the elements $Local(\chi\psi_K,\Sigma,\delta)$ are almost equal to Gauss sums. In particular that implies that they can generate over E only extentions that are included in $E\mathbb{Q}^{ab}$ (see also the comment in [28] page 109). Hence in general the two "periods" of χ are not equal up to elements in $K(\chi)^\times$. That implies, that in general the measures $\frac{1}{\Omega_p(E)^g}\mu^{KHT}_{\psi_K,\delta}$ and $\frac{1}{\Omega_p(E)^{g'}}\mu^{KHT}_{\psi_{K'},\delta'}$ are not elements of the Iwasawa algebras $\mathbb{Z}_p[[G_K]]$ and $\mathbb{Z}_p[[G_{K'}]]$ respectively. However if χ is cyclotomic i.e. $\chi(\tau g\tau^{-1}) = \chi(g)$ for all $g\in G_K$ then we have the following

Lemma 3.4. *For χ cyclotomic we have*

$$\frac{\int_{G_K}\chi(g)\mu^{KHT}_{\psi_K,\delta}(g)}{\Omega_p(E)^g} \in \mathbb{Z}_p[\chi]. \tag{3.38}$$

Proof. From the interpolation properties of the measure $\mu^{KHT}_{\psi_K,\delta}$ we have

$$\frac{\int_{G_K}\chi(g)\mu^{KHT}_{\psi_K,\delta}(g)}{\Omega_p(E)^g} = (\mathfrak{R}^\times:\mathfrak{r}^\times)Local(\Sigma,\chi\psi_K^{-1},\delta)\frac{(-1)^{kg}\Gamma(k)^g}{\sqrt{D_F}\Omega_\infty(E)^{p\Sigma}}L(0,\chi\psi_K^{-1})\times$$

$$\prod_{\mathfrak{q}|\mathfrak{F}\mathfrak{I}}(1-\check{\chi}\check{\psi}_K^{-1}(\bar{\mathfrak{q}}))\prod_{\mathfrak{q}|\mathfrak{F}}(1-\chi\psi_K^{-1}(\bar{\mathfrak{q}}))\prod_{\mathfrak{p}\in\Sigma_p}(1-\chi\psi_K^{-1}(\bar{\mathfrak{p}}))(1-\check{\chi}\check{\psi}_K^{-1}(\bar{\mathfrak{p}})). \tag{3.39}$$

As the measure is integral valued [25, theorem (5.3.0)] we have only to show that

$$\frac{L(0,\chi\psi_K^{-1})}{\sqrt{D_F}\Omega_\infty(E)^{p\Sigma}}Local(\Sigma,\chi\psi_K^{-1},\delta).\in\mathbb{Q}_p(\chi) \tag{3.40}$$

From the discussion above we have that $Local(\Sigma,\chi\psi_K^{-1},\delta)$ is equal to $\prod_{\mathfrak{p}\in\Sigma_p}e_{\mathfrak{p}}(\chi^{-1}\psi_K)\prod_{\mathfrak{q}|\mathfrak{F}\mathfrak{I}}e_{\mathfrak{q}}(\chi^{-1}\psi_K)$ up to elements in $K(\chi)$. But then, if we write $\mathfrak{f}_{\psi_K}$ for the conductor of ψ_K, we have that $\prod_{\mathfrak{q}|\mathfrak{f}_{\psi_K}}e_{\mathfrak{q}}(\psi_K)=\pm1$ as this is the sign of the functional equation of E/F. In particular, up to elements in $K(\chi)$ (as ψ_K is unramified above p and $(cond(\chi),cond(\psi_K))=1$) we have that $\prod_{\mathfrak{p}\in\Sigma_p}e_{\mathfrak{p}}(\chi^{-1}\psi_K)\prod_{\mathfrak{q}|\mathfrak{F}\mathfrak{I}}e_{\mathfrak{q}}(\chi^{-1}\psi_K)=\prod_{\mathfrak{p}\in\Sigma_p}e_{\mathfrak{p}}(\chi^{-1})\prod_{\mathfrak{q}|\mathfrak{F}\mathfrak{I}}e_{\mathfrak{q}}(\chi^{-1})$. We write now f_{ψ_K} for the Hilbert modular form over F that is induced by automorphic induction from ψ_K (i.e. the one that corresponds to the modular elliptic curve E/F) and $\tilde{\chi}$ for the finite character over F whom χ is the base change of from F to K. Then, up to elements in $K(\chi)$, $\prod_{\mathfrak{p}\in\Sigma_p}e_{\mathfrak{p}}(\chi^{-1})\prod_{\mathfrak{q}|\mathfrak{F}\mathfrak{I}}e_{\mathfrak{q}}(\chi^{-1})=e(\tilde{\chi}^{-1})$ where $e(\tilde{\chi}^{-1})$ the global epsilon factor of $\tilde{\chi}^{-1}$. Moreover we have that $L(\chi\psi_K^{-1},0)=L(f_{\psi_K},\tilde{\chi}^{-1},1)$ (here is crucial that χ is cyclotomic). But it is known, as for example is proved in [20, page 435 Theorem I], that

$$\frac{L(f_{\psi_K},\tilde{\chi}^{-1},1)}{\sqrt{D_F}\Omega_\infty(E)^{p\Sigma}}e(\tilde{\chi}^{-1})\in\mathbb{Q}_p(\chi), \tag{3.41}$$

which allows us to conclude the proof of the lemma. □

Actually, using the full force of the results in [20], we have that

$$\left(\frac{L(f_{\psi_K},\tilde{\chi}^{-1},1)}{\sqrt{D_F}\Omega_\infty(E)^{p\Sigma}}e(\tilde{\chi}^{-1})\right)^\sigma=\frac{L(f_{\psi_K},\tilde{\chi}^{-\sigma},1)}{\sqrt{D_F}\Omega_\infty(E)^{p\Sigma}}e(\tilde{\chi}^{-\sigma}) \tag{3.42}$$

for all $\sigma\in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, which can be easily seen to imply that

$$\left(\frac{\int_{G_K}\chi(g)\mu^{KHT}_{\psi_K,\delta}(g)}{\Omega_p(E)^g}\right)^\sigma=\frac{\int_{G_K}(\chi(g))^\sigma\mu^{KHT}_{\psi_K,\delta}(g)}{\Omega_p(E)^g} \tag{3.43}$$

for all $\sigma\in Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

The Twisted Katz-Hida-Tilouine Measure: Now we modify the KHT-measure in the case where the relative different is principal. The interpolation properties of the twisted measure are going to be different with respect with the "epsilon" factors. We explain now this modification. We follow the construction that we presented above. We still consider the relative situation F'/F and the corresponding K'/K extension and we remind the reader that we consider extensions that are unramified outside p. Under our assumption we have that $(\xi)=\theta_{F'/F}$ where ξ is a totally positive element in F'. Moreover our assumptions on F'/F imply that $\theta_{F'/F}$ splits in K' to $\mathfrak{P}\bar{\mathfrak{P}}$.

Over K' we define the KHT-measure by picking instead of δ' the element $\delta \in K \hookrightarrow K'$. Note that, since the CM type (K', Σ') is a lift of (K, Σ), this is a valid choice. The polarization that the element δ induces to the lattice $\mathfrak{R}'$ is

$$\bigwedge_{\mathfrak{r}'}^{2}(\mathfrak{R}') \cong \theta_F^{-1}\mathfrak{c}^{-1}\mathfrak{r}' \tag{3.44}$$

if the same element, seen as an element in K, induces the polarization

$$\bigwedge_{\mathfrak{r}}^{2}(\mathfrak{R}) \cong \theta_F^{-1}\mathfrak{c}^{-1}. \tag{3.45}$$

Indeed, under our assumptions about the ramification of F' and F and K_0 and the fact that F'/F and K/F are disjoint, we have that $\mathfrak{R}' = \mathfrak{r}' \otimes_{\mathbb{Z}} \mathfrak{R}_0$ and similarly $\mathfrak{R} = \mathfrak{r} \otimes_{\mathbb{Z}} \mathfrak{R}_0$, from which we obtain $\mathfrak{R}' = \mathfrak{R} \otimes_{\mathfrak{r}} \mathfrak{r}'$ and the above claim follows. With respect to this polarization we have, for fractional ideals of K' of the form $\mathfrak{U} \otimes \xi^{-1} = \mathfrak{U} \otimes \theta_{F'/F}^{-1}$, the polarization

$$\bigwedge_{\mathfrak{r}'}^{2}(\mathfrak{U} \otimes \xi^{-1}) \cong \theta_F^{-1}\mathfrak{c}^{-1}\mathfrak{U}\mathfrak{U}^c\theta_{F'/F}^{-2} = \theta_{F'}^{-1}\mathfrak{c}^{-1}\mathfrak{U}\mathfrak{U}^c\theta_{F'/F}^{-1}. \tag{3.46}$$

The twisted triples: Our twisted measure is going to be defined again by evaluating Eisenstein series on the very CM abelian varieties as the measure of Katz-Hida-Tilouine but we will twist them by ξ^{-1} and use the above mentioned polarization. In particular the triples that we consider are

(1) The abelian varieties are $X(\mathfrak{U}_j^{\xi}) := X(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}) \cong X(\mathfrak{U}_j)/X(\mathfrak{U}_j)[\theta_{F'/F}]$.
(2) The polarization $\lambda_\delta^{\xi}(\mathfrak{U}_j \otimes (\theta_{F'/F}^{-1})) := \lambda_\delta(\mathfrak{U}_j \otimes (\theta_{F'/F}^{-1}))$ i.e. the one defined above and
(3) The $p^\infty\mathfrak{f}^2$-arithmetic structure is explained below after stating proposition 3.5.

We then define the twisted measure as follows

$$\int_{G'} \phi(g)\mu_{\delta,\xi}^{KHT,tw}(g) := \sum_j \int_T \tilde{\phi}_j dE_j :=$$

$$\sum_j E_1(\phi_j, \mathfrak{c}_j^{\xi})(X(\mathfrak{U}_j^{\xi}), \lambda_\delta^{\xi}(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^{\xi}(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1})) \tag{3.47}$$

with $\mathfrak{c}_j^{\xi} := \mathfrak{c}(\mathfrak{U}_j\mathfrak{U}_j^c)^{-1}\theta_{F'/F}$. For a character ϕ of infinite type $-k\Sigma$ we have

$$\int_{G'} \phi(g)\mu_\delta^{KHT,tw}(g) = \sum_j E_1(\phi_j, \mathfrak{c}_j^{\xi})\left(X(\mathfrak{U}_j^{\xi}), \lambda_\delta^{\xi}(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^{\xi}(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1})\right) =$$

$$\sum_j \phi\left([\mathfrak{U}_j^{-1}]\right) E_1\left(\mathbf{N}(x)^{k-1}\phi_{finite}(x^{-1}, a^{-1}, y, b), \mathfrak{c}_j^{\xi}\right)\left(X(\mathfrak{U}_j^{\xi}), \lambda_\delta^{\xi}(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^{\xi}(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1})\right) =$$

$$\sum_j \phi\left([\mathfrak{U}_j^{-1}]\right) E_k\left(\phi_{finite}(x^{-1},a^{-1},y,b),\mathfrak{c}_j^{\xi}\right)\left(X(\mathfrak{U}_j^{\xi}),\lambda_\delta^{\xi}(\mathfrak{U}_j\otimes\theta_{F'/F}^{-1}),\imath^{\xi}(\mathfrak{U}_j\otimes\theta_{F'/F}^{-1}),\omega^{can}(\mathfrak{U}_j^{\xi})\right),$$

where for the last equation we note that since the character is of infinite type $-k\Sigma'$ we do not need the theory of p-adic differential operators used in one of the equations of [25, page 277, (5.5.7)] (the theory of p-adic differential operators is developed under the assumption that the polarization ideal $\mathfrak{c}_j$ is relative prime to p). Similarly to the classical (untwisted) Katz-Hida-Tilouine measure we have for a locally constant function ϕ on G' and a character χ of infinite type $-k\Sigma'$

$$\int_{G'}\phi(g)\chi(g)\mu_\delta^{KHT,tw}(g)=\sum_j E_k\left((\phi\chi)_{j,finite},\mathfrak{c}_j^{\xi}\right)\left(X(\mathfrak{U}_j^{\xi}),\lambda_\delta^{\xi}(\mathfrak{U}_j\otimes\theta_{F'/F}^{-1}),\imath^{\xi}(\mathfrak{U}_j\otimes\theta_{F'/F}^{-1}),\omega^{can}(\mathfrak{U}_j^{\xi})\right).$$

We next study the interpolation properties of the twisted measure. Let us write $cond(\chi)_p=\prod_{\mathfrak{p}_j\in\Sigma'_p}\mathfrak{p}_j^{a_j}\bar{p}_j^{b_j}$ for the p-part of the conductor of χ. We define $e_j := ord_{\mathfrak{p}_j}\xi$ for all $\mathfrak{p}_j\in\Sigma'_p$. We have already described a decomposition $\mathfrak{C}=\mathfrak{F}\mathfrak{F}_c\mathfrak{I}$.

Proposition 3.5 (Interpolation Properties of the "twisted" Katz-Hida-Tilouine measure). *For a character χ of $G' := Gal(K'(\mathfrak{C}p^\infty)/K')$ of infinite type $-k\Sigma'$ we have*

$$\frac{\int_{G'}\chi(g)\mu_{\delta,\xi}^{KHT,tw}(g)}{\Omega_p^{k\Sigma'}}=(\mathfrak{R}'^{\times}:\mathfrak{r}'^{\times})Local(\Sigma',\chi,\delta,\xi)\prod_{a_j=0}\chi(\mathfrak{p}_j)^{-e_j}\times$$

$$\prod_{\mathfrak{q}_j|\mathfrak{I}}(1-\check{\chi}(\mathfrak{q}_j))\left(\prod_{\mathfrak{q}_j|\mathfrak{F}}(1-\check{\chi}(\bar{\mathfrak{q}}_j))(1-\chi(\bar{\mathfrak{q}}_j))\right)\left(\prod_{\mathfrak{p}_j\in\Sigma'_p}(1-\check{\chi}(\bar{\mathfrak{p}}_j))(1-\chi(\bar{\mathfrak{p}}_j))\right)$$

$$\frac{(-1)^{kg'}\Gamma(k)^{g'}}{\sqrt{D_{F'}}\Omega_\infty^{k\Sigma'}}\times L(0,\chi). \tag{3.48}$$

Here the factor $Local(\Sigma',\chi,\delta,\xi)=\prod_{\mathfrak{q}|\mathfrak{C}p}Local(\Sigma',\chi,\delta,\xi)_{\mathfrak{q}}$ is a modification of the local factor of the measure of Katz-Hida-Tilouine and it will be defined in the proof of the proposition. However the modification will be only at the primes above p, that is for $\mathfrak{q}|\mathfrak{C}$ we have $Local(\Sigma',\chi,\delta,\xi)_{\mathfrak{q}}=Local(\Sigma',\chi,\delta')_{\mathfrak{q}}$. We now explain shortly how the periods $\Omega_\infty^{k\Sigma'}$ and $\Omega_p^{k\Sigma'}$ appearing in the proposition are defined. These periods will be studied more closely in section 6. We first define $\Omega_\infty\in(\mathfrak{r}'\otimes\mathbb{C})^{\times}$ (resp. $\Omega_p\in(\mathfrak{r}'\otimes D_p)^{\times}$) by $\omega(\mathfrak{U}_j^{\xi})=\Omega_\infty\omega_{trans}(\mathfrak{U}_j^{\xi})$ (resp. $\omega(\mathfrak{U}_j^{\xi})=\Omega_p\omega_{can}(\mathfrak{U}_j^{\xi})$). That this is well-defined, independently of j, will become clear in section 6. Then we define $\Omega_\infty^{k\Sigma'}$ as the image with respect to the character $(-)^{k\Sigma'}:(\mathfrak{r}'\otimes\mathbb{C})^{\times}\to\mathbb{C}^{\times}$ of $\Omega_\infty\in(\mathfrak{r}'\otimes\mathbb{C})^{\times}$. Similarly we define $\Omega_p^{k\Sigma'}$ as the image with respect to the character $(-)^{k\Sigma'}:(\mathfrak{r}'\otimes D_p)^{\times}\to D_p^{\times})$ of $\Omega_p\in(\mathfrak{r}'\otimes D_p)^{\times}$.

The $p^\infty\mathfrak{f}^2$-arithmetic structure: Before we proceed to the proof of the above proposition we must explain the arithmetic structure of the twisted HBAV used in the above proposition. Note that since $(\mathfrak{f},\theta_{F'/F})=1$ holds, the $\mathfrak{f}^2$-structure can be defined exactly as in [17, page 211]. We now explain the p^∞ part. As in Katz we use the ordinary type Σ_p to obtain an isomorphism

$$\mathfrak{R}' \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{\mathfrak{p} \in \Sigma_p} \mathfrak{R}'_{\mathfrak{p}} \times \prod_{\mathfrak{p} \in \bar{\Sigma}_p} \mathfrak{R}'_{\mathfrak{p}} \cong \mathfrak{r}'_p \times \mathfrak{r}'_p. \tag{3.49}$$

And similarly, for any fractional ideal $\mathfrak{U}$ of $\mathfrak{R}'$ relative prime to p, we can identify $\mathfrak{U} \otimes \mathbb{Z}_p = \mathfrak{R}' \otimes \mathbb{Z}_p$ in $K' \otimes \mathbb{Z}_p$. In particular we have an isomorphism for such ideals

$$\mathfrak{U} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{\mathfrak{p} \in \Sigma_p} \mathfrak{R}'_{\mathfrak{p}} \times \prod_{\mathfrak{p} \in \bar{\Sigma}_p} \mathfrak{R}'_{\mathfrak{p}} \cong \mathfrak{r}'_p \times \mathfrak{r}'_p. \tag{3.50}$$

Then, as Katz explains (see [25, page 265 and lemma 5.7.52]), the p^∞ structure of $X(\mathfrak{U})$ is defined by picking the isomorphism

$$\mathfrak{r}'_p \cong \theta_{F'}^{-1} \otimes \mathbb{Z}_p, \tag{3.51}$$

which is given by $x \mapsto \delta_0 x$, where δ_0 is the image of $(2\delta')^{-1}$ in K'_p, and using it to define the injection

$$\theta_{F'}^{-1} \otimes \mathbb{Z}_p \hookrightarrow \mathfrak{U} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathfrak{r}'_p \times \mathfrak{r}'_p \tag{3.52}$$

by means of the isomorphism in the first component. Now the p^∞ structure of the twisted varieties $\mathfrak{U} \otimes \xi^{-1}$ is defined using the isomorphisms

$$(\mathfrak{U} \otimes \xi^{-1}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{\mathfrak{p} \in \Sigma_p} \frac{1}{\xi} \mathfrak{R}'_{\mathfrak{p}} \times \prod_{\mathfrak{p} \in \bar{\Sigma}_p} \frac{1}{\xi} \mathfrak{R}'_{\mathfrak{p}} \cong \frac{1}{\xi} \mathfrak{r}'_p \times \frac{1}{\xi} \mathfrak{r}'_p \tag{3.53}$$

and picking the isomorphism

$$\frac{1}{\xi} \mathfrak{r}'_p = \theta_{F'/F}^{-1} \otimes \mathfrak{r}'_p \cong \theta_{F'}^{-1} \otimes \mathbb{Z}_p, \tag{3.54}$$

given by $x \mapsto x\delta_0^{-1}$ where δ_0 is the image of δ in $\prod_{\mathfrak{p} \in \Sigma'_p} K'_{\mathfrak{p}} \cong \prod_{\mathfrak{p}} F'_{\mathfrak{p}}$. Now we proceed to the proof of the proposition on the interpolation properties of the twisted Katz-Hida-Tilouine measure.

Proof (of Proposition 3.5). We will follow closely the proof of Katz in [25]. Actually we will mainly indicate the differences of our setting from his setting. We are going to prove the proposition in the case of $\mathfrak{C} = 1$, which is enough in order to demonstrate the "new" features of the twisted measure. One could generalize the calculations below (in the same way that Hida and Tilouine [17] generalized the calculations of Katz [25]) to the more general case of non trivial $\mathfrak{C}$. We only note here that since we are assuming that $(\xi, \mathfrak{C}) = 1$ (i.e. F'/F ramifies only above p) the local factors at primes that divide $\mathfrak{C}$ that appear in the interpolation properties of the twisted measure are the same with those that appear in the interpolation properties of the untwisted measure, that is we make no modifications "outside p". Hence, after setting $\tilde{F}(x, a, y, b) := \chi_{finite}(x^{-1}, a^{-1}, y, b)$, we have

$$\frac{\int_{G'} \chi(g) \mu_{\delta,\xi}^{KHT,tw}(g)}{\Omega_p^{k\Sigma'}} = \frac{1}{\Omega_p^{k\Sigma'}} \sum_j \chi([\mathfrak{U}_j]^{-1}) E_k(\tilde{F}, \mathfrak{c}_j^{\xi}) \left(X(\mathfrak{U}_j^{\xi}), \lambda_\delta^{\xi}(\mathfrak{U}_j^{\xi}), i^{\xi}(\mathfrak{U}_j^{\xi}), \omega^{can}(\mathfrak{U}_j^{\xi}) \right)$$

$$= \sum_j \chi([\mathfrak{U}_j]^{-1}) E_k(\tilde{F}, \mathfrak{c}_j^\xi) \left(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j^\xi), i^\xi(\mathfrak{U}_j^\xi), \omega(\mathfrak{U}_j^\xi) \right)$$

$$= \frac{1}{\Omega_\infty^{k\Sigma'}} \sum_j \chi([\mathfrak{U}_j]^{-1}) E_k(\tilde{F}, \mathfrak{c}_j^\xi) \left(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j^\xi), i^\xi(\mathfrak{U}_j^\xi), \omega^{trans}(\mathfrak{U}_j^\xi) \right),$$

where the second and the third equalities follow from the fact that E_k has parallel weight k. But now

$$E_k(\tilde{F}, \mathfrak{c}_j^\xi) \left(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j^\xi), i^\xi(\mathfrak{U}_j^\xi), \omega^{trans}(\mathfrak{U}_j^\xi) \right) = E_k(\tilde{F}, \mathfrak{c}_j^\xi) \left(\mathfrak{U}_j^\xi, < \cdot, \cdot >_{\lambda_{\delta\xi}}, i^\xi \right).$$

By the definition of the Eisenstein series E_k we conclude

$$\frac{\int_{G'} \chi(g) \mu_{\delta,\xi}^{KHT,tw}(g)}{\Omega_p^{k\Sigma'}} = \frac{(-1)^{kg'} \Gamma(k+s)^{g'}}{\Omega_\infty^{k\Sigma'} \sqrt{D_{F'}}} \sum_{j=1} \chi(\mathfrak{U}_j)^{-1} \sum_{a \in \mathfrak{U}_j(\xi^{-1})[\frac{1}{p}] \cap PV_p(\mathfrak{U}_j(\xi^{-1}))} \frac{P\tilde{F}(a)}{\prod_\sigma \sigma(a)^k |N_{\mathbb{Q}}^{K'}(a)|^s} |_{s=0}$$

Now we split the proof in two cases. We first consider the case where the character χ is ramified in all primes $\mathfrak{p} \in \Sigma_p'$ and then we generalize. We start by writing the conductor of the character χ as $cond(\chi) = \prod_i \mathfrak{p}_i^{a_i} \bar{\mathfrak{p}}_i^{b_i}$. At this point we would like to warn the reader that we use the notation $\mathfrak{p}_i$ instead of $\mathfrak{P}_i$ of Katz. This is going to be the only significant discrepancy from Katz's notation in this paragraph. We now define an $\alpha \in {K'}^\times$ by

$$\prod_{a_i \geq 1} \mathfrak{p}_i^{a_i} (\prod_{a_j \geq 1} \mathfrak{p}_j^{e_j}) = (\alpha)\mathfrak{B},$$

where $\mathfrak{B}$ prime to p and $e_j := ord_{\mathfrak{p}_j} \xi$. We also decompose $(\xi) = \mathfrak{P}\bar{\mathfrak{P}}$ as ideals in K' with $\mathfrak{P} = \prod_j \mathfrak{p}_j^{e_j}$.

Special Case: χ ramified at all $\mathfrak{p}$ in Σ_p': We follow Katz [25] as in page 279. In this case we have $\mathfrak{P} \prod_i \mathfrak{p}_i^{a_i} = (\alpha)\mathfrak{B}$ since $a_i \geq 1$ for all i. From the definition of the p^∞-structure we have that the function $P\tilde{F}$ is supported in

$$(\prod_i \mathfrak{p}_i^{-a_i}) \mathfrak{U}_j \mathfrak{P}^{-1} = (\alpha^{-1}) \mathfrak{B}^{-1} \mathfrak{U}_j. \tag{3.55}$$

In particular the computations of Katz for the twisted values now read,

$$\sum_{j=1}^h \chi(\mathfrak{U}_j)^{-1} \sum_{a \in \mathfrak{U}_j(\xi^{-1})[\frac{1}{p}] \cap PV_p(\mathfrak{U}_j(\xi^{-1}))} \frac{P\tilde{F}(a)}{\prod_\sigma \sigma(a)^k |N_{\mathbb{Q}}^{K'}(a)|^s} = \tag{3.56}$$

$$= \sum_{j=1}^h \chi(\mathfrak{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1}\mathfrak{U}_j} \frac{P\tilde{F}(\alpha^{-1}a)}{\prod_\sigma \sigma(\alpha^{-1}a)^k |N_{\mathbb{Q}}^{K'}(\alpha^{-1}a)|^s} = \tag{3.57}$$

$$= \sum_{j=1}^h \chi(\mathfrak{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1}\mathfrak{U}_j} \frac{P_\delta F(\alpha^{-1}) \chi_{finite}(a)}{\prod_\sigma \sigma(\alpha^{-1}a)^k |N_{\mathbb{Q}}^{K'}(\alpha^{-1}a)|^s} = \tag{3.58}$$

$$= \left(P_\delta F(\alpha^{-1}) |N^{K'}_{\mathbb{Q}}(\alpha)|^s \prod_\sigma \sigma(\alpha)^k \right) \sum_{j=1}^{h} \chi(\mathfrak{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1}\mathfrak{U}_j} \frac{\chi_{finite}(a)}{\prod_\sigma \sigma(a)^k |N^{K'}_{\mathbb{Q}}(a)|^s}. \tag{3.59}$$

There is a special case where it is easy to see the difference of the new factors with those of Katz. Let us assume that for the decomposition $\theta_{F'/F} = \mathfrak{P}\bar{\mathfrak{P}}$ there exists $\zeta \in K'$ so that $\mathfrak{P} = (\zeta)$. We define $\alpha' \in K'^{\times}$ as in Katz by $\prod_i \mathfrak{p}_i^{a_i} = (\alpha')\mathfrak{B}'$ for $\mathfrak{B}'$ prime to p and we compare

$$Local(\Sigma', \chi, \delta, \xi)_p := \frac{P_\delta F(\alpha^{-1})}{\chi(\mathfrak{B})} \prod_\sigma \sigma(\alpha)^k \tag{3.60}$$

against the local factor of Katz

$$\frac{P_{\delta'} F(\alpha'^{-1})}{\chi(\mathfrak{B}')} \prod_\sigma \sigma(\alpha')^k. \tag{3.61}$$

We consider

$$\frac{\frac{P_\delta F(\alpha^{-1})}{\chi(\mathfrak{B})} \prod_\sigma \sigma(\alpha)^k}{\frac{P_{\delta'} F(\alpha'^{-1})}{\chi(\mathfrak{B}')} \prod_\sigma \sigma(\alpha')^k} = \frac{P_\delta F(\alpha^{-1})}{P_{\delta'} F(\alpha'^{-1})} \times \chi(\mathfrak{B}'\mathfrak{B}^{-1}) \times \prod_\sigma \sigma \left(\frac{\alpha}{\alpha'}\right)^k. \tag{3.62}$$

Note that from our assumptions $\xi = \zeta\bar{\zeta}$ hence we have $\alpha = \alpha'\zeta$. This implies

$$\frac{P_\delta F(\alpha^{-1})}{P_{\delta'} F(\alpha'^{-1})} = \frac{\prod_{\mathfrak{p} \in \Sigma'_p} \hat{F}_{\mathfrak{p},\delta}(\alpha^{-1}) F_{\bar{\mathfrak{p}}}(\alpha^{-1})}{\prod_{\mathfrak{p} \in \Sigma'_p} \hat{F}_{\mathfrak{p},\delta'}(\alpha'^{-1}) F_{\bar{\mathfrak{p}}}(\alpha'^{-1})} = \prod_{\mathfrak{p} \in \Sigma'_p} \chi_{\mathfrak{p}}(\bar{\zeta}) \chi_{\bar{\mathfrak{p}}}(\zeta^{-1}). \tag{3.63}$$

$\mathfrak{B} = \mathfrak{B}'$ and $\prod_\sigma \sigma\left(\frac{\alpha}{\alpha'}\right)^k = \prod_\sigma \sigma(\zeta)^k$.

The general case: Now we consider the case where some of the a_i's in $cond(\chi) = \prod_i \mathfrak{p}_i^{a_i} \bar{\mathfrak{p}}_i^{b_i}$ may be zero. We start by stating the following (see [25] page 282 or [17] page 209),

$$\int_{\mathfrak{R}_{\mathfrak{p}}^\times} \psi_{\delta'}(xy) dy = I_{\mathfrak{R}_{\mathfrak{p}}}(x) - \frac{1}{N\mathfrak{p}} I_{\mathfrak{p}^{-1}\mathfrak{R}_{\mathfrak{p}}}(x), \tag{3.64}$$

where $\psi_{\delta'}$ is the additive character of $K_{\mathfrak{p}}$ given by

$$\psi_{\delta'}(x) := exp \circ Tr_{\mathfrak{p}} \left(\frac{x}{\delta'}\right). \tag{3.65}$$

In particular, if we denote by ψ_δ the additive character

$$\psi_\delta(x) := exp \circ Tr_{\mathfrak{p}} \left(\frac{x}{\delta}\right) \tag{3.66}$$

we have

$$\int_{\mathfrak{R}_{\mathfrak{p}}^\times} \psi_\delta(xy) dy = I_{\mathfrak{R}_{\mathfrak{p}}}(x\xi) - \frac{1}{N\mathfrak{p}} I_{\mathfrak{p}^{-1}\mathfrak{R}_{\mathfrak{p}}}(x\xi), \tag{3.67}$$

where we recall $\xi = \frac{\delta'}{\delta}$ up to elements in ${\mathfrak{R}'_{\mathfrak{p}}}^{\times}$. Now we follow the computations of Katz as in ([25] page 281-282) and use the same notation. In our setting, after the observation above, we have that the function $P\tilde{F}$ is supported in

$$\prod_{a_i \geq 1} \mathfrak{p}_i^{-a_i} (\prod_{a_j \geq 1} \mathfrak{p}_j^{-e_j})(\prod_{a_j=0} \mathfrak{p}_j^{-1-e_j})\mathfrak{U}_i = (\alpha^{-1})\mathfrak{B}^{-1}(\prod_{a_j=0} \mathfrak{p}_j^{-1-e_j})\mathfrak{U}_i. \tag{3.68}$$

Now for $a \in \mathfrak{B}^{-1}(\prod_{a_j=0} \mathfrak{p}_j^{-1-e_j})\mathfrak{U}_i$ we have,

$$P\tilde{F}(\alpha^{-1}a) = P_\delta F(\alpha^{-1})\chi_{2,finite}(a) \prod_{a_j=0} \widehat{char}(\mathfrak{p}_j^{1+e_j})(a), \tag{3.69}$$

where

$$\widehat{char}(\mathfrak{p}_j^{1+e_j})(a) = \begin{cases} 1 - \frac{1}{N\mathfrak{p}_j}, & \text{if } ord_{\mathfrak{p}_j}(a) \geq -e_j; \\ -\frac{1}{N\mathfrak{p}_j}, & \text{if } ord_{\mathfrak{p}_j}(a) = -e_j - 1. \end{cases} \tag{3.70}$$

Following Katz (note a typo in Katz's definition! compare 5.5.31 with 5.5.35), we extend the above function to the set $\mathbf{I}$ of fractional ideals I of K' of the form

$$I = (\prod_{a_j=0} \mathfrak{p}_j^{-1-e_j})\mathfrak{Q}, \tag{3.71}$$

where $\mathfrak{Q}$ is an integral ideal prime to those $\mathfrak{p}_i$ with $a_i \neq 0$ and to all $\bar{\mathfrak{p}}_k$, by

$$\widehat{char}(\mathfrak{p}_j^{1+e_j})(I) = \begin{cases} 1 - \frac{1}{N\mathfrak{p}_j}, & \text{if } I\mathfrak{p}_j^{e_j} \text{ is integral;} \\ -\frac{1}{N\mathfrak{p}_j}, & \text{if not.} \end{cases} \tag{3.72}$$

Following Katz's computations we have that the values that we are interested in are

$$\sum_{j=1}^{h} \chi(\mathfrak{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1}(\prod_{a_j=0}(\mathfrak{p}_j^{-1-e_j}))\mathfrak{U}_j} \frac{P\tilde{F}(\alpha^{-1}a)}{\prod_\sigma \sigma(\alpha^{-1}a)^k |N_{\mathbb{Q}}^{K'}(\alpha^{-1}a)|^s} = \tag{3.73}$$

$$\left(\frac{P_\delta F(\alpha^{-1})}{\chi(\mathfrak{B})} \prod_\sigma \sigma(\alpha)^k\right) \sum_{I_0 \in \mathbf{I}(p)} \frac{\chi(I_0)}{N(I_0)^s} \prod_{a_j=0} \sum_{n \geq -1-e_j} \frac{\chi_2(\mathfrak{p}_j)^n}{N(p_j)^{ns}} \widehat{char}(\mathfrak{p}_j^{1+e_j})(\mathfrak{p}_j^n). \tag{3.74}$$

We now set

$$Local(\Sigma', \chi, \delta, \xi) := Local(\Sigma, \chi, \delta, \xi)_p := \frac{P_\delta F(\alpha^{-1})}{\chi(\mathfrak{B})} \prod_\sigma \sigma(\alpha)^k \tag{3.75}$$

As in Katz, we compute now the inner sum

$$\sum_{n=-1-e_j}^{\infty} \frac{\chi_2(\mathfrak{p}_j)^n}{N(\mathfrak{p}_j)^{ns}} \widehat{char}(\mathfrak{p}_j^{1+e_j})(\mathfrak{p}_j^n) = \frac{-1}{N(\mathfrak{p}_j)} \frac{\chi_2(\mathfrak{p}_j)^{-1-e_j}}{N(\mathfrak{p}_j)^{(-1-e_j)s}} + \left(1 - \frac{1}{N(\mathfrak{p}_j)}\right) \sum_{n=-e_j}^{\infty} \frac{\chi_2(\mathfrak{p}_j)^n}{N(\mathfrak{p}_j)^{ns}} \tag{3.76}$$

$$= \sum_{n=-e_j}^{\infty} \frac{\chi_2(\mathfrak{p}_j)^n}{N(\mathfrak{p}_j)^{ns}} - \frac{1}{N(\mathfrak{p}_j)} \left(\frac{\chi_2(\mathfrak{p}_j)^{-1-e_j}}{N(\mathfrak{p}_j)^{(-1-e_j)s}} + \sum_{n=-e_j}^{\infty} \frac{\chi_2(\mathfrak{p}_j)^n}{N(\mathfrak{p}_j)^{ns}}\right) \tag{3.77}$$

$$= \sum_{n=-e_j}^{\infty} \frac{\chi_2(\mathfrak{p}_j)^n}{N(\mathfrak{p}_j)^{ns}} - \frac{1}{N(\mathfrak{p}_j)} \sum_{n=-1-e_j}^{\infty} \frac{\chi_2(\mathfrak{p}_j)^n}{N(\mathfrak{p}_j)^{ns}} \tag{3.78}$$

$$= \left(1 - \frac{1}{N(\mathfrak{p}_j)} \frac{\chi_2(\mathfrak{p}_j)^{-1}}{N(\mathfrak{p}_j)^{-s}}\right) \sum_{n=-e_j}^{\infty} \frac{\chi_2(\mathfrak{p}_j)^n}{N(\mathfrak{p}_j)^{ns}} \tag{3.79}$$

$$= \left(1 - \frac{N(\mathfrak{p}_j)^s}{\chi_2(\mathfrak{p}_j)N(\mathfrak{p}_j)}\right) \frac{\chi_2(\mathfrak{p}_j)^{-e_j}}{N(\mathfrak{p}_j)^{-e_j s}} \sum_{n=0}^{\infty} \frac{\chi_2(\mathfrak{p}_j)^n}{N(\mathfrak{p}_j)^{ns}} \tag{3.80}$$

$$= \left(1 - \frac{N(\mathfrak{p}_j)^s}{\chi_2(\mathfrak{p}_j)N(\mathfrak{p}_j)}\right) \frac{\chi_2(\mathfrak{p}_j)^{-e_j}}{N(\mathfrak{p}_j)^{-e_j s}} \left(1 - \chi_2(\mathfrak{p}_j)N(\mathfrak{p}_j)^{-s}\right)^{-1} \tag{3.81}$$

$$= (1 - N(\mathfrak{p}_j)^s \check{\chi}_2(\bar{\mathfrak{p}}_j)) \frac{\chi_2(\mathfrak{p}_j)^{-e_j}}{N(\mathfrak{p}_j)^{-e_j s}} \left(1 - \chi_2(\mathfrak{p}_j)N(\mathfrak{p}_j)^{-s}\right)^{-1}. \tag{3.82}$$

So we conclude,

$$\sum_{j=1}^{h} \chi(\mathfrak{U}_j)^{-1} \sum_{a \in \mathfrak{B}^{-1}(\prod_{a_j=0}(\mathfrak{p}_j^{-1-e_j}))\mathfrak{U}_j} \frac{P\tilde{F}(\alpha^{-1}a)}{\prod_\sigma \sigma(\alpha^{-1}a)^k |N_{\mathbb{Q}}^{K'}(\alpha^{-1}a)|^s} = \tag{3.83}$$

$$= Local(\Sigma', \chi, \delta, \xi) \times L(s, \chi_1) \prod_{a_j=0} \left(\frac{1 - N(\mathfrak{p}_j)^s \check{\chi}_2(\bar{\mathfrak{p}}_j)}{(1 - \chi_2(\mathfrak{p}_j)N(\mathfrak{p}_j)^{-s})} \times \frac{\chi_2(\mathfrak{p}_j)^{-e_j}}{N(\mathfrak{p}_j)^{-e_j s}} \right),$$

whose value at $s = 0$ is equal to

$$Local(\Sigma', \chi, \delta, \xi) \times L(0, \chi_1) \prod_{a_j=0} \left(\frac{1 - \check{\chi}_2(\bar{\mathfrak{p}}_j)}{(1 - \chi_2(\mathfrak{p}_j))} \times \chi_2(\mathfrak{p}_j)^{-e_j} \right). \tag{3.84}$$

But $L(s, \chi_1) = L(s, \chi) \prod_{\mathfrak{p}_i} (1 - \chi(\mathfrak{p}_i)N(\mathfrak{p}_i)^{-s})(1 - \chi(\bar{\mathfrak{p}}_i)N(\bar{\mathfrak{p}}_i)^{-s})$, which allow us to conclude that the values are equal to

$$Local(\Sigma', \chi, \delta, \xi) \times L(0, \chi) \left(\prod_{\mathfrak{p}_j \in \Sigma'_p} (1 - \check{\chi}(\bar{\mathfrak{p}}_j))(1 - \chi(\bar{\mathfrak{p}}_j)) \right) \prod_{a_j=0} \chi(\mathfrak{p}_j)^{-e_j}. \qquad \square$$

4. The Relative Setting: Congruences between Eisenstein Series.

Now we consider the following relative setting. We consider as in the introduction the totally real field extension F' of F of degree p and write $\Gamma = Gal(F'/F)$. In particular we recall that $\theta_{F'/F}$ is the relative different and that F'/F is ramified only at primes above p. We have fixed ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{f}$ of F and consider also the corresponding ideals in F', that is their natural image under $F \hookrightarrow F'$. We write $T'_1 \times T'_2$ for the corresponding spaces in the F' setting that we have introduced for the F setting. We note that Γ operates naturally on this space. Moreover, the embedding $F \hookrightarrow F'$ induces a natural diagonal embedding $\mathbb{H}^{[F:\mathbb{Q}]} \hookrightarrow \mathbb{H}^{[F':\mathbb{Q}]}$ with the property that the pull back of a Hilbert modular form of F' is a

Hilbert modular form of F. We need to make this last remark more explicit.

The Tate-Abelian Scheme and the modular interpretation of the diagonal embedding: We follow the book of Hida [19] as in chapter 4 (and especially section 4.1.5) and the notation there. For fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of the totally real field F and a ring R we define the ring $R[[(\mathfrak{a}\mathfrak{b})_+]]$ with $(\mathfrak{a}\mathfrak{b})_+ := \mathfrak{a}\mathfrak{b} \cap F_+$ to be the ring of formal series

$$R[[(\mathfrak{a}\mathfrak{b})_+]] := \{a_0 \sum_{\xi \in (\mathfrak{a}\mathfrak{b})_+} a_\xi q^\xi |\ a_\xi \in R\}. \tag{4.1}$$

We pick the multiplicative set $q^{(\mathfrak{a}\mathfrak{b})_+} := \{q^\xi | \xi \in (\mathfrak{a}\mathfrak{b})_+\}$ and define $R\{\mathfrak{a}\mathfrak{b}\}$ as the localization of $R[[(\mathfrak{a}\mathfrak{b})_+]]$ to this multiplicative set. Then, as explained in Hida, the Tate semi-abelian scheme $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ is defined over the ring $R\{\mathfrak{a}\mathfrak{b}\}$ (with R depending on the extra level structure that we impose) by the algebraization of the rigid analytic variety

$$(\mathbf{G}_m \otimes \mathfrak{a}^{-1}\theta_F^{-1})/q^{\mathfrak{b}}. \tag{4.2}$$

Let X be a HBAV over a ring R with real multiplication by $\mathfrak{r}$. We may define a HBAV X' over R with real multiplication by $\mathfrak{r}'$ by considering the functor from schemes S over R to $\mathfrak{r}'$ modules defined by

$$S \mapsto X'(S) := X(S) \otimes_{\mathfrak{r}} \theta_{F'/F}^{-1}. \tag{4.3}$$

We define the map

$$\Delta : M(\mathfrak{c}, \Gamma_{00}(p^\infty \mathfrak{f}^2)) \to M(\mathfrak{c}\theta_{F'/F}, \Gamma_{00}(p^\infty \mathfrak{f}^2 \mathfrak{r}'))$$

$$(X, \lambda, \omega, \imath) \mapsto (X \otimes_{\mathfrak{r}} \theta_{F'/F}^{-1}, \lambda \otimes_{\mathfrak{r}} \theta_{F'/F}^{-1}, \omega \otimes_{\mathfrak{r}} \theta_{F'/F}^{-1}, \imath \otimes_{\mathfrak{r}} \theta_{F'/F}^{-1}),$$

where we are using the notation $M(\mathfrak{c}, \Gamma_{00}(p^\infty \mathfrak{f}^2))$ (resp. $M(\mathfrak{c}\theta_{F'/F}^{-1}, \Gamma_{00}(p^\infty \mathfrak{f}^2)))$ for the moduli stack of $\mathfrak{c}$ (resp. $\mathfrak{c}\theta_{F'/F}$) polarized HBAV of F (resp. of F') with a $p^\infty \mathfrak{f}^2$-arithmetic structure (resp. $p^\infty \mathfrak{f}^2 \mathfrak{r}'$-arithmetic structure). Before we proceed further, we would like to see this map from the complex point of view. For fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of F (resp. $\mathfrak{a}'$ and $\mathfrak{b}'$ of F') with $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$ (resp. $\mathfrak{a}'\mathfrak{b}'^{-1} = \mathfrak{c}' = \mathfrak{c}\theta_{F'/F}$) and $(\mathfrak{a}, p\mathfrak{f}) = 1$ (resp. $(\mathfrak{a}', p\mathfrak{f}\mathfrak{r}') = 1$) we set $\mathbb{H}_{(\mathfrak{a},\mathfrak{b})} := \mathbb{H}^{[F:\mathbb{Q}]}$ (resp. $\mathbb{H}_{(\mathfrak{a}',\mathfrak{b}')} := \mathbb{H}^{[F':\mathbb{Q}]}$). The embedding $F \hookrightarrow F'$ induces a natural diagonal embedding

$$\Delta : \mathbb{H}_{(\mathfrak{a},\mathfrak{b})} \hookrightarrow \mathbb{H}_{(\mathfrak{a}',\mathfrak{b}')}$$

by $\Delta((z_\sigma)_{\sigma \in \Sigma}) := (z_{\sigma'})_{\sigma' \in \Sigma'}$ with $z_{\sigma'} := z_\sigma$ for $\sigma'_{|F} = \sigma$. For any natural number n we introduce the following congruence subgroups which are relevant to the moduli problem that we consider (see [12, pages 259 and 262])

$$\Gamma_{00}(p^n \mathfrak{f}^2; \mathfrak{a}, \mathfrak{b}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{r} & \mathfrak{a}^{-1}\mathfrak{b}^{-1}\theta_F^{-1} \\ p^n \mathfrak{f}^2 \mathfrak{a}\mathfrak{b}\theta_F & \mathfrak{r} \end{pmatrix} | ad - bc = 1, a - 1 \in p^n \mathfrak{f}^2 \mathfrak{r}, d - 1 \in p^n \mathfrak{f}^2 \mathfrak{r} \right\}.$$

Similarly we define for F',

$$\Gamma_{00}(p^n \mathfrak{f}^2; \mathfrak{a}', \mathfrak{b}') := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{r}' & \mathfrak{a}'^{-1}\mathfrak{b}'^{-1}\theta_{F'}^{-1} \\ p^n \mathfrak{f}^2 \mathfrak{a}'\mathfrak{b}'\theta_{F'} & \mathfrak{r}' \end{pmatrix} | ad - bc = 1, a - 1 \in p^n \mathfrak{f}^2 \mathfrak{r}', d - 1 \in p^n \mathfrak{f}^2 \mathfrak{r}' \right\}.$$

We now take above $\mathfrak{a}' := \mathfrak{a}\mathfrak{r}'$ and $\mathfrak{b}' := \mathfrak{b}\theta_{F'/F}^{-1}$. In particular we have $\mathfrak{c}' = \mathfrak{a}'\mathfrak{b}'^{-1} = \mathfrak{a}\mathfrak{b}^{-1}\theta_{F'/F} = \mathfrak{c}\theta_{F'/F}$. Then we note that the embedding $SL_2(F) \hookrightarrow SL_2(F')$ induced by $F \hookrightarrow F'$ gives an embedding

$$\Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a},\mathfrak{b}) \hookrightarrow \Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a}',\mathfrak{b}') = \Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1})$$

since in this case we have

$$\Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{r}' & \mathfrak{a}^{-1}\mathfrak{b}^{-1}\theta_F^{-1}\mathfrak{r}' \\ p^n\mathfrak{f}^2\mathfrak{a}\mathfrak{b}\theta_F\mathfrak{r}' & \mathfrak{r}' \end{pmatrix} |ad-bc=1, a-1 \in p^n\mathfrak{f}^2\mathfrak{r}', d-1 \in p^n\mathfrak{f}^2\mathfrak{r}' \right\},$$

because $\theta_{F'} = \theta_F\theta_{F'/F}$. In particular, this implies that the map Δ induces by pull-back a map

$$res_\Delta : M_k(\mathfrak{c}\theta_{F'/F},\Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1}),\chi) \to M_{pk}(\mathfrak{c},\Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a},\mathfrak{b}),\chi\circ ver), \quad f \mapsto f\circ\Delta,$$

i.e. from the space of $\mathfrak{c}\theta_{F'/F}$- polarized complex Hilbert modular forms over F' of the congruences group $\Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1})$ of parallel weight k and Nebentype χ to the space of $\mathfrak{c}$-polarized complex Hilbert modular forms over F of the congruences group $\Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a},\mathfrak{b})$ of parallel weight pk and Nebentype $\chi\circ ver$. As this holds for any n, we obtain a map

$$res_\Delta : M_k(\mathfrak{c}\theta_{F'/F},\Gamma_{00}(p^\infty\mathfrak{f}^2;\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1}),\chi) \to M_{pk}(\mathfrak{c},\Gamma_{00}(p^\infty\mathfrak{f}^2;\mathfrak{a},\mathfrak{b}),\chi\circ ver),$$

where

$$M_k(\mathfrak{c}\theta_{F'/F},\Gamma_{00}(p^\infty\mathfrak{f}^2;\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1}),\chi) := \bigcup_{n\geq 0} M_k(\mathfrak{c}\theta_{F'/F},\Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1}),\chi)$$

and similarly

$$M_{pk}(\mathfrak{c},\Gamma_{00}(p^\infty\mathfrak{f}^2;\mathfrak{a},\mathfrak{b}),\chi\circ ver) := \bigcup_{n\geq 0} M_{pk}(\mathfrak{c},\Gamma_{00}(p^n\mathfrak{f}^2;\mathfrak{a},\mathfrak{b}),\chi\circ ver).$$

The map Δ that we described above, that is $X \mapsto X' := X \otimes_{\mathfrak{r}} \theta_{F'/F}^{-1}$, agrees with the diagonal embedding when X is a complex HBAV. Indeed, in this case, X corresponds to the lattice of the form $\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau) := 2\pi i(\theta_F^{-1}\mathfrak{a}^{-1} + \mathfrak{b}\tau)$ for some $\tau \in \mathbb{H}_{(\mathfrak{a},\mathfrak{b})}$ (see [25, page 215]). Then the lattice $\mathcal{L}_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1}}(\tau')$ that corresponds to the HBAV X' is given by $2\pi i(\theta_{F'}^{-1}\mathfrak{a}^{-1} + \mathfrak{b}\theta_{F'/F}^{-1}\tau')$ with $\tau' = \Delta(\tau) \in \mathbb{H}_{(\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1})}$. Further, the polarization λ on $\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)$ corresponds to an alternating $\mathfrak{r}$-pairing (see [25, page 214])

$$<\cdot,\cdot>: \bigwedge_{\mathfrak{r}}^2 \mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau) \cong \theta_F^{-1}\mathfrak{c}^{-1}$$

and hence $\lambda \otimes_{\mathfrak{r}} \theta_{F'/F}^{-1}$ to an alternating $\mathfrak{r}'$-pairing

$$<\cdot,\cdot>': \bigwedge_{\mathfrak{r}'}^2 \mathcal{L}_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta_{F'/F}^{-1}}(\Delta(\tau)) = \bigwedge_{\mathfrak{r}'}^2 (\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)\otimes_r\theta_{F'/F}^{-1}) \cong \theta_F^{-1}\mathfrak{c}^{-1}\theta_{F'/F}^{-2} = \theta_{F'}^{-1}\mathfrak{c}^{-1}\theta_{F'/F}^{-1}.$$

Further, ω gives an isomorphism [25, page 214]

$$\omega : Lie(X^{an}) \cong \theta_F^{-1} \otimes \mathbb{C} = F \otimes \mathbb{C}$$

and hence

$$\omega \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} : Lie(X'^{an}) \cong \theta^{-1}_{F'} \otimes \mathbb{C} = F' \otimes \mathbb{C}.$$

Finally, the effect on the arithmetic structure follows from the fact that the p^{∞} induces an exact sequence of free $\mathfrak{r}_p$ modules,

$$0 \to \theta^{-1}_{F} \otimes_{\mathfrak{r}} \mathfrak{r}_p \to \mathcal{L} \otimes_{\mathfrak{r}} \mathfrak{r}_p \to \mathfrak{c}^{-1} \otimes_{\mathfrak{r}} \mathfrak{r}_p \to 0$$

and hence, after tensoring with $\theta^{-1}_{F'/F}$, we get

$$0 \to \theta^{-1}_{F'} \otimes_{\mathfrak{r}'} \mathfrak{r}'_p \to \mathcal{L}' \otimes_{\mathfrak{r}'} \mathfrak{r}'_p \to \mathfrak{c}^{-1}\theta^{-1}_{F'/F} \otimes_{\mathfrak{r}'} \mathfrak{r}'_p \to 0,$$

where $\mathcal{L} = \mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)$ and $\mathcal{L}' = \mathcal{L}_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(\tau')$. Similarly follows the $\mathfrak{f}^2$-structure.

Next we study the effect of the diagonal map on the q-expansion of Hilbert modular forms at particular cusps. If we assume that the Hilbert modular form $f \in M_k(\mathfrak{c}\theta_{F'/F}, \Gamma_{00}(p^n\mathfrak{f}^2; \mathfrak{a}\mathfrak{r}', \mathfrak{b}\theta^{-1}_{F'/F}))$ has at the cusp $(\mathfrak{a}\mathfrak{r}', \mathfrak{b}\theta^{-1}_{F'/F})$ the Fourier expansion

$$f(z') = \sum_{\xi' \geq 0, \xi' \in \mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}} a(\xi', f) {q'}^{\xi'}$$

with $q' = exp(2\pi i \sum_{\sigma \in \Sigma'} z'_{\sigma'}\sigma(\xi'))$, then the form $res_{\Delta}(f)$ has at the cusp $(\mathfrak{a}, \mathfrak{b})$ the Fourier expansion

$$res_{\Delta}(f)(z) = \sum_{\xi \geq 0, \xi \in \mathfrak{a}\mathfrak{b}} \left(\sum_{\xi', Tr_{F'/F}(\xi') = \xi} a(\xi', f) \right) q^{\xi}$$

with $q = exp(2\pi i \sum_{\sigma \in \Sigma} z_{\sigma}\sigma(\xi))$. Note that, as pointed also above, $Tr_{F'/F}(\xi') \in \mathfrak{a}\mathfrak{b}$ for $\xi' \in \mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}$. Algebraically these considerations can be expressed with the help of the Tate HBAV. We consider the effect of our map on the Tate HBAV $Tate_{\mathfrak{a},\mathfrak{b}}(q)$. That is we consider the HBAV with real multiplication by $\mathfrak{r}'$ defined by $Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} = (\mathbf{G}_m \otimes \theta^{-1}_{F}/q^{\mathfrak{b}}) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$. We consider the map $tr_{F'/F} : R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\} \to R\{\mathfrak{a}\mathfrak{b}\}$ given by $q^{\alpha} \mapsto q^{tr_{F'/F}(\alpha)}$.

Lemma 4.1.

$$\left(Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q), \lambda'_{can}, \omega'_{can}, i'_{can}\right) \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\}$$

$$= \left(Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}, \lambda_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}, \omega_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}, i_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}\right)$$

Proof. Even though the lemma holds in general, we are going to use it while working over number fields. Hence, after fixing embeddings in the complex numbers, we may just prove it over $\mathbb{C}$. Over the complex numbers this follows easily by observing that $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ corresponds to that lattice $2\pi i(\mathfrak{b}z + \mathfrak{a}^{-1}\theta^{-1}_{F})$ for $z \in \mathbb{H}_{(\mathfrak{a},\mathfrak{b})}$ and hence $Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$ to the lattice

$$2\pi i(\mathfrak{b}z + \mathfrak{a}^{-1}\theta^{-1}_{F}) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} = 2\pi i(\mathfrak{b}\theta^{-1}_{F'/F}z' + \mathfrak{a}^{-1}\theta^{-1}_{F'}), \qquad (4.4)$$

with $z' \in \mathbb{H}_{(\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F})}$ the image of z under the map Δ introduced above. Moreover in this case the map $tr_{F'/F} : R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\} \to R\{\mathfrak{a}\mathfrak{b}\}$ given by $q^{\alpha} \mapsto q^{tr_{F'/F}(\alpha)}$ corresponds to setting the indeterminate $q := exp(Tr_{F'}(z')) := exp(\sum_{\sigma\in\Sigma'} z'_{\sigma})$ (where $\sigma \in \Sigma'$ the embeddings $\sigma : F' \hookrightarrow \mathbb{C}$ and $z' = (z'_{\sigma}) \in \mathbb{H}_{(\mathfrak{a},\mathfrak{b}\theta^{-1}_{F'/F})}$) equal to the indeterminate $q = exp(Tr_{F'}(\Delta(z)))$. In particular that implies that the complex points of $Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q) \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\}$ correspond to the lattice $2\pi i(\mathfrak{b}\theta^{-1}_{F'/F}z' + \mathfrak{a}^{-1}\theta^{-1}_{F'})$ for $z' = \Delta(z)$.

Now we prove the statement about the polarization. We have that

$$\lambda_{can} : Tate_{\mathfrak{a},\mathfrak{b}}(q)^t \cong Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes_{\mathfrak{r}} \mathfrak{c}$$

and hence

$$\lambda_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} : Tate_{\mathfrak{a},\mathfrak{b}}(q)^t \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} \cong Tate_{\mathfrak{a},\mathfrak{b}} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} \otimes_{\mathfrak{r}} \mathfrak{c}.$$

But we know that we can identify $Tate_{\mathfrak{a},\mathfrak{b}}(q)^t = Tate_{\mathfrak{b},\mathfrak{a}}(q)$ (see for example [19, page 117]). We obtain

$$\lambda_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} : Tate_{\mathfrak{b},\mathfrak{a}}(q) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} \cong Tate_{\mathfrak{a},\mathfrak{b}} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} \otimes_{\mathfrak{r}} \mathfrak{c}.$$

Similarly we have that

$$\lambda'_{can} : Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q)^t \cong Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q) \otimes_{\mathfrak{r}'} \mathfrak{c}\theta_{F'/F}$$

and after identifying $Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q)^t = Tate_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(q)$ we obtain

$$\lambda'_{can} : Tate_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(q) \cong Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q) \otimes_{\mathfrak{r}'} \mathfrak{c}\theta_{F'/F},$$

or equivalently

$$\lambda'_{can} : Tate_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(q) \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F} \cong Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q) \otimes_{\mathfrak{r}'} \mathfrak{r}'\mathfrak{c}.$$

Applying base change to the last isomorphism and denoting $\lambda'_{can} \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\}$ with $\tilde{\lambda}_{can}$ we obtain

$$\tilde{\lambda}_{can} : Tate_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(q) \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\} \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F} \cong Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q) \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\} \otimes_{\mathfrak{r}'} \mathfrak{r}'\mathfrak{c}.$$

But we have seen that

$$Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q) \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\} = Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$$

and similarly we have that

$$Tate_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(q) \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\} = Tate_{\mathfrak{b},\mathfrak{a}}(q) \otimes_{\mathfrak{r}} \mathfrak{r}'.$$

That is,

$$\tilde{\lambda}_{can} : Tate_{\mathfrak{b},\mathfrak{a}}(q) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} \cong Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} \otimes_{\mathfrak{r}'} \mathfrak{r}'\mathfrak{c}.$$

Now we claim that $\tilde{\lambda}_{can} = \lambda_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$. We show this in the case of interest, namely when the characteristic is zero (we are going to apply everything over number fields) and hence we can consider the complex analytic case. In this case it is easy to see what the canonical polarizations λ'_{can} and λ_{can} are.

Indeed over $\mathbb{C}$ the HBAV $Tate_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(q)(\mathbb{C})$ corresponds to the lattice $\mathcal{L}_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(\tau') = 2\pi i(\theta^{-1}_{F'}\mathfrak{b}^{-1}\theta_{F'/F} + \mathfrak{a}\mathfrak{r}'\tau')$ with $q = exp(Tr_{F'}(\tau'))$ and $(Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q) \otimes_{\mathfrak{r}'} \mathfrak{c}\theta_{F'/F})(\mathbb{C})$ to the lattice $\mathcal{L}_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(\tau') \otimes_{\mathfrak{r}'} \mathfrak{c}\theta_{F'/F}$. But it is easily seen that

$$\mathcal{L}_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(\tau') \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F} = \mathcal{L}_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(\tau') \otimes_{\mathfrak{r}'} \mathfrak{r}'\mathfrak{c}.$$

In particular after all these identifications we have that the map λ'_{can} can be described by the trivial map

$$z \mod \mathcal{L}_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(\tau') \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F} \mapsto z \mod \mathcal{L}_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(\tau') \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F},$$

where $z \in \mathbb{C}^{[F':\mathbb{Q}]}$. Similarly, it can be seen that the map $\lambda_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$ can be described by

$$z \mod \mathcal{L}_{\mathfrak{b}\mathfrak{r},\mathfrak{a}\mathfrak{r}}(\tau) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} \mapsto z \mod \mathcal{L}_{\mathfrak{b}\mathfrak{r},\mathfrak{a}\mathfrak{r}}(\tau) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}.$$

But, as we have already seen, taking base change in this setting is nothing else than setting $\tau' := \Delta(\tau)$. That is, the map $\tilde{\lambda}_{can}$ is nothing else than

$$z \mod \mathcal{L}_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(\Delta(\tau)) \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F} \mapsto z \mod \mathcal{L}_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(\Delta(\tau)) \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F}.$$

Now we notice that $\mathcal{L}_{\mathfrak{b}\theta^{-1}_{F'/F},\mathfrak{a}\mathfrak{r}'}(\Delta(\tau)) = \mathcal{L}_{\mathfrak{b}\mathfrak{r},\mathfrak{a}\mathfrak{r}}(\tau) \otimes_{\mathfrak{r}} \mathfrak{r}'$ and hence $\tilde{\lambda}_{can} = \lambda_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$ or using the previous notation $\lambda'_{can} \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\} = \lambda_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$.

For the statement about the differentials one has simply to observe that (see also [25, page 210])

$$Lie(Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q) \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\}) \cong Lie(\mathbb{G}_m \otimes \mathfrak{a}^{-1}\theta^{-1}_{F'}) \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\}$$

$$= \theta^{-1}_{F'}\mathfrak{a}^{-1} \otimes R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\} \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\} = \theta^{-1}_{F'}\mathfrak{a}^{-1} \otimes R\{\mathfrak{a}\mathfrak{b}\}.$$

On the other hand we have

$$Lie(Tate_{\mathfrak{a},\mathfrak{b}}(q)) \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F} = R\{\mathfrak{a}\mathfrak{b}\} \otimes \mathfrak{a}^{-1}\theta^{-1}_{F} \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F} = R\{\mathfrak{a}\mathfrak{b}\} \otimes \mathfrak{a}^{-1}\theta^{-1}_{F'}.$$

That is,

$$Lie(Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q) \times_{R\{\mathfrak{a}\mathfrak{b}\theta^{-1}_{F'/F}\}} R\{\mathfrak{a}\mathfrak{b}\}) \cong Lie(Tate_{\mathfrak{a},\mathfrak{b}}(q)) \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F},$$

which concludes the statement for the canonical differentials. Finally, the statement about the numerical structures follows in the same way as we indicated above. $\square$

We can use the above lemma to study the effect of the diagonal embedding to the the q-expansion, that is to the values of Hilbert modular forms on the Tate abelian scheme. For a $\mathfrak{c}\theta_{F'/F}$-HMF ϕ of F' we have that

$$\phi\left(Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}, \lambda_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}, \omega_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}, i_{can} \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}\right)$$

$$= tr_{F'/F}\left(\phi\left(Tate_{\mathfrak{a}\mathfrak{r}',\mathfrak{b}\theta^{-1}_{F'/F}}(q), \lambda_{can}, \omega_{can}, i_{can}\right)\right). \quad (4.5)$$

Note that this over the complex numbers follows directly from the description of the map Δ as the diagonal embedding $\Delta : \mathbb{H}_{\mathfrak{c}} \hookrightarrow \mathbb{H}_{\mathfrak{c}\theta_{F'/F}}$.

The next question that we need to clarify is what is happening under this diagonal map for an HBAV with real multiplication by $\mathfrak{r}$ that has CM by $\mathfrak{R}$, the ring of integers of a totally imaginary quadratic extension K of F. It is well known that, up to isomorphism, these are given by the fractional ideals of K. Let us write $\mathfrak{U}$ for one of these and $X(\mathfrak{U})$ for the corresponding HBAV with CM by $\mathfrak{R}$. We see that the above map gives us the HBAV $X(\mathfrak{U}) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$ with real multiplication by $\mathfrak{r}'$. We set $K' = KF'$ and write $\mathfrak{R}'$ for its ring of integers. Then we have,

Lemma 4.2. *Assume that $\mathfrak{R}' = \mathfrak{R} \otimes_{\mathfrak{r}} \mathfrak{r}'$. Then the HBAV $X(\mathfrak{U}) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$ has CM by $\mathfrak{R}'$ and it corresponds to the fractional ideal $\mathfrak{U}\theta^{-1}_{F'/F}$.*

Proof. We write $K = F(d)$ and then $K' = F'(d)$. In particular, since $X(\mathfrak{U})$ has CM by K, we conclude that $X(\mathfrak{U}) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F}$ has CM by K', as we have $d \in End(X(\mathfrak{U})) \hookrightarrow End(X(\mathfrak{U})) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F})$. Moreover, we have

$$X(\mathfrak{U}) \otimes_{\mathfrak{r}} \theta^{-1}_{F'/F} = X(\mathfrak{U}) \otimes_{\mathfrak{r}} \mathfrak{r}' \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F} = X(\mathfrak{U}\mathfrak{R}') \otimes_{\mathfrak{r}'} \theta^{-1}_{F'/F} = X(\mathfrak{U}\mathfrak{R}')/(X(\mathfrak{U}\mathfrak{R}')[\theta_{F'/F}]).$$

But we have that $X(\mathfrak{U}\mathfrak{R}')/(X(\mathfrak{U}\mathfrak{R}')[\theta_{F'/F}]) = X(\mathfrak{U}\theta^{-1}_{F'/F}\mathfrak{R}')$, which concludes the proof as a fractional ideal of K' has CM by $\mathfrak{R}'$. □

We remark that the condition of the lemma, $\mathfrak{R}' = \mathfrak{R}\mathfrak{r}'$ holds in our setting. Indeed we know that $\mathfrak{R} = \mathfrak{R}_0\mathfrak{r}$ as $[K : \mathbb{Q}] = [K_0 : \mathbb{Q}][F : \mathbb{Q}]$ and $F/\mathbb{Q}$ and $K_0/\mathbb{Q}$ have disjoint ramification. Similarly we have $\mathfrak{R}' = \mathfrak{R}_0\mathfrak{r}'$. But then we have $\mathfrak{R}' = \mathfrak{R}_0\mathfrak{r}' = \mathfrak{R}_0\mathfrak{r}\mathfrak{r}' = \mathfrak{R}\mathfrak{r}'$.

In order to proceed further we need to consider the action of $Frob_p$ on modular forms. We refer to [25, page 224 (1.11.21)] for its definition. Here we simply recall its effect on the q-expansion. It is

$$Frob_p(f)(q) = f(q^p),$$

which is explained in [25, page 224, (1.11.22)]. We then have the following proposition.

Proposition 4.3. *(Congruences) Let $\mathfrak{c}$ be a fractional ideal of F relative prime to $\mathfrak{f}p$ and assume that the prime factors of $\theta_{F'/F}$ appear in the prime factors dividing $p\mathfrak{f}$. Then we have the congruences of Eisenstein series*

$$res_{\Delta}(E_k(\phi', \mathfrak{c}\theta_{F'/F})) \equiv Frob_p(E_{pk}(\phi, \mathfrak{c})) \mod p \quad (4.6)$$

where $\phi := \phi' \circ ver$ and ϕ' is a locally constant $\mathbb{Z}_p$-valued function on $\{\mathfrak{r}'^{\times}_p \times (\mathfrak{r}'/\mathfrak{f})^{\times}\} \times \{\mathfrak{r}'^{\times}_p \times (\mathfrak{r}'/\mathfrak{f})^{\times}\}$, extended trivially by zero to $\{\mathfrak{r}'_p \times (\mathfrak{r}'/\mathfrak{f})\} \times \{\mathfrak{c}^{-1}\theta^{-1}_{F'/F} \otimes_{\mathfrak{r}'} \mathfrak{r}'_p \times (\mathfrak{r}'/\mathfrak{f})\}$, which satisfy $\phi'^{\gamma} = \phi'$ for all $\gamma \in \Gamma$.

Proof. We consider the cusp $(\mathfrak{r}', \mathfrak{b}\theta_{F'/F}^{-1})$ for $\mathfrak{b}$ a fractional ideal of F equal to $\mathfrak{c}^{-1}$. From Proposition 3.1 we know that the q-expansion of the Eisenstein series $E_k(\phi', \mathfrak{c}\theta_{F'/F})$ at the cusp $(\mathfrak{r}', \mathfrak{b}\theta_{F'/F}^{-1})$ is given by

$$E_k(\phi', \mathfrak{c}\theta_{F'/F})(Tate_{\mathfrak{r}', \mathfrak{b}\theta_{F'/F}^{-1}}(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 \ll \xi' \in \mathfrak{b}\theta_{F'/F}^{-1}} a(\xi', \phi', k))q^{\xi'}, \tag{4.7}$$

with

$$a(\xi', \phi', k) = \sum_{(a,b) \in (\mathfrak{r}' \times \mathfrak{b}\theta_{F'/F}^{-1})/\mathfrak{r}'^{\times}, ab = \xi'} \phi'(a, b) sgn(N(a))N(a)^{k-1}. \tag{4.8}$$

As the function ϕ' is supported on $\mathfrak{r}'^{\times}_p \times (\mathfrak{r}/\mathfrak{f})^{\times}$ with respect to the second variable (i.e. the b's above), then our assumptions on the ramification of F'/F (i.e. that the prime factors of $\theta_{F'/F}$ appear in the prime factors dividing $p\mathfrak{f}$) imply that the above q-expansion with respect to the selected cusp is given by

$$E_k(\phi', \mathfrak{c}\theta_{F'/F})(Tate_{\mathfrak{r}', \mathfrak{b}\theta_{F'/F}^{-1}}(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 \ll \xi' \in \mathfrak{b}\mathfrak{r}'} a(\xi', \phi', k))q^{\xi'}, \tag{4.9}$$

with

$$a(\xi', \phi', k) = \sum_{(a,b) \in (\mathfrak{r}' \times \mathfrak{b}\mathfrak{r}')/\mathfrak{r}'^{\times}, ab = \xi'} \phi'(a, b) sgn(N(a))N(a)^{k-1}. \tag{4.10}$$

From Lemma 4.1 and the discussion after that it follows that the q-expansion of the restricted Eisenstein series $res_{\Delta}(E_k(\phi', \mathfrak{c}\theta_{F'/F}))$ at the cusp $(\mathfrak{r}, \mathfrak{b})$ is given by

$$res_{\Delta}(E_k(\phi', \mathfrak{c}\theta_{F'/F}))(Tate_{\mathfrak{r}, \mathfrak{b}}(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 \ll \xi \in \mathfrak{b}} a(\xi, \phi', k)q^{\xi}, \tag{4.11}$$

where

$$a(\xi, \phi', k) = \sum_{\xi' \in \mathfrak{b}\mathfrak{r}', Tr_{F'/F}(\xi') = \xi} a(\xi', \phi', k). \tag{4.12}$$

The q-expansion of the Eisenstein series $E_{pk}(\phi, \mathfrak{c})$ at the cusp $(\mathfrak{r}, \mathfrak{b})$ is given by

$$E_{pk}(\phi, \mathfrak{c})(Tate_{\mathfrak{r}, \mathfrak{b}}(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 \ll \xi \in \mathfrak{b}} a(\xi, \phi, pk))q^{\xi}, \tag{4.13}$$

with

$$a(\xi, \phi, pk) = \sum_{(a,b) \in (\mathfrak{r} \times \mathfrak{b})/\mathfrak{r}^{\times}, ab = \xi} \phi(a, b) sgn(N(a))N(a)^{pk-1} \tag{4.14}$$

and hence that of $Frob_p(E_{pk}(\phi, \mathfrak{c}))$ is given by

$$Frob_p(E_{pk}(\phi, \mathfrak{c})(Tate_{\mathfrak{r}, \mathfrak{b}}(q), \lambda_{can}, \omega_{can}, i_{can}) = \sum_{0 \ll \xi \in \mathfrak{b}} a(\xi, \phi, pk))q^{p\xi}. \tag{4.15}$$

We note here that, by the support assumptions on ϕ' and hence also on ϕ, only the terms $a(\xi, \phi, pk)$ with $(\xi, p) = 1$ in the sum above are non-zero. In order to establish the congruences of the Eisenstein series it is enough, thanks to the q-expansion principle, to establish the congruences between the q-expansions at the selected cusp $(\mathfrak{r}, \mathfrak{b})$.

We start by observing that the Eisenstein series $Frob_p(E_{pk}(\phi, \mathfrak{c}))$ has non-zero terms only at terms divisible by p, as we assume that the ideal $\mathfrak{b}$ is prime to p. We consider the ξ^{th}-term of $res_{\Delta} E_k(\phi', \mathfrak{c}\theta_{F'/F})$. It is equal to

$$a(\xi, \phi', k) = \sum_{\xi' \in \mathfrak{b}\mathfrak{r}', Tr_{F'/F}(\xi') = \xi} \ \sum_{(a,b) \in (\mathfrak{r}' \times \mathfrak{b}\mathfrak{r}')/\mathfrak{r}'^{\times}, ab = \xi'} \phi'(a, b) sgn(N(a)) N(a)^{k-1}. \tag{4.16}$$

We observe that the group $\Gamma = Gal(F'/F)$ acts on the triples (ξ', a, b) of the summation above by $(\xi', a, b)^{\gamma} := (\xi'^{\gamma}, a^{\gamma}, b^{\gamma})$, as $\mathfrak{b}$ is an ideal of F hence is preserved by Γ, where the action on a and b is modulo the units in $\mathfrak{r}'$ to understand. We write γ for a generator of Γ. We consider two cases, the case where (ξ', a, b) is fixed by γ and the case where it is not.

Let S_{ξ} be the set of triples $(\xi', a, b) \in \mathfrak{b}\mathfrak{r}' \times (\mathfrak{r}' \times \mathfrak{b}\mathfrak{r}')/\mathfrak{r}'^{\times}$ with $Tr_{F'/F}(\xi') = \xi$ and $ab = \xi'$. As Γ acts on S_{ξ}, we may write

$$a(\xi, \phi', k) = \sum_{(\xi', a, b) \in S_{\xi}} \phi'(a, b) sgn(N(a)) N(a)^{k-1}$$

$$= \sum_{j} \sum_{(\xi', a, b) \in (\xi'_j, a_j, b_j)\Gamma} \phi'(a, b) sgn(N(a)) N(a)^{k-1},$$

where we write $S_{\xi} = \coprod_j (\xi'_j, a_j, b_j)\Gamma$, for a set $\{(\xi'_j, a, b)\}$ of representatives of the Γ-action on S_{ξ}. Since Γ is cyclic of order p, for every j we have that all the triples $(\xi', a, b) \in (\xi'_j, a_j, b_j)\Gamma$ are conjugated and no one of them is fixed, or $(\xi'_j, a_j, b_j)\Gamma = \{(\xi', a, b)\}$ and (ξ', a, b) is fixed. We set

$$S_j := \sum_{(\xi', a, b) \in (\xi'_j, a_j, b_j)\Gamma} \phi'(a, b) sgn(N(a)) N(a)^{k-1}.$$

We first consider the case where $(\xi', a, b) \in (\xi'_j, a_j, b_j)\Gamma$ is not fixed by Γ. In this case we notice that, as ϕ' is fixed under Γ, we have that $\phi'(a^{\gamma}, b^{\gamma}) = \phi'(a, b)$. Hence we have

$$S_j = \sum_{i=0}^{p-1} \phi'(a_j^{\gamma^i}, b_j^{\gamma^i}) sgn(N(a_j^{\gamma^i})) N(a_j^{\gamma^i})^{k-1} = p\ \phi'(a_j, b_j) sgn(N(a_j)) N(a_j)^{k-1} \equiv 0 \mod p. \tag{4.17}$$

If (ξ', a, b) is fixed by γ then that implies that (i) $\xi' \in F$ and (ii) the ideals generated by a and b in $\mathfrak{r}'$ are coming from ideals in $\mathfrak{r}$ since they are relative prime to $\theta_{F'/F}$, i.e. to the primes where the extension is ramified. The last fact follows from the support assumption on ϕ'. Moreover, as we assume that $Cl_F \hookrightarrow Cl_{F'}$, we have that actually the elements themselves are (up to units) equal to elements from F. In this case we first notice that $\xi = Tr_{F'/F}(\xi') = p\xi'$ and, as $\xi' \in \mathfrak{b}\mathfrak{r}'$ with $\mathfrak{b}$ prime to p, we have that ξ is also divisible by p in the sense that is of the form $p\xi'$ for $\xi' \in \mathfrak{b}$. Further, we have the congruences modulo p

$$S_j = \phi'(a_j, b_j)sgn(N_{F'}(a_j))N_{F'}(a_j)^{k-1} = \phi(a_j, b_j)sgn(N_F(a_j)^p)N_F(a_j)^{p(k-1)}$$

$$\equiv \phi(a_j, b_j)sgn(N_F(a_j))N_F(a_j)^{pk-1} \mod p. \tag{4.18}$$

We already remarked that, whenever there exists (ξ', a, b) which is fixed, then $\xi = p\xi'$ with $\xi' \in F \cap \mathfrak{b}\mathfrak{r}' = \mathfrak{b}$, (this latter equality as we assume that $(\mathfrak{b}\mathfrak{r}', \theta_{F'/F}) = 1$). Conversely, whenever $\xi = p\xi'$ with $\xi' \in \mathfrak{b}$, such a fixed triple exists (for example $(\xi', 1, \xi')$). Hence, if we do not have $\xi = p\xi'$ for some $\xi' \in \mathfrak{b}$, all triples are not fixed, $S_j \equiv 0 \mod p$ for every j by the congruences in equation 4.17 and then $a(\xi, \phi', k) \equiv 0 \mod p$. On the other hand the ξ-Fourier coefficient in equation 4.15 is zero. Suppose now that $\xi = p\xi'$ with $\xi' \in \mathfrak{b}$. By the congruences in equations 4.17 and 4.18 we have

$$a(\xi, \phi', k) \equiv \sum_{j\ fixed} S_j \equiv \sum_{(\xi', a, b)\ fixed} \phi(a, b)sgn(N_F(a))N_F(a)^{pk-1} \tag{4.19}$$

Now we note that, in order to conclude the proposition, it is enough to show that, for all $\xi \in F$ of the form $\xi = p\xi'$ with ξ' relative prime to p, we have

$$S_\xi^\Gamma = \{(a, b) \in (\mathfrak{r} \times \mathfrak{b})/\mathfrak{r}^\times : ab = \xi'\}, \tag{4.20}$$

since then the right hand side of equation 4.19 is $a(\xi', \phi, pk)$, which is indeed the ξ-Fourier coefficient in 4.15. For the proof of equation 4.20 we first observe that the inclusion

$$S_\xi^\Gamma \supseteq \{(a, b) \in (\mathfrak{r} \times \mathfrak{b})/\mathfrak{r}^\times : ab = \xi'\} \tag{4.21}$$

follows directly. For the other direction we have to observe the following. As we have already remarked, from our assumption that $Cl_F \hookrightarrow Cl_{F'}$, it follows that for an element $(\xi', a, b) \in S_\xi^\Gamma$ there exists $(a', b') \in \mathfrak{r} \times \mathfrak{b}$ and $u_a, u_b \in \mathfrak{r}'^\times$ such that $a = u_a a'$ and $b = u_b b'$. In particular, we have that $\xi' = ab = u_a u_b a' b'$. Note that, since $\xi' \in F$ and $a'b' \in F$, it follows that $u_a u_b \in F \cap \mathfrak{r}'^\times = \mathfrak{r}^\times$. But then we have that

$$(\xi', a, b) = (\xi', u_a a', u_b b') = (\xi', u_b^{-1} u_b u_a a', u_b b') \sim (\xi', u_a u_b a', b'), \tag{4.22}$$

since the equivalence relation $(\xi', x, y) \sim (\xi', x', y')$ is given by $x = e^{-1}x'$ and $y = ey'$ for $e \in \mathfrak{r}'^\times$. But then we are done since

$$(u_a u_b a', b') \in \{(a, b) \in (\mathfrak{r} \times \mathfrak{b})/\mathfrak{r}^\times : ab = \xi'\}$$

as $u_a u_b a' b' = ab = \xi'$, $u_a u_b \in \mathfrak{r}^\times$ and $(a', b') \in \mathfrak{r} \times \mathfrak{b}$. □

5. Using the Theory of Complex Multiplication

Before we prove our main theorem we need to make some preparation. In this section we explain how we can use the theory of complex multiplication to understand how Frobenious operates on values of Eisenstein series at CM points. We recall that we consider the CM types (K_0, Σ_0) and its lift (K, Σ). Moreover by our setting we have that the reflex field for both of these CM types is simply (K_0, Σ_0). We first note that, since we assume that p is unramified in F, then the triples $(X(\mathfrak{U}), \lambda(\mathfrak{U}), i(\mathfrak{U})))$ are defined over the ring of integers of $W = W(\bar{\mathbb{F}}_p)$ (see [18] page 69). We write Φ for the extension of the Frobenious in $Gal(\mathbb{Q}_p^{nr}/\mathbb{Q}_p)$ to W. In this section we prove the following proposition, which is just a reformulation of what is done in [24] (page 539) in the case of quadratic imaginary fields.

Proposition 5.1. *(Reciprocity law on CM points) For every fractional ideal $\mathfrak{U}$ of the CM field K and ϕ a $\mathbb{Z}_p$ valued locally constant function, we have the reciprocity law*

$$Frob_p(E_{pk}(\phi, \mathfrak{c})(X(\mathfrak{U}), \lambda(\mathfrak{U}), i(\mathfrak{U}))) = (E_{pk}(\phi, \mathfrak{c})(X(\mathfrak{U}), \lambda(\mathfrak{U}), i(\mathfrak{U})))^{\Phi}. \tag{5.1}$$

Proof. Let us write $\mathcal{R}$ for the ring of integers of W. As we are assuming that ϕ is $\mathbb{Z}_p$ valued and we know from above that the triple $(X(\mathfrak{U}), \lambda(\mathfrak{U}), i(\mathfrak{U}))$ is defined over $\mathcal{R}$, then we have by [25, page 247 (3.2.3)] that the value of the Eisenstein series is in $\mathcal{R}$. From the compatibility of p-adic modular forms with ring extensions and the fact that the Eisenstein series is defined over $\mathbb{Z}_p$, we have that

$$(E_{pk}(\phi, \mathfrak{c})(X(\mathfrak{U}), \lambda(\mathfrak{U}), i(\mathfrak{U})))^{\Phi} = E_{pk}(\phi, \mathfrak{c})((X(\mathfrak{U}), \lambda(\mathfrak{U}), i(\mathfrak{U})) \otimes_{\mathcal{R},\Phi} \mathcal{R}), \tag{5.2}$$

where the tensor product is with respect to the map $\Phi : \mathcal{R} \to \mathcal{R}$, i.e. the base change of the triple $(X(\mathfrak{U}), \lambda(\mathfrak{U}), i(\mathfrak{U}))$ with respect to the Frobenious map. But then, from the theory of complex multiplication see [26] (Lemma 3.1 in page 61 and Theorem 3.4 in page 66), the fact that the reflex field of (K, Σ) is (K_0, Σ_0) and that p is ordinary we have that

$$(X(\mathfrak{U}), \lambda(\mathfrak{U}), i(\mathfrak{U})) \otimes_{\mathcal{R},\Phi} \mathcal{R} \cong (X'(\mathfrak{U}), \lambda'(\mathfrak{U}), i'(\mathfrak{U})), \tag{5.3}$$

where $(X'(\mathfrak{U}), \lambda'(\mathfrak{U}), i'(\mathfrak{U}))$ is the quotient obtained by X/H_{can}, with $H_{can} := i(\theta_F^{-1} \otimes \mu_p)$ as explained in Katz [25, page 222]. Moreover, as in Katz, we have that the Tate HBAV $(Tate'_{\mathfrak{a},\mathfrak{b}}(q), \lambda'_{can}, i'_{can})$ is obtained from $(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, i_{can})$ by the map $q \mapsto q^p$, from which we conclude the proposition. □

6. Complex and p-adic Periods.

In this section we study the various periods (archimedean and p-adic) that appear in the interpolation properties of the KHT-measure. We also consider the relative situation and we focus especially in the case of interest with $(K_0, \Sigma_0) < (K, \Sigma) < (K', \Sigma')$.

The periods of Katz: We start by recalling the periods defined by Katz and then showing that in the case of the twisted measure the periods used remain unchanged. We follow Katz (see [25] page 268 and for a more detailed explanation see [13]) and fix a nowhere vanishing differential over $A := \{a \in \bar{\mathbb{Q}} : incl(p)(a) \in D_p\}$ (here $incl(p)$ is the embedding of $\bar{\mathbb{Q}}$ in D_p induced from our fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$),

$$\omega : Lie(X(\mathfrak{R})) \cong \theta_F^{-1} \otimes A. \tag{6.1}$$

Then for any fractional ideal $\mathfrak{U}$ of K that is relative prime to the place induced by $incl(p)$ we have an identification $Lie(X((\mathfrak{U})) = Lie(X(\mathfrak{R}))$ and hence one may use the very same ω to fix a nowhere differential of $X(\mathfrak{U})$ by

$$\omega(\mathfrak{U}) : Lie(X((\mathfrak{U})) = Lie(X(\mathfrak{R})) \cong \theta_F^{-1} \otimes A. \tag{6.2}$$

We use $incl(\infty) : A \hookrightarrow \mathbb{C}$ to define the standard complex nowhere vanishing differential $\omega_{trans}(X(\mathfrak{U}))$ associated to the torus $\mathbb{C}^{\Sigma}/\Sigma(\mathfrak{U})$. Then as in Katz ([25, Lemma 5.1.45]) we have an element $\Omega_K^{Katz} = (\ldots, \Omega(\sigma), \ldots)) \in (\mathbb{C}^{\times})^{\Sigma}$ such that, for all fractional ideals $\mathfrak{U}$ of K relative prime to p, we have

$$\omega(\mathfrak{U}) = \Omega_K^{Katz} \omega_{trans}(\mathfrak{U}). \tag{6.3}$$

Of course the same considerations hold for K_0 and K'. Especially for K' we want to compute also the periods for the twisted HBAV $X(\mathfrak{U} \otimes \xi)$. From the isomorphism $\imath$: $X(\mathfrak{U}) \cong X(\mathfrak{U} \otimes \xi^{-1})$ we have that we can pick the invariant differentials $\omega(\mathfrak{U} \otimes \xi^{-1})$ and $\omega_{trans}(\mathfrak{U} \otimes \xi^{-1})$ as $\xi \cdot \omega(\mathfrak{U})$ and $\xi \cdot \omega_{trans}(\mathfrak{U})$ respectively. That is,

$$\omega(\mathfrak{U} \otimes \xi^{-1}) : Lie(X((\mathfrak{U} \otimes \xi^{-1})) \rightarrow Lie(X(\mathfrak{U})) = Lie(X(\mathfrak{R}')) \cong \theta_{F'}^{-1} \otimes A \tag{6.4}$$

and

$$\omega_{trans}(\mathfrak{U} \otimes \xi^{-1}) : Lie(X((\mathfrak{U} \otimes \xi^{-1})^{an}) \rightarrow Lie(X(\mathfrak{U})^{an}) = Lie(X(\mathfrak{R}')^{an}) \cong \theta_{F'}^{-1} \otimes \mathbb{C}, \tag{6.5}$$

where the first map is the isomorphism induced by $\imath$ and is given by multiplication by ξ. In particular, we have that the selected periods are equal to $\Omega_{K'}^{Katz}$. Similarly Katz ([25] Lemma 5.1.47) defines p-adic periods in $(D_p^{\times})^{\Sigma}$ relating the invariant differential $\omega(\mathfrak{U})$ to the invariant differential $\omega_{can}(\mathfrak{U})$ obtained from the p^{∞}-structure. As above we obtain that the p-adic periods for the twisted HBAV are the same.

Picking the periods compatible:(See also [13] page 195 on the properties of the periods defined by Katz). Now we consider the more specific setting where (K, Σ) and (K', Σ') are lifted from the type (K_0, Σ_0). Moreover, as we assume that K_0 is the CM field of an elliptic curve defined over $\mathbb{Q}$, we have that $\mathfrak{R}_0$ has class number one, i.e. it is a P.I.D. That means that the ring of integers $\mathfrak{R}$ and $\mathfrak{R}'$ are free over $\mathfrak{R}_0$. In particular we have

$$X_{\Sigma}(\mathfrak{R}) = X_{\Sigma}(\mathfrak{R} \otimes_{\mathfrak{R}_0} \mathfrak{R}_0) = X_{\Sigma_0}(\mathfrak{R}_0) \otimes_{\mathfrak{R}_0} \mathfrak{R}$$

and similarly

$$X_{\Sigma'}(\mathfrak{R}') = X_{\Sigma'}(\mathfrak{R}' \otimes_{\mathfrak{R}_0} \mathfrak{R}_0) = X_{\Sigma_0}(\mathfrak{R}_0) \otimes_{\mathfrak{R}_0} \mathfrak{R}'.$$

These imply that we have

$$Lie(X_{\Sigma}(\mathfrak{R})) = \oplus_{j=1}^{g} Lie(X_{\Sigma_0}(\mathfrak{R}_0)) \tag{6.6}$$

and similarly

$$Lie(X_{\Sigma'}(\mathfrak{R}')) = \oplus_{j=1}^{g'} Lie(X_{\Sigma_0}(\mathfrak{R}_0)). \tag{6.7}$$

In particular that implies that

$$\Omega_K^{Katz} = (\ldots, \Omega(E), \ldots), \ and \ \Omega_{K'}^{Katz} = (\ldots, \Omega(E), \ldots). \tag{6.8}$$

Similarly for the p-adic periods we observe that $X(\mathfrak{R}) \cong E \times \ldots \times E$ and hence $X(\mathfrak{R})[p^{\infty}] \cong E[p^{\infty}] \times \ldots \times E[p^{\infty}]$, where E is the elliptic curve defined over $\mathbb{Q}$ that corresponds to the ideal $\mathfrak{R}_0$ with respect to the CM type (K_0, Σ_0). These considerations imply that

$$\Omega_{p,K}^{Katz} = (\ldots, \Omega_p(E), \ldots), \;\; and \;\; \Omega_{p,K'}^{Katz} = (\ldots, \Omega_p(E), \ldots). \tag{6.9}$$

In particular recalling the notation used in Theorem 3.2 and Proposition 3.5 we have that $\Omega_\infty^{k\Sigma} = \Omega_\infty(E)^{kg}$, $\Omega_p^{k\Sigma} = \Omega_p(E)^{kg}$,$\Omega_\infty^{k\Sigma'} = \Omega_\infty(E)^{kg'} = \Omega_\infty(E)^{kgp}$ and $\Omega_p^{k\Sigma} = \Omega_p(E)^{kgp}$. Finally, we note that the definition of the periods of Katz is in general independent of the Grössencharacter, since they depend only on its infinite type. This is why it is important to pick the differentials $\omega(\mathfrak{R})$ and $\omega(\mathfrak{R}')$ properly. And actually in our setting we have a very natural choice by considering the elliptic curve $E/\mathbb{Q}$ to whom the Grössencharacter ψ_0 is attached (recall that $\psi_K = \psi_{K_0} \circ N_{K/K_0}$ and $\psi_{K'} = \psi_{K_0} \circ N_{K'/K_0}$).

7. Congruences of Measures

We start this section by recalling various notations in force during this work. We recall that in section 3 we have fixed an integral ideal $\mathfrak{C}$ of K and a decomposition $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ such that

$$\mathfrak{F} + \mathfrak{F}_c = \mathfrak{R}, \;\; \mathfrak{F} + \mathfrak{F}^c = \mathfrak{R}, \;\; \mathfrak{F}_c + \mathfrak{F}_c^c = \mathfrak{R}, \;\; \mathfrak{F}_c \supset \mathfrak{F}^c$$

and $\mathfrak{I}$ consists of ideals that are inert or ramify in K/F. We set $\mathfrak{f}' := \mathfrak{F}\mathfrak{I} \cap F$ and $\mathfrak{f}'' := \mathfrak{F}_c\mathfrak{I} \cap F$, $\mathfrak{f} := \mathfrak{f}' \cap \mathfrak{f}'' = \mathfrak{f}'$, $\mathfrak{s} = \mathfrak{F}_c \cap F$ and $\mathfrak{j} := \mathfrak{I} \cap F$.

Then we have considered various fractional ideals of K'. In particular, we have set $\mathfrak{F}' := \mathfrak{R}'\mathfrak{F}$, $\mathfrak{F}_c' := \mathfrak{R}'\mathfrak{F}_c$, $\mathfrak{I}' := \mathfrak{R}'\mathfrak{I}$ so that $\mathfrak{C}' := \mathfrak{R}'\mathfrak{C} = \mathfrak{F}'\mathfrak{F}_c'\mathfrak{I}'$ with

$$\mathfrak{F}' + \mathfrak{F}_c' = \mathfrak{R}', \;\; \mathfrak{F}' + \mathfrak{F}'^c = \mathfrak{R}', \;\; \mathfrak{F}_c' + \mathfrak{F}_c'^{\,c} = \mathfrak{R}', \;\; \mathfrak{F}_c' \supset \mathfrak{F}'^c$$

Similarly we define ideal of K' as $(\mathfrak{f}')' := \mathfrak{F}'\mathfrak{I}' \cap F'$ and $(\mathfrak{f}'')' := \mathfrak{F}_c'\mathfrak{I}' \cap F'$, $(\mathfrak{f})' := (\mathfrak{f}')' \cap (\mathfrak{f}'')' = (\mathfrak{f}')'$, $\mathfrak{s}' = \mathfrak{F}_c' \cap F'$ and $\mathfrak{j}' := \mathfrak{I}' \cap F'$, from which it follows that $(\mathfrak{f}')' := \mathfrak{f}'\mathfrak{R}'$, $(\mathfrak{f}'')' = \mathfrak{f}''\mathfrak{R}'$, $\mathfrak{s}' = \mathfrak{s}\mathfrak{R}'$ and $\mathfrak{j}' = \mathfrak{j}\mathfrak{R}'$.

In order to simplify our notation we are going to abuse the " $'$ " symbol used in the notation of the ideals over K' when it is clear over which field (K or K') we are working. For example we are going to write $Cl_{K'}(\mathfrak{I})$ for $Cl_{K'}(\mathfrak{I}')$ or $(\mathfrak{U}_j \otimes \xi^{-1})(\mathfrak{f}p)$ for $(\mathfrak{U}_j \otimes \xi^{-1})(\mathfrak{f}'p)$ and so on. Finally we inform the reader that the symbol ψ (or ψ') will denote a Grössecharacter (see below) and not the additive character that was used in the definition of the epsilon factors.

The goal of this section is to prove our main theorem. We recall that this amounts to proving the following

Theorem 7.1. *If (i) $Cl_K^-(\mathfrak{I}) \cong Cl_{K'}^-(\mathfrak{I})^\Gamma$ (ii) $Cl_F(1) \hookrightarrow Cl_{F'}(1)$ and (iii) $\theta_{F'/F} = (\xi)$ with $\xi \gg 0$, then we have the congruences*

$$\frac{\int_{G_K} \epsilon \circ ver \;\; d\mu_{\psi_K,\delta}^{KHT}}{\Omega_p(E)^g} \equiv \frac{\int_{G_{K'}} \epsilon \;\; d\mu_{\psi_{K'},\delta,\xi}^{KHT,tw}}{\Omega_p(E)^{pg}} \mod p\mathbb{Z}_p \tag{7.1}$$

for all ϵ locally constant $\mathbb{Z}_p$-valued functions on $G_{K'}$ with $\epsilon^\gamma = \epsilon$ and belonging to the cyclotomic part of it, i.e. when they are written as a sum of finite order characters they are of the form $\epsilon = \sum c_\chi \chi$ with $\chi^\tau = \chi$.

The strategy for proving the above theorem is as follows. By definition we have that the twisted KHT-measure is given as

$$\int_{G'} \phi(g)\mu_{\delta,\xi}^{KHT,tw}(g) := \sum_j \int_T \tilde{\phi}_j dE_j := \sum_j E_1(\phi_j, \mathfrak{c}_j^\xi)(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1})).$$

We consider the set of representatives $\{\mathfrak{U}_j\}$ of $Cl_{K'}^-(\mathfrak{J})$. If we consider the map

$$\rho : Cl_K^-(\mathfrak{J}) \to Cl_{K'}^-(\mathfrak{J})^\Gamma, \tag{7.2}$$

we may pick representatives of $Im(\rho)$ to be fractional ideals $\mathfrak{U}_j$ with the property $\mathfrak{U}_j^\gamma = \mathfrak{U}_j$ for all $\gamma \in \Gamma$. Moreover we may pick the other representatives of $Cl_{K'}^-(\mathfrak{J})$ such that, if $\mathfrak{U}_j$ is a representative, then if $\mathfrak{U}_j^\gamma$ is not in the same equivalent class as $\mathfrak{U}_j$ then it is also a representative (and this must hold for all $\gamma \in \Gamma$). We may split the twisted measure as follows,

$$\int_{G'} \phi(g)\mu_{\delta,\xi}^{KHT,tw}(g) = \sum_{\mathfrak{U}_j \in Im(\rho)} E_1(\phi_j, \mathfrak{c}_j^\xi)(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}))$$

$$+ \sum_{\mathfrak{U}_j \notin Im(\rho)} E_1(\phi_j, \mathfrak{c}_j^\xi)(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1})). \tag{7.3}$$

Our strategy is to compare the first summand (i.e those CM points that are coming from K) with the KHT-measure of K through the diagonal embedding that we have worked above. For the other part we will prove directly that, under the assumptions of our theorem, it is in $p\mathbb{Z}_p$. We start with the following proposition

Proposition 7.2. *Let $\mathfrak{U}_j$ be a fractional ideal of K'. Then for ϕ a locally constant function invariant under Γ we have,*

$$E_k(\phi, (\mathfrak{c}_j^\xi)^\gamma)(X(\mathfrak{U}_j^{\gamma(\xi)}), \lambda_\delta^\xi(\mathfrak{U}_j^\gamma \otimes \theta_{F'/F}^{-1}), \imath^\xi(\mathfrak{U}_j^\gamma \otimes \theta_{F'/F}^{-1}), \omega^{can}(\mathfrak{U}^{\gamma}{}_j^{(\xi)})) = E_k(\phi, \mathfrak{c}_j^\xi)(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^\xi(\mathfrak{U}_j \otimes \theta_F^{-}$$

for $\gamma \in \Gamma$.

Proof. The first thing that we note is that the following equality holds

$$E_k(\phi, (\mathfrak{c}_j^\xi)^\gamma)(X(\mathfrak{U}_j^{\gamma(\xi)}), \lambda_\delta^\xi(\mathfrak{U}_j^\gamma \otimes \theta_{F'/F}^{-1}), \imath^\xi(\mathfrak{U}_j^\gamma \otimes \theta_{F'/F}^{-1}), \omega^{can}(\mathfrak{U}^{\gamma}{}_j^{(\xi)})) = E_k(\phi, (\mathfrak{c}_j^\xi)^\gamma)(X(\mathfrak{U}_j^{\gamma(\xi^\gamma)}), \lambda_\delta^\xi(\mathfrak{U}_j^\gamma \otimes \theta_{F'/F}^{-1}), \imath^\xi($$

for all $\gamma \in \Gamma$. Indeed it is enough to observe that $\frac{\xi^\gamma}{\xi} \in \mathfrak{R}^\times$ and hence we have the equality of ideals $\mathfrak{U}_j^\gamma \otimes \xi^{-1} = \mathfrak{U}_j^\gamma \otimes (\xi^{-\gamma})$. We now have, from the definition of the Eisenstein series,

$$E_k(\phi, \mathfrak{c}_j^\xi)(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \omega^{can}(\mathfrak{U}_j^\xi)) =$$

$$\left(\frac{\Omega_p}{\Omega_\infty}\right)^{k\Sigma'} \frac{(-1)^{kg'}\Gamma(k+s)^{g'}}{\sqrt{(D_{F'})}} \sum_{w \in (\mathfrak{U}_j \otimes (\xi)^{-1})(\mathfrak{f}p)/\mathfrak{r}'^\times} \frac{P\phi(w)}{N(w)^k |N(w)^{2s}|} \,|_{s=0} \,. \tag{7.4}$$

As we assume that $\phi^\gamma = \phi$ for all $\gamma \in \Gamma$ we have that $P\phi(w^\gamma) = P\phi(w)$. Indeed, from the definition of the partial Fourier transform, we have

$$P\phi(x,y) = p^{\alpha[F':\mathbb{Q}]N(\mathfrak{f})^{-1}} \sum_{a \in X_\alpha} \phi(a,y) e_{F'}(ax), \tag{7.5}$$

for ϕ factoring through $X_\alpha \times \mathfrak{r}'_p \times (\mathfrak{r}'/\mathfrak{f})$ with $X_\alpha := \mathfrak{r}'_p/\alpha\mathfrak{r}'_p \times (\mathfrak{r}'/\mathfrak{f})$ with $\alpha \in \mathbb{N}$. But then

$$P\phi(x^\gamma, y^\gamma) = p^{\alpha[F':\mathbb{Q}]N(\mathfrak{f})^{-1}} \sum_{a \in X_\alpha} \phi(a, y^\gamma) e_{F'}(ax^\gamma). \tag{7.6}$$

As γ permutes X_α we have

$$\sum_{a \in X_\alpha} \phi(a,y^\gamma) e_{F'}(ax^\gamma) = \sum_{a \in X_\alpha} \phi(a^\gamma, y^\gamma) e_{F'}(a^\gamma x^\gamma) = \sum_{a \in X_\alpha} \phi(a,y) e_{F'}(ax), \tag{7.7}$$

which concludes our claim.

Back to our considerations we have that

$$\sum_{w \in (\mathfrak{U}_j \otimes (\xi))(\mathfrak{f}p)/\mathfrak{r}'^\times} \frac{P\phi(w)}{N(w)^k |N(w)^{2s}|} |_{s=0} = \sum_{w \in (\mathfrak{U}_j \otimes (\xi)^{-1})(\mathfrak{f}p)/\mathfrak{r}'^\times} \frac{P\phi(w^\gamma)}{N(w^\gamma)^k |N(w^\gamma)^{2s}|} |_{s=0} .$$

But the last sum is equal to $\sum_{w \in (\mathfrak{U}_j^\gamma \otimes (\xi^{-\gamma}))(\mathfrak{f}p)/\mathfrak{r}^\times} \frac{P\phi(w)}{N(w)^k |N(w)^{2s}|} |_{s=0}$, which concludes the proof. □

We know consider the measure $\mu^{KHT,tw}_{\psi_{K'},\delta,\xi}$. We recall that $\psi_{K'}$ is a Grössencharacter of type $-1\Sigma'$. To simplify our notation we now write $\psi' := \psi_{K'}$ and $\psi := \psi_K$. We write ψ'_{finite} for its finite part. From the computations that we have already done before theorem 3.5 we have

$$\int_{G'} \phi(g) \mu^{KHT}_{\psi_{K'},\delta,\xi} = \sum_j E_1\left((\phi\chi)_{j,finite}, \mathfrak{c}^\xi_j\right) \left(X(\mathfrak{U}^\xi_j), \lambda^\xi_\delta(\mathfrak{U}_j \otimes \theta^{-1}_{F'/F}), \imath^\xi(\mathfrak{U}_j \otimes \theta^{-1}_{F'/F}), \omega^{can}(\mathfrak{U}^\xi_j)\right)$$

$$= \sum_j E_{\psi'}(\phi_j, \mathfrak{c}^\xi_j) \left(X(\mathfrak{U}^\xi_j), \lambda^\xi_\delta(\mathfrak{U}_j \otimes \theta^{-1}_{F'/F}), \imath^\xi(\mathfrak{U}_j \otimes \theta^{-1}_{F'/F}), \omega^{can}(\mathfrak{U}^\xi_j)\right),$$

where we set

$$E_{\psi'}(\phi_j, \mathfrak{c}^\xi_j) := E_1((\phi\psi'^{-1})_{j,finite}, \mathfrak{c}^\xi_j).$$

Moreover we define the subset S of the selected representatives of $Cl^-_{K'}(\mathfrak{J})$ as the set of ideals that represent classes in $Cl^-_{K'}(\mathfrak{J})^\Gamma$ but not in $Im(\rho)$. Of course, under our assumptions that ρ is an isomorphism, we have that this set is empty, but we introduce this notation in order to prove the second form of the main theorem, where we relax the assumption that ρ is an isomorphism.

Corollary 7.3. *For the twisted KHT-measure we have the congruences*

$$\int_{G'} \phi(g) \mu^{KHT,tw}_{\psi',\delta,\xi}(g) \equiv \sum_{\mathfrak{U}_j \in Im(\rho)} E_{\psi'}(\phi_j, \mathfrak{c}^\xi_j)(X(\mathfrak{U}^\xi_j), \lambda^\xi_\delta(\mathfrak{U}_j \otimes \theta^{-1}_{F'/F}), \imath^\xi(\mathfrak{U}_j \otimes \theta^{-1}_{F'/F}), \omega^{can}(\mathfrak{U}^\xi_j))$$

$$+\sum_{\mathfrak{U}_j\in S} E_{\psi'}(\phi_j,\mathfrak{c}_j^{\xi})(X(\mathfrak{U}_j^{\xi}),\lambda_\delta^{\xi}(\mathfrak{U}_j\otimes\theta_{F'/F}^{-1}),\imath^{\xi}(\mathfrak{U}_j\otimes\theta_{F'/F}^{-1}),\omega^{can}(\mathfrak{U}_j^{\xi}))\mod p, \quad (7.8)$$

for all $\mathbb{Z}_p$-valued locally constant functions ϕ of G' such that $\phi^\gamma=\phi$ for all $\gamma\in\Gamma$.

Proof. It follows directly from the fact that $|\Gamma|=p$ and that $\phi^\gamma=\phi$ for all $\gamma\in\Gamma$. □

Our next aim is to prove the following proposition

Proposition 7.4. *Under our assumption, for all $\mathbb{Z}_p$-valued locally constant ϕ with $\phi^\gamma=\phi$ for all $\gamma\in\Gamma$, we have the congruences*

$$\Phi(\int_G(\phi\circ ver)(g)\mu_{\psi^p,\delta}^{KHT}(g))\equiv\sum_{\mathfrak{U}_j\in Im(\rho)}E_{\psi'}(\phi_j,\mathfrak{c}_j^{\xi})(X(\mathfrak{U}_j^{\xi}),\lambda_\delta^{\xi}(\mathfrak{U}_j\otimes\theta_{F'/F}^{-1}),\imath^{\xi}(\mathfrak{U}_j\otimes\theta_{F'/F}^{-1}),\omega^{can}(\mathfrak{U}_j^{\xi}))\mod p,$$

where Φ was the extension of the Frobenious element from its action on $\mathbb{Q}_p^{nr}$ to its p-adic completion $\mathcal{J}_\infty$.

Proof. By definition we have that

$$\int_G(\phi\circ ver)(g)\mu_{\psi^p,\delta}^{KHT}(g)=\sum_j E_p\left(((\phi\circ ver)\psi^{-p})_{j,finite},\mathfrak{c}_j\right)\left(X(\mathfrak{U}_j),\lambda_\delta(\mathfrak{U}_j),i(\mathfrak{U}_j),\omega^{can}(\mathfrak{U}_j)\right)$$

$$=E_p\left(((\phi\psi^{-p}\circ ver))_{j,finite},\mathfrak{c}_j\right)\left(X(\mathfrak{U}_j),\lambda_\delta(\mathfrak{U}_j),i(\mathfrak{U}_j),\omega^{can}(\mathfrak{U}_j)\right)$$

$$=\sum_j E_{(\psi^p)}(\phi\circ ver_j,\mathfrak{c}_j)(X(\mathfrak{U}_j),\lambda_\delta(\mathfrak{U}_j),\imath(\mathfrak{U}_j),\omega^{can}(\mathfrak{U}_j),$$

where the sum runs over a set of representatives of $Cl_K^-(\mathfrak{J})$ and

$$E_{(\psi^p)}(\phi\circ ver_j,\mathfrak{c}_j):=E_p((\phi{\psi'}^{-1}\circ ver)_{j,finite},\mathfrak{c}_j),$$

where we note that $\psi'\circ ver=\psi^p$ as $\psi'=\psi\circ N_{K'/K}$. Now we claim that $(\phi{\psi'}^{-1}\circ ver)_{j,finite}=(\phi{\psi'}^{-1})_{j,finite}\circ ver$. Indeed from the definition of $(\phi{\psi'}^{-1})_{j,finite}$ we have

$$(\phi{\psi'}^{-1})_{j,finite}(x',a',y',b')=\tilde{\phi}_j({x'}^{-1},{a'}^{-1},y',b'){\psi'}^{-1}_{finite}({x'}^{-1},{a'}^{-1},y',b'){\psi'}^{-1}({\mathfrak{U}'_j}^{-1}).$$

Since the map $ver(\mathfrak{U}_j)=\mathfrak{U}_j\mathfrak{R}'=\mathfrak{U}'_j$ we have $\tilde{\phi}_j\circ ver=(\widetilde{\phi\circ ver})_j$ (we recall that $\tilde{\phi}_j(t)=\phi(t[\mathfrak{U}'_j]^{-1})$). We also have that $({\psi'}^{-1}\circ ver)(\mathfrak{U}_j^{-1})={\psi'}^{-1}(\mathfrak{U}_j^{-1}\mathfrak{R}')$. Hence in order to conclude our claim we have to check that ${\psi'}^{-1}_{finite}\circ ver=({\psi'}^{-1}\circ ver)_{finite}$. But this last equality follows from the definition of the finite part as

$$\psi'(x',a',y',b')=\psi'_{finite}(x',a',y',b')\mathbf{N}'(x'),$$

where on the left hand-side of the equation is the restriction of ψ' to identity component (i.e. $\mathfrak{U}_j=\mathfrak{R}'$) and $\mathbf{N}'$ has the obvious meaning. But since ψ' is of type $-\Sigma'$ we have that $\psi'\circ ver$ is a character of infinity type $-p\Sigma$ hence

$$(\psi'\circ ver)(x,a,y,b)=(\psi'\circ ver)_{finite}(x,a,y,b)\mathbf{N}^p(x).$$

But by definition $(\psi' \circ ver)(x,a,y,b) = \psi'(x,a,y,b)$ and $\mathbf{N}^p(x) = (\mathbf{N}' \circ ver)(x) = \mathbf{N}'(x)$ from which we conclude our claim. Now we have,

$$Frob_p\left(E_{\psi^p}(\phi \circ ver_j, \mathfrak{c}_j)\right) = Frob_p\left(E_p\left(\left(\phi\psi'^{-1} \circ ver\right)_{j,finite}, \mathfrak{c}_j\right)\right)$$

$$= Frob_p\left(E_p\left((\phi\psi'^{-1})_{j,finite} \circ ver, \mathfrak{c}_j\right)\right).$$

From the congruences between the Eisenstein series that we have proved in Proposition 4.3 we have that

$$Frob_p(E_{(\psi^p)}((\phi \circ ver)_j, \mathfrak{c}_j))(X(\mathfrak{U}_j), \lambda_\delta(\mathfrak{U}_j), \imath(\mathfrak{U}_j), \omega^{can}(\mathfrak{U}_j)) \equiv$$

$$E_{\psi'}(\phi_j, \mathfrak{c}_j^\xi)(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \omega^{can}(\mathfrak{U}_j^\xi)) \mod p,$$

where of course in the right hand side $\mathfrak{U}_j$ is understood as $\mathfrak{U}_j\mathfrak{R}'$. We sum over all representatives of $Cl_K^-(\mathfrak{J})$ and then, from Proposition 5.1 and our assumption that ρ is injective, we obtain

$$\Phi(\int_G (\phi \circ ver)(g)\mu_{\psi^p,\delta}^{KHT}(g)) \equiv$$

$$\equiv \sum_{\mathfrak{U}_j \in Im(\rho)} E_{\psi'}(\phi_j, \mathfrak{c}_j^\xi)(X(\mathfrak{U}_j^\xi), \lambda_\delta^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \imath^\xi(\mathfrak{U}_j \otimes \theta_{F'/F}^{-1}), \omega^{can}(\mathfrak{U}_j^\xi)) \mod p.$$ □

Lemma 7.5. *Let ϕ be a locally constant $\mathbb{Z}_p$-valued function of G_K that is cyclotomic i.e. ϕ is the restriction to G_K of a locally constant function on G_F. Then we have that*

$$\frac{\int_G \phi(g)\mu_{\psi^k,\delta}^{KHT}(g)}{\Omega_p(E)^{gk}} \in \mathbb{Z}_p \tag{7.9}$$

for all $k \in \mathbb{N}$

Proof. This follows almost directly Lemma 3.4 and the discussion after it. Indeed we may write $\phi = \sum_\chi c_\chi \chi$, where χ are cyclotomic i.e. $\chi \circ c = \chi$. For such characters it is known (see for example [20]) that, for all $\sigma \in Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, we have

$$\left(\frac{\int_G \chi(g)\mu_{\psi^k,\delta}^{KHT}(g)}{\Omega_p(E)^{gk}}\right)^\sigma = \frac{\int_G (\chi(g))^\sigma \mu_{\psi^k,\delta}^{KHT}(g)}{\Omega_p(E)^{gk}}. \tag{7.10}$$

For all $\sigma \in G_{\mathbb{Q}_p}$ and ϕ's cyclotomic we have

$$\left(\frac{\int_G \phi(g)\mu_{\psi^k,\delta}^{KHT}(g)}{\Omega_p(E)^{gk}}\right)^\sigma = \sum_\chi c_\chi^\sigma \left(\frac{\int_G \chi(g)\mu_{\psi^k,\delta}^{KHT}(g)}{\Omega_p(E)^{gk}}\right)^\sigma = \sum_\chi c_\chi^\sigma \frac{\int_G (\chi(g))^\sigma \mu_{\psi^k,\delta}^{KHT}(g)}{\Omega_p(E)^{gk}}.$$

But then as $\phi(g) = (\phi(g))^\sigma = \sum_\chi c_\chi^\sigma \chi(g)^\sigma$ the last sum is equal to $\frac{\int_G \phi(g)\mu_{\psi^k,\delta}^{KHT}(g)}{\Omega_p(E)^{gk}}$ which finishes the proof. □

Note that a direct corollary of the proposition is

Corollary 7.6. *If ϕ is cyclotomic then,*

$$\int_{G'}\phi(g)\mu^{KHT}_{\psi',\delta,\xi}(g) - u^g\int_G(\phi\circ ver)(g)\mu^{KHT}_{\psi^p,\delta}(g)\equiv$$

$$\equiv\sum_{\mathfrak{U}_j\in S}E_{\psi'}(\phi_j,\mathfrak{c}^\xi_j)(X(\mathfrak{U}^\xi_j),\lambda^\xi_\delta(\mathfrak{U}_j\otimes\theta^{-1}_{F'/F}),\imath^\xi(\mathfrak{U}_j\otimes\theta^{-1}_{F'/F}))\mod p,\tag{7.11}$$

where $u\in\mathbb{Z}_p^\times$ is the element that has been fixed in section 2 by the property that $\frac{\Omega_p(E)^\Phi}{\Omega_p(E)}=u$.

Proof. We have

$$\Phi\left(\int_G(\phi\circ ver)(g)\mu^{KHT}_{\psi^p,\delta}(g)\right)=\Phi\left(\Omega_p(E)^{gp}\frac{\int_G(\phi\circ ver)(g)\mu^{KHT}_{\psi^p,\delta}(g)}{\Omega_p(E)^{gp}}\right)=u^{gp}\int_G(\phi\circ ver)(g)\mu^{KHT}_{\psi^p,\delta}(g)$$

since $\frac{\Omega_p(E)^\Phi}{\Omega_p(E)}=u$ and, from the assumption on ϕ, we have by lemma 7.5 that $\frac{\int_G(\phi\circ ver)(g)\mu^{KHT}_{\psi^p,\delta}(g)}{\Omega_p(E)^{gp}}\in\mathbb{Z}_p$. But as $u:=\psi_0(\bar\pi)\in\mathbb{Z}_p$ we have $u^p\equiv u\mod p$. Then the proof follows from Corollary 7.3 and Proposition 7.4. □

Lemma 7.7. *We have the congruences*

$$u^g\frac{\int_G\phi(g)\mu^{KHT}_{\psi^p,\delta}(g)}{\Omega_p(E)^{gp}}\equiv\frac{\int_G\phi(g)\mu^{KHT}_{\psi,\delta}(g)}{\Omega_p(E)^g}\mod p\tag{7.12}$$

for all locally constant $\mathbb{Z}_p$-valued functions ϕ of G.

Proof. As $\psi^p\equiv\psi\mod p$ we have that

$$\int_G\phi(g)\mu^{KHT}_{\psi^p,\delta}(g)=\int_G\phi(g)\hat\psi^{-p}\,\mu^{KHT}_\delta(g)\equiv\int_G\phi(g)\hat\psi^{-1}\,\mu^{KHT}_\delta(g)=\int_G\phi(g)\mu^{KHT}_{\psi,\delta}(g)\mod p.$$

Dividing by the unit $\Omega_p(E)^{pg}$ and observing that $u=\frac{\Omega_p(E)^\Phi}{\Omega_p(E)}\equiv\frac{\Omega_p(E)^p}{\Omega_p(E)}\mod p$, we have

$$\frac{\int_G\phi(g)\mu^{KHT}_{\psi^p,\delta}(g)}{\Omega_p(E)^{gp}}\equiv\frac{\int_G\phi(g)\mu^{KHT}_{\psi,\delta}(g)}{\Omega_p(E)^g}\times\frac{\Omega_p(E)^g}{\Omega_p(E)^{pg}}\mod p,$$

which concludes the proof. □

Now our assumptions of the main theorem imply that $S=\emptyset$. Then the last two statements conclude the proof of the main theorem. Note that if we do not assume that $S=\emptyset$ then we obtain the congruences

$$\int_{G_F}\epsilon\circ ver\quad d\mu_{E/F}\equiv\int_{G_{F'}}\epsilon\quad d\mu_{E/F'}-\Delta(\epsilon)\mod p\mathbb{Z}_p,\tag{7.13}$$

where

$$\Delta(\epsilon):=\frac{1}{\Omega_p(E)^{pg}}\sum_{\mathfrak{U}_j\in S}E_{\psi'}(\phi_j,\mathfrak{c}^\xi_j)(X(\mathfrak{U}^\xi_j),\lambda^\xi_\delta(\mathfrak{U}_j\otimes\theta^{-1}_{F'/F}),\imath^\xi(\mathfrak{U}_j\otimes\theta^{-1}_{F'/F}),\omega^{can}(\mathfrak{U}^\xi_j)).\tag{7.14}$$

The Fukaya-Kato conjecture and the measure of Katz: We would like to finish this work by stating the question of whether the p-adic interpolation properties of the Katz-Hida-Tilouine measure are canonical. In [15] (page 67, theorem 4.2.22) Fukaya and Kato conjecture a general formula for p-adic L functions for motives over any field. Does this formula agree with Katz-Hida-Tilouine's formula in the case where the motive considered is the one attached to a Grössencharacter over a CM field? We remark that our question is more concerning the p-adic and archimedean periods that appear in the two formulas.

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Appendix A. Evidences for the Needed Modification.

In this appendix we would like to indicate that the twisting of the Katz-Hida-Tiluine measure is needed, at least when the extension is ramified at p, in order to establish the congruences. We assume for simplicity that the character ψ_K is unramified (otherwise we have to consider also epsilon factors at primes besides those above p but this will not modify the main argument as these are always p-adic units) and we pick, with notation as in the introduction, $\mathfrak{n} = \mathfrak{r}$. Let us pick as the locally constant function ϵ that appear in the congruences the character $\tilde{\phi} := \phi \circ N_{K'/K}$ for some finite $\mathbb{Z}_p^\times$-valued character of G_K, which we assume cyclotomic (for example $\phi := \mathbf{1}$ or some of the $p-1$ order characters factorizing through the torsion of G_F, base changed to G_K). Then by the interpolation properties of the measure we have

$$\frac{\int_{G_{K'}} \tilde{\phi} \quad d\mu_{\psi_{K'}}}{\Omega_p(E)^{pg}} = \prod_{\mathfrak{p}\in\Sigma'_p} Local_{\mathfrak{p}}(\tilde{\phi}\psi_{K'}, \Sigma', \delta')(1-\tilde{\phi}\check{\psi}_{K'}(\bar{\mathfrak{p}}))(1-\tilde{\phi}\psi_{K'}(\bar{\mathfrak{p}}))\frac{L(0,\tilde{\phi}\psi_{K'})}{\sqrt{|D_{F'}|}\Omega(E)^{pg}} =$$

$$\frac{\prod_{\mathfrak{p}\in\Sigma'_p} Local_{\mathfrak{p}}(\tilde{\phi}\psi_{K'}, \Sigma', \delta')}{\sqrt{|D_{F'}|}} \prod_{\chi}\prod_{\mathfrak{p}\in\Sigma_p}(1-\phi\check{\psi}_K\chi(\bar{\mathfrak{p}}))(1-\phi\psi_K\chi(\bar{\mathfrak{p}}))\frac{L(0,\phi\psi_K\chi)}{\Omega(E)^g}, \tag{A.1}$$

where χ runs over the characters of the extension K'/K. Now we note that $\chi \equiv 1 \mod (\zeta_p - 1)$ and, since $Gal(K'/K)$ is a quotient of G_K, we have that

$$\frac{\int_{G_K} \phi\chi \quad d\mu_{\psi_K}}{\Omega_p(E)^g} \equiv \frac{\int_{G_K} \phi \quad d\mu_{\psi_K}}{\Omega_p(E)^g} \mod (\zeta_p - 1) \tag{A.2}$$

or equivalently, by [25, page 274, (5.3.5)]

$$\prod_{\mathfrak{p}\in\Sigma_p} Local_{\mathfrak{p}}(\phi\chi\psi_K, \Sigma, \delta)\prod_{\mathfrak{p}\in\Sigma_p}(1-\phi\check{\psi}_K\chi(\bar{\mathfrak{p}}))(1-\phi\psi_K\chi(\bar{\mathfrak{p}}))\frac{L(0,\phi\psi_K\chi)}{\sqrt{|D_F|}\Omega(E)^g} \equiv$$

$$\equiv \prod_{\mathfrak{p}\in\Sigma_p} Local_{\mathfrak{p}}(\phi\psi_K,\Sigma,\delta)(1-\phi\check{\psi}_K(\bar{\mathfrak{p}}))(1-\phi\psi_K(\bar{\mathfrak{p}}))\frac{L(0,\phi\psi_K)}{\sqrt{|D_F|}\Omega(E)^g} \mod (\zeta_p-1). \tag{A.3}$$

Taking the product over all χ's we obtain

$$\frac{\sqrt{|D_{F'}|}}{\sqrt{|D_F|^p}}\frac{\prod_\chi(\prod_{\mathfrak{p}\in\Sigma_p} Local_{\mathfrak{p}}(\phi\chi\psi_K,\Sigma,\delta))}{\prod_{\mathfrak{p}\in\Sigma'_p} Local_{\mathfrak{p}}(\tilde{\phi}\psi_{K'},\Sigma',\delta')}\frac{\int_{G_{K'}}\tilde{\phi}\quad d\mu_{\psi_{K'}}}{\Omega_p(E)^{pg}} \equiv \left(\frac{\int_{G_K}\phi\quad d\mu_{\psi_K}}{\Omega_p(E)^g}\right)^p \mod (\zeta_p-1)$$

Now we note that

$$\left(\frac{\int_{G_K}\phi\quad d\mu_{\psi_K}}{\Omega_p(E)^g}\right)^p \equiv \frac{\int_{G_K}\phi^p\quad d\mu_{\psi_K}}{\Omega_p(E)^g} \mod p \tag{A.4}$$

as the values of the integrals are in $\mathbb{Z}_p$ since we assume that ϕ is cyclotomic. In particular we have

$$\frac{\int_{G_{K'}}\tilde{\phi}d\mu_{\psi_{K'}}}{\Omega_p(E)^{gp}}\cdot Diff \equiv \frac{\int_{G_K}\tilde{\phi}\circ verd\mu_{\psi_K}}{\Omega_p(E)^g} \mod p. \tag{A.5}$$

Our aim is to show that $Diff \neq 1$ in general. First we need to understand the factor $\frac{\sqrt{|D_{F'}|}}{\sqrt{|D_F|^p}}\frac{\prod_\chi(\prod_{\mathfrak{p}\in\Sigma_p} Local_{\mathfrak{p}}(\phi\chi\psi_K,\Sigma,\delta))}{\prod_{\mathfrak{p}\in\Sigma'_p} Local_{\mathfrak{p}}(\tilde{\phi}\psi_{K'},\Sigma',\delta')}$ and where the quantity $\frac{\int_{G_{K'}}\tilde{\phi}\quad d\mu_{\psi_{K'}}}{\Omega_p(E)^{pg}}$ lies. We start with the local factors. From Lemma 3.3 we have that

$$Local(\phi\chi\psi_K,\Sigma,\delta)_{\mathfrak{p}} = c_{\mathfrak{p}}^{(\chi)}(\delta)e_{\mathfrak{p}}(\phi^{-1}\chi^{-1},\psi,dx_1)\left(\frac{\psi_K^{-1}(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\phi\chi)+n_{\mathfrak{p}}(\psi)} \tag{A.6}$$

and

$$Local(\tilde{\phi}\psi_K,\Sigma',\delta')_{\mathfrak{p}} = c'_{\mathfrak{p}}(\delta')e_{\mathfrak{p}}(\tilde{\phi}^{-1},\psi',dx_1)\left(\frac{\psi_{K'}^{-1}(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\tilde{\phi})+n_{\mathfrak{p}}(\psi')}, \tag{A.7}$$

where $c_{\mathfrak{p}}^{(\chi)}$ the local part of $\phi\chi\psi_K$ and dx_1 is the Haar measure that assigns measure 1 to the ring of integers of $K_{\mathfrak{p}}$ (with similar notations for the second expression). Now we note that (as easily seen from the functional equation and the fact that $Ind_K^{K'}\mathbf{1} = \oplus_\chi\chi$) we have that

$$\prod_{\mathfrak{p}\in\Sigma'} e_{\mathfrak{p}}(\tilde{\phi},\psi',dx'_\psi) = \prod_\chi\prod_{\mathfrak{p}\in\Sigma} e_{\mathfrak{p}}(\phi\chi,\psi,dx_\psi) \tag{A.8}$$

where we follow Tate's notation as in [29] for the Tamagawa measures dx_ψ and $dx_{\psi'}$. The relation between the Tamagawa measure dx_ψ and the normalized measure dx_1 of a place $\mathfrak{p}$ is given by $dx_\psi = N(\mathfrak{p})^{-n_{\mathfrak{p}}(\psi)/2}dx_1$ (There is a typo in Tate's [29] page 17, but see the same article in page 18 or Lang's Algebraic Number Theory page 277). That implies

$$\prod_{\mathfrak{p}\in\Sigma'} e_{\mathfrak{p}}(\tilde{\phi},\psi',dx'_\psi) = \prod_{\mathfrak{p}\in\Sigma'} e_{\mathfrak{p}}(\tilde{\phi},\psi',dx_1)N(\mathfrak{p})^{-n_{\mathfrak{p}}(\psi')/2} \tag{A.9}$$

and

$$\prod_\chi\prod_{\mathfrak{p}\in\Sigma} e_{\mathfrak{p}}(\phi\chi,\psi,dx_\psi) = \prod_\chi\prod_{\mathfrak{p}\in\Sigma} e_{\mathfrak{p}}(\phi\chi,\psi,dx_1)N(\mathfrak{p})^{-n_{\mathfrak{p}}(\psi)/2} = \prod_{\mathfrak{p}\in\Sigma} N(\mathfrak{p})^{-pn_{\mathfrak{p}}(\psi)/2}\prod_\chi e_{\mathfrak{p}}(\phi\chi,\psi,dx_1).$$

So we conclude the equation

$$\prod_{\mathfrak{p}\in\Sigma'} e_{\mathfrak{p}}(\tilde{\phi},\psi',dx_1)N(\mathfrak{p})^{-n_{\mathfrak{p}}(\psi')/2} = \prod_{\mathfrak{p}\in\Sigma} N(\mathfrak{p})^{-pn_{\mathfrak{p}}(\psi)/2}\prod_{\chi} e_{\mathfrak{p}}(\phi\chi,\psi,dx_1) \tag{A.10}$$

or, equivalently,

$$\prod_{\chi}\prod_{\mathfrak{p}\in\Sigma} e_{\mathfrak{p}}(\phi\chi,\psi,dx_1) = \frac{\prod_{\mathfrak{p}\in\Sigma'} N(\mathfrak{p})^{-n_{\mathfrak{p}}(\psi')/2}}{\prod_{\mathfrak{p}\in\Sigma} N(\mathfrak{p})^{-pn_{\mathfrak{p}}(\psi)/2}}\prod_{\mathfrak{p}\in\Sigma'} e_{\mathfrak{p}}(\tilde{\phi},\psi',dx_1). \tag{A.11}$$

As we assume that Σ and Σ' are ordinary and, for simplicity, we take the extension to be ramified only at p, we have that $\frac{\prod_{\mathfrak{p}\in\Sigma'} N(\mathfrak{p})^{n_{\mathfrak{p}}(\psi')/2}}{\prod_{\mathfrak{p}\in\Sigma} N(\mathfrak{p})^{pn_{\mathfrak{p}}(\psi)/2}} = \frac{\sqrt{|D_{F'}|}}{\sqrt{|D_F|^p}}$. Putting everything together we see that the discrepancy factor in the congruences,

$$Diff := \frac{\sqrt{|D_{F'}|}}{\sqrt{|D_F|^p}} \times \frac{\prod_{\chi}(\prod_{\mathfrak{p}\in\Sigma_p} Local_{\mathfrak{p}}(\phi\chi\psi_K,\Sigma,\delta))}{\prod_{\mathfrak{p}\in\Sigma'_p} Local_{\mathfrak{p}}(\tilde{\phi}\psi_{K'},\Sigma',\delta')}, \tag{A.12}$$

is equal to

$$Diff = \frac{\prod_{\chi}\prod_{\mathfrak{p}\in\Sigma_p} c_{\mathfrak{p}}^{(\chi)}(\delta)\left(\frac{\psi_K^{-1}(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\phi\chi)+n_{\mathfrak{p}}(\psi)}}{\prod_{\mathfrak{p}\in\Sigma'_p} c'_{\mathfrak{p}}(\delta')\left(\frac{\psi_{K'}^{-1}(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\tilde{\phi})+n_{\mathfrak{p}}(\psi')}}. \tag{A.13}$$

Now we claim that the factor

$$\frac{\prod_{\chi}\prod_{\mathfrak{p}\in\Sigma_p}\left(\frac{\psi_K^{-1}(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\phi\chi)+n_{\mathfrak{p}}(\psi)}}{\prod_{\mathfrak{p}\in\Sigma'_p}\left(\frac{\psi_{K'}^{-1}(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\tilde{\phi})+n_{\mathfrak{p}}(\psi')}} = 1. \tag{A.14}$$

Indeed we have

$$\prod_{\mathfrak{p}\in\Sigma'_p}\left(\frac{\psi_{K'}^{-1}(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\tilde{\phi})+n_{\mathfrak{p}}(\psi')} = \prod_{\mathfrak{p}\in\Sigma'_p}\left(\frac{\psi_K^{-1}\circ N_{K'/K}(\pi_{\mathfrak{p}})}{N_K\circ N_{K'/K}(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\tilde{\phi})+n_{\mathfrak{p}}(\psi')}. \tag{A.15}$$

For those $\mathfrak{p}'\in\Sigma'_p$ that are not ramified we have $n_{\mathfrak{p}'}(\psi') = n_{\mathfrak{p}}(\psi)$ for $\mathfrak{p}\in\Sigma_p$, the prime below $\mathfrak{p}'$. Similarly $n_{\mathfrak{p}'}(\tilde{\phi}) = n_{\mathfrak{p}}(\phi\chi) = n_{\mathfrak{p}}(\phi)$ for all the χ, as these are ramified only at the primes that ramify in K'/K. Then we have

$$\prod_{\mathfrak{p}\in\Sigma'_p,\ unram.}\left(\frac{\psi_K^{-1}\circ N_{K'/K}(\pi_{\mathfrak{p}})}{N_K\circ N_{K'/K}(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\tilde{\phi})+n_{\mathfrak{p}}(\psi')} = \prod_{\mathfrak{p}\in\Sigma_p,\ unram.}\left(\frac{\psi_K^{-1}(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{p(n_{\mathfrak{p}}(\phi)+n_{\mathfrak{p}}(\psi))} =$$

$$=\prod_{\chi}\prod_{\mathfrak{p}\in\Sigma_p,\ unram.}\left(\frac{\psi_K^{-1}(\pi_{\mathfrak{p}})}{N(\mathfrak{p})}\right)^{n_{\mathfrak{p}}(\chi\phi)+n_{\mathfrak{p}}(\psi)}. \tag{A.16}$$

Now we consider the ramified primes. We have

$$\prod_{\mathfrak{p}\in\Sigma'_p,\ ram.} \left(\frac{\psi_K^{-1}\circ N_{K'/K}(\pi_\mathfrak{p})}{N_K\circ N_{K'/K}(\mathfrak{p})}\right)^{n_\mathfrak{p}(\tilde{\phi})+n_\mathfrak{p}(\psi')} = \prod_{\mathfrak{p}\in\Sigma_p} \left(\frac{\psi_K^{-1}(\pi_\mathfrak{p})}{N(\mathfrak{p})}\right)^{n_\mathfrak{p}(\tilde{\phi})+n_\mathfrak{p}(\psi')}. \tag{A.17}$$

For every $\mathfrak{p}' \in \Sigma'_p$ that is ramified (totally as we consider a p-order extension), we have from the conductor-discriminant formula that

$$n_{\mathfrak{p}'}(\psi') = \sum_\chi n_\mathfrak{p}(\chi) + pn_\mathfrak{p}(\psi), \tag{A.18}$$

for the prime $\mathfrak{p} \in \Sigma_p$ below $\mathfrak{p}'$. Moreover, as the conductor-function $n_\mathfrak{p}(\cdot)$ is additive and inductive in degree zero, we have that

$$n_{\mathfrak{p}'}(\tilde{\phi}) = n_{\mathfrak{p}'}(Res(\phi)) = n_{\mathfrak{p}'}(Res(\phi)) - n_{\mathfrak{p}'}(\mathbf{1}) = n_{\mathfrak{p}'}(Res(\phi)\ominus\mathbf{1}) = n_\mathfrak{p}(Ind(Res(\phi))\ominus Ind(\mathbf{1})) =$$

$$= n_\mathfrak{p}(IndRes(\phi)) - n_\mathfrak{p}(Ind(\mathbf{1})) = n_\mathfrak{p}(\oplus_\chi \phi\chi) - n_\mathfrak{p}(\oplus_\chi\chi) = \sum_\chi n_\mathfrak{p}(\phi\chi) - \sum_\chi n_\mathfrak{p}(\chi). \tag{A.19}$$

Putting all together we conclude our claim. Hence we have that

$$Diff = \frac{\prod_\chi \prod_{\mathfrak{p}\in\Sigma_p} c_\mathfrak{p}^{(\chi)}(\delta) \left(\frac{\psi_K^{-1}(\pi_\mathfrak{p})}{N(\mathfrak{p})}\right)^{n_\mathfrak{p}(\phi\chi)+n_\mathfrak{p}(\psi)}}{\prod_{\mathfrak{p}\in\Sigma'_p} c'_\mathfrak{p}(\delta') \left(\frac{\psi_{K'}^{-1}(\pi_\mathfrak{p})}{N(\mathfrak{p})}\right)^{n_\mathfrak{p}(\tilde{\phi})+n_\mathfrak{p}(\psi')}} = \frac{\prod_\chi \prod_{\mathfrak{p}\in\Sigma_p} c_\mathfrak{p}^{(\chi)}(\delta)}{\prod_{\mathfrak{p}\in\Sigma'_p} c'_\mathfrak{p}(\delta')}. \tag{A.20}$$

Now we observe that

$$\prod_\chi \prod_{\mathfrak{p}\in\Sigma_p} c_\mathfrak{p}^{(\chi)}(\delta) = \prod_\chi \prod_{\mathfrak{p}\in\Sigma_p} (\phi\chi\psi_K)_\mathfrak{p}(\delta) = \prod_{\mathfrak{p}\in\Sigma_p} (\phi\psi_K)_\mathfrak{p}(\delta^p) \prod_\chi \chi_\mathfrak{p}(\delta)$$

$$= \prod_{\mathfrak{p}\in\Sigma_p} (\phi\psi_K)_\mathfrak{p}(\delta) \prod_\chi \chi_\mathfrak{p}(\delta) = \prod_{\mathfrak{p}\in\Sigma_p} (\phi\psi_K)_\mathfrak{p}(\delta^p), \tag{A.21}$$

since $\prod_\chi \chi_\mathfrak{p}(\delta) = 1$ because we multiply over all elements of the multiplicative group of characters of $Gal(K'/K)$ and we know that $\chi \neq \chi^{-1}$ for all $\chi \neq 1$ as these are p-order characters. Also we have that

$$\prod_{\mathfrak{p}\in\Sigma'_p} c'_\mathfrak{p}(\delta') = \prod_{\mathfrak{p}\in\Sigma'_p} (\phi\circ N_{K'/K})_\mathfrak{p}(\psi_K\circ N_{K'/K})_\mathfrak{p}(\delta') = \prod_{\mathfrak{p}\in\Sigma_p} (\phi\psi_K)_\mathfrak{p}(N_{K'/K}\delta'). \tag{A.22}$$

In particular, we observe that in general we have that

$$\prod_\chi \prod_{\mathfrak{p}\in\Sigma_p} c_\mathfrak{p}^{(\chi)}(\delta) \neq \prod_{\mathfrak{p}\in\Sigma'_p} c'_\mathfrak{p}(\delta'), \tag{A.23}$$

as $N_{K'/K}(\delta') \neq \delta^p$ when the extension K'/K is ramified at p. Actually, the two expressions may not even have the same valuation.

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