GEOMETRIC PROPERTIES OF RANK ONE ASYMPTOTICALLY HARMONIC MANIFOLDS

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Abstract

In this article we consider asymptotically harmonic manifolds which are simply connected complete Riemannian manifolds without conjugate points such that all horospheres have the same constant mean curvature h. We prove the following equivalences for asymptotically harmonic manifolds X under the additional assumption that their curvature tensor together with its covariant derivative are uniformly bounded: (a) X has rank one; (b) X has Anosov geodesic flow; (c) X is Gromov hyperbolic; (d) X has purely exponential volume growth with volume entropy equals h. This generalizes earlier results by G. Knieper for noncompact harmonic manifolds and by A. Zimmer for asymptotically harmonic manifolds admitting compact quotients.

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1. Introduction

Let (X, g) be a complete simply connected Riemannian manifold without conjugate points and SX its unit tangent bundle. For $v \in SX$ we denote by $c_v : \mathbb{R} \to X$ the corresponding geodesic with $c_v'(0) = v$ and $b_v : X \to \mathbb{R}$, $b_v(q) = \lim_{t\to\infty} d(c_v(t), q) - t$ be the associated Busemann function.

Let $S_{v,r}$ and $U_{v,r}$ be the orthogonal Jacobi tensors along c_v, defined by $S_{v,r}(0) = U_{v,r}(0) = \mathrm{id}$ and $S_{v,r}(r) = 0$ and $U_{v,r}(-r) = 0$. Note that we have $U_{v,r}(t) = S_{-v,r}(-t)$. The stable and unstable Jacobi tensors S_v and U_v are defined as the Jacobi tensors along c_v with initial conditions $S_v(0) = U_v(0) = \mathrm{id}$ and $S_v'(0) = \lim_{r\to\infty} S_{v,r}'(0)$ and $U_v'(0) = \lim_{r\to\infty} U_{v,r}'(0)$. We define $U(v) = U_v'(0)$ and $S(v) = S_v'(0)$. For a general introduction into Jacobi tensors see [**Kn1**].

Important for this paper will be the notion of rank which in nonpositive curvature has been defined in [**BBE**] as the dimension of the parallel Jacobi fields along geodesics, and is one of the central concepts in rigidity theory. In the case of no conjugate points it is due to Knieper [**Kn2**] and generalizes this concept.

Definition 1.1. *Let (X, g) be a complete simply connected Riemannian manifold without conjugate points. For $v \in SX$ let $D(v) = U(v) - S(v)$ and we define*

$$\mathrm{rank}(v) = \dim(\ker D(v)) + 1$$

and

$$\mathrm{rank}(X) = \min\{\mathrm{rank}(v) \mid v \in SX\}$$

It is easy to see that the function $v \to \mathrm{rank}(v)$ is invariant under the geodesic flow.

As already observed in [**Kn2**] the notion of rank is very important in the study of *harmonic manifolds*. After Szabo's proof [**Sz**] of the Lichnerowicz conjecture for *compact* simply connected harmonic manifolds, the classification of *noncompact* harmonic manifolds is still wide

open, even though there have been interesting new developments in the last decade (see, e.g., [**RaSh**, **Ni**, **He**]). In this paper we consider the more general class of asymptotically harmonic manifolds, originally introduced by Ledrappier [**Le**, Thm 1] in connection with rigidity of measures related to the Dirichlet problem (harmonic measure) and the dynamics of the geodesic flow (Bowen-Margulis measure).

Definition 1.2. *An* asymptotically harmonic manifold (X, g) *is a complete, simply connected Riemannian manifold without conjugate points such that for all* $v \in SX$ *we have* $\operatorname{tr} U(v) = h$ *for a constant* $h \geq 0$.

Our first main result is the following:

Theorem 1.3. *Let* (X, g) *be an asymptotically harmonic manifold such that* $\|R\| \leq R_0$ *and* $\|\nabla R\| \leq R_0'$ *with suitable constants* $R_0, R_0' > 0$. *Then* $v \mapsto \det D(v)$ *is a constant function on* SX.

Moreover, if X *has rank one, there exists* $\rho > 0$ *such that* $D(v) \geq \rho \cdot \mathrm{id}$ *for all* $v \in SX$.

For harmonic manifolds, this theorem is a consequence of the relation between $\det D(v)$ and the volume density function (see [**Kn2**, Cor. 2.5]). For asymptotically harmonic manifolds this theorem was proved in [**HKS**, Cor. 2.1] under the additional condition of strictly negative curvature bounded away from zero. Zimmer [**Zi**, Proof of Prop. 3.3] provides a proof under the additional assumption of the existence of a compact quotient, using dynamical arguments. The proof of the general case without negative curvature or compact quotient requires new subtle estimates for second fundamental forms of spheres and horospheres which are presented in Section 2 of this article.

For the next result about asymptotic geometric and dynamical properties equivalent to the rank one condition we first need to introduce the notion of volume entropy.

Definition 1.4. *The* volume entropy $h_{vol}(X)$ *of a connected Riemannian manifold* X *is defined as*

$$h_{vol}(X) = \limsup_{r \to \infty} \frac{\log \operatorname{vol} B_r(p)}{r}, \tag{1.1}$$

where $B_r(p) \subset X$ *is the open ball of radius* r *around* $p \in X$.

Note that (1.1) does not depend on the choice of reference point p and $h_{vol}(X)$ is therefore well defined.

Theorem 1.3 is essential in the proof of our second main result.

Theorem 1.5. *Let* (X, g) *be an asymptotically harmonic manifold such that* $\|R\| \leq R_0$ *and* $\|\nabla R\| \leq R_0'$ *with suitable constants* $R_0, R_0' > 0$. *Let* $h \geq 0$ *be the mean curvature of its horospheres, i.e.* $h = \operatorname{tr} U(v)$. *Then the following properties are equivalent.*

(a) *X has rank one.*
(b) *X has Anosov geodesic flow $\phi^t : SX \to SX$.*
(c) *X is Gromov hyperbolic.*
(d) *X has purely exponential volume growth with growth rate $h_{vol} = h$.*

This equivalence has been obtained in the case of *noncompact harmonic manifolds* by Knieper in [**Kn2**]. In the case that (X, g) is an asymptotically harmonic manifold *with compact quotient*, this equivalence has been derived by Zimmer [**Zi**]. Since for harmonic manifolds the curvature tensor and its covariant derivative are bounded ([**Be**, Props. 6.57 and 6.68]), the current article generalizes these results in both papers to asymptotically harmonic manifolds (without a compact quotient condition).

In a subsequent article [**KnPe**] we use the main results of this article to derive results about harmonic functions (solution of the Dirichlet problem at infinity and mean value property of harmonic functions at infinity) on rank one asymptotically harmonic manifolds.

2. Manifolds without conjugate points: general results

In this section, (X, g) always denotes a complete simply connected Riemannian manifold without conjugate points. Let $\pi : SX \to X$ be the footpoint projection and $v \in SX$. The associated curvature tensor $R_v(t) : \phi^t(v)^\perp \to \phi^t(v)^\perp$ along c_v is defined by

$$R_v(t)w = R(w, \phi^t(v))\phi^t(v).$$

The stable and unstable manifolds through $v \in SX$ are defined as $W^s(v) = \{-\operatorname{grad} b_v(q) \mid b_v(q) = 0\}$ and $W^u(v) = \{\operatorname{grad} b_{-v}(q) \mid b_{-v}(q) = 0\}$. The footpoint projections $\pi W^s(v)$ and $\pi W^u(v)$ are level sets of Busemann functions and, therefore, horospheres. Horospheres are usually denoted by $\mathcal{H}$. Observe that $S(v)$ and $U(v)$ are the associated second fundamental forms.

2.1. A formula for the difference of second fundamental forms in horospheres. Of importance is the following result which is based on an formula of E. Hopf [**Ho**, (7.2)] for surfaces.

Proposition 2.1. *Let $\gamma : [0, 1] \to W^s(v)$ be a smooth curve with $\gamma(0) = v$ and $\gamma(1) = \widetilde{v}$. Let $e_1(s), \dots, e_{n-1}(s)$ be an orthonormal frame in $\mathcal{H} = \pi W^s(v)$ along $\beta = \pi\gamma$ which is parallel in $\mathcal{H}$ with the induced connection. Let $e_i(s, t)$ be the parallel translation along the geodesic $c_{\gamma(s)}$. Then we have*

$$S'_{\widetilde{v},r}(0) - S'_{v,r}(0) = \int_0^1 \int_0^r S^*_{\gamma(s),r}(t) \left(\frac{\partial}{\partial s} R_{\gamma(s)}(t) \right) S_{\gamma(s),r}(t) dt\, ds \tag{2.1}$$

and

(2.2)
$$U'_{\tilde{v},r}(0) - U'_{v,r}(0) = -\int_0^1 \int_{-r}^0 U^*_{\gamma(s),r}(t) \left(\frac{\partial}{\partial s} R_{\gamma(s)}(t)\right) U_{\gamma(s),r}(t) dt\, ds,$$

where all tensors are expressed with respect to the frame $e_1(s,t), \ldots, e_{n-1}(s,t)$.

Proof. We only prove (2.1), the second identity is proved analogously. We start with the Jacobi equation

$$S''_{\gamma(s),r}(t) + R_{\gamma(s)}(t) S_{\gamma(s),r}(t) = 0$$

and define

$$Z_{\gamma(s),r}(t) = \frac{\partial}{\partial s} S_{\gamma(s),r}(t).$$

Then we have

$$\begin{aligned} Z''_{\gamma(s),r}(t) &= \frac{\partial}{\partial s}\frac{\partial^2}{\partial^2 t} S_{\gamma(s),r}(t) = -\frac{\partial}{\partial s}\left(R_{\gamma(s)}(t) S_{\gamma(s),r}(t)\right) \\ &= -\left(\frac{\partial}{\partial s} R_{\gamma(s)}(t)\right) S_{\gamma(s),r}(t) - R_{\gamma(s)}(t)\left(\frac{\partial}{\partial s} S_{\gamma(s),r}(t)\right), \end{aligned}$$

and therefore,

$$Z''_{\gamma(s),r}(t) = -R_{\gamma(s)}(t) Z_{\gamma(s),r}(t) - \left(\frac{\partial}{\partial s} R_{\gamma(s)}(t)\right) S_{\gamma(s),r}(t).$$

Differentiating the Wronskian of $Z_{\gamma(s),r}$ and $S_{\gamma(s),r}$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\left((Z^*_{\gamma(s),r})'(t) S_{\gamma(s),r}(t) - Z^*_{\gamma(s),r}(t) S'_{\gamma(s),r}(t)\right) = \\ &\qquad (Z^*_{\gamma(s),r})''(t) S_{\gamma(s),r}(t) - Z^*_{\gamma(s),r}(t) S''_{\gamma(s),r}(t) = \\ &- Z^*_{\gamma(s),r}(t) R_{\gamma(s)}(t) S_{\gamma(s),r}(t) - S^*_{\gamma(s),r}(t)\left(\frac{\partial}{\partial s} R_{\gamma(s)}(t)\right) S_{\gamma(s),r}(t) \\ &+ Z^*_{\gamma(s),r}(t) R_{\gamma(s)}(t) S_{\gamma(s),r}(t) = -S^*_{\gamma(s),r}(t)\left(\frac{\partial}{\partial s} R_{\gamma(s)}(t)\right) S_{\gamma(s),r}(t). \end{aligned}$$

Integration with respect to t from 0 to r yields

$$\frac{\partial}{\partial s} S'_{\gamma(s),r}(0) = (Z^*_{\gamma(s),r})'(0) = \int_0^r S^*_{\gamma(s),r}(t)\left(\frac{\partial}{\partial s} R_{\gamma(s)}(t)\right) S_{\gamma(s),r}(t) dt.$$

Integration with respect to s from 0 to 1 leads finally to

$$S'_{\tilde{v},r}(0) - S'_{v,r}(0) = \int_0^1 \int_0^r S^*_{\gamma(s),r}(t)\left(\frac{\partial}{\partial s} R_{\gamma(s)}(t)\right) S_{\gamma(s),r}(t) dt\, ds,$$

proving (2.1). q.e.d.

In order to make use of the formulas in Proposition 2.1, we need to have estimates for $\|S_{\gamma(s),r}\|$, $\|U_{\gamma(s),r}\|$ and $\|\frac{\partial}{\partial s} R_{\gamma(s)}(t)\|$. These estimates are derived in the following two subsections.

2.2. Estimates for $\|S_{v,r}\|$ and $\|U_{v,r}\|$. We recall the following facts from [**Kn1**, Chapter 1, Cor. 2.12 and Lemma 2.17] (choosing $r = \infty$ there):

Lemma 2.2. *Assume that there exists a constant $R_0 > 0$ such that $-R_0 \,\mathrm{id} \leq R_v(t)$ for all $v \in SX$ and $t \in \mathbb{R}$. Let A_v be the orthogonal Jacobi tensor along c_v with $A_v(0) = 0$ and $A_v'(0) = \mathrm{id}$. Then we have*

$$-\sqrt{R_0} \leq A_v'(t)A_v^{-1}(t) \leq \sqrt{R_0}\coth(t\sqrt{R_0})$$

for all $t > 0$. Furthermore, we have

$$-\sqrt{R_0} \leq S_v'(0) \leq U_v'(0) \leq \sqrt{R_0}$$

for all $v \in SX$.

Note that A_v and $S_{v,r}$ are related by $S_{v,r}(t) = A_v(r-t)A_v^{-1}(r)$. Therefore, Lemma 2.2 has the following consequence.

Corollary 2.3. *Let $r_0 > 1$ and $T \leq r_0$. If $\|R_v(t)\| \leq R_0$ for all $v \in SX$ and $t \in \mathbb{R}$ with a constant $R_0 > 0$, we have for all $r \geq r_0$*

$$\|S_{v,r}(t)\| \leq C_1(R_0, r_0, T) \qquad \text{for all } 0 \leq t \leq T,$$

with $C_1(R_0, r_0, T) > 0$ only depending on r_0, R_0 and T.

Proof. We conclude from Lemma 2.2 for all $r \geq r_0$,

$$\|S_{v,r}'(0)\| = \|A_v'(r)A_v^{-1}(r)\| \leq \sqrt{R_0}\coth(r_0\sqrt{R_0}).$$

Let $y(t) = (y_1(t), y_2(t))^\top$ with $y_1(t) = S_{v,r}(t)$ and $y_2(t) = S_{v,r}'(t)$. Then

$$y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -R_v(t) & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = C(t)y(t),$$

i.e.,

$$y(t) = \exp\left(\int_0^t C(s)ds\right) y(0).$$

Note that $\|C(t)\| \leq \sqrt{{R_0}^2 + 1}$ and $\|y(0)\|^2 \leq 1 + R_0\coth^2(r_0\sqrt{R_0})$. This yields

$$\|S_{v,r}(t)\| \leq \|y(t)\| \leq \exp\left(T\sqrt{{R_0}^2+1}\right)\sqrt{1 + R_0\coth^2(r_0\sqrt{R_0})}$$

for all $0 \leq t \leq T$, finishing the proof. q.e.d.

Next, we present some useful Jacobi tensor identities.

Lemma 2.4. *For all $v \in SX$ and $t < r$ we have*

$$S_{\phi^t(v),r-t}'(0) = S_{v,r}'(t)S_{v,r}^{-1}(t), \tag{2.3}$$

and

$$\begin{aligned} U'_{\phi^t(v)}(0) - S'_{\phi^t(v),r-t}(0) &= (U_v^*)^{-1}(t)(U'_v(0) - S'_{v,r}(0))S_{v,r}^{-1}(t) \\ &= (S_{v,r}^*)^{-1}(t)(U'_v(0) - S'_{v,r}(0))U_v^{-1}(t). \end{aligned} \tag{2.4}$$

Furthermore,

$$U'_{\phi^t(v)}(0) - S'_{\phi^t(v),r-t}(0) = (S_{v,r}^*)^{-1}(t)\left(\int_{-\infty}^{t}(S_{v,r}^*S_{v,r})^{-1}(u)du\right)^{-1} S_{v,r}^{-1}(t). \tag{2.5}$$

Proof. Notice first that

$$S_{\phi^t v,x}(y) = S_{v,t+x}(y+t)S_{v,t+x}^{-1}(t), \tag{2.6}$$

since, for fixed x and t, both sides define Jacobi tensors in y which agree at $y = 0$ and $y = x$. Differentiating at $y = 0$ yields for $x = r - t$ the first identity (2.3). Using the fact that the Wronskian of two Jacobi tensors is constant, we have

$$W(U_v, S_{v,r})(t) = (U_v^*)'(t)S_{v,r}(t) - U_v^*(t)S'_{v,r}(t) = W(U_v, S_{v,r})(0) = U'_v(0) - S'_{v,r}(0).$$

This yields

$$(U'_v(t)U_v^{-1}(t))^* - S'_{v,r}(t)S_{v,r}^{-1}(t) = (U_v^*)^{-1}(t)(U'_v(0) - S'_{v,r}(0))S_{v,r}^{-1}(t).$$

Since

$$(U'_v(t)U_v^{-1}(t)) = U'_{\phi^t(v)}(0) \text{ and } S'_{v,r}(t)S_{v,r}^{-1}(t) = S'_{\phi^t(v),r-t}(0)$$

are symmetric, we obtain the first and second identity of (2.4).

To prove the last assertion we note that for $0 \le t \le r$ we have (see [**Kn2**, (7.8)])

$$\left(\int_{-\infty}^{t}(S_{v,r}^*S_{v,r})^{-1}(u)du\right)^{-1} S_{v,r}(t)^{-1}U_v(t) = U'_v(0) - S'_{v,r}(0).$$

Inserting this into (2.4) yields (2.5). q.e.d.

Recall from the introduction that $S(v) = S'_v(0)$ and $U(v) = U'_v(0)$. A key role plays the positive symmetric operators

$$D(v) = U(v) - S(v),$$

since their kernels determine the rank of the manifold X.

Proposition 2.5. *Assume there exists $R_0 > 0$ such that $\|R_v(t)\| \le R_0$ for all $t \in \mathbb{R}$. Then we have the following estimates for $S_{v,r}$ and $U_{v,r}$.*

(a) *There exists* $a_1 = a_1(R_0)$ *such that for all* $r > 1$ *and* $t \geq 0$,

$$\|S_{v,r}(-t)\| \leq a_1 e^{\sqrt{R_0}t}, \qquad \|U_{v,r}(t)\| \leq a_1 e^{\sqrt{R_0}t}.$$

(b) *Under the additional assumption* $D(\phi^t(v)) \geq \rho \cdot \mathrm{id}$ *for all* $t \in \mathbb{R}$ *and some constant* $\rho > 0$, *there exists* $a_2 = a_2(R_0, \rho)$ *such that for all* $r > 1$ *and* $0 \leq t < r$,

$$\|S_{v,r}(t)\| \leq a_2 e^{-\frac{\rho}{2}t}, \qquad \|U_{v,r}(-t)\| \leq a_2 e^{-\frac{\rho}{2}t}.$$

Proof. Rauch's comparison estimate (see, e.g., [**Kn1**, Chapter 1, Prop. 2.11]) implies that $\|A(t)x\|/\sinh\sqrt{R_0}t$ is monotone decreasing. Since $S_{v,r}(-t) = A_v(r+t)A_v^{-1}(r)$ we conclude

$$\|S_{v,r}(-t)\frac{A_v(r)x}{\|A_v(r)x\|}\| = \frac{\|A(r+t)x\|}{\|A(r)x\|} \leq \frac{\sinh\sqrt{R_0}(r+t)}{\sinh\sqrt{R_0}r} \leq a_1(R_0)e^{\sqrt{R_0}t}.$$

This together with $U_{v,r}(t) = S_{-v,r}(-t)$ proves (a).

Using the monotonicity $S'_{w,r}(0) \nearrow S'_w(0)$, we have by assumption

$$U'_{\phi^t(v)}(0) - S'_{\phi^t(v),r-t}(0) \geq U'_{\phi^t(v)}(0) - S'_{\phi^t(v)}(0) = D(\phi^t(v)) \geq \rho\,\mathrm{id}\,.$$

Using (2.5), this yields for all $x \in (\phi^t v)^\perp$ with $\|x\| = 1$,

$$\begin{aligned}\rho &\leq \left\langle \left(\int_{-\infty}^{t} (S^*_{v,r}S^{-1}_{v,r}(u)du\right)^{-1} S^{-1}_{v,r}(t)x, S^{-1}_{v,r}(t)x \right\rangle \\ &\leq \left\| \left(\int_{-\infty}^{t} (S^*_{v,r}S_{v,r})^{-1}(u)du\right)^{-1} \right\| \cdot \left\|S^{-1}_{v,r}(t)x\right\|^2.\end{aligned}$$

Furthermore, we have

$$\left\| \left(\int_{-\infty}^{t} (S^*_{v,r}S_{v,r})^{-1}(u)du\right)^{-1} \right\| = \frac{1}{\min\left\{\int_{-\infty}^{t} \langle (S^*_{v,r}S_{v,r})^{-1}(u)y_u, y_u\rangle du : \ y \in v^\perp, \|y\| = 1\right\}}.$$

Therefore,

$$\rho \min\left\{\int_{-\infty}^{t} \langle (S^*_{v,r})^{-1}(u)y_u, (S^*_{v,r})^{-1}(u)y_u\rangle du : \ y \in v\perp, \|y\| = 1)\right\} \leq \|S^{-1}_{v,r}(t)x\|^2$$

for all $x \in (\phi^t v)^\perp$ with $\|x\| = 1$. Defining

$$\begin{aligned}\varphi(u) &:= \min\left\{\|(S^*_{v,r})^{-1}(u)y\|^2 : \ y \in (\phi^u v)^\perp, \|y\| = 1\right\} \\ &= \min\left\{\|S^{-1}_{v,r}(u)y\|^2 : \ y \in (\phi^u v)^\perp, \|y\| = 1\right\},\end{aligned}$$

we obtain

$$\rho \int_0^t \varphi(u)du \le \rho \int_{-\infty}^t \varphi(u)du \le \varphi(t)$$

and, hence,

$$\rho F(t) \le F'(t)$$

for $F(t) := \int_0^t \varphi(u)du$. This means that $\rho \le (\log F)'(t)$, which implies $F(t) \ge \frac{F(1)}{e} e^{\rho t}$ for all $t \ge 1$ and, therefore,

$$\varphi(t) = F'(t) \ge \rho F(t) \ge \frac{\rho F(1)}{e} e^{\rho t} \tag{2.7}$$

for all $r > t \ge 1$. Corollary 2.3 implies for $0 \le t \le 1 < r$

$$\begin{aligned}\varphi(t) &= \min\left\{\|S^{-1}_{v,r}(t)y\|^2 : \ y \in (\phi^t v)^\perp, \|y\| = 1\right\} \\ &= \frac{1}{\|S_{v,r}(t)\|} \ge \frac{1}{C_1(R_0, 1, 1)},\end{aligned}$$

i.e., $F(1) \ge \frac{1}{C_1(R_0,1,1)}$. Plugging this into (2.7), we obtain

$$\varphi(t) \ge \frac{\rho}{eC_1(R_0, 1, 1)} e^{\rho t}$$

for all $r > t \ge 1$. Choosing $a_2 = \left(\frac{C_1(R_0,1,1)}{\min\{\rho/e, e^{-\rho}\}}\right)^{1/2}$, this implies that we have

$$\frac{\|S^{-1}_{v,r}(t)y\|^2}{\|y\|^2} \ge \varphi(t) \ge \frac{1}{C_1(R_0, 1, 1)} \min\{\frac{\rho}{e}, e^{-\rho}\} e^{\rho t} = \frac{1}{{a_2}^2} e^{\rho t}$$

for all $r > t \ge 0$ and $y \in (\phi^t v)^\perp$, $y \ne 0$. Since $S_{v,r}(t) : (\phi^t v)^\perp \to (\phi^t v)^\perp$ is an isomorphism, we obtain for all $x \in (\phi^t v)^\perp$, $r > 1$ and $t \in [0, r)$

$$\|S_{v,r}(t)x\| \le a_2 e^{-\frac{\rho}{2}t} \|x\|,$$

finishing the proof of (b). q.e.d.

Remark. *The special case of Proposition* 2.5*(b) for stable and unstable Jacobi tensors was obtained by Bolton (see* [**Bo**, Lemma 2]*).*

The following corollary summarizes the facts which we will need further on in this chapter.

Corollary 2.6. *Let $\|R_v(t)\| \leq R_0$ for all $v \in SX$ and $t \in \mathbb{R}$ with a constant $R_0 > 0$. Let $\gamma : [0,1] \to W^s(v)$ be a smooth curve and $\rho > 0$ such that*

$$D(\phi^t(\gamma(s))) \geq \rho \cdot \mathrm{id}$$

for all $s \in [0,1]$ and $t \in \mathbb{R}$. Then there exists a function $b : \mathbb{R} \to (0,\infty)$, only depending on R_0 and ρ, such that we have for all $r > 1$ and all $-\infty < t < r$,

$$\|S_{\gamma(s),r}(t)\| \leq b(t), \qquad \|U_{\gamma(s),r}(-t)\| \leq b(t). \tag{2.8}$$

For $t \geq 0$ we have

$$b(t) \leq a_2 e^{-\frac{\rho}{2}t}. \tag{2.9}$$

Moreover, if $\beta = \pi\gamma$ and $\beta_t = \pi(\phi^t\gamma)$, we have

$$\|\beta_t'(s)\| \leq b(t)\,\|\beta'(s)\| \tag{2.10}$$

$$\|(\phi^t\gamma)'(s)\| \leq b(t)\sqrt{1+R_0}\,\|\beta'(s)\| \tag{2.11}$$

for all $s \in [0,1]$ and $t \in \mathbb{R}$.

Proof. The inequalities (2.8) and (2.9) are straightforward consequences of Proposition 2.5. The same inequalities hold also for the stable and unstable Jacobi tensors $S_{\gamma(s)}$ and $U_{\gamma(s)}$. Note that $J_s(t) = \beta_t'(s) = \frac{\partial}{\partial s} c_{\gamma(s)}(t)$ is the stable Jacobi field along $c_{\gamma(s)}$ with initial values

$$J_s(0) = \beta'(s) \quad \text{and} \quad J_s'(0) = S_{\gamma(s)}'(0) J_s(0).$$

Then $J_s(t) = S_{\gamma(s)}(t)(J_s(0))_t$, which implies

$$\|\beta_t'(s)\| \leq \|S_{\gamma(s)}(t)(\beta'(s))_t\| \leq b(t)\|\beta'(s)\|.$$

Furthermore we have

$$\|(\phi^t\gamma)'(s)\|^2 = \|\frac{d}{ds}\beta_t(s)\|^2 + \|\frac{D}{ds}\phi^t\gamma(s)\|^2 = \|\frac{d}{ds}\beta_t(s)\|^2 + \|\nabla_{\beta_t'(s)}\phi^t\gamma(s)\|^2.$$

Since $\nabla_{\beta_t'(s)}\phi^t\gamma(s)$ is the second fundamental form of the horosphere $\pi W^s(\phi^t(\gamma(s)))$, we have with Lemma 2.2

$$\|(\phi^t\gamma)'(s)\|^2 = \|\frac{d}{ds}\beta_t(s)\|^2 + \|S_{\phi^t\gamma(s)}'(0)\beta_t'(s)\|^2 \leq b(t)^2(1+R_0)\|\beta'(s)\|^2.$$

This implies (2.11). q.e.d.

2.3. Estimate for $\|\frac{\partial}{\partial s}R_{\gamma(s)}(t)\|$. Our next goal is to derive an estimate for $\|\frac{\partial}{\partial s}R_{\gamma(s)}(t)\|$ in terms of $\beta'(s)$. Henceforth, we assume that the curvature tensor and its covariant derivative of X are bounded, i.e.,

$$\|R\| \leq R_0 \quad \text{and} \quad \|\nabla R\| \leq R_0'$$

with suitable constants $R_0, R_0' > 0$. Moreover, let $\gamma : [0,1] \to W^s(v)$ denote a smooth curve such that

$$D(\phi^t(\gamma(s))) \geq \rho \cdot \mathrm{id}$$

for all $s \in [0,1]$ and $t \in \mathbb{R}$ with a suitable constant $\rho > 0$. Let $e_i(s)$ and $e_i(s,t)$ be defined as in Proposition 2.1 and $\beta = \pi\gamma$ and $\beta_t = \pi(\phi^t\gamma)$.

Lemma 2.7. *Let $r > 1$. Then there exists a constant $C_2(R_0, \rho, r)$, only depending on R_0, ρ and r, such that*

$$\left\| \frac{D}{ds} e_i(s,t) \right\| \leq C_2(R_0, \rho, r) \|\beta'(s)\| \qquad \textit{for all } t \in (-r, r).$$

Proof. First of all, note that the second fundamental form of all horospheres is bounded by $\sqrt{R_0}$. Let N be a unit normal vector field of $\mathcal{H} = \pi W^s(v)$. Since e_i is parallel in $\mathcal{H}$ with respect to the induced connection, we have

$$\frac{D}{ds} e_i(s) = \left\langle \frac{D}{ds} e_i(s), (N \circ \beta)(s) \right\rangle \, (N \circ \beta)(s).$$

Therefore,

$$\begin{aligned} \left\| \frac{D}{ds} e_i(s) \right\|^2 &= \left\langle \frac{D}{ds} e_i(s), (N \circ \beta)(s) \right\rangle^2 = \left\langle e_i(s), \frac{D}{ds} N \circ \beta(s) \right\rangle^2 \\ &\leq \|e_i(s)\|^2 \, \|\nabla N \circ \beta(s)\|^2 \, \|\beta'(s)\|^2 \\ &\leq R_0 \, \|\beta'(s)\|^2. \end{aligned} \tag{2.12}$$

Let $P^t_{\gamma(s)}$ be the parallel transport along $c_{\gamma(s)}$. Define

$$f(s,t) = \left\| \frac{D}{ds} e_i(s,t) \right\| = \left\| \frac{D}{ds} P^t_{\gamma(s)} e_i(s) \right\|.$$

Differentiation yields

$$\frac{\partial}{\partial t} f^2(s,t) = 2 \left\langle \frac{D}{dt} \frac{D}{ds} P^t_{\gamma(s)} e_i(s), \frac{D}{ds} e_i(s,t) \right\rangle. \tag{2.13}$$

Note that

$$\begin{aligned} \frac{D}{dt} \frac{D}{ds} P^t_{\gamma(s)} e_i(s) &= \underbrace{\frac{D}{ds} \frac{D}{dt} P^t_{\gamma(s)} e_i(s)}_{=0} + R\left(c'_{\gamma(s)}(t), \frac{\partial}{\partial s} c_{\gamma(s)}(t) \right) e_i(s,t) \\ &= R\left(c'_{\gamma(s)}(t), \frac{\partial}{\partial s} c_{\gamma(s)}(t) \right) e_i(s,t). \end{aligned}$$

Plugging this into (2.13) we conclude

$$\left| \frac{\partial f}{\partial t}(s,t) \right| \leq R_0 \, \|c'_{\gamma(s)}(t)\| \, \left\| \frac{\partial}{\partial s} c_{\gamma(s)}(t) \right\| \, \|e_i(s,t)\| = R_0 \, \|\beta'_t(s)\|,$$

which implies

$$\begin{aligned} f(s,t) &\leq f(s,0) + \int_{\min\{0,t\}}^{\max\{0,t\}} \left| \frac{\partial f}{\partial t}(s,\tau) \right| d\tau \\ &\leq \left\| \frac{D}{ds} e_i(s) \right\| + R_0 \int_{-r}^{r} \|\beta'(s)\| d\tau \\ &\overset{(2.12)}{\leq} \sqrt{R_0} \|\beta'(s)\| + R_0 \int_{-r}^{r} \left\| S_{\gamma(s)}(\tau)(\beta'(s))_\tau \right\| d\tau \\ &\leq \sqrt{R_0}\, \|\beta'(s)\| + R_0 \int_{-r}^{r} b(t) dt\, \|\beta'(s)\|. \end{aligned}$$

This finishes the proof. q.e.d.

The estimate for $\|\frac{\partial}{\partial s} R_{\gamma(s)}(t)\|$ is derived from the components. The (i,j)-th component of $R_{\gamma(s)}(t)$ is

$$\langle R_{\gamma(s)} e_i(s,t), e_j(s,t) \rangle = \langle R(e_i(s,t), \phi^t(\gamma(s)))\phi^t(\gamma(s)), e_j(s,t) \rangle.$$

This implies that we have

$$\left(\frac{\partial}{\partial s} R_{\gamma(s)}(t) \right)_{i,j} = \left\langle \frac{D}{ds} \left(R_{\gamma(s)}(t) e_i(s,t) \right), e_j(s,t) \right\rangle + \left\langle R_{\gamma(s)}(t) e_i(s,t), \frac{D}{ds} e_j(s,t) \right\rangle.$$

Using

$$\begin{aligned} &\nabla_J (R(Z,W)W) = \\ &(\nabla_J R)(Z,W)W + R(\nabla_J Z, W)W + R(Z, \nabla_J W)W + R(Z,W)\nabla_J W \end{aligned}$$

and the bounds $\|R\| \leq R_0$ and $\|\nabla R\| \leq R_0'$, we obtain

$$\begin{aligned} &\left\| \frac{D}{ds} \left(R_{\gamma(s)}(t) e_i(s,t) \right) \right\| \leq \\ &\qquad R_0' \parallel \beta_t'(t) \| + R_0 \left(\left\| \frac{D}{ds} e_i(s,t) \right\| + 2 \left\| \frac{D}{ds} \left(\phi^t(\gamma(s)) \right) \right\| \right) \leq \\ &\qquad R_0'\, b(t)\, \|\beta'(s)\| + R_0 \left(C_2(R_0,\rho,r) \|\beta'(s)\| + 2 \left\| \frac{D}{ds} \left(\phi^t(\gamma(s)) \right) \right\| \right), \end{aligned}$$

where we used (2.10) and Lemma 2.7. Since $\frac{D}{ds}\phi^t(\gamma(s)) = \nabla_{\beta_t'(s)} \phi^t(\gamma(s))$ is the second fundamental form of the horosphere $\pi W^s(\phi^t(\gamma(s)))$ which is bounded in norm by $\sqrt{R_0}\|\beta_t'(s)\|$, we finally obtain

$$\left\| \frac{D}{ds} \left(R_{\gamma(s)}(t) e_i(s,t) \right) \right\| \leq C_3(R_0, R_0', \rho, r) \|\beta'(s)\|$$

with $C_3(R_0, R_0', \rho, r) = R_0' b(t) + R_0 C_2(R_0, \rho, r) + 2R_0^{3/2} b(t)$. This implies that

(2.14)
$$\left| \left(\frac{\partial}{\partial s} R_{\gamma(s)}(t) \right)_{i,j} \right| \leq \big(C_3(R_0, R_0', \rho, r) + R_0 C_2(R_0, \rho, r) \big) \, \|\beta'(s)\|.$$

2.4. An estimate for the difference of second fundamental forms in horospheres. Combining the results in the first three subsections, we are now able to prove the following result.

Theorem 2.8. *Let (X, g) be a complete simply connected Riemannian manifold without conjugate points. Assume that $\|R_0\| \leq R_0$ and $\|\nabla R\| \leq R_0'$ with suitable constants $R_0, R_0' > 0$. Let $\gamma : [0,1] \to W^s(v)$ be a smooth curve and $\beta = \pi\gamma$. Assume that $D(\phi^t(\gamma(s))) \geq \rho \cdot \mathrm{id}$ for all $s \in [0,1]$ and $t \in \mathbb{R}$ and some constant $\rho > 0$. Let $r > 1$. Then there exists a constant $C_5(R_0, R_0', \rho, r) > 0$, only depending on R_0, R_0', ρ and r, such that*

$$\|S'_{\gamma(1),r}(0) - S'_{\gamma(0),r}(0)\| \leq C_5(R_0, R_0', \rho, r) \; \ell(\beta)$$

and

$$\|U'_{\gamma(1),r}(0) - U'_{\gamma(0),r}(0)\| \leq C_5(R_0, R_0', \rho, r) \; \ell(\beta),$$

where $\ell(\beta)$ denotes the length of the curve β.

Proof. We only give the proof of the second estimate, the first estimate is proved analogously. Let $v = \gamma(0)$ and $\widetilde{v} = \gamma(1)$. Inequality (2.14) implies that there is a constant $C_4 = C_4(R_0, R_0', \rho, r) > 0$ such that

$$\left\| \frac{\partial}{\partial s} R_{\gamma(s)}(t) \right\| \leq C_4 \|\beta'(s)\|.$$

We conclude from Proposition 2.1 and Corollary 2.6 that

$$\begin{aligned} \|U'_{\gamma(1),r}(0) - U'_{\gamma(0),r}(0)\| &\leq \int_0^1 \int_{-r}^0 \|U_{\gamma(s),r}(t)\|^2 \left\| \frac{\partial}{\partial s} R_{\gamma(s)}(t) \right\| dt \, ds \\ &\leq \int_0^1 C_4 \int_{-r}^0 b(-t)^2 dt \; \|\beta'(s)\| ds \\ &\leq C_5(R_0, R_0', \rho, r) \; \ell(\beta) \end{aligned}$$

with $C_5(R_0, R_0', \rho, r) = C_4 \int_{-r}^r b(t)^2 dt$. q.e.d.

3. The function $\det D(v)$ is constant

From now on, we assume that (X, g) is asymptotically harmonic. Recall that we introduced the positive symmetric operator $D(v) = U(v) - S(v)$. Our aim is to prove Theorem 1.3 in the Introduction.

Note first that in the case of asymptotically harmonic manifolds the stable and unstable Jacobi tensors are continuous in the sense of [**Es**, p. 242]. This property is also called *continuous asymptote*.

Lemma 3.1. *Let (X,g) be an asymptotically harmonic manifold. Then $v \mapsto U(v)$ and $v \mapsto S(v)$ are continuous maps on SX.*

Proof. Since $U'_{v,t}(0) - U'_v(0)$ is positive, we have for $t > 0$

$$\|U'_{v,t}(0) - U'_v(0)\| \leq \operatorname{tr}(U'_{v,t}(0) - U'_v(0)) = \operatorname{tr}(U'_{v,t}(0)) - h.$$

Since $\operatorname{tr}(U'_{v,t}(0))$ converges pointwise monotonically to h as $t \to \infty$, we conclude from Dini that the convergence is uniformly on all compact subsets of SX. Since the maps $v \mapsto U'_{v,t}(0)$ is continuous for all $t > 0$ and $U'_{v,t}(0) \to U'_v(0) = U(v)$ uniformly on compact sets, we conclude continuity of $v \mapsto U(v)$. The continuity of $v \mapsto S(v)$ follows immediately from $S(v) = -U(-v)$. q.e.d.

As a start, it is easy to see that $\det D(v) = \det D(-v)$:

$$\det D(-v) = \det(U(-v) - S(-v)) = \det(-S(v) + U(v)) = \det D(v).$$

Now we work towards the result that $\det D(v)$ is constant on all of SX.

3.1. $\det D(v)$ **is constant along the geodesic flow.** The arguments in this section follow the arguments given in the proof of [**HKS**, Lemma 2.2].

Proposition 3.2. *Let (X,g) be asymptotically harmonic. Then for all $v \in SX$, the map $t \mapsto \det(D(\phi^t v))$ is constant.*

Proof. For the proof we need besides $D(v)$ the symmetric tensor $H(v) = -\frac{1}{2}(U(v) + S(v))$. Note that U and therefore also S are solutions of the Ricatti equation

$$\frac{d}{dt}U(\phi^t v) + U(\phi^t v)^2 + R_{\phi^t v} = 0.$$

Hence, a straightforward calculation yields for all $v \in SX$

$$(HD + DH)(\phi^t v) = S(\phi^t v)^2 - U(\phi^t v)^2 = \frac{d}{dt}D(\phi^t v). \tag{3.1}$$

In the case $\det D(\phi^t v) = 0$ for all $t \in \mathbb{R}$, there is nothing to prove. If $\det D(\phi^t v) \neq 0$ for some $t \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{d}{dt}\log\det D(\phi^t v) &= \frac{1}{\det D(\phi^t v)}\operatorname{tr}\left(\left(\frac{d}{dt}D(\phi^t v)\right)D^{-1}(\phi^t v)\right) \\
&= \frac{1}{\det D(\phi^t v)}\operatorname{tr}\big((HD + DH)(\phi^t v)D^{-1}(\phi^t v)\big) \\
&= \frac{2}{\det D(\phi^t v)}\operatorname{tr}\big(H(\phi^t v)\big) = 0,
\end{aligned}$$

since $\operatorname{tr} H(w) = -\frac{1}{2}(\operatorname{tr} U(w) + \operatorname{tr} S(w)) = -\frac{1}{2}(\operatorname{tr} U(w) - \operatorname{tr} U(-w)) = 0$. This implies that $t \mapsto \det D(\phi^t v)$ is constant for all $t \in \mathbb{R}$. q.e.d.

3.2. $\det D(v)$ **is constant along stable and unstable manifolds.** Note that the key ingredients here are Proposition 2.5(b) and Theorem 2.8. We first prove the following lemma.

Lemma 3.3. *Assume there is a constant $R_0 > 0$ such that $\|R\| \le R_0$. Let $v \in SX$. Assume there is a constant $\rho > 0$ such that*

$$D(\phi^t(v)) \ge \rho\,\mathrm{id}$$

for all $t \in \mathbb{R}$. Then there exists a constant $a \ge 1$, depending only on R_0 such that

$$0 < S'_{\phi^t(v)}(0) - S'_{\phi^t(v),r}(0) \le \frac{a}{r} \quad \text{and} \quad 0 < U'_{\phi^t(v),r}(0) - U'_{\phi^t(v)}(0) \le \frac{a}{r}$$

for all $r > 0$ and all $t \in \mathbb{R}$.

Proof. Since we have $U'_{w,r}(0) = -S_{-w,r}(0)$ for all $w \in SX$, it suffices to prove the first assertion. Proposition 2.5(b) yields for all $t \ge 0$

$$\|S_w(t)\| \le a_2 = a_2(R_0, \rho),$$

where $w = \phi^s(v)$ for some $s \in \mathbb{R}$. Recall from [**Kn2**, Lemma 2.3] that

$$S'_w(0) - S'_{w,r}(0) = \left(\int_0^r (S_w^* S_w)^{-1}(u)du\right)^{-1}.$$

This implies for all $x \in S_{c_v(s)}X$

$$\begin{aligned}
\langle (S'_w(0) - S'_{w,r}(0))x, x\rangle &\le \left\|\left(\int_0^r (S_w^* S_w)^{-1}(u)du\right)^{-1}\right\| \\
&\le \left(\int_0^r \|(S_w^* S_w)\|^{-1}(u)du\right)^{-1} \\
&\le \left(\int_0^r \frac{1}{a_2^2}du\right)^{-1} = \frac{a_2^2}{r},
\end{aligned}$$

which yields the required estimate. q.e.d.

Now we assume that (X, g) has rank one, i.e., we have $\det D(w) > 0$ for all $w \in SX$. It suffices to show that $w \mapsto \det D(w)$ is locally constant on $W^s(v)$. Let $v \in SX$ and $\rho > 0$ such that $\det D(v) = 2\rho$. Since $w \mapsto \det D(w)$ is continuous on SX by Lemma 3.1, we find an open neighbourhood $U \subset SX$ of v such that $\det D(w) \ge \rho$ for all $w \in U$. Let $v, \widetilde{v} \in U \cap W^s(v)$ and $\gamma : [0,1] \to U \cap W^s(v)$ be a smooth curve with $\gamma(0) = v$ and $\gamma(1) = \widetilde{v}$. We need to show that for every $\epsilon > 0$ we have $|\det D(v) - \det D(\widetilde{v})| < \epsilon$. We have

$$|\det D(v) - \det D(\widetilde{v})| = |\det D(\phi^t(v)) - \det D(\phi^t(\widetilde{v}))|$$

for all $t \in \mathbb{R}$ and

$$\lim_{t\to\infty} d(\phi^t(v), \phi^t(\widetilde{v})) = 0, \tag{3.2}$$

and the convergence is exponentially because of (2.9) and (2.11).

Since our operators $D(w) = U(w) - S(w) \geq 0$ are uniformly bounded by $2\sqrt{R_0}$ and the determinant is a differentiable function, there is a uniform Lipschitz constant $A > 0$ such that

$$|\det D(w_1) - \det D(w_2)| \leq A \, \|D(w_1) - D(w_2)\|.$$

Therefore, it suffices to show that, for every $\delta > 0$, there exists $t > 0$ such that

$$\|D(\phi^t(v)) - D(\phi^t(\widetilde{v}))\| < \delta. \tag{3.3}$$

Let $D_r(w) = U'_{w,r}(0) - S'_{w,r}(0)$. Lemma 3.3 implies

$$\|D(\phi^t(w)) - D_r(\phi^t(w))\| \leq \frac{2a}{r}$$

for all $w \in \gamma([0,1])$ and $t \in \mathbb{R}$. Therefore, we can choose $r > 1$ large enough such that we have

$$\|D(\phi^t(w)) - D_r(\phi^t(w))\| \leq \frac{\delta}{3}$$

for all $w \in \gamma([0,1])$ and $t \in \mathbb{R}$. This implies that (3.3) holds if

$$\|D_r(\phi^t(v)) - D_r(\phi^t(\widetilde{v}))\| < \frac{\delta}{3}$$

for some $t > 0$. But this is a direct consequence of (3.2) and Theorem 2.8.

This shows that $w \to \det D(w)$ is locally and therefore also globally constant on $W^s(v)$. Note that $w \to \det D(w)$ is also constant on $W^u(v)$: Let $w \in W^u(v)$. Then $-w \in W^s(-v)$ and

$$\det D(w) = \det D(-w) = \det D(-v) = \det D(v).$$

3.3. $\det D(v)$ **is constant on** SX**.** In the case $\det D(v) = 0$ for all $v \in SX$ there is nothing to prove. Therefore, we assume that there exists $v \in SX$ with $\det D(v) \neq 0$.

For $v \in SX$, let

$$\begin{aligned} W^{0s}(v) &= \bigcup_{t\in\mathbb{R}} \phi^t(W^s(v)) = \{-\operatorname{grad} b_v(q) \mid q \in X\}, \\ W^{0u}(v) &= \bigcup_{t\in\mathbb{R}} \phi^t(W^u(v)) = \{\operatorname{grad} b_{-v}(q) \mid q \in X\}. \end{aligned}$$

Observe that $W^{0u}(v) = -W^{0s}(-v)$.

We define a vector $w \in SX$ to be *asymptotic to* $v \in SX$ if $w \in W^{0s}(v)$. Since X has continuous asymptote, being asymptotic is an equivalence relation (see [**Es**, Prop. 3]). We write $v \sim w$ for asymtotic vectors $v, w \in SX$. Note that a flow line $\phi^{\mathbb{R}}(v_1)$ can intersect a leaf $W^u(v_2)$ in at most one vector, since the footpoint sets of these leafs are level sets of Busemann functions and $b_v(\pi(\phi^t(w))) = b_v(\pi(w)) - t$ for asymptotic vectors $v, w \in SX$.

Lemma 3.4. *Let $v, v' \in SX$ with $\det D(v) \neq 0$. Assume that $W^u(v) = W^u(v')$ and $v' \in W^{0s}(v)$. Then $v = v'$.*

Proof. $v' \in W^{0s}(v)$ implies that v and v' are asymptotic. Since $W^u(v) = -W^s(-v)$, $W^u(v) = W^u(v')$ implies that $-v$ and $-v'$ are also asymptotic. Therefore, v and v' are bi-asymptotic. We have $v' \notin \phi^{\mathbb{R}}(v)$, since both v and v' lie in the same unstable manifold $W^u(v)$. By [**Es**, Thm. 1](iv), there exists a central Jacobi field along c_v, i.e., $\ker D(v) \neq 0$. But this contradicts to $\det D(v) \neq 0$. q.e.d.

The assumption $\|R\| \leq R_0$ implies that the intrinsic sectional curvatures of all horospheres are also uniformly bounded in absolute value, by the Gauss equation. Therefore, there exists $\delta > 0$ such that for all horospheres $\mathcal{H}$ and all $p \in \mathcal{H}$, the intrinsic exponential map $\exp_{p,\mathcal{H}} : T_p\mathcal{H} \to \mathcal{H}$ is a diffeomorphism on the ball $B_{p,\mathcal{H}}(\delta) = \{v \in T_p\mathcal{H} \mid \|v\| < \delta\}$.

Assume that $n = \dim X$. Let $v \in SX$ be a fixed vector with $\det D(v) \neq 0$. Now, we define the following continuous map (see Figure 1)

$$\varphi_v : X \times B_\delta(0) \to SX,$$

where $B_\delta(0) = \{y \in \mathbb{R}^{n-1} \mid \|y\| < \delta\}$: Choose a smooth global orthonormal frame $Z_1 = -\operatorname{grad} b_v, Z_2, \ldots, Z_n$ on X. Define

$$\varphi_v(q, y) = \psi^u_{Z_1(q)}\left[\exp_{q,\pi W^u(Z_1(q))}\left(\sum_{i=2}^{n} y_i Z_i(q)\right)\right] \in W^u(Z_1(q)),$$

where $\psi^u_w : \pi W^u(w) \to W^u(w)$ is defined by $\psi^u_w(q) = \operatorname{grad} b_{-w}(q)$.

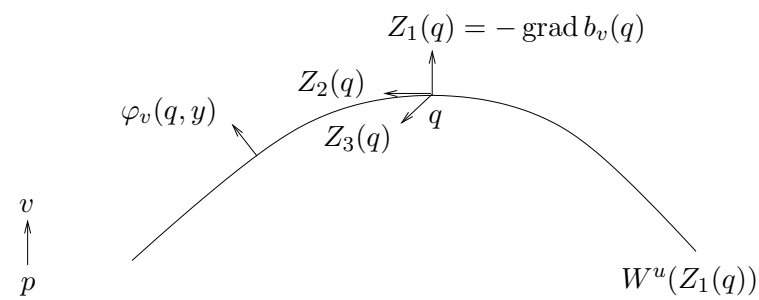

Figure 1. Illustation of the map $\varphi_v : X \times B_\delta(0) \to SX$

We now show that φ_v is injective: Let $\varphi_v(q, y) = \varphi_v(q', y')$. Then $W^u(Z_1(q)) = W^u(Z_1(q'))$ and

$$Z_1(q') = -\operatorname{grad} b_v(q') \sim v \sim -\operatorname{grad} b_v(q) = Z_1(q),$$

which implies $Z_1(q') \in W^{0s}(Z_1(q))$. We conclude from the previous subsections that $\det D(Z_1(q)) = \det D(v) \neq 0$. Using Lemma 3.4, we obtain $Z_1(q) = Z_1(q')$, i.e., $q = q'$. The equality $y = y'$ follows now from the injectivity of the exponential maps and ψ^u_w.

Since $\dim X \times B_\delta(0) = 2n-1 = \dim SX$, we conclude that $U = \varphi_v(X \times B_\delta(0)) \subset SX$ is an open neighborhood of v, by Brouwer's domain invariance. Moreover, $\det D(w) = \det D(v) \neq 0$ for all $w \in U$, using that $\det D$ is constant along unstable manifolds, as well.

Now we are able to prove Theorem 1.3.

Proof. Without loss of generality, we assume that there exists a vector $v_0 \in SX$ with $\det D(v_0) = \alpha \neq 0$. Let $SX_\alpha = \{w \in SX \mid \det D(w) = \alpha\}$. By continuity of $w \mapsto \det D_w$, the set $SX_\alpha \subset SX$ is closed. Since $v_0 \in SX_\alpha$, we know that SX_α is non-empty. The above arguments and Sections 3.1 and 3.2 show for every vector $v \in SX_\alpha$ that the open neighbourhood $\varphi_v(U)$ is contained in SX_α, i.e., SX_α is open. Since SX is connected, we conclude that $SX_\alpha = SX$.

Since $\|R\| \leq R_0$ implies that X has bounded sectional curvature, the second fundamental forms of horospheres are bounded and therefore the eigenvalues of the positive endomorphism $D(v) = U_v'(0) - S_v'(0)$ are also uniformly bounded from above.The rank one assumption implies $\det D(v) = \text{const} > 0$. Both facts together imply that the smallest eigenvalue of $D(v)$ is uniformly bounded from below by a constant $\rho > 0$.

q.e.d.

4. Proof of the equivalences

From now on, we assume that (X, g) is asymptotically harmonic with $\|R\| \leq R_0$ and $\|\nabla R\| \leq R_0'$. Our goal is to prove Theorem 1.5. We prove each of the implications separately.

4.1. Rank one implies Anosov geodesic flow. Observe first that $h = 0$ implies $\operatorname{tr} D(v) = \operatorname{tr} U(v) - \operatorname{tr} S(v) = h - h = 0$. Since $D(v)$ is positive, this implies $D(v) = 0$ and $\det D(v) = 0$ which contradicts to $\operatorname{rank}(X) = 1$. Now we assume that $\operatorname{rank}(X) = 1$ and, therefore, $D(v) \geq \rho > 0$, by Theorem 1.3. By [**Bo**, Theorem, p. 107] this implies that the geodesic flow is Anosov.

4.2. Anosov geodesic flow implies Gromov hyperbolicity. Recall that a geodesic metric space is called *Gromov hyperbolic* if there exists $\delta > 0$ such that every geodesic triangle is δ-thin, i.e., every side of the triangle is contained in the union of the δ-tubular neighborhoods of the other two sides.

Assume now that the geodesic flow $\phi^t : SX \to SX$ is Anosov with respect to the Sasaki metric. For $v \in SX$ consider the normal Jacobi tensor along c_v with $A_v(0) = 0$ and $A_v'(0) = \text{id}$. Then the Anosov property implies (see [**Bo**, p. 113])

$$\|A_v(t)x\| \geq ce^{\alpha t}\|x\|$$

for $t \geq 1$ with suitable constants $c, \alpha > 0$. Consider two distinct geodesic rays $c_1 : [0, \infty) \to X$ and $c_2 : [0, \infty) \to X$ with $c_1(0) = c_2(0) = p$ and

define

$$\begin{aligned} d_t^q(c_1(t), c_2(t)) \quad := \quad & \inf\{L(\gamma) \mid \ \gamma : [a,b] \to X \setminus B(p,t) \\ & \text{a piecewise smooth curve joining } \ c_1(t) \text{ and } c_2(t)\}, \end{aligned}$$

where $B(p,t) = \{q \in X \mid d(p,q) < t\}$. Let $t \geq 1$ and $\gamma : [0,1] \to X \setminus B(p,t)$ be a curve connecting $c_1(t)$ and $c_2(t)$. Let $v_1 = c_1'(0) \in S_pX$ and $v_2 = c_2'(0) \in S_pX$. Then $\gamma(s) = \exp_p(r(s)v(s))$ with $r : [0,1] \to [t, \infty)$ and $v : [0,1] \to S_pX$ and

$$\begin{aligned} \gamma'(s) &= D\exp_p(r(s)v(s))\left(r'(s)v(s) + r(s)v'(s)\right) \\ &= r'(s)c'_{v(s)}(r(s)) + A_v(r(s))v'(s). \end{aligned}$$

Since $c'_{v(s)}(r) \perp A_v(r)v'(s)$, we conclude that

$$\|\gamma'(s)\| \geq ce^{\alpha r(s)}\|v'(s)\|.$$

This implies that

$$L(\gamma) = \int_0^1 \|\gamma'(s)\| ds \geq ce^{\alpha t} \angle(v_1, v_2),$$

and therefore

$$\liminf_{t\to\infty} \frac{\log d_t^q(c_1(t), c_2(t))}{t} \geq c_0\alpha \tag{4.1}$$

with a suitable constant $c_0 > 0$. This implies, using [**BH**, Chapter III, Prop. 1.26] that X is Gromov hyperbolic. (Note that the condition there is $\liminf_{t\to\infty} \frac{d_t^q(c_1(t),c_2(t))}{t} = \infty$, which is a priori weaker than (4.1). In fact, both conditions are equivalent to Gromov hyperbolicity, see [**BH**, Chapter III, Prop. 1.25].)

4.3. Gromov hyperbolicity implies purely exponential volume growth with $h = h_{vol}$. We like to note first that simply connected Riemannian manifolds X without conjugate points which are Gromov hyperbolic spaces *admitting compact quotients* have purely exponential volume growth (see [**Coor**, Thm. 7.2]). Here we consider the special case of an asymptotic harmonic manifold without the additional assumption that X admits a compact quotient.

Definition 4.1. *Let X be a Riemannian manifold with $h_{vol} = h_{vol}(X) > 0$. Then X has* purely exponential volume growth with growth rate h_{vol} *if, for every $p \in X$, there exists a constant $C = C(p) \geq 1$ with*

$$\frac{1}{C}e^{h_{vol}r} \leq \operatorname{vol} B_r(p) \leq Ce^{h_{vol}r} \quad \textit{for all } r \geq 1.$$

We first prove the following general lemma.

Lemma 4.2. *Let X be a δ-hyperbolic space without conjugate points and bounded curvature. Then the volume of any geodesic sphere grows exponentially. In particular, we have $h_{vol}(X) > 0$.*

Proof. Fix $p \in X$ and geodesic rays $c_1, c_2 : [0, \infty) \to X$ with $c_1(0) = c_2(0)$. As remarked above, Gromov hyperbolicity implies

$$\liminf_{t\to\infty} \frac{\log d_t^q(c_1(t), c_2(t))}{t} \geq c(\delta),$$

where $c(\delta) > 0$ depends only on the Gromov constant δ. In particular, there exists $t_0 > 0$ such that for all $t \geq t_0$

$$d_{S_p(t)}(c_1(t), c_2(t)) \geq e^{tc(\delta)/2},$$

where $d_{S_p(t)}$ is the intrinsic distance in the sphere $S_p(t) \subset X$. Let $\gamma_t : [0, l(t)] \to S_p(t)$ be a minimal geodesic in $S_p(t)$ connnecting $c_1(t)$ and $c_2(t)$. The 1/4-balls in $S_p(t)$ with centers $\gamma_t(k)$ and $k \in \mathbb{Z} \cap [0, l(t)]$ are pairwise disjoint. Lemma 2.2 implies that the second fundamental forms of $S_p(t)$ are bounded by a universal constant for all $t \geq t_0 > 0$. Using the Gauss equation, this implies that the curvatures of the spheres $S_p(t)$ are uniformly bounded for $t \geq t_0$, as well. Therefore, the 1/4-balls in $S_p(t)$ have a uniform lower volume bound $A_0 > 0$. Hence, we have

$$\mathrm{vol}(S_t(p)) \geq A_0(e^{tc(\delta)/2} - 1)$$

for all $t \geq t_0$. This finishes the proof of the lemma. q.e.d.

Lemma 4.3. *Let (X, g) be an asymptotically harmonic manifold. Then, for all $p \in X$, there exists a constant $C_1(p) > 0$ such that*

$$\frac{\mathrm{vol}\, S_r(p)}{e^{hr}} \geq C_1(p) \quad \textit{for all } r \geq 1.$$

In particular, we have

$$h \leq h_{vol}(X).$$

Proof. As in the proof of [**Kn2**, Cor. 25], we have for all $v \in SX$

$$\frac{\det A_v(t)}{\det U_v(t)} = \frac{\det A_v(t)}{e^{ht}} = \frac{1}{\det(U(v) - S'_{v,t}(0))},$$

which implies

$$\frac{\mathrm{vol}\, S_r(p)}{e^{hr}} = \int_{S_pX} \frac{1}{\det(U(v) - S'_{v,r}(0))} d\theta_p(v). \tag{4.2}$$

Using $U(v) - S'_{v,t_1}(0) \geq U(v) - S'_{v,t_2}(0) > 0$ for all $0 < t_1 < t_2$, we obtain

$$\frac{\mathrm{vol}\, S_r(p)}{e^{hr}} \geq \int_{S_pX} \frac{1}{\det(U(v) - S'_{v,1}(0))} d\theta_p(v).$$

Continuous asymptote implies the continuity of $v \mapsto U(v) - S'_{v,1}(0)$. This yields the existence of a constant $a > 0$ such that $\det(U(v) - S'_{v,1}(0)) \geq a$ for all $v \in S_pX$ and implies the statement in the lemma. q.e.d.

Recall the following result in [**Kn2**, Cor. 4.6].

Proposition 4.4. *Let X be a simply connected δ-hyperbolic manifold without conjugate points. Consider for $v \in S_pX$, $\ell = \delta + 1$ and $r > 0$ the spherical cone in X given by*

$$A_{v,\ell}(r) := \{c_w(t) \mid 0 \le t \le r, w \in S_pX, d(c_v(\pm\ell), c_w(\pm\ell)) \le 1\}.$$

Then, for $\rho = 4\delta + 2$ the set $A_{v,\ell}(r)$ is contained in

$$H_{v,\rho}(r) := \{c_q(t) \mid -\rho/2 \le t \le r,\ c_q \text{ is an integral curve of } \operatorname{grad} b_{-v} \text{ with } c_q(0) = q \in b_{-v}^{-1}(0) \cap B(p,\rho)\}.$$

This useful result has the following consequence.

Corollary 4.5. *Let (X, g) be a Gromov hyperbolic asymptotically harmonic manifold and $p \in X$. Then there exists a constant $C_2(p) > 0$ such that*

$$\operatorname{vol} B_r(p) \le C_2(p) \int_{-\rho/2}^{r} e^{hs} ds,$$

where ρ is defined as in Proposition 4.4. In particular, we have

$$h_{vol}(X) \le h.$$

Proof. Let $p \in X$. Choose $l = \delta + 1$. Then we have

$$S_pX = \bigcup_{v \in S_pX} U_{v,\ell}(r),$$

with the open sets

$$U_{v,\ell}(r) = \{w \in S_pX \mid d(c_v(\pm\ell), c_w(\pm\ell)) < 1\}.$$

Since S_pX is compact, we find finitely many vectors $v_1, \dots, v_k \in S_pX$ with

$$S_pX = \bigcup_{j=1}^{k} U_{v_j,\ell}(r),$$

which implies for $\rho = 4\delta + 2$

$$B_r(p) \subset \bigcup_{j=1}^{k} A_{v_j,\ell}(r) \subset \bigcup_{j=1}^{k} H_{v_j,\rho}(r).$$

Using

$$\operatorname{vol}(H_{v,\rho}(r)) = \int_{-\rho/2}^{r} e^{hs} ds \operatorname{vol}_0(b_v^{-1}(0) \cap B_\rho(p))$$

where vol_0 denotes the induced volume on the horosphere $b_v^{-1}(0)$, we conclude

$$\operatorname{vol} B_r(p) \le \left(\sum_{j=1}^{k} \operatorname{vol}_0(b_{v_j}^{-1}(0) \cap B_\rho(p)) \right) \int_{-\rho/2}^{r} e^{hs} ds.$$

Setting $C_2(p) = \sum_{j=1}^k \mathrm{vol}_0(b_{v_j}^{-1}(0) \cap B_\rho(p))$ proves the first part of the corollary. The inequality $h_{vol}(X) \leq h$ follows then from the definition of $h_{vol}(X)$. q.e.d.

Now we prove the implication claimed in this subsection.

Proposition 4.6. *Let (X, g) be a Gromov hyperbolic asymptotically harmonic space with with bounded curvature. Then X has purely exponential volume growth with $h = h_{vol}$.*

Proof. Gromov hyperbolicity implies $h_{vol}(X) > 0$, by Lemma 4.2. Lemma 4.3 and Corollary 4.5 together yield $h = h_{vol}(X)$. Moreover, we derive from Corollary 4.5 that

$$\mathrm{vol}\, B_r(p) \leq \frac{C_2(p)}{h} e^{hr}.$$

The lower volume estimate follows from Lemma 4.3: For $r \geq 2$ we have

$$\frac{\mathrm{vol}\, B_r(p)}{e^{hr}} \geq \frac{\int_{r-1}^r \mathrm{vol}\, S_t(p) dt}{e^{hr}} \geq \frac{\mathrm{vol}\, S_{t_0}(p)}{e^{hr}} \geq \frac{C_1(p)}{e^h},$$

for some $t_0 \in [r-1, r]$. This finishes the proof of purely exponential volume growth. q.e.d.

4.4. Purely exponential volume growth with $h = h_{vol}$ implies rank one. Finally, we show the remaining implication of Theorem 1.5. This closes the chain of implications and finishes the proof that all four properties listed in (a), (b), (c) and (d) are equivalent.

Assume that (X, g) is asymptotically harmonic with purely exponential volume growth $h = h_{vol}$. We have

$$\int_{r-1}^r e^{ht} \int_{S_pX} \frac{1}{\det(U(v) - S'_{v,t}(0))} d\theta_p(v) dt \leq \mathrm{vol}(B_r(p)).$$

This implies

$$\frac{1}{e} \int_{r-1}^r \int_{S_pX} \frac{1}{\det(U(v) - S'_{v,t}(0))} d\theta_p(v) dt \leq \frac{\mathrm{vol}(B_r(p))}{e^{hr}} \leq C(p)$$

for some constant $C(p) > 0$. Assume that $\det(U(v) - S'_{v,t}(0)) \to 0$ for all $v \in S_pX$. Then, because of monotonicity and Dini, we know that this convergence is uniform. This is in contradiction to the above inequality. Therefore, there exist $v \in S_pX$ with $\det(U(v) - S(v)) \neq 0$ and (X, g) has rank one.

5. Asymptotically harmonic manifolds with bounded asymptote

The notion of bounded asymptote was first introduced by Eschenburg in [**Es**, Section 4]. Examples of manifolds of bounded asymptote are manifolds with nonpositive curvature or, more generally, manifolds with no focal points.

Definition 5.1. *Let (X, g) be a complete, simply connected Riemannian manifold without conjugate points. X is called a manifold* of bounded asymptote *if there exists a constant $A \geq 1$ such that*

$$\|S_v(t)\| \leq A \quad \forall\, t \geq 0, \quad \forall\, v \in SX. \tag{5.1}$$

Lemma 5.2. *The bounded asymptote property* (5.1) *implies*

$$\|U_v(t)\| \geq \frac{1}{A} \quad \forall\, t \geq 0, \quad \forall\, v \in SX.$$

Proof. Letting $x \to \infty$, we conclude from (2.6)

$$S_{\phi^t v}(y) = S_v(y+t) S_v^{-1}(t).$$

Using $S_v(t) = U_{-v}(-t)$, we obtain

$$S_{-\phi^t v}(s) = U_v(t-s) U_v^{-1}(t).$$

This implies

$$1 = \|S_{-\phi^t v}(t) U_v(t)\| \leq A\, \|U_v(t)\|,$$

finishing the proof. q.e.d.

Remark. *Rank one asymptotically harmonic manifolds with $\|R\| \leq R_0$ and $\|\nabla R\| \leq R_0'$ are manifolds of bounded asymptote by Proposition* 2.5.

Next, we discuss relations between the geometrically defined constants $h, h_{vol}(X)$ and the *Cheeger constant* $h_{Cheeg}(X)$, defined as

$$h_{Cheeg}(X) = \inf_{K \subset X} \frac{\operatorname{area}(\partial K)}{\operatorname{vol}(K)},$$

where K ranges over all connected, open submanifolds of X with compact closure and smooth boundary.

Proposition 5.3. *Let (X, g) be an asymptotically harmonic manifold. Then we have*

$$h_{vol}(X), h_{Cheeg}(X) \geq h.$$

Proof. The inequality $h_{vol}(X) \geq h$ was already stated in Lemma 4.3. For the proof of $h_{Cheeg}(X) \geq h$ let $K \subset X$ be a set as described above. Choosing a Busemann function b_v, we have $\Delta b_v = h$ and obtain via Gauss' divergence theorem and $\|\operatorname{grad} b_v\| = 1$,

$$h \operatorname{vol}(K) = \int_K \Delta b_v(x) dx = \int_{\partial K} \langle \operatorname{grad} b_v, \nu \rangle dx \leq \operatorname{area}(\partial K),$$

where ν is the outward unit normal vector of ∂K in X. q.e.d.

Even though we proved in the previous section that $h = h_{vol}(X)$ for Gromov hyperbolic asymptotically harmonic spaces X with bounded curvature, we do not know whether this holds for general asymptotically harmonic manifolds. However, a sufficient condition for $h =$

$h_{vol}(X) = h_{Cheeg}(X)$ is that X is asymptotically harmonic and has bounded asymptote.

Theorem 5.4. *Let (X, g) be asymptotically harmonic and of bounded asymptote. Then we have*

$$h = h_{vol}(X) = h_{Cheeg}(X).$$

In particular, this equality holds for all rank one asymptotically harmonic manifolds with $\|R\| \leq R_0$ and $\|\nabla R\| \leq R_0'$.

Proof. The bounded asymptote property implies that we have

$$\frac{1}{A^2 t} \leq \langle (U(v) - S'_{v,t}(0))x, x\rangle,$$

for all unit vectors $x \in v^\perp$ (see the proof of [**Kn2**, Prop. 5.2]). This implies

$$\det(U(v) - S'_{v,t}(0)) \geq \frac{1}{A^{2n-2}t^{n-1}}.,$$

and we obtain with (4.2)

$$\frac{\operatorname{vol} S_r(p)}{e^{hr}} \leq \int_{S_pX} A^{2n-2} r^{n-1} d\theta_p(v) = \omega_{n-1} A^{2n-2} r^{n-1},$$

where ω_{n-1} is the volume of the Euclidean unit sphere of dimension $n-1$.This implies $\operatorname{vol} S_r(p) \leq C r^{n-1} e^{hr}$ and, therefore, $h_{vol}(X) \leq h$. Together with Lemma 4.3 we obtain $h = h_{vol}(X)$.

Next we prove $h_{Cheeg}(X) \leq h$: Let $g(r) = \frac{\operatorname{vol} S_r(p)}{\operatorname{vol} B_r(p)}$. We will show that $g(r) \to h$ for $r \to \infty$ which implies $h_{Cheeg}(X) \leq h$. We have with l'Hospital

$$\begin{aligned}
\lim_{r\to\infty} g(r) &= \lim_{r\to\infty} \frac{\int_{S_pX} \det A_v(r)\, d\theta_p(v)}{\int_0^r \int_{S_pX} \det A_v(s)\, d\theta_p(v)\, ds} \\
&= \lim_{r\to\infty} \frac{\int_{S_pX} \operatorname{tr}(A_v'(r) A_v^{-1}(r)) \det A_v(r)\, d\theta_p(v)}{\int_{S_pX} \det A_v(r)\, d\theta_p(v)},
\end{aligned}$$

provided the last limit exists. (We used the notion $d\theta$ for the canonical volume element of the unit sphere S_pX). Note that $A_v'(r)A_v^{-1}(r) = U'_{\phi^r v,r}(0)$. Since $\|U'_{w,r}(0) - U'_w(0)\| \leq \frac{A^2}{r}$ (see, for instance, [**Kn2**, bottom of p. 686]), we conclude

$$0 \leq \operatorname{tr} U'_{w,r}(0) - h \leq (n-1)\frac{A^2}{r}$$

for all $w \in SX$ and $r \geq 0$. Therefore, $\operatorname{tr}(A_v'(r)A_v^{-1}(r)) \to h$ and the convergence is uniformly, which implies that the last limit above exists and is equal to h. This, together with Proposition 5.3 above, implies that $h_{Cheeg}(X) = h$. q.e.d.

Remark It was shown by Zimmer in the proof of [**Zi**, Cor. 49] that $h_{vol}(X) = h$ also holds in the case that (X, g) is asymptotically harmonic *admitting compact quotients*. Equality of $h, h_{vol}(X)$ and $h_{Cheeg}(X)$ also holds for all *noncompact harmonic manifolds* X without additional conditions (see [**PeSa**, Theorem 5.1]). Moreover, the agreement of these three geometric constants implies (see [**PeSa**, Corollary 5.2]) that the bottom of the spectrum and of the essential spectrum of the Laplacian Δ_X coincide and $\lambda_0(X) = \lambda_0^{\text{ess}}(X) = \frac{h^2}{4}$.

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