On the size and shape of drumlins

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Abstract

We provide a mechanistic explanation for observed metrics for drumlins, which represent their sizes and shapes. Our explanation is based on a concept of drumlin growth occurring through a process of instability, whereby small amplitude wave forms first grow as ice slides over a bed of deformable sediments, following by a coarsening process, in which the wavelength as well as the relief of the drumlins continues to grow. The observations then provide inferences about the growth process itself.

Keywords: Drumlins, size and shape, instability theory.

1 Introduction

Recent work on the size and shape of British and Irish drumlins (Clark *et al.* 2009, Spagnolo *et al.* 2012) has shown that both their lengths and widths have well-defined smooth distributions, which are approximately lognormal. A lognormal random variable X takes positive values $x > 0$, and has the probability density function

$$f_X(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left[\frac{-(\ln x - \mu)^2}{2\sigma^2}\right]. \tag{1.1}$$

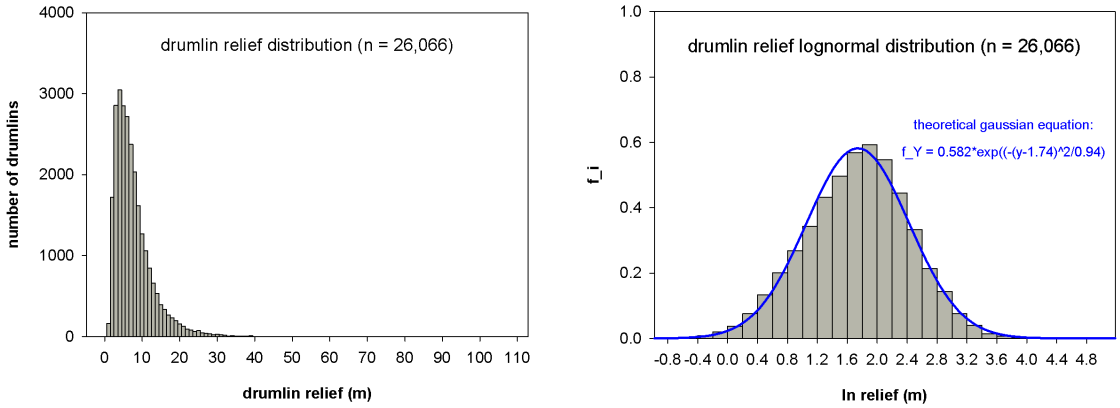

Figure 1: Distribution of drumlin relief h, plotted on the left as a histogram of the number of drumlins in bins of equal width in relief, and on the right as a histogram of the frequency f_i of $\ln h$ (see (1.4)) with equal bin widths in $\ln h$. The vertical axis on the right gives the frequency f_i in the i-th bin. If the distribution is lognormal, the right histogram will be fit by a Gaussian, given by (1.2), as explained in the text.

The name lognormal is associated with the fact that the random variable $Y = \ln X$, taking values $y = \ln x \in (-\infty, \infty)$ has the normal distribution

$$f_Y(y) = x f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(y-\mu)^2}{2\sigma^2}\right], \tag{1.2}$$

thus μ and σ^2 are the mean and variance of the distribution of $\ln X$.

Estimates of the density can be made by binning a sample population into equal sized bins and plotting the results as a histogram. If unequal bins are used, then the number in each bin must be divided by the bin width in order to provide a true measure of the frequency. Thus the histograms using equal bin widths of length L and width W provided by Clark *et al.* (2009), and of relief (Spagnolo *et al.* 2012), provide an estimate of the probability density. Similarly, if we plot histograms of $\ln X$ using equal bin widths of $\ln x$, we gain an estimate of the density of $\ln X$ (but not of X). In figures 1, 2 and 3, we show estimates of the densities of $\ln H$, $\ln L$ and $\ln W$, where also H is drumlin relief, together with Gaussian densities computed from the formula (1.2), using the unbiased estimators $\bar{\mu}$ and $\bar{\sigma}$ for μ and σ defined by

$$\begin{aligned} \bar{\mu} &= \frac{1}{n}\sum_i \ln x_i, \\ \bar{\sigma}^2 &= \frac{1}{n-1}\sum_i (\ln x_i - \bar{\mu})^2. \end{aligned} \tag{1.3}$$

More specifically, if n_i is the number of drumlins in each bin, the plots represent the histogram of the frequency f_i, which is an approximation of the probability density f_Y, and is given by

$$f_i = \frac{n_i}{n\Delta} \approx f_Y(y), \tag{1.4}$$

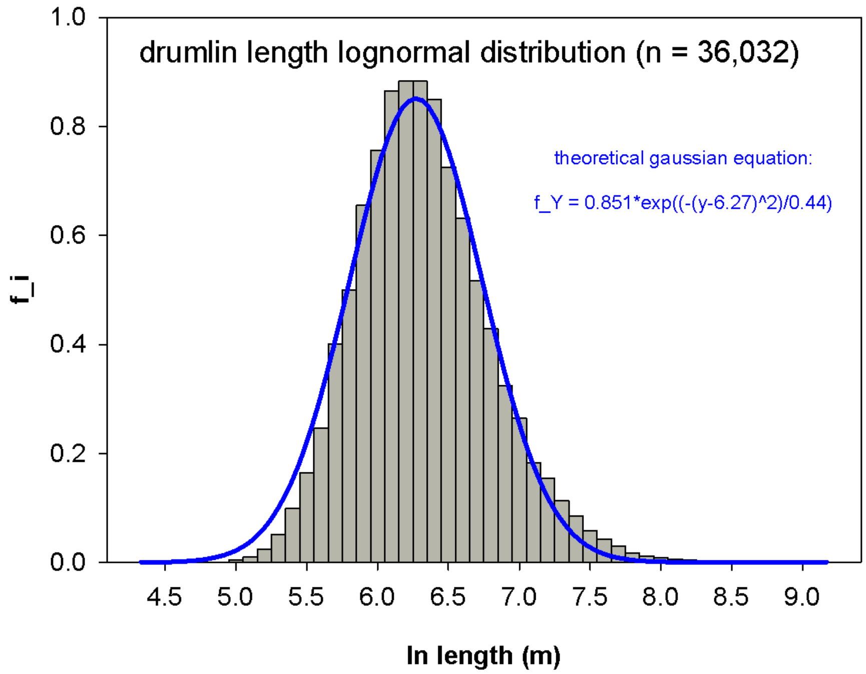

Figure 2: Distribution of drumlin lengths l, plotted as a histogram of frequency f_i of $\ln l$ with equal bin widths in $\ln l$.

where Δ is the bin width in $y = \ln x$, for $x = H, L, W$, and n is the total number of the sample; f_Y is the Gaussian given by (1.2). It can be seen that the data are very well fitted by the Gaussian curves, and this naturally raises the question as to why this should be so.

2 Log-normal distributions

An elementary discussion of lognormal distributions is given by Limpert *et al.* (2001), who also point to many different sciences where such distributions arise. For example in linguistics, the number of letters per word and the number of words per sentence fit a log-normal distribution. Kolmogorov (1941) provided a basic mechanism which can explain their occurrence, and his argument is nicely summarised by Kondolf and Adhikari (2000). Kolmogorov considers rock fragments of dimension r_i, which are successively fractured in events indexed by the subscript i, so that $r_{i+1} = F_i r_i$, where $F_i \in (0, 1)$ is the fraction of the particle which remains. It follows that

$$\ln r_n = \sum_0^n \ln F_i + \ln r_0, \tag{2.1}$$

and assuming that F_i is drawn from some appropriate distribution (for example,

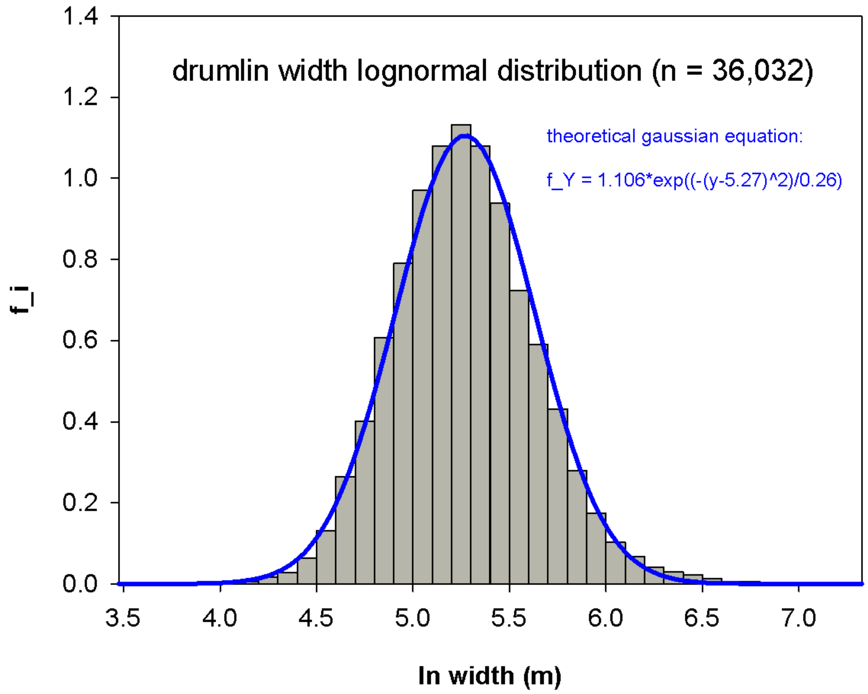

Figure 3: Distribution of drumlin widths w, plotted as a histogram of frequency f_i of $\ln w$ with equal bin widths in $\ln w$.

uniformly distributed on (0,1)) it follows from the central limit theorem that $\ln r$ will be normally distributed, i. e., r is lognormal. The key to the production of a lognormal distribution is thus the combination of a large number of independent events in which incremental (multiplicative) growth or decay occur.

Let us suppose that a deterministic theory of drumlin growth provides an evolution equation for quantities like the relief h of the form

$$\dot{h} = rh. \tag{2.2}$$

The growth rate r will in practice depend on conditions of the ice flow and subglacial water pressure, and is unlikely to be uniform in space or time. If we suppose that in fact drumlins are built in a series of *events*, each of which we might characterise as a relatively short term period of rapid flow, for example, or excess water flow, as for example in a sheet flood such as those found in Antarctica by Goodwin (1988) and Wingham *et al.* (2006), then we might correspondingly expect drumlin relief to grow in spurts, consistent with observations of King *et al.* (2009) and Smith *et al.* (2007). In that case we might expect that each measurable quantity x, such as width, length or relief, then evolves in the following way:

$$\Delta x \approx rx\Delta t. \tag{2.3}$$

The equation represents the increment in x due to a supposed instability, in which r

is the growth rate and Δt is the duration of the event. The quantity

$$\Delta w = r\Delta t \tag{2.4}$$

is considered to be a random variable, drawn from a suitable distribution.

Now we suppose that these events occur randomly in space, so that at every point where a drumlin forms, the sequence of events represents a series of independent trials, each of which has some probability p of causing an increment of growth given by (2.3). The central limit theorem then indicates that the total number of successes is described by a normal distribution; since $\Delta x/x = \Delta \ln x$, the sum of the increments is normally distributed, and specifically we have, for the random variable X taking the value x,

$$\ln\left(\frac{X}{x_0}\right) \sim N(\mu_E n_E, \sigma_E^2 n_E), \tag{2.5}$$

where N represents a normal distribution of mean μn_E and variance $\sigma^2 n_E$: n_E is the number of events, x_0 is the initial value for x, and μ_E and σ_E^2 are the mean and variance of the distribution of Δw. If, for example (like coin tossing), an event either occurs ($\Delta w = r\Delta t$ with probability p) or not ($\Delta w = 0$ with probability $1-p$), then

$$\mu_E = pr\Delta t, \quad \sigma_E^2 = p(1-p)\,(r\Delta t)^2\,. \tag{2.6}$$

Thus h (and similarly width and length) can be expected to have a log-normal distribution. We comment more on the physical basis of this supposition below.

A discussion similar to the above follows from the precepts of stochastic dynamics (Gardiner 2009, Van Kampen 2007), where we would associate the stochastic exponential growth of a quantity S with the stochastic differential equation

$$dS = rS\,dW, \tag{2.7}$$

where W is a Wiener process, that is to say $W(t)$ is continuous, non-differentiable and is a random variable with a Gaussian distribution and variance t. From this, we deduce a Fokker–Planck equation for the evolution of the density of $\ln S$, which is simply a diffusion equation for $\ln S$: hence S is lognormally distributed.

3 Non-lognormal distributions

While the probability densities in figures 1 to 3 are close to lognormal, they are not precisely so, and one may ask what the significance of these variations is. Particularly, the difference in the tails of the densities for length and width could be important. Fortunately, it is simple to modify the discussion above to link any unimodal density function to an hypothesised deterministic predictive equation.

Suppose now that a deterministic model for x is

$$\dot{x} = rg(x), \tag{3.1}$$

where we will assume $g > 0$ and $g \sim x$ as $x \to 0$. Exponential growth is recovered if $g = x$.

The discrete stochastic analogue is

$$\Delta x = g(x)\Delta w, \tag{3.2}$$

and the consequence of the central limit theorem is that

$$I(Y) \sim N(\mu_E n_E, \sigma_E^2 n_E), \tag{3.3}$$

with the same notation as in (2.5), and where

$$I(y) = \int_{x_0}^{x} \frac{dx}{g(x)}, \quad y = \ln\left(\frac{x}{x_0}\right), \tag{3.4}$$

as a consequence of which

$$g(x) = \frac{x}{I'(y)}. \tag{3.5}$$

The central limit theorem implies

$$P[y < I(Y) < y + \Delta] \approx G(y)\Delta, \tag{3.6}$$

where G is the Gaussian

$$G(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(y-\mu)^2}{2\sigma^2}\right], \tag{3.7}$$

and we use the sample mean and variance from (1.3) in the definition of G; thus also

$$\begin{aligned} G(y)\Delta &\approx P[I^{-1}(y) < Y < I^{-1}(y+\Delta)] \\ &\approx P\left[I^{-1}(y) < Y < I^{-1}(y) + \Delta \frac{d}{dy} I^{-1}(y)\right], \end{aligned} \tag{3.8}$$

from which it follows that

$$G(y) \approx f_i\left[I^{-1}(y)\right] \frac{d}{dy} I^{-1}(y). \tag{3.9}$$

If we define the cumulative distribution functions of the Gaussian and the sample as

$$\begin{aligned} F_G(I) &= \int_{-\infty}^{I} G(I)\, dI = \tfrac{1}{2}\left[1 + \operatorname{erf}\left(\frac{I-\mu}{\sigma\sqrt{2}}\right)\right], \\ F_Y(\eta) &= \int_{-\infty}^{\eta} f_i(\eta)\, d\eta, \end{aligned} \tag{3.10}$$

each of which is a monotonically increasing function of its argument, with $F_G(-\infty) = F_Y(-\infty) = 0$, $F_G(\infty) = F_Y(\infty) = 1$, then we determine $I(y)$ through the identification

$$F_G[I(y)] = F_Y(y). \tag{3.11}$$

This is illustrated in figure 4.

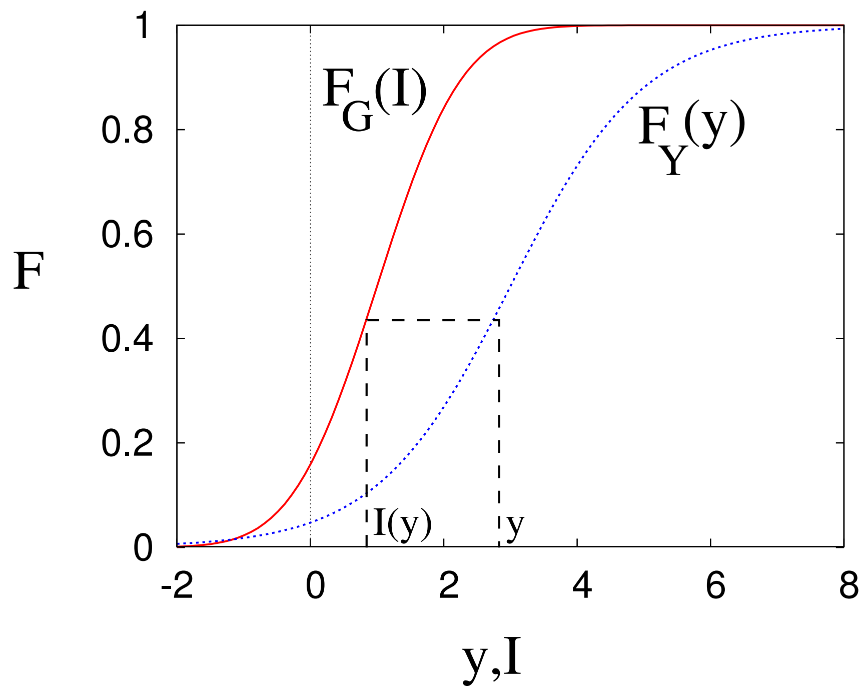

Figure 4: Illustration of the method of determining $I(y)$ via (3.11). The distributions illustrated are F_G in (3.10) and $F_Y = \dfrac{e^{y-3}}{1+e^{y-3}}$, and the mean and variance of F_G are $\mu = \sigma^2 = 1$.

In practice, we do the following. For any quantity x and with $y = \ln\left(\dfrac{x}{x_0}\right)$, we order each drumlin in the database as a pair (j, y_j), where $y_1 \leq y_2 \leq y_3 \ldots$, etc. For a total sample of size n, the numerical definition of F_Y is then just

$$F_j = F_Y(y_j) = \frac{j}{n}. \tag{3.12}$$

The computational step of calculating $I_j = I(y_j)$ consists of inverting (3.11), but (unless one has the inverse error function available) it is easier to calculate $y(I)$. Thus given I_i (in practice we select a monotonic sequence $I_1, \ldots, I_K$: note that the sequences $\{i\}$ and $\{k\}$ are distinct), we may calculate $F_G(I_i)$ explicitly via (3.10). For a y sample of size n, we define

$$j = \text{int}\,[nF_G(I_i)], \tag{3.13}$$

which gives $j(i)$ and thus also the corresponding y_j as a stair-like function of I. Note that some of the j values may be repeated, depending on the fineness of the partition for I. In practice we want the I partition to be sufficiently coarse that $j(i+1) - j(i)$ is reasonably large.

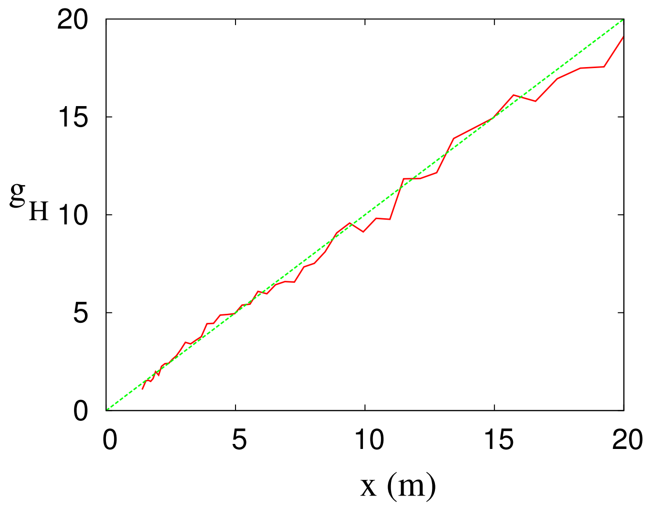

Figure 5: The function $g_H(x)$ computed as described in the text. For pure exponential growth, $g_H(x) = x$.

To calculate $g(x_j)$, we note that $x_j = x_0 e^{y_j}$, and then from (3.5) that

$$g_j = x_j y'(I_i) = \frac{x_j F'_G(I_i)}{F'_Y(y_j)}; \tag{3.14}$$

using (3.11). To compute $F'_Y(y_j)$, which is just the density of Y, we approximate

$$F'_Y(y_j) \approx \frac{F_Y(y_{j(i+1)}) - F_Y(y_{j(i)})}{y_{j(i+1)} - y_{j(i)}}, \tag{3.15}$$

and using (3.12), this implies

$$\frac{1}{F'_Y(y_j)} \approx \frac{n(y_{j(i+1)} - y_{j(i)})}{j(i+1) - j(i)}. \tag{3.16}$$

Thus (3.14) is just

$$g_j = \frac{n x_j \{y_{j(i+1)} - y_{j(i)}\}}{\sigma\sqrt{2\pi}\{j(i+1) - j(i)\}} \exp\left[\frac{-(I_i - \mu)^2}{2\sigma^2}\right]. \tag{3.17}$$

Note that the denominator in (3.16) is just n_i, the number of drumlins in the i-th bin, while the numerator is $n\Delta_i$, with Δ_i being the binwidth, so (3.16) is consistent with (1.4). As a simple illustration, if $\{y_j\}$ is exactly lognormal, then $F_Y = F_G$, so that $I(y) = y$ and (3.14) implies $g = x$.

Because of the irregularity of the sequence y_j and the sparseness of the data far from the mean, we restrict ourselves to values of y within $\pm 2\bar{\sigma}$ of the logarithmic mean. Additionally, the approximation (3.15) becomes very noisy if K is too large. For the figures below, we use $K = 50$.

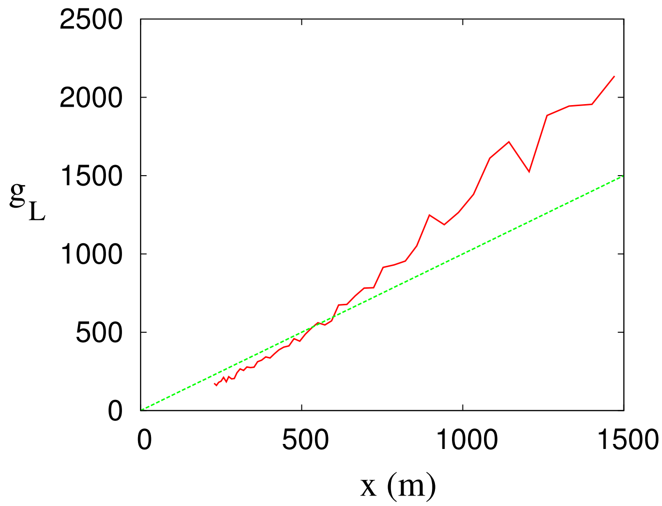

Figure 6: The function $g_L(x)$.

Figures 5, 6 and 7 show the results of carrying out this algorithm for the three data sets which give the height (or relief) H, length L and width W. The plots show the form of the deterministic equation

$$\dot{x} = g_M(x), \tag{3.18}$$

where the metric M is H, L or W. It can be seen that, as evident in figure 1, the relief growth rate is almost exactly exponential, with a slight tendency to saturate at higher relief. The length and width both show accelerated growth rate at higher values, corresponding to the noticeably fatter tails in figures 2 (in particular) and 3.

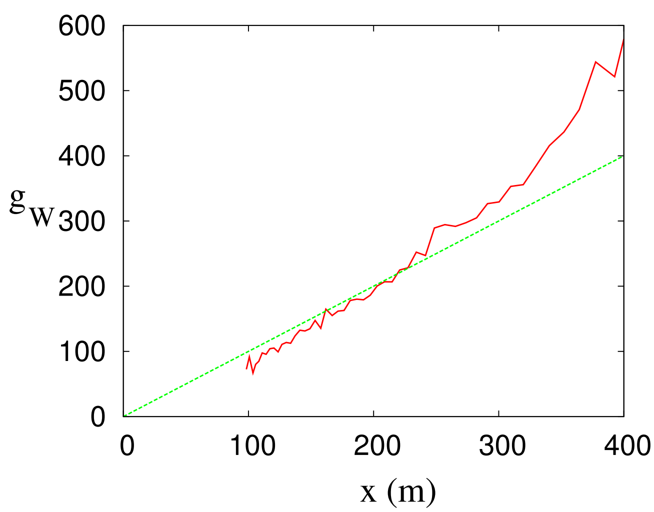

Figure 7: The function $g_W(x)$.

4 Discussion

A striking feature of all morphometric characteristics of drumlins is the regularity of the frequency distributions (Clark *et al.* 2009, Spagnolo *et al.* 2012). This regularity is very suggestive of a robust mechanism. We know of two ways in which robust distributions can form, although in essence they boil down to the same thing. The first is the central limit theorem, which when applied to large sequences of independent events, inevitably leads to the normal distribution. When the events consist of relative growth or fragmentation, the consequence is a lognormal distribution. In the case of drumlins, their lognormal distribution of size thus suggests that these landforms are a consequence of an incremental growth process. The second way is through a description of the dynamics as a stochastic process, where the stochastic term is usually taken to be a Wiener process; but in essence this is the same thing, since the representation of a Wiener process is as a Gaussian, essentially as a consequence of a large number of independent events, as for example in Brownian motion.

Therefore, a possible explanation of the very closely lognormal distributions of length, height and width which drumlins exhibit is to be found in an interpretation of their growth as arising through a combination of near exponential growth, together with a time history in which the growth phases occur randomly, or for random durations. Because conditions for the unstable growth of drumlins require the deformation of subglacial sediment, and thus low effective pressure, we might suppose such events are associated with, for example, transient ice spurts, or transient high subglacial discharge events, assuming a distributed system in which effective pressure decreases as water flux increases; and indeed such events appear to occur (Wingham *et al.* 2006).

In this view, the lognormal distributions of drumlin metrics are consistent with the instability theory, but the interpretation of the events as a jerky progression is not the only one. Indeed, in their extensive investigation of stability characteristics in terms of different flow parameters, Dunlop *et al.* (2008) (figure 7, middle) found that a random spread of flow conditions could also lead to a lognormal distribution of drumlin wavelengths. Admittedly this was only a linear analysis, but it suggests that an alternative source of the stochasticity may lie in the distribution of ice velocities, effective pressures, and so on. Nor indeed is there much to separate these two views at a conceptual level.

While the exponential growth of relief in stop-start time is consistent with instability theory, the evolution of length and width is at first sight not, since instabilities grow at preferred wavelengths. However, in common with other pattern-forming processes, Chapwanya *et al.* (2011) showed that coarsening occurred in finite amplitude calculations of drumlin evolution, and this is consistent with the behaviour seen in figures 6 and 7.

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