ALGEBRAICITY OF L-VALUES FOR ELLIPTIC CURVES IN A FALSE TATE CURVE TOWER

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Abstract. Let E be an elliptic curve over $\mathbb{Q}$, and τ an Artin representation over $\mathbb{Q}$ that factors through the non-abelian extension $\mathbb{Q}(\sqrt[p^n]{m}, \mu_{p^n})/\mathbb{Q}$, where p is an odd prime and n, m are positive integers. We show that $L(E,\tau,1)$, the special value at $s=1$ of the L-function of the twist of E by τ, divided by the classical transcendental period $\Omega_+^{d^+}|\Omega_-^{d^-}|\epsilon(\tau)$ is algebraic and Galois-equivariant, as predicted by Deligne's conjecture.

1. Introduction

Recently, there has been great interest in generalising classical Iwasawa theory of elliptic curves to a non-commutative setting. That is, instead of considering cyclotomic extensions of number fields, one considers infinite extensions K_∞/K, whose Galois group is a compact p-adic Lie group (see [4]). A fundamental concept in the theory is that of a p-adic L-function. Roughly speaking, this should interpolate the special values at $s=1$ of the L-functions of twists of the elliptic curve E by Artin representations that factor through K_∞.

One of the most basic non-abelian p-adic Lie extension is the false Tate curve extension over $\mathbb{Q}$, that is

$$\mathbb{Q}_{FT} = \cup_{n\geq 1}\mathbb{Q}(\mu_{p^n},\ \sqrt[p^n]{m}),$$

where p is an odd prime, $m>1$ is an integer that is not a p-th power, and where we write μ_k for the set of k-th roots of 1. The Galois group of $\mathbb{Q}_{FT}$ over $\mathbb{Q}$ is a semidirect product of $\mathbb{Z}_p^*$ by $\mathbb{Z}_p$. In this paper, we prove Deligne's period conjecture (see [5]) for the L-function of an elliptic curve over $\mathbb{Q}$ twisted by any Artin representation that factors through $\mathbb{Q}_{FT}$ (Theorem 4.2). In particular, we show (Corollary 4.3) that if E is an elliptic curve over $\mathbb{Q}$, and if F is a number field contained in $\mathbb{Q}_{FT}$, then

$$\frac{L(E/F,1)\sqrt{|\Delta_F|}}{\Omega_+(E)^{r_1+r_2}|\Omega_-(E)|^{r_2}} \in \mathbb{Q}\,,$$

where r_1 (resp. r_2) is the number of real (resp. complex) places of F, Δ_F is the discriminant of F, and $\Omega_\pm$ are the usual periods attached to E (see §2.2). Note that this is precisely the quotient that appears in the Birch–Swinnerton-Dyer conjecture.

The essential characteristic of the false Tate curve extension is that every irreducible Artin representation is induced from a 1-dimensional representation over $\mathbb{Q}(\mu_{p^n})$, for some $n\geq 0$ (see [6]). Note that, for $n>0$, this is a CM field, i.e. it is a totally imaginary quadratic extension of the totally real field $\mathbb{Q}(\mu_{p^n})^+$. This enables us to write the L-function of the twist of E by such an irreducible Artin

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representation as a Rankin-Selberg product over $\mathbb{Q}(\mu_{p^n})^+$. In [12], Shimura has established results concerning the algebraicity of the corresponding L-values. The periods appearing in Shimura's work involve the Petersson inner product of Hilbert modular forms, while the periods in non-commutative Iwasawa theory involve the classical periods $\Omega_\pm(E)$. The main issue in the proof of Theorem 4.2 is how to relate these periods (Theorem 3.4).

2. Notation and background results

2.1. **Fields and Artin representations.** We fix, once and for all, an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$, and an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_l$ and of $\overline{\mathbb{Q}}_l$ into $\mathbb{C}$ for each prime l, in such a way that the composition $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_l \to \mathbb{C}$ agrees with $\overline{\mathbb{Q}} \to \mathbb{C}$.

For every prime v of a number field K, we fix a copy of the decomposition group at v. We write I_v for the inertia group at v, and Φ_v for a geometric Frobenius at v (i.e. its image modulo I_v is the inverse of arithmetic Frobenius).

An Artin representation ρ over a number field K will always be assumed to take algebraic values, i.e. $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}})$. We write $\mathbb{Q}(\rho)$ for the finite Galois extension of $\mathbb{Q}$ generated by the coefficients of ρ. We will abuse notation and also write ρ for $\rho \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_\ell$, the representation with scalars extended to $\overline{\mathbb{Q}}_\ell$ for a prime ℓ.

If χ is a Hecke character of finite order over K, we also write χ for the corresponding ideal and Galois characters. (For the latter, we set our local reciprocity maps in unramified extensions to take a uniformiser to the geometric Frobenius element.)

We fix the following notation:

Δ_K	absolute discriminant of the number field K.
$\Delta_{K/K'}$	relative discriminant of K over K'.
$\mathbb{Q}_{ab}$	maximal abelian extension of $\mathbb{Q}$.
$\mathbb{Q}_{ab}^{\Sigma}$	for a finite set of rational primes Σ, this is the maximal abelian extension of $\mathbb{Q}$, unramified outside $\Sigma \cup \{\infty\}$.
K^+	for an abelian extension K of $\mathbb{Q}$, this is its maximal real subfield.
$\rho^\pm$	subspace of the Artin representation ρ over $\mathbb{Q}$ on which complex conjugation acts by ± 1. Its dimension does not depend on the choice of the complex conjugation element.
ρ^σ	for $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}})$ an Artin representation over K, and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, this is the Artin representation $\sigma \circ \rho$, i.e. σ acts on the coefficients of ρ.

2.2. **Elliptic curves and their L-functions.** Let E be an elliptic curve over $\mathbb{Q}$. We write $N(E)$ for the conductor of E. Fix a minimal Weierstrass equation for E, and write ω_E for the Néron differential. Pick a generator γ_+ (respectively γ_-) of the subspace of $H_1(E(\mathbb{C}), \mathbb{Z})$ on which complex conjugation acts trivially (respectively by -1). Set

$$\Omega_+(E) = \int_{\gamma_+} \omega_E \qquad\qquad \Omega_-(E) = \int_{\gamma_-} \omega_E \,.$$

Classically, one chooses $\gamma_\pm$ so that $\Omega_+(E)$ and $-i\Omega_-(E)$ are positive real numbers. We write $T_\ell(E)$ for the ℓ-adic Tate module of E, and set $V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $H^1_\ell(E) = \mathrm{Hom}(V_\ell(E), \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$.

We now define the Euler factors of E, ρ and the twist of E by ρ as follows. For a prime v of K, set

$$\begin{aligned} P_v(E/K, X) &= \det(1 - X\Phi_v | H^1_\ell(E)^{I_v}) \,, \\ P_v(\rho, X) &= \det(1 - X\Phi_v | \rho^{I_v}) \,, \\ P_v(E/K, \rho, X) &= \det(1 - X\Phi_v | (H^1_\ell(E) \otimes_{\mathbb{Q}_l} \rho)^{I_v}) \,, \end{aligned}$$

where ℓ is any prime not divisible by v. These definitions are independent of the choice of ℓ. Moreover, the coefficients of $P_v(E/K, X)$ are integral, and those of $P_v(\rho, X)$ and $P_v(E/K, \rho, X)$ lie in $\mathbb{Q}(\rho)$.

We define the L-functions

$$L(E/K, s) = \prod_v P_v(E/K, (N_{K/\mathbb{Q}} v)^{-s})^{-1} \,,$$

$$L(\rho, s) = \prod_v P_v(\rho, (N_{K/\mathbb{Q}} v)^{-s})^{-1} \,,$$

$$L(E/K, \rho, s) = \prod_v P_v(E/K, \rho, (N_{K/\mathbb{Q}} v)^{-s})^{-1} \,,$$

where the products are taken over the primes of K. These Euler products converge for $\mathrm{Re}(s)$ sufficiently large. The L-function of ρ is known to have meromorphic continuation to $\mathbb{C}$. The L-functions of E and its twists are conjectured to have analytic continuation to $\mathbb{C}$.

2.3. **Modular forms.** For Hilbert modular forms we follow the conventions of Shimura's article [12]. Let F be a totally real field, abelian over $\mathbb{Q}$. Let $\mathbf{g}$ be a Hilbert modular form of parallel weight l, level $\mathbf{n}$ and character ϕ. We write $L(\mathbf{g}, s) = \sum_{\mathbf{m}} C(\mathbf{m}, \mathbf{g}) N(\mathbf{m})^{-s}$ for the Dirichlet series attached to $\mathbf{g}$, where $\mathbf{m}$ will always be supposed to run over the non-zero integral ideals of $\mathcal{O}_F$. For a Hecke character $\chi : \mathbf{A}_F^*/F^* \to \mathbb{C}^*$, we write

$$D(\mathbf{g}, \chi, s) = \sum_{\mathbf{m}} \chi(\mathbf{m}) C(\mathbf{m}, \mathbf{g}) N(\mathbf{m})^{-s} \,.$$

Let $\mathbf{f}$ be a cusp form of parallel weight k, level $\mathbf{n}$ and character ψ. We write $\mathbf{D}(\mathbf{f}, \mathbf{g}, s)$ for the Rankin-Selberg product of $\mathbf{f}$ and $\mathbf{g}$, which is defined by

$$\mathbf{D}(\mathbf{f}, \mathbf{g}, s) = L_{\mathbf{n}}(\psi\phi, 2s + 2 - k - l) \sum_{\mathbf{m}} C(\mathbf{m}, \mathbf{f}) C(\mathbf{m}, \mathbf{g}) N(\mathbf{m})^{-s},$$

where $L_{\mathbf{n}}(\psi\phi, s)$ denotes the classical L-function over F of $\psi\phi$, with the Euler factors at the primes dividing $\mathbf{n}$ removed. Whenever we do not specify the level of $\mathbf{f}$ and $\mathbf{g}$, we always consider $\mathbf{n}$ to be least common multiple of their levels. We write $\langle \mathbf{f}, \mathbf{g} \rangle_F$ for the Petersson inner product of $\mathbf{f}$ and $\mathbf{g}$, normalised as in [12] §2.

For an elliptic curve E over $\mathbb{Q}$ of conductor $N(E)$, we write f_E for the associated rational new form (over $\mathbb{Q}$) of weight 2, trivial character and level $N(E)$ (using the modularity of elliptic curves, [15, 14, 3]). We write $\mathbf{f}_E$ for the base change of f_E to F. Such a base change exists as $F/\mathbb{Q}$ is abelian, [7, 1]. It is a Hilbert modular form that is a new form of parallel weight 2, trivial character, and of conductor dividing $N(E)\mathcal{O}_F$.

Remark 2.1. Note that, following Shimura's terminology in [12], $\mathbf{f}_E$ is primitive, in the sense that it is a normalised new form of some level.

For a Hilbert modular form $\mathbf{g}$, we write $\mathbb{Q}(\mathbf{g})$ for the extension of $\mathbb{Q}$ generated by the $C(\mathbf{m},\mathbf{g})$, defined above. For E an elliptic curve over $\mathbb{Q}$, the field $\mathbb{Q}(\mathbf{f}_E)$ is just $\mathbb{Q}$. If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\mathbf{g}$ is a Hilbert modular form of parallel weight and with $C(\mathbf{m},\mathbf{g}) \in \overline{\mathbb{Q}}$, we write $\mathbf{g}^\sigma$ for the Hilbert modular form which has $C(\mathbf{m},\mathbf{g}^\sigma) = C(\mathbf{m},\mathbf{g})^\sigma$.

Remark 2.2. If $\mathbf{f}_E$ comes from an elliptic curve over $\mathbb{Q}$, and if $\mathbf{g}$ corresponds to a 2-dimensional Artin representation ρ over F, the Rankin-Selberg product $D(\mathbf{f}_E,\mathbf{g},s)$ is the L-function $L(E/F,\rho,s)$ up to a finite number of Euler factors. It has an Euler product

$$D(\mathbf{f}_E,\mathbf{g},s) = \prod_v \det(1-(N_{F/\mathbb{Q}}v)^{-s}\Phi_v|H^1_\ell(E)^{I_v}\otimes\rho^{I_v})^{-1}\ .$$

The Euler factor at v can only differ from that of $L(E,\rho,s)$ when both $H^1_\ell(E)$ and ρ are ramified at v. Moreover, the functions $D(\mathbf{f}_E,\mathbf{g},s)$ and $L(E,\rho,s)$ coincide provided that at every prime of bad reduction of E,

$$(H^1_\ell(E)\otimes\rho)^{I_v} = H^1_\ell(E)^{I_v}\otimes\rho^{I_v}\ .$$

There is a similar Euler product if we replace $\mathbf{g}$ by a character χ of finite order. Thus, once again, $L(E/F,\chi,s)$ may differ from $D(\mathbf{f}_E,\chi,s)$ at finitely many Euler factors, and there is a similar criterion for when $D(\mathbf{f}_E,\chi,s) = L(E,\chi,s)$.

Remark 2.3. With notation as in Remark 2.2, the functions $D(\mathbf{f}_E,\mathbf{g},s)$ and $D(\mathbf{f}_E,\chi,s)$ are analytic on $\mathbb{C}$ (see [12] Proposition 4.13). As the local polynomials $P_v(E/F,\rho,X)$ and $P_v(E/F,\chi,X)$ do not vanish at $X = (N_{F/\mathbb{Q}}v)^{-1}$, it follows that the L-functions $L(E/F,\rho,s)$ and $L(E/F,\chi,s)$ are meromorphic on $\mathbb{C}$, and analytic at $s = 1$.

2.4. **Gauss sums and ϵ-factors.** Let K be a number field, and ρ an Artin representation over K. We write $\epsilon_K(\rho)$ for the global ϵ-factor of ρ (see [13, 5]). Recall that it can be written as a product of local ϵ-factors. We write $\epsilon_{K,\infty}(\rho)$ for the contribution from the archimedean places of K, where we have chosen the standard additive character and measure at these places (i.e. $\exp(2\pi i x), dx$ for real places and $\exp(2\pi i\,\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\,z), |dz\wedge d\bar{z}|$ for complex ones). Global ϵ-factors are inductive, in the sense that, whenever K/L is a finite extension and ρ an Artin representation over K,

$$\epsilon_K(\rho) = \epsilon_L(\mathrm{Ind}\,\rho)\ .$$

For ψ a character over K of finite order we define

$$\tau_K(\psi) = \sqrt{|\Delta_K|}^{-1}\,\frac{\epsilon_K(\psi)}{\epsilon_{K,\infty}(\psi)}\ .$$

This is Tate's $\epsilon_1(\psi)$ (see [13] §3.6), with the local epsilon factors at infinity removed. The quantity τ_K coincides with that used by Shimura in [12], where τ_K is defined for totally real fields K.

Remark 2.4. With our conventions, the local ϵ-factor at a complex place of an Artin representation is 1. In particular, if K is totally imaginary, then $\epsilon_{K,\infty}(\psi) = 1$. The local ϵ-factor at a real place v of an Artin representation ρ is $i^{\dim\rho^-}$, where ρ^- is the subspace on which complex conjugation at v acts by -1.

We have already mentioned that ϵ_K is inductive. On the other hand, τ_K satisfies the following Galois equivariance property:

Lemma 2.5. (see also [12] Lemma 4.12) *Let K be a number field, and let ψ_1, ψ_2 be characters over K of finite order. Then for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,*

$$\frac{\tau_K(\psi_1)^\sigma \tau_K(\psi_2)^\sigma}{\tau_K(\psi_1\psi_2)^\sigma} = \frac{\tau_K(\psi_1^\sigma)\tau_K(\psi_2^\sigma)}{\tau_K(\psi_1^\sigma\psi_2^\sigma)}.$$

Proof. The formula holds prime-by-prime when we view τ_K as a product of local non-archimedean ϵ-factors (see [13] (3.2.3) and (3.2.6)). □

2.5. **Results from the theory of Hilbert modular forms.** The following two results from the theory of modular forms, concerning the special values of L-functions, are fundamental to our study.

Theorem 2.6. (Shimura [12]**)** *Let F be a totally real field. Write $d = [F : \mathbb{Q}]$. Let $\mathbf{f}$ be a primitive Hilbert modular form of parallel weight 2 and with trivial character. Let $\mathbf{g}$ be a Hilbert modular of parallel weight 1 and character ϕ. Define*

$$T(\mathbf{f}, \mathbf{g}) = \frac{D(\mathbf{f}, \mathbf{g}, 1)}{(2i)^d \pi^{3d} \langle \mathbf{f}, \mathbf{f} \rangle_F \, \tau_F(\phi)}.$$

Then $T(\mathbf{f}, \mathbf{g}) \in \overline{\mathbb{Q}}(\mathbf{g})$. Moreover, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$T(\mathbf{f}, \mathbf{g})^\sigma = T(\mathbf{f}^\sigma, \mathbf{g}^\sigma).$$

Theorem 2.7. (Shimura [11]**, see also Birch** [2]**)** *Let E be an elliptic curve over $\mathbb{Q}$, and let $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^*$ be a Dirichlet character. Define*

$$S(E, \psi) = \frac{L(E/\mathbb{Q}, \psi, 1)}{\Omega_{\mathrm{sign}(\psi)}(E)\tau_{\mathbb{Q}}(\psi)} .$$

Then $S(E, \psi) \in \overline{\mathbb{Q}}$. Moreover, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$S(E, \psi)^\sigma = S(E, \psi^\sigma) .$$

3. Periods of elliptic curves and the Petersson inner product

In this section we establish a relation between the quantity $\Omega_+\Omega_-$, which appears in the Birch–Swinnerton-Dyer conjecture and its generalisations, and $\langle \mathbf{f}_E, \mathbf{f}_E \rangle_F$, which appears in the theory of Hilbert modular forms (Theorem 3.4).

The method is the following. Suppose that $E/\mathbb{Q}$ is an elliptic curve, K an abelian extension of $\mathbb{Q}$ not contained in $\mathbb{R}$, and F its maximal totally real subfield. Theorem 2.6 gives an expression for $L(E/K, 1)$ involving the Petersson inner product $\langle \mathbf{f}_E, \mathbf{f}_E \rangle_F$. On the other hand, we can decompose the L-function $L(E/K, 1)$ as a product of L-functions over $\mathbb{Q}$ of 1-dimensional twists of E. Theorem 2.7 then provides an expression for $L(E/K, 1)$ involving the classical periods $\Omega_\pm(E)$.

The problem with the above method as it stands, is that the L-value $L(E/K, 1)$ may be zero. To correct this, we replace $L(E/K, s)$ in the above procedure by $L(E/K, \chi, s)$, for a suitable 1-dimensional Artin representation χ over K. Our proof uses the following well-known result of Rohrlich,

Theorem 3.1. (Rohrlich [8, 9]**)** *Let E be an elliptic curve over $\mathbb{Q}$. Let Σ be a finite set of primes. Then for all but finitely many Dirichlet characters ψ unramified outside Σ and infinity,*

$$L(E/\mathbb{Q}, \psi, 1) \neq 0 .$$

Corollary 3.2. *Let E be an elliptic curve over $\mathbb{Q}$. Let Σ be a finite set of primes of $\mathbb{Q}$. Let $F \subset \mathbb{Q}_{ab}^{\Sigma}$. Then there exist an extension F_0/F, with $F_0 \subset \mathbb{Q}_{ab}^{\Sigma}$, such that for every character $\psi : \mathrm{Gal}(\mathbb{Q}_{ab}^{\Sigma}/F) \to \mathbb{C}^*$ of finite order that does not factor through F_0/F,*

$$L(E/F, \psi, 1) \neq 0.$$

Proof. Let $S = \{\phi_1, \dots, \phi_k\}$ be the set of all Dirichlet characters over $\mathbb{Q}$ that are unramified outside $\Sigma \cup \{\infty\}$ and satisfy $L(E/\mathbb{Q}, \phi_j, 1) = 0$. Note that by Rohrlich's theorem 3.1, this set is finite.

Let us pick $F \subset F_0 \subset \mathbb{Q}_{ab}^{\Sigma}$ such that all the ϕ_j factor through F_0. We claim that this field satisfies the conclusion of the corollary. Indeed, let $\psi : \mathrm{Gal}(\mathbb{Q}_{ab}^{\Sigma}/F) \to \mathbb{C}^*$ be a character of finite order that does not factor through F_0/F. Then, by the inductive properties of L-functions,

$$L(E/F, \psi, s) = \prod_j L(E/\mathbb{Q}, \psi_j, s)\,,$$

where the $\bigoplus_j \psi_j$ is the representation of $\mathrm{Gal}(\mathbb{Q}_{ab}^{\Sigma}/\mathbb{Q})$ induced from ψ. By Frobenius Reciprocity, the ψ_j are precisely the characters of $\mathrm{Gal}(\mathbb{Q}_{ab}^{\Sigma}/\mathbb{Q})$ that restrict to ψ. In particular, none of them factor through $F_0/\mathbb{Q}$, and hence $\psi_j \notin S$ for every j. By the definition of S we conclude that

$$L(E/F, \psi, 1) = \prod_j L(E/\mathbb{Q}, \psi_j, 1) \neq 0\,.$$

□

Corollary 3.3. *Let E be an elliptic curve over $\mathbb{Q}$. Let F be any abelian extension of $\mathbb{Q}$. Let $\epsilon : \mathrm{Gal}(\mathbb{Q}_{ab}/F) \to \mathbb{C}^*$ be any character of finite order. Then there exists a character of finite order $\tilde{\chi} : \mathrm{Gal}(\mathbb{Q}_{ab}/F) \to \mathbb{C}^*$, such that for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have*

$$L(E/F, \tilde{\chi}^{\sigma}, 1) \neq 0, \qquad L(E/F, \tilde{\chi}^{\sigma}\epsilon, 1) \neq 0$$

and

$$L(E/F, \tilde{\chi}^{\sigma}, s) = D(\mathbf{f}_E, \tilde{\chi}^{\sigma}, s), \qquad L(E/F, \tilde{\chi}^{\sigma}\epsilon, s) = D(\mathbf{f}_E, \tilde{\chi}^{\sigma}\epsilon, s)\,.$$

Proof. Let Σ be a set of primes containing all primes of bad reduction of E, the divisors of Δ_F, and the primes of $\mathbb{Q}$ below those dividing the conductor of ϵ. Let F_0 be the field given by Corollary 3.3 for the field F and the set Σ.

Pick a character $\tilde{\chi} : \mathrm{Gal}(\mathbb{Q}_{ab}^{\Sigma}/F) \to \mathbb{C}^*$, such that for every prime v of F above Σ,

$$\mathrm{ord}_v\, N(\tilde{\chi}) > \max(\mathrm{ord}_v(\Delta_{F_0/F}), \mathrm{ord}_v\, N(\epsilon), \mathrm{ord}_v\, N(E/F))\,,$$

where $N(\xi)$ denotes the conductor of ξ. Such a character can always be found in $F(\mu_m)/F$ for $m = \prod_{p\in\Sigma} p^{k_p}$, with the k_p sufficiently large (see e.g. [10], chapter IV).

We now have, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and every such v,

$$\mathrm{ord}_v\, N(\tilde{\chi}^{\sigma}) = \mathrm{ord}_v\, N(\tilde{\chi}^{\sigma}\epsilon) > \max(\mathrm{ord}_v(\Delta_{F_0/F}), \mathrm{ord}_v\, N(E/F))$$

Therefore, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the characters $\tilde{\chi}^{\sigma}$ and $\tilde{\chi}^{\sigma}\epsilon$ do not factor through F_0/F. Moreover, for every prime v of F where E has bad reduction,

$$(H_l(E) \otimes \tilde{\chi}^{\sigma})^{I_v} = 0 = H_l(E)^{I_v} \otimes (\tilde{\chi}^{\sigma})^{I_v}$$

and

$$(H_l(E) \otimes \tilde{\chi}^\sigma \epsilon)^{I_v} = 0 = H_l(E)^{I_v} \otimes (\tilde{\chi}^\sigma \epsilon)^{I_v} .$$

Thus $\tilde{\chi}$ satisfies the requirements of the corollary. $\square$

Theorem 3.4. *Let E be an elliptic curve over $\mathbb{Q}$. Let F be a totally real field, with $F/\mathbb{Q}$ abelian. Write $d = [F : \mathbb{Q}]$. Then*

$$\frac{(2i)^d \pi^{3d} \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F}{(\Omega_+(E)\Omega_-(E))^d} \in \mathbb{Q} .$$

Proof. Pick $K \subset \mathbb{Q}_{ab}$ to be any totally imaginary quadratic extension of F (for instance $K = F(i)$ will do). Write ϵ for the quadratic character of K/F. Take $\tilde{\chi} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{C}^*$ to be the character of finite order provided by Corollary 3.3 in this setting.

We write χ for the restriction of $\tilde{\chi}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Note that the induction of χ to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ is $\tilde{\chi} \oplus \tilde{\chi}\epsilon$. We let $\mathbf{g}_\chi$ be the Hilbert modular form of parallel weight 1 and character $\epsilon\tilde{\chi}^2$ that is the automorphic induction of χ, see e.g. [12] §5. (This is the Hilbert modular form associated to the Artin representation $\tilde{\chi} \oplus \tilde{\chi}\epsilon$.)

With notation as in Theorem 2.6, with $\mathbf{f} = \mathbf{f}_E$ and $\mathbf{g} = \mathbf{g}_\chi$,

$$D(\mathbf{f}_E, \mathbf{g}_\chi, 1) = T(\mathbf{f}_E, \mathbf{g}_\chi)(2i)^d \pi^{3d} \tau_F(\tilde{\chi}^2\epsilon)\langle \mathbf{f}_E, \mathbf{f}_E \rangle_F.$$

By our choice of $\tilde{\chi}$

$$D(\mathbf{f}_E, \mathbf{g}_\chi, 1) = L(E/F, \tilde{\chi} \oplus \tilde{\chi}\epsilon, 1) = L(E/F, \tilde{\chi}, 1)L(E/F, \tilde{\chi}\epsilon, 1) .$$

Let $\theta_1^{\tilde{\chi}}, \dots, \theta_d^{\tilde{\chi}}$ and $\theta_1^{\tilde{\chi}\epsilon}, \dots, \theta_d^{\tilde{\chi}\epsilon}$ be the characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that restrict to $\tilde{\chi}$ and $\tilde{\chi}\epsilon$, respectively. In particular, by Frobenius reciprocity, the induction of $\tilde{\chi}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is $\oplus_j \theta_j^{\tilde{\chi}}$, and similarly for $\tilde{\chi}\epsilon$. Inductive properties of L-functions allow us to rewrite the above formula as

$$D(\mathbf{f}_E, \mathbf{g}_\chi, 1) = \prod_j L(E/\mathbb{Q}, \theta_j^{\tilde{\chi}}, 1) \prod_j L(E/\mathbb{Q}, \theta_j^{\tilde{\chi}\epsilon}, 1) .$$

We obtain, using the notation of 2.7,

$$\frac{T(\mathbf{f}_E, \mathbf{g}_\chi)\tau_F(\tilde{\chi}^2\epsilon) \cdot (2i)^d \pi^{3d} \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F}{\prod_j \Omega_{\mathrm{sign}(\theta_j^{\tilde{\chi}\epsilon})}(E)\Omega_{\mathrm{sign}(\theta_j^{\tilde{\chi}})}(E)} = \prod_j S(E, \theta_j^{\tilde{\chi}})\tau_{\mathbb{Q}}(\theta_j^{\tilde{\chi}}) \prod_j S(E, \theta_j^{\tilde{\chi}\epsilon})\tau_{\mathbb{Q}}(\theta_j^{\tilde{\chi}\epsilon}) .$$

Note that, by the choice of $\tilde{\chi}$, the above does not read $0 = 0$, since the corresponding L-functions do not vanish at $s = 1$. We rewrite the above as

$$\frac{(2i)^d \pi^{3d} \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F}{(\Omega_+(E)\Omega_-(E))^d} = \frac{\prod_j \tau_{\mathbb{Q}}(\theta_j^{\tilde{\chi}})\tau_{\mathbb{Q}}(\theta_j^{\tilde{\chi}\epsilon})}{\tau_F(\tilde{\chi}^2\epsilon)} \frac{\prod_j S(E, \theta_j^{\tilde{\chi}})S(E, \theta_j^{\tilde{\chi}\epsilon})}{T(\mathbf{f}_E, \mathbf{g}_\chi)} ,$$

where we have used the fact that exactly d of $\theta_1^{\tilde{\chi}}, \dots, \theta_d^{\tilde{\chi}}, \theta_1^{\tilde{\chi}\epsilon}, \dots, \theta_d^{\tilde{\chi}\epsilon}$ have sign $+1$.

Using the definition of τ, the inductive properties of global epsilon factors and the formula for the local ϵ-factor at a real place (see Remark 2.4), we obtain

$$\frac{(2i)^d \pi^{3d} \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F}{(\Omega_+(E)\Omega_-(E))^d} = |\Delta_F| \frac{\tau_F(\tilde{\chi})\tau_F(\tilde{\chi}\epsilon)}{\tau_F(\tilde{\chi}^2\epsilon)} \frac{\prod_j S(E, \theta_j^{\tilde{\chi}})S(E, \theta_j^{\tilde{\chi}\epsilon})}{T(\mathbf{f}_E, \mathbf{g}_\chi)} .$$

Let us call the quantity on the right-hand-side $R(\tilde{\chi})$. To obtain the above equality we have only used the fact that $L(E/F, \tilde{\chi} \oplus \tilde{\chi}\epsilon, 1)$ coincides with $D(\mathbf{f}_E, \mathbf{g}_\chi, 1)$, and that this L-value is non-zero. Thus, by our choice of $\tilde{\chi}$, the above equality also holds if we replace $\tilde{\chi}$ by $\tilde{\chi}^\sigma$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and so we have $R(\tilde{\chi}) = R(\tilde{\chi}^\sigma)$.

By Theorems 2.6 and 2.7, $R(\tilde{\chi}) \in \overline{\mathbb{Q}}$. To prove our theorem, we need to show that $R(\tilde{\chi})$ actually lies in $\mathbb{Q}$. So let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By Lemma 2.5, Theorem 2.6 and Theorem 2.7,

$$\left(\frac{\tau_F(\tilde{\chi})\tau_F(\tilde{\chi}\epsilon)}{\tau_F(\tilde{\chi}^2\epsilon)}\right)^{\sigma} = \frac{\tau_F(\tilde{\chi}^{\sigma})\tau_F(\tilde{\chi}^{\sigma}\epsilon)}{\tau_F((\tilde{\chi}^2)^{\sigma}\epsilon)},$$

and

$$\left(\frac{\prod_j S(E,\theta_j^{\tilde{\chi}})S(E,\theta_j^{\tilde{\chi}\epsilon})}{T(\mathbf{f}_E,\mathbf{g}_{\chi})}\right)^{\sigma} = \frac{\prod_j S(E,(\theta_j^{\tilde{\chi}})^{\sigma})S(E,(\theta_j^{\tilde{\chi}\epsilon})^{\sigma})}{T(\mathbf{f}_E,\mathbf{g}_{\chi}^{\sigma})}.$$

Observe that $\{\theta_1^{\tilde{\chi}^{\sigma}},\ldots,\theta_d^{\tilde{\chi}^{\sigma}}\} = \{(\theta_1^{\tilde{\chi}})^{\sigma},\ldots,(\theta_d^{\tilde{\chi}})^{\sigma}\}$, and similarly for $\theta_j^{\tilde{\chi}\epsilon}$. Furthermore, as $\mathbf{g}_{\chi}^{\sigma} = \mathbf{g}_{\chi^{\sigma}}$, we deduce that $R(\tilde{\chi})^{\sigma} = R(\tilde{\chi}^{\sigma}) = R(\tilde{\chi})$. As σ was arbitrary, it follows that

$$\frac{(2i)^d\pi^{3d}\langle\mathbf{f}_E,\mathbf{f}_E\rangle_F}{(\Omega_+(E)\Omega_-(E))^d} \in \mathbb{Q}.$$

□

Corollary 3.5. *Let E be an elliptic curve over $\mathbb{Q}$. Let K be an abelian extension of $\mathbb{Q}$, which is not a subfield of $\mathbb{R}$. Let $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^*$ be a character of finite order, and write ρ for the Artin representation over $\mathbb{Q}$ induced by ψ. Then*

$$\frac{L(E,\rho,1)}{\Omega_+(E)^d\Omega_-(E)^d} \in \overline{\mathbb{Q}},$$

where $d = [K:\mathbb{Q}]/2$.

Proof. This follows from Theorems 2.6 and 3.4, and the inductive properties of L-functions. For $\mathbf{g}$ in Theorem 2.6 we take the Hilbert modular form over K^+ induced from ψ (as a Hecke character). Notice that the Rankin-Selberg product $D(\mathbf{f},\mathbf{g},s)$ may differ from $L(E,\rho,s)$ only at finitely many Euler factors, that take algebraic values and do not vanish at $s=1$. □

Remark 3.6. In view of Theorems 2.6 and 2.7, the result in Theorem 3.4 is equivalent to saying that $\langle\mathbf{f}_E,\mathbf{f}_E\rangle_F/\langle f_E,f_E\rangle_{\mathbb{Q}}^d \in \mathbb{Q}$. This result can be easily generalised to deal with any primitive cusp form f over $\mathbb{Q}$ that has even weight, trivial character and $C(m,f) \in \mathbb{Q}$. Indeed, the results of Rohrlich and Shimura that we used in the proof apply in this setting. The rôle of $\Omega_{\pm}$ can be played by the quantities $u_1^{\pm}$ in [11].

4. The false Tate curve extension

Let p be an odd prime, and let $m > 1$ be an integer, that is not a p-th power. We consider the false Tate curve extension of $\mathbb{Q}$,

$$\mathbb{Q}_{FT} = \cup_n \mathbb{Q}(\sqrt[p^n]{m}, \mu_{p^n}).$$

We write $K_n = \mathbb{Q}(\mu_{p^n})$. For a discussion of the representation theory in the false Tate curve extension see [6].

Let ρ_n be the irreducible Artin representation over $\mathbb{Q}$, that is induced from any 1-dimensional representation of $\mathrm{Gal}(K_n(\sqrt[p^n]{m})/K_n)$ of exact order p^n. Every irreducible Artin representation that factors through $\mathbb{Q}_{FT}$ has the form $\rho_n\phi$, where ϕ is a 1-dimensional Artin representation over $\mathbb{Q}$ that factors through K_k, for some $k \geq 0$. If ρ_n is induced from $\chi : \mathrm{Gal}(K_n(\sqrt[p^n]{m})/K_n) \to \mathbb{C}^*$, then $\rho_n\phi$ is induced from $\chi\,\mathrm{Res}\,\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/K_n) \to \mathbb{C}^*$, Note that, in particular, every irreducible representation of dimension larger than 1 is induced from a 1-dimensional Artin representation over an abelian extension of $\mathbb{Q}$, not contained in $\mathbb{R}$. We immediately obtain from Corollary 3.5 and Theorem 2.7

Corollary 4.1. *Let E be an elliptic curve over $\mathbb{Q}$. Let p be an odd prime and let $m > 1$ be an integer that is not a p-th power. Let ρ be an Artin representation over $\mathbb{Q}$ that factors through $\mathbb{Q}(\sqrt[p^n]{m}, \mu_{p^n})$, for some $n \geq 0$. Then*

$$\frac{L(E,\rho,1)}{\Omega_+(E)^{\dim\rho^+}\Omega_-(E)^{\dim\rho^-}} \in \overline{\mathbb{Q}}\,.$$

Our aim is to make a more precise statement about the field of definition of the special value in the above corollary. First, we recall some more terminology.

Let F be a totally real field of degree d, and let K/F be a totally imaginary quadratic extension. We fix an embedding $K \to \mathbb{C}$, and write c for the complex conjugation element. We say that a character of finite order $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^*$ is cyclotomic if $\psi(cgc) = \psi(g)$ for all $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. We call ψ anticyclotomic if $\psi(cgc) = \psi(g)^{-1}$. Note that when ψ is cyclotomic, it can be extended to a character $\tilde{\psi}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Write ϵ for the quadratic character of K/F. If ψ is cyclotomic over K, then the representation induced by ψ to F is $\tilde{\psi} \oplus \tilde{\psi}\epsilon$. Moreover, if χ is anticyclotomic and ψ is cyclotomic, then the determinant of the representation induced by $\chi\psi$ to F is $\tilde{\psi}^2\epsilon$.

Theorem 4.2. *Let E be an elliptic curve over $\mathbb{Q}$. Let p be an odd prime and let $m > 1$ be an integer, that is not a p-th power. Let ρ be an Artin representation over $\mathbb{Q}$ that factors through $\mathbb{Q}(\sqrt[p^n]{m}, \mu_{p^n})$, for some $n \geq 0$. Define*

$$R(E,\rho) = \frac{L(E,\rho,1)\epsilon(\rho)^{-1}}{\Omega_+(E)^{\dim\rho^+}|\Omega_-(E)|^{\dim\rho^-}}\,.$$

Then $R(E,\rho) \in \overline{\mathbb{Q}}$. Moreover, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$R(E,\rho)^\sigma = R(E,\rho^\sigma)\,.$$

In particular,

$$R(E,\rho) \in \mathbb{Q}(\rho)\,.$$

Proof. It is sufficient to prove the Galois-equivariance formula for irreducible representations ρ. If ρ is 1-dimensional, the result follows from Theorem 2.7, observing that in this case $i^{\dim\rho^-}\tau_{\mathbb{Q}}(\rho) = \epsilon_{\mathbb{Q}}(\rho)$.

Otherwise ρ is induced from a character $\chi\phi$ over K_n for some $n \geq 1$, where χ factors through $\mathrm{Gal}(K_n(\sqrt[p^n]{m})/K_n)$ and ϕ factors through $\mathrm{Gal}(K_k/K_n)$, for some $k \geq n$. In this case $\dim\rho^- = \dim\rho^+ = p^{n-1}(p-1)/2$. The field K_n is a quadratic totally imaginary extension of the totally real field $F = K_n^+$. Note that χ is anticyclotomic and ϕ is cyclotomic. We write ϵ for the quadratic character of K_n/F, and $\tilde{\phi}$ for an extension of ϕ to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Let $\mathbf{g}$ be the Hilbert modular form over F of parallel weight 1 and character $\epsilon\tilde{\phi}^2$ that is the automorphic induction of $\chi\phi$ to F (see e.g. [12] §5). This Hilbert modular form corresponds to the Artin representation $\mathrm{Ind}\,\chi\phi$ over F. We apply Theorem 2.6 to the pair $\mathbf{f}_E$, $\mathbf{g}$. Eliminating the Petersson inner product $\langle \mathbf{f}_E, \mathbf{f}_E\rangle_F$ using Theorem 3.4, we deduce that the expression

$$\frac{D(\mathbf{f}_E,\mathbf{g},1)}{\Omega_+(E)^{\dim\rho^+}\Omega_-(E)^{\dim\rho^-}\tau_F(\epsilon\tilde{\phi}^2)}$$

is algebraic and Galois-equivariant. The term $D(\mathbf{f}_E, \mathbf{g}, 1)$ can be replaced by the L-value $L(E, \rho, 1)$ by Remark 2.2 and Lemma 4.4, and the inductive property of L-functions. Finally, the Gauss sum $i^{\dim \rho^-} \tau_F(\epsilon\tilde{\phi}^2)$ can be replaced by $\epsilon_{K_n}(\chi\phi) = \epsilon_{\mathbb{Q}}(\rho)$ by Proposition 4.5.

□

Corollary 4.3. *Let E be an elliptic curve over $\mathbb{Q}$. Let p be an odd prime and let $m, n > 1$ be integers. For any subfield F of $\mathbb{Q}(\sqrt[p^n]{m}, \mu_{p^n})$,*

$$\frac{L(E/F, 1)\sqrt{|\Delta_F|}}{\Omega_+(E)^{r_1+r_2}|\Omega_-(E)|^{r_2}} \in \mathbb{Q}\,,$$

where r_1 (resp. r_2) is the number of real (resp. complex) places of F.

Proof. Take ρ to be the Artin representation induced from the trivial representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Then ρ can be realised over $\mathbb{Q}$, and $\epsilon(\rho) = \pm\sqrt{|\Delta_F|}$. The result follows from Theorem 4.2. Note that in the resulting formula, $\sqrt{|\Delta_F|}^{-1}$ can be replaced by $\sqrt{|\Delta_F|}$, as their quotient is rational. □

The Rankin-Selberg product. In the lemma below, we justify the assertion in the proof of Theorem 4.2, that the quotient of the Rankin-Selberg product by the classical L-function is Galois equivariant. More precisely, if $E/\mathbb{Q}$ is an elliptic curve, F a totally real field and ρ a (suitable) two dimensional Artin representation over F, then $\mathbf{D}(\mathbf{f}_E, \mathbf{g}, 1)/L(E/F, \rho, 1)$ is Galois equivariant, where we view this quotient as the finite product over primes where the local Euler factors of $\mathbf{D}$ and L differ (so that it is well-defined even when both L-values are zero.)

Lemma 4.4. *Let K be a number field, and let E be an elliptic curve over K. Let ρ be an Artin representation over K with coefficients in $\overline{\mathbb{Q}}$. Let v be a prime of K, and let ℓ be a rational prime not divisible by v. Then, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,*

$$\det(1 - X\Phi_v|(H^1_\ell(E) \otimes \rho)^{I_v})^\sigma = \det(1 - X\Phi_v|(H^1_\ell(E) \otimes \rho^\sigma)^{I_v})$$

and

$$\det(1 - X\Phi_v|(H^1_\ell(E)^{I_v} \otimes \rho^{I_v}))^\sigma = \det(1 - X\Phi_v|(H^1_\ell(E)^{I_v} \otimes (\rho^\sigma)^{I_v})\,.$$

Proof. Recall that the above characteristic polynomials have algebraic coefficients. Let L be a finite Galois extension of $\mathbb{Q}$ that contains these coefficients, and such that ρ is realised over L. It is clearly sufficient to prove the assertion in the lemma for $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$.

Let $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$. Write M for the subfield of L fixed by σ. The coefficients in the above characteristic polynomials are independent of the choice of ℓ, with $v \nmid \ell$. We can therefore take ℓ to be a rational prime not divisible by v, that splits in $M/\mathbb{Q}$ and is inert in L/M. Such primes exist by Chebotarev's density theorem.

Let α be an element in L with $L = M(\alpha)$. For a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, write $\tilde{\alpha} \in \overline{\mathbb{Q}}_\ell$ for the image of α. We obtain an explicit isomorphism $\mathrm{Gal}(L/M) \cong \mathrm{Gal}(\mathbb{Q}_\ell(\tilde{\alpha})/\mathbb{Q}_\ell)$. We write σ also for its image in $\mathrm{Gal}(\mathbb{Q}_\ell(\tilde{\alpha})/\mathbb{Q}_\ell)$ under this isomorphism.

If R is any representation of the Weil group at v with coefficients in $\mathbb{Q}_\ell(\tilde{\alpha})$, we have

$$\det(1 - \Phi_v X|(R^\sigma)^{I_v}) = \det(1 - \Phi_v X|R^{I_v})^\sigma\,.$$

Indeed, letting V be the $\mathbb{Q}_\ell(\tilde{\alpha})$-vector space on which R acts, if $v_1, \ldots, v_n$ is a basis of $V^{R(I_v)}$ then $v_1^\sigma, \ldots, v_n^\sigma$ is a basis of $V^{R^\sigma(I_v)}$. If the matrix of $R(\Phi_v)$ with respect to $v_1, \ldots, v_n$ is (λ_{ij}), then the matrix of $R^\sigma(\Phi_v)$ with respect to $v_1^\sigma, \ldots, v_n^\sigma$ is (λ_{ij}^σ).

Applying this observation with $R = H^1_\ell(E), \rho$ and $H^1_\ell(E) \otimes \rho$ we obtain the asserted Galois-equivariance. $\square$

Comparing Gauss sums and ϵ-factors. We now complete the proof of Theorem 4.2 by showing that the quotient of the Gauss sum τ_F, that appears in the definition of $T(E/F, \rho)$, by the ϵ-factor that appears in Theorem 4.2, is Galois equivariant.

Proposition 4.5. *Let F be a totally real field of degree d, and let K be a totally imaginary quadratic extension of F. Write ϵ for the quadratic character of K/F. Let χ be an anticyclotomic character of K, and let ψ be a cyclotomic character. Then, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have*

$$\left(\frac{\epsilon_K(\chi\psi)}{i^d\,\tau_F(\tilde{\psi}^2\epsilon)}\right)^\sigma = \frac{\epsilon_K(\chi^\sigma\psi^\sigma)}{i^d\,\tau_F((\tilde{\psi}^\sigma)^2\epsilon)}\ ,$$

where $\tilde{\psi}$ denotes an extension of ψ to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

Proof. This is a straight-forward computation using the definition of τ and Lemma 2.5. We have

$$\left(\frac{\epsilon_K(\chi\psi)}{i^d\,\tau_F(\tilde{\psi}^2\epsilon)}\right)^\sigma = \left(\frac{\sqrt{|\Delta_K|}\tau_K(\chi\psi)}{i^d\,\tau_F(\tilde{\psi}^2\epsilon)}\right)^\sigma$$

Applying Lemma 2.5, we rewrite the latter expression as

$$\frac{\sqrt{|\Delta_K|}^\sigma}{(i^d)^\sigma\,\tau_F(\tilde{\psi}^2\epsilon)^\sigma}\frac{\tau_K(\chi^\sigma\psi^\sigma)}{\tau_K(\chi^\sigma)\tau_K(\psi^\sigma)}\tau_K(\chi)^\sigma\tau_K(\psi)^\sigma =$$

$$= \frac{\sqrt{|\Delta_K|}^\sigma}{(i^d)^\sigma\sqrt{|\Delta_K|}}\cdot\frac{\tau_F(\tilde{\psi}^\sigma\epsilon)\tau_F(\tilde{\psi}^\sigma)}{\tau_F(\tilde{\psi}\epsilon)^\sigma\tau_F(\tilde{\psi})^\sigma}\cdot\frac{\epsilon_K(\chi^\sigma\psi^\sigma)}{\tau_F((\tilde{\psi}^2)^\sigma\epsilon)}\cdot\frac{\tau_K(\chi)^\sigma}{\tau_K(\chi^\sigma)}\cdot\frac{\tau_K(\psi)^\sigma}{\tau_K(\psi^\sigma)}\,.$$

As χ is anticyclotomic, the term $\tau_K(\chi)^\sigma/\tau_K(\chi^\sigma)$ in the above expression is 1. We now rewrite the last term in the formula

$$\frac{\tau_K(\psi)^\sigma}{\tau_K(\psi^\sigma)} = \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}}\frac{\epsilon_K(\psi)^\sigma}{\epsilon_K(\psi^\sigma)} = \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}}\frac{\epsilon_F(\tilde{\psi}\oplus\tilde{\psi}\epsilon)^\sigma}{\epsilon_F(\tilde{\psi}^\sigma\oplus\tilde{\psi}^\sigma\epsilon)} = \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}}\frac{\epsilon_F(\tilde{\psi})^\sigma\epsilon_F(\tilde{\psi}\epsilon)^\sigma}{\epsilon_F(\tilde{\psi}^\sigma)\epsilon_F(\tilde{\psi}^\sigma\epsilon)} =$$

$$= \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}}\cdot\frac{\tau_F(\tilde{\psi})^\sigma\tau_F(\tilde{\psi}\epsilon)^\sigma}{\tau_F(\tilde{\psi}^\sigma)\tau_F(\tilde{\psi}^\sigma\epsilon)}\cdot\frac{\epsilon_{F,\infty}(\tilde{\psi})^\sigma\epsilon_{F,\infty}(\tilde{\psi}\epsilon)^\sigma}{\epsilon_{F,\infty}(\tilde{\psi}^\sigma)\epsilon_{F,\infty}(\tilde{\psi}^\sigma\epsilon)} = \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}}\cdot\frac{\tau_F(\tilde{\psi})^\sigma\tau_F(\tilde{\psi}\epsilon)^\sigma}{\tau_F(\tilde{\psi}^\sigma)\tau_F(\tilde{\psi}^\sigma\epsilon)}\cdot\frac{(i^d)^\sigma}{i^d}.$$

The proposition follows. $\square$

References

[1] J.Arthur, L.Clozel, *Simple algebras, base change and the advanced theory of the trace formula*, Annals of Math. Studies 120, Princeton University Press 1989.

[2] B.J.Birch, 'Elliptic curves, a progress report', Proceedings of the 1969 Summer Institute on Number Theory, Stony Brook, New York AMS, pp 396–400, 1971.

[3] C.Breuil, B.Conrad, F.Diamond, R.Taylor, 'On the modularity of elliptic curves over $\mathbb{Q}$', *J.A.M.S.* 14 (2001), 843–939.

[4] J.Coates, T.Fukaya, K.Kato, R.Sujatha, O.Venjakob, 'The GL_2 main conjecture for elliptic curves without complex multiplication', *Inst. Hautes Études Sci. Publ. Math.* 101 (2005), 163–208.

[5] P.Deligne, 'Valeur de fonctions L et périodes d'intégrales', *Automorphic forms, representations and L-function* (ed A. Borel and W. Casselman), Proceedings of Symposia in Pure Mathematics 33, Part 2 (American Mathematical Society, Providence, RI, 1979) 313–346.
[6] V.Dokchitser, 'Root numbers of non-abelian twists of elliptic curves', *Proc. London Math. Soc.* (3) 91 (2005) 300–324.
[7] R.P.Langlands, *Base change for* GL(2), Annals of Math. Studies 96, Princeton Univ. Press, Princeton, 1980.
[8] D. Rohrlich, 'On L-functions of elliptic curves and cyclotomic towers', *Invent. Math.* 75 (1984), 404–423.
[9] D. Rohrlich, 'L-functions and division towers', *Math. Ann.* 281 (1988), 611–632.
[10] J-P.Serre, *Local fields*, GTM 67, Springer-Verlag 1979.
[11] G.Shimura, 'On the periods of modular forms', *Math. Ann.* 229, 211–221 (1977), Springer-Verlag 1977.
[12] G.Shimura, 'The special values of the zeta functions associated with Hilbert modular forms', *Duke Mathematical Journal*, Vol 45 No 3, 1978.
[13] J.Tate, 'Number theoretic background', *Automorphic forms, representations and L-function* (ed A. Borel and W. Casselman), Proceedings of Symposia in Pure Mathematics 33, Part 2 (American Mathematical Society, Providence, RI, 1979) 3–26.
[14] R.Taylor, A.Wiles, 'Ring theoretic properties of certain Hecke algebras', *Annals of math.* 141 (1995), 553–572.
[15] A.Wiles, 'Modular elliptic curves and Fermat's last theorem', *Annals of math.* 141 (1995), 443–551.

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