

Toric Duality, Seiberg Duality and Picard-Lefschetz Transformations

Sebastián Franco and Amihay Hanany

Center for Theoretical Physics,
Massachusetts Institute of Technology,
Cambridge, MA 02139, USA.

Abstract: Toric Duality arises as an ambiguity in computing the quiver gauge theory living on a D3-brane which probes a toric singularity. It is reviewed how, in simple cases Toric Duality is Seiberg Duality. The set of all Seiberg Dualities on a single node in the quiver forms a group which is contained in a larger group given by a set of Picard-Lefschetz transformations. This leads to elements in the group (sometimes called fractional Seiberg Duals) which are not Seiberg Duality on a single node, thus providing a new set of gauge theories which flow to the same universality class in the Infra Red.

1 Introduction

In [3], it was realized that the gauge theory living on the world volume of a D3-brane probing a toric singularity is sometimes non-uniquely determined. Thus, by considering D3-branes probing non-compact, toric, singular Calabi-Yau manifolds we are lead to more than one gauge theory with the same toric geometry as its moduli space. This phenomenon is the essence of Toric Duality. It is a full equivalence between distinct $\mathcal{N} = 1$, $d = 4$ gauge theories in the IR limit. Microscopic theories with different matter content and interactions become indistinguishable when we consider the long distance physics they describe. This short note is a summary of recent talks given by the authors which describes the main features of this phenomenon as well as describing a formalism to conveniently compute various features of gauge theories of branes on a class of singular manifolds¹.

The organization of this note is as follows. In Section 2, we present some examples of toric dual theories. Section 3 gives a brief introduction to (p, q) webs, which are useful in defining 5d fixed points as well as studying dynamics of 5d gauge theories but also in describing toric varieties and their associated 4d gauge theories. In Section 4, we explain how local mirror symmetry enables the computation of the gauge theory on the world volume of a D3-brane probing a toric singularity using the geometric information encoded in a (p, q) web. We also exemplify how the (p, q) web machinery can be used to derive toric duals. Based on the mirror Type IIA picture, we show in Section 5 how Picard-Lefschetz monodromy transformations point toward generalizations of Seiberg duality. Section 6

¹Similar ideas have been used recently in the construction of phenomenological models by wrapping D6-branes on compact, intersecting 3-cycles of Calabi-Yau manifolds [13, 14].

describes the construction of invariants for the singularities under study that generate Diophantine equations encoding the whole set of dual theories.

2 Toric duality

The determination of the gauge theory on a D3-brane probing a toric singularity was systematized in [3, 4], by the development of the *Inverse Algorithm*. This procedure is based on the realization of toric varieties as partial resolutions of Abelian orbifolds. Using this technique, the cases of cones over the Zeroth Hirzebruch and toric del Pezzo surfaces have been extensively studied. These are toric singularities with a shrinking compact 4-cycle.

In various cases, the resulting gauge theory is non-unique. Let us consider the example of the Zeroth Hirzebruch surface (F_0) for which the corresponding theories are displayed in Figure 1.

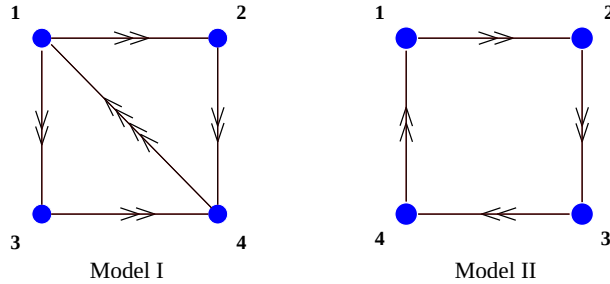


Figure 1: Quivers for the two phases of F_0 . Nodes represent $U(N)$ gauge theories, with N the number of D3 branes. Each arrow represents a bi-fundamental field transforming under the two gauge groups associated to the nodes it connects.

Accompanying the different matter contents summarized in their quivers, these models also display very distinct interactions. These are given by the following superpotentials, which correspond to a sum over a subset of all closed polygons in the quiver

$$\begin{aligned} W_I &= \epsilon_{ij}\epsilon_{mn}X_{12}^iX_{23}^mX_{31}^{jn} - \epsilon_{ij}\epsilon_{mn}X_{41}^iX_{23}^mX_{31}^{jn} \\ W_{II} &= \epsilon_{ij}\epsilon_{mn}X_{12}^iX_{23}^mX_{34}^jX_{41}^n \end{aligned} \quad (2.1)$$

Finding the right subset is a difficult task in general and presents a technical challenge. An overall re-scaling of the gauge group ranks by a common factor N , as well as tracing over gauge indices is understood in the previous quivers and superpotentials.

3 (p, q) webs

Five dimensional gauge theories can be engineered by 5-brane webs in type IIB string theory [1, 2]. In these constructions, the 5d gauge theories live on the $4 + 1$ common dimensions of the branes. The non-trivial intersections of the branes take place on a 2-dimensional transverse plane. The 5d theories are fully determined by the structure of the webs on this (x, y) plane.

Every brane has an associated (p, q) charge that dictates its tension

$$T_{p,q} = |p + \tau q|T_{D_5} \quad (3.2)$$

and its slope on the (x, y) plane

$$\Delta x : \Delta y = p : q \quad (3.3)$$

where T_{D5} is the D5-brane tension and τ is the complex scalar of type IIB (which we have set to be equal to i in 3.3). Condition 3.3 determines that 8 supercharges are preserved, leading to $\mathcal{N} = 1$ in five dimensions. Furthermore, (p, q) charge has to be conserved at each brane intersection

$$\sum_i p_i = \sum_i q_i = 0 \quad (3.4)$$

The reader is referred to [1, 2] for a detailed discussion of (p, q) webs, their use in engineering five dimensional theories and explicit examples. Gauge couplings, masses of gauge bosons and quarks, BPS spectrum and monopole tension can be computed straightforwardly from the geometry of the (p, q) web [1, 2].

Alternatively, (p, q) webs can be viewed as toric skeletons defining toric varieties [9] (see also [10] for applications of this idea along the lines that will be discussed in this note). In this interpretation, 5-branes correspond to the loci of points at which some 1-cycles of the T^2 fibrations of the toric varieties shrink to zero radius.

4 4d theories via local mirror symmetry

Each factor in the product gauge group of the theory on the D-brane world-volume is given by a fractional brane. These are bound states of D3, D5 and D7-branes, sharing four non-compact dimensions. D3-branes are located at points (i.e. 0-cycles) on the Calabi-Yau, while D5 and D7-branes wrap compact 2 and 4 cycles respectively.

The corresponding quiver can be obtained by looking at the mirror Type IIA geometry, in which D3-branes transverse to the original non-compact Calabi-Yau map to D6-branes wrapping a T^3 [8]. From the homology class of the T^3

$$[T^3] = \sum_{i=1}^n n_i S_i \quad n_i \in \mathbb{Z} \quad (4.5)$$

we can compute the gauge group and matter content of the $\mathcal{N} = 1$, $d = 4$ gauge theory produced by a wrapped D6-brane

$$G = \prod_{i=1}^n U(n_i) \quad I_{ij} = \#(S_i \cdot S_j) \quad (4.6)$$

Each 3-cycle S_i wraps a 1-cycle C_i of a smooth elliptic fiber that degenerates at a point z_i . The intersection numbers between the S_i 's are thus equal to the ones of the C_i 's which can be easily computed from their (p_i, q_i) charges

$$\#(S_i \cdot S_j) = \#(C_i \cdot C_j) = \det \begin{pmatrix} p_i & q_i \\ p_j & q_j \end{pmatrix} \quad (4.7)$$

Let us study how these concepts come together in the explicit example of dP_0 . The corresponding (p, q) web is presented in Figure 2.a, from where we read the following (p, q) charges

$$(p_1, q_1) = (-1, 2) \quad (p_2, q_2) = (2, -1) \quad (p_3, q_3) = (-1, -1) \quad (4.8)$$

Using 4.7, we compute the following intersection numbers

$$\#(C_1.C_2) = -3 \quad \#(C_2.C_3) = -3 \quad \#(C_3.C_1) = -3 \quad (4.9)$$

which can be conveniently summarized in the quiver diagram presented in Figure 2.b.

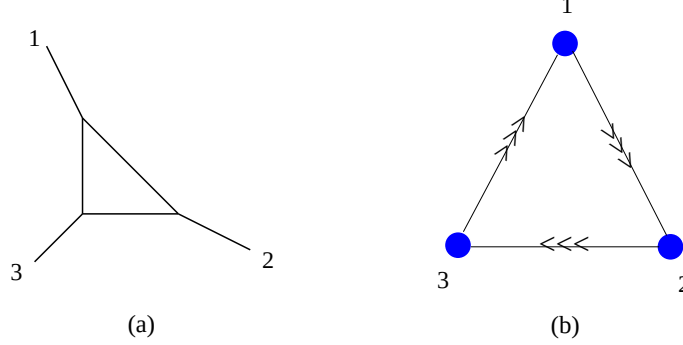


Figure 2: (a) (p,q) web and (b) quiver diagram for dP_0 .

It is interesting to note that in this case, the $SU(3)$ isometry of $\mathbb{P}^2$ appears as a flavor symmetry in the gauge theory, the X_{ij} chiral fields transforming in the fundamental representation for each pair of indices (i,j) . Invariance under this $SU(3)$ fixes the superpotential uniquely

$$W = \epsilon_{\alpha\beta\gamma} X_{12}^{(\alpha)} X_{23}^{(\beta)} X_{31}^{(\gamma)} \quad (4.10)$$

which is the singlet in $X_{12}X_{23}X_{31} = 3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$. This superpotential is also invariant under the $\mathbb{Z}_3$ cyclic permutations of the nodes (123). Let us demonstrate how one can compute the moduli space of vacua for this model and reproduce the right manifold we started with. The set of gauge invariant operators is given by 27 invariants, $a^{\alpha\beta\gamma} = X_{12}^{(\alpha)} X_{23}^{(\beta)} X_{31}^{(\gamma)}$. Using the F-term equations we find that any antisymmetric combination of the indices vanishes. Therefore $a^{\alpha\beta\gamma}$ is in the completely symmetric 10 dimensional representation of $SU(3)$. Furthermore, due to this symmetry we find the set of equations

$$(a^{\alpha\beta\gamma})^3 = a^{\alpha\alpha\alpha} a^{\beta\beta\beta} a^{\gamma\gamma\gamma}, \quad (4.11)$$

which is a set of 7 equations for 10 variables. A quick inspection verifies that this is the set of equations for the orbifold $C^3/\mathbb{Z}_3$, the manifold we started with.

In fact, (p,q) webs are powerful computational tools in deriving toric dual theories [10]. Del Pezzo surfaces are constructed by blowing-up up to eight generic points on $\mathbb{P}^2$. A blow-up corresponds to the replacement of a point by a 2-sphere. Since the toric $((p,q)$ web) representation of a 2-sphere is a segment, the blow-up of a vertex of a given (p,q) web corresponds to its replacement by a segment. At the same time the external leg that was originally attached to the blown-up vertex is replaced by a pair of legs, whose (p,q) charges are dictated by (p,q) charge conservation at the new vertices. Using the $SL(3, \mathbb{C})$ symmetry of $\mathbb{P}^2$, the positions of up to three generic points can be mapped to vertices of the web. In this way, we see that, starting from the (p,q) web for dP_0 given in Figure 2, we can construct all the toric duals for del Pezzo surfaces up to dP_3 , by blowing-up vertices of the webs in every possible way. We apply this technique to dP_1 in Figure 3 to obtain the two toric phases of dP_2 . It has been shown in [5, 6] that for the Zeroth Hirzebruch and del Pezzo surfaces, Toric dual theories are indeed Seiberg duals.

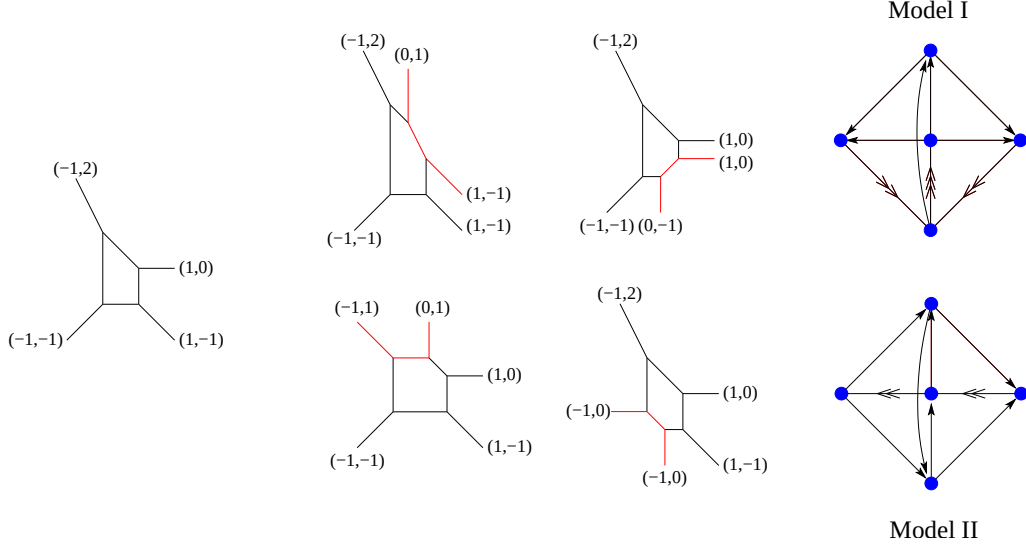


Figure 3: Possible blowups of dP_1 . They correspond to two inequivalent phases of dP_2 .

The discussion in this section, in addition to the one in Section 3, allows the translation of four dimensional quantities and processes into five dimensional ones. This approach was pursued in [10] to link the relocation of blown-up points associated to Toric Duality to the crossing of curves of marginal stability in related five dimensional theories.

5 Picard-Lefschetz transformations

We have seen in Section 4 how to exploit local mirror symmetry to compute four dimensional quiver theories from the intersections of 3-cycles in the type IIA mirror picture. This geometric realization of the theories suggests how to generalize Seiberg duality by using Picard-Lefschetz (PL) monodromy transformations.

PL monodromy corresponds to the reordering of vanishing cycles. When moving a vanishing cycle S_j around another one S_i , S_j gets a contribution proportional to S_i , weighted by the mutual intersection number

$$S_j \rightarrow S_j + (S_j \cdot S_i) S_i \quad (5.12)$$

while S_i remains invariant. The 3-cycles S_i can be represented by $[p_i, q_i]$ 7-branes with wrapping numbers n_i and we can recast (5.12) in terms of $[p, q]$ charges (Figure 4). PL monodromies corresponds in this language to the motion of a 7-brane across the branch cut of another one

The set of Seiberg dual theories associated to a given singularity can be obtained by starting from a configuration of $[p, q]$ 7-branes for the geometry, acting on them with PL monodromy transformations according to the rules in (5.12) and computing the resulting quiver as explained in Section 4. In fact, the group of PL transformations is larger than the one of Seiberg dualities, and there are gauge theories attainable with monodromies that cannot be reached by performing any chain of Seiberg dualities on nodes of the quiver [11]. Since further action with PL transformations results in Seiberg duals, these theories have been named *Fractional Seiberg duals*.

Figure 5 exhibits three theories related by PL transformations and Seiberg dualities for the Zeroth Hirzebruch surface. In this case, Model 2 is a new fractional Seiberg dual

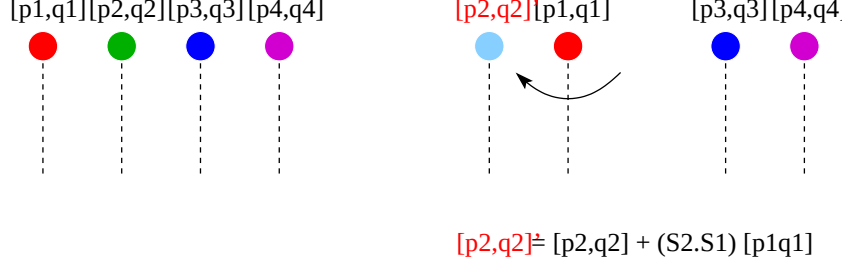


Figure 4: Picard-Lefschetz monodromy as reordering of a configuration of 7-branes. We have indicated in the figure the expression of (5.12) as a transformation of $[p, q]$ charges.

theory that could not have been computed by means of traditional Seiberg duality.

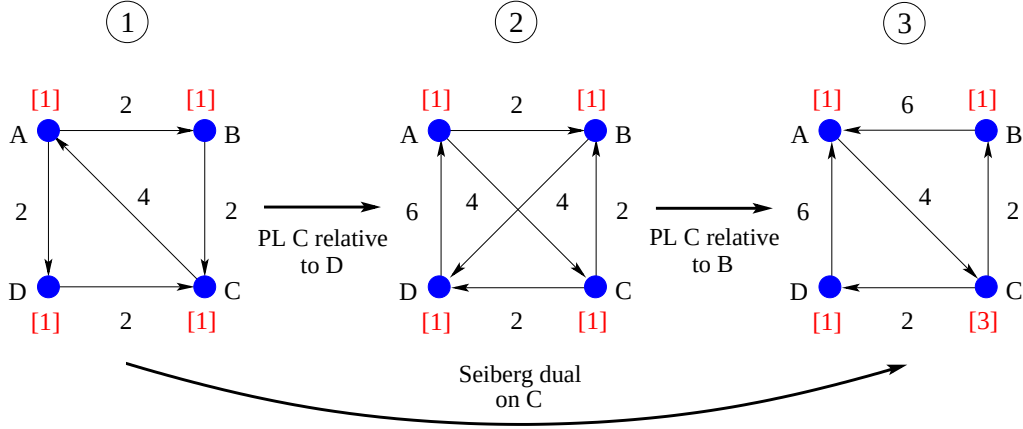


Figure 5: A sequence of Picard-Lefschetz transformations for F_0 . In this case, model 2 cannot be obtained by any combination of Seiberg dualities. The ranks of the gauge groups (up to an overall rescaling) are denoted in red.

6 Diophantine equations from invariant traces

In the previous section, we have described how dual theories can be generated by performing Picard-Lefschetz transformations on the set of degenerate fibers. All along the dualization process there is a set of quantities that remain invariant, thus providing us with a powerful tool in characterizing dual theories. The existence of these invariants was studied in detail in [12]. They are the number of the degenerate fibers, the greatest common divisor of the intersection numbers and the trace of the total monodromy matrix,

$$K = K_{z_1} K_{z_2} \dots K_{z_n} \quad (6.13)$$

For the configurations of degenerate fibers under study,

$$\text{Tr} K = 2 \quad (6.14)$$

This equality corresponds, for each geometry, to a Diophantine equation in the intersection numbers that completely encodes all the gauge theories that can be generated by performing Picard-Lefschetz transformations. As an example, let us consider the case of dP_0 . Using the (p, q) charges in Figure 2 we arrive at

$$I_{12}^2 + I_{23}^2 + I_{32}^2 - I_{12}I_{23}I_{32} = 0 \quad (6.15)$$

Since for this specific example $N_i = 3I_{jk}$, with $j, k \neq i$, (6.15) can be turned into an equation for the allowed ranks of the gauge groups

$$N_1^2 + N_2^2 + N_3^2 - 3N_1N_2N_3 = 0 \quad (6.16)$$

Acknowledgement We would like to thank Bo Feng, Yang-Hui He, Amer Iqbal and Angel Uranga for collaborations in the material presented in this note. A.H. Would also like to thank the organizers of the "35th International Symposium Ahrenshoop on the Theory of Elementary Particles" for their hospitality. Research supported in part by the CTP and the LNS of MIT and the U.S. Department of Energy under cooperative agreement #DE-FC02-94ER40818. A.H. is also supported by the Reed Fund Award and a DOE OJI ward.

References

- [1] Ofer Aharony , Amihay Hanany , "Branes, superpotentials and superconformal fixed points", Nucl. Phys. B **504**, 239 (1997), hep-th/9704170.
- [2] Ofer Aharony , Amihay Hanany , Barak Kol, "Webs of (p,q) five-branes, five-dimensional field theories and grid diagrams", JHEP **9801**, 002 (1998), hep-th/9710116.
- [3] Bo Feng, Amihay Hanany and Yang-Hui He, "D-Brane Gauge Theories from Toric Singularities and Toric Duality" Nucl. Phys. B **595**, 165 (2001), hep-th/0003085.
- [4] Bo Feng, Amihay Hanany and Yang-Hui He, "Phase Structure of D-brane Gauge Theories and Toric Duality", JHEP 0108 (2001) 040, hep-th/0104259.
- [5] Bo Feng, Amihay Hanany, Yang-Hui He and Angel M. Uranga, "Toric Duality as Seiberg Duality and Brane Diamonds", JHEP **0112**, 035 (2001), hep-th/0109063.
- [6] C. E. Beasley and M. R. Plesser, "Toric Duality Is Seiberg Duality", JHEP **0112**, 001 (2001), hep-th/0109053.
- [7] Bo Feng, Sebastian Franco, Amihay Hanany and Yang-Hui He, "Symmetries of toric duality", hep-th/0205144.
- [8] Amihay Hanany, Amer Iqbal, "Quiver Theories from D6-branes via Mirror Symmetry", JHEP **0204**, 009 (2002), hep-th/0108137.
- [9] N.C. Leung and C. Vafa, "Branes and Toric Geometry", Adv. Theor. Math. Phys. **2**, 91 (1998), hep-th/9711013.
- [10] Sebastian Franco and Amihay Hanany and Yang-Hui He, "Geometric Dualities in 4d Field Theories and their 5-d Interpretation", hep-th/0207006.
- [11] Bo Feng, Amihay Hanany, Yang-Hui He and Amer Iqbal, "Quiver Theories, Soliton Spectra and Picard-Lefschetz Transformations", hep-th/0206152.
- [12] O.DeWolfe, T. Hauer, A. Iqbal and B. Zwiebach, "Uncovering infinite symmetries on (p,q) 7-branes: Kac-Moody algebras and beyond", Adv. Theor. Math. Phys. **3**, 1835 (1999), hep-th/9812209.

- [13] A. M. Uranga, “Local models for intersecting brane worlds”, JHEP **0212**, 058 (2002), hep-th/0208014.
- [14] R. Blumenhagen, V. Braun, B. Kors and D. Lust, “Orientifolds of K3 and Calabi-Yau manifolds with intersecting D-branes”, JHEP **0207**, 026 (2002), hep-th/0206038.