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Solving the BFKL equation in the next-to-leading approximation

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Abstract

The Balitsky–Fadin–Kuraev–Lipatov equation in the next-to-leading logarithmic approximation is solved using an iterative method. We derive the solution for forward scattering with all conformal spins. A discussion of the infrared finiteness of the results is included.

1. Introduction

The Balitsky–Fadin–Kuraev–Lipatov (BFKL) [1] formalism enables the resummation of logarithms appearing in scattering amplitudes, which are large in the Regge limit, where the center of mass energy $\sqrt{s}$ is large and the momentum transfer $\sqrt{-t}$ fixed. In this approach the high energy cross-section for the process $A + B \to A' + B'$ is factorised as

$$\sigma(s) = \int \frac{d^2\mathbf{k}_a}{2\pi \mathbf{k}_a^2} \int \frac{d^2\mathbf{k}_b}{2\pi \mathbf{k}_b^2} \Phi_A(\mathbf{k}_a)\Phi_B(\mathbf{k}_b) f(\mathbf{k}_a, \mathbf{k}_b, \Delta = \ln s/s_0), \tag{1}$$

where $\Phi_{A,B}$ are process-dependent impact factors and $f(\mathbf{k}_a, \mathbf{k}_b, \Delta)$ is the process-independent gluon Green's function describing the interaction between two Reggeised gluons exchanged in the t-channel with transverse momenta $\mathbf{k}_{a,b}$. In this Letter we use the Regge scale $s_0 = |\mathbf{k}_a||\mathbf{k}_b|$, a different choice would modify the impact factors in such a way that the prediction for the cross-section remains unchanged.

The representation in (1) is valid [2] in the leading logarithmic approximation (LLA), where terms of the form $(\alpha_s \Delta)^n$ are resummed, and in the next-to-leading logarithmic approximation (NLLA) [3], where contributions of the type $\alpha_s(\alpha_s \Delta)^n$ are also taken into account. This formalism is valid in both the forward and non-forward cases [4]. In this Letter we deal with the former but the proposed method is also applicable to the latter.

In recent years there have been many studies of the behaviour of the gluon Green's function in the NLLA [5]. This Green's function is obtained as the solution of an integral equation where radiative corrections enter through its kernel. The main difficulty in solving the equation analytically in the NLLA stems from the logarithmic dependence

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introduced by the running of the coupling. In this work we show how it is possible to solve the equation using an iterative method directly in energy space, considering the full kernel with scale invariant and running coupling terms. In this approach we keep all the angular information from the BFKL evolution, solving the equation for a general conformal spin without relying on any asymptotic expansion.

In Section 2 we present the BFKL equation in the NLLA in dimensional regularisation and show how, introducing a cut-off in the real emissions, it is possible to write it in a form suitable for iteration. In Section 3 we solve the equation using an iterative method and compare with previous results in the literature. In Section 4 we present our conclusions.

2. The BFKL equation in the NLLA

To write the BFKL equation for the gluon Green's function, $f(\mathbf{k}_a, \mathbf{k}_b, \Delta)$, it is convenient to introduce the Mellin transform, $f_\omega(\mathbf{k}_a, \mathbf{k}_b)$, in Δ space as follows

$$f(\mathbf{k}_a, \mathbf{k}_b, \Delta) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\omega\, e^{\omega\Delta} f_\omega(\mathbf{k}_a, \mathbf{k}_b). \tag{2}$$

In this way the BFKL equation in the NLLA for forward scattering can be written in dimensional regularisation $(D = 4 + 2\epsilon)$ as

$$\omega f_\omega(\mathbf{k}_a, \mathbf{k}_b) = \delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}_b) + \int d^{2+2\epsilon}\mathbf{k}'\, \mathcal{K}(\mathbf{k}_a, \mathbf{k}') f_\omega(\mathbf{k}', \mathbf{k}_b). \tag{3}$$

The kernel in the NLLA is expressed in terms of the gluon Regge trajectory [6], $\omega^{(\epsilon)}(\mathbf{k}_a^2)$, and the real emission component [7], $\mathcal{K}_r(\mathbf{k}_a, \mathbf{k})$, in such a way that

$$\mathcal{K}(\mathbf{k}_a, \mathbf{k}) = 2\omega^{(\epsilon)}\big(\mathbf{k}_a^2\big)\delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}) + \mathcal{K}_r(\mathbf{k}_a, \mathbf{k}). \tag{4}$$

It is convenient to split the kernel $\mathcal{K}_r$ into two parts: a ϵ-dependent, $\mathcal{K}_r^{(\epsilon)}$, and a ϵ-independent, $\widetilde{\mathcal{K}}_r$. In the integral over real emission we also perform a shift in the variable of integration over transverse degrees of freedom of the form $\mathbf{k} = \mathbf{k}' - \mathbf{k}_a$ to write

$$\begin{aligned}
\omega f_\omega(\mathbf{k}_a, \mathbf{k}_b) = {} & \delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}_b) + \int d^{2+2\epsilon}\mathbf{k}\, 2\omega^{(\epsilon)}\big(\mathbf{k}_a^2\big)\delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}) f_\omega(\mathbf{k}, \mathbf{k}_b) \\
& + \int d^{2+2\epsilon}\mathbf{k}\, \mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k}) f_\omega(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b) \\
& + \int d^{2+2\epsilon}\mathbf{k}\, \widetilde{\mathcal{K}}_r(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k}) f_\omega(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b).
\end{aligned} \tag{5}$$

In order to explicitly show the cancellation of the poles at $\epsilon = 0$ we split the integral over transverse phase space for $\mathcal{K}_r^{(\epsilon)}$ into two regions separated by a cut-off λ, i.e.,

$$\begin{aligned}
\omega f_\omega(\mathbf{k}_a, \mathbf{k}_b) = {} & \delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}_b) + \int d^{2+2\epsilon}\mathbf{k}\, 2\omega^{(\epsilon)}\big(\mathbf{k}_a^2\big)\delta^{(2+2\epsilon)}(\mathbf{k}_a - \mathbf{k}) f_\omega(\mathbf{k}, \mathbf{k}_b) \\
& + \int d^{2+2\epsilon}\mathbf{k}\, \mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k})\big(\theta(\mathbf{k}^2 - \lambda^2) + \theta(\lambda^2 - \mathbf{k}^2)\big) f_\omega(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b) \\
& + \int d^{2+2\epsilon}\mathbf{k}\, \widetilde{\mathcal{K}}_r(\mathbf{k}_a, \mathbf{k}_a + \mathbf{k}) f_\omega(\mathbf{k}_a + \mathbf{k}, \mathbf{k}_b).
\end{aligned} \tag{6}$$

For any finite λ we now introduce an arbitrary dependence of $\mathcal{O}(\lambda^2/\mathbf{k}_a^2)$ by using the approximation $f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)\simeq f_\omega(\mathbf{k}_a,\mathbf{k}_b)$ for $|\mathbf{k}|<\lambda$. This dependence is negligible when the external scale $\mathbf{k}_a^2$ is large. It is possible to introduce the cut-off in several different ways but all the choices must have the same $\lambda\to 0$ limit. In this way we can approximate Eq. (6) by

$$\omega f_\omega(\mathbf{k}_a,\mathbf{k}_b)=\delta^{(2+2\epsilon)}(\mathbf{k}_a-\mathbf{k}_b)+\left\{2\omega^{(\epsilon)}\left(\mathbf{k}_a^2\right)+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\theta(\lambda^2-\mathbf{k}^2)\right\}f_\omega(\mathbf{k}_a,\mathbf{k}_b)$$
$$+\int d^{2+2\epsilon}\mathbf{k}\left\{\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\theta(\mathbf{k}^2-\lambda^2)+\widetilde{\mathcal{K}}_r(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\right\}f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b). \quad (7)$$

In dimensional regularisation the gluon Regge trajectory can be written as [3]

$$2\omega^{(\epsilon)}(\mathbf{q}^2)=-\bar{\alpha}_s\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\left(\frac{1}{\epsilon}+\ln\frac{q^2}{\mu^2}\right)$$
$$-\frac{\bar{\alpha}_s^2}{8}\frac{\Gamma^2(1-\epsilon)}{(4\pi)^{2\epsilon}}\left\{\frac{\beta_0}{N_c}\left(\frac{1}{\epsilon^2}+\ln^2\frac{q^2}{\mu^2}\right)+\left(\frac{4}{3}-\frac{\pi^2}{3}+\frac{5}{3}\frac{\beta_0}{N_c}\right)\left(\frac{1}{\epsilon}+2\ln\frac{q^2}{\mu^2}\right)\right.$$
$$\left.-\frac{32}{9}+2\zeta(3)-\frac{28}{9}\frac{\beta_0}{N_c}\right\}, \quad (8)$$

where $\beta_0\equiv\frac{11}{3}N_c-\frac{2}{3}n_f$, $\bar{\alpha}_s\equiv\frac{\alpha_s(\mu)N_c}{\pi}$ and $\alpha_s(\mu)=\frac{g_\mu^2}{4\pi}$. μ is the renormalisation scale in the $\overline{\mathrm{MS}}$ scheme.

The ϵ-dependent part of the real emission kernel reads [3]

$$\mathcal{K}_r^{(\epsilon)}(\mathbf{q},\mathbf{q}+\mathbf{k})=\frac{\bar{\alpha}_s\mu^{-2\epsilon}}{\pi^{1+\epsilon}(4\pi)^\epsilon}\frac{1}{\mathbf{k}^2}\left\{1+\frac{\bar{\alpha}_s}{4}\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\left[\frac{\beta_0}{N_c}\frac{1}{\epsilon}\left(1-\left(\frac{\mathbf{k}^2}{\mu^2}\right)^\epsilon\left(1-\epsilon^2\frac{\pi^2}{6}\right)\right)\right.\right.$$
$$\left.\left.+\left(\frac{\mathbf{k}^2}{\mu^2}\right)^\epsilon\left(\frac{4}{3}-\frac{\pi^2}{3}+\frac{5}{3}\frac{\beta_0}{N_c}+\epsilon\left(-\frac{32}{9}+14\zeta(3)-\frac{28}{9}\frac{\beta_0}{N_c}\right)\right)\right]\right\}. \quad (9)$$

For our purposes we are interested in the integration of this piece of the kernel over the phase space limited by the cut-off, i.e.,

$$\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}(\mathbf{q},\mathbf{q}+\mathbf{k})\theta(\lambda^2-\mathbf{k}^2)$$
$$=\frac{1}{\Gamma(1+\epsilon)}\frac{\bar{\alpha}_s}{(4\pi)^\epsilon}\frac{1}{\epsilon}\left(\frac{\lambda^2}{\mu^2}\right)^\epsilon\left\{1+\frac{\bar{\alpha}_s}{4}\frac{\Gamma(1-\epsilon)}{(4\pi)^\epsilon}\left[\frac{\beta_0}{N_c}\frac{1}{\epsilon}\left(1-\frac{1}{2}\left(\frac{\lambda^2}{\mu^2}\right)^\epsilon\left(1-\epsilon^2\frac{\pi^2}{6}\right)\right)\right.\right.$$
$$\left.\left.+\frac{1}{2}\left(\frac{\lambda^2}{\mu^2}\right)^\epsilon\left(\frac{4}{3}-\frac{\pi^2}{3}+\frac{5}{3}\frac{\beta_0}{N_c}+\epsilon\left(-\frac{32}{9}+14\zeta(3)-\frac{28}{9}\frac{\beta_0}{N_c}\right)\right)\right]\right\}. \quad (10)$$

When this term is combined with the trajectory, the poles in ϵ cancel and we obtain a finite expression depending on λ:

$$\omega_0(\mathbf{q}^2,\lambda^2)\equiv\lim_{\epsilon\to 0}\left\{2\omega^{(\epsilon)}(\mathbf{q}^2)+\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}(\mathbf{q},\mathbf{q}+\mathbf{k})\theta(\lambda^2-\mathbf{k}^2)\right\}$$
$$=-\bar{\alpha}_s\left\{\ln\frac{\mathbf{q}^2}{\lambda^2}+\frac{\bar{\alpha}_s}{4}\left[\frac{\beta_0}{2N_c}\ln\frac{\mathbf{q}^2}{\lambda^2}\ln\frac{\mu^4}{\mathbf{q}^2\lambda^2}+\left(\frac{4}{3}-\frac{\pi^2}{3}+\frac{5}{3}\frac{\beta_0}{N_c}\right)\ln\frac{\mathbf{q}^2}{\lambda^2}-6\zeta(3)\right]\right\}. \quad (11)$$

To simplify our formulae we introduce the notation

$$\omega_0(\mathbf{q}^2,\lambda^2)\equiv-\xi\left(|\mathbf{q}|\lambda\right)\ln\frac{\mathbf{q}^2}{\lambda^2}+\eta \quad (12)$$

with

$$\xi(\mathrm{X}) \equiv \bar{\alpha}_s + \frac{\bar{\alpha}_s^2}{4}\left[\frac{4}{3} - \frac{\pi^2}{3} + \frac{5}{3}\frac{\beta_0}{N_c} - \frac{\beta_0}{N_c}\ln\frac{\mathrm{X}}{\mu^2}\right], \tag{13}$$

$$\eta \equiv \bar{\alpha}_s^2 \frac{3}{2}\zeta(3). \tag{14}$$

In this way we can write

$$\lim_{\epsilon\to 0}\int d^{2+2\epsilon}\mathbf{k}\,\mathcal{K}_r^{(\epsilon)}(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\theta(\mathbf{k}^2-\lambda^2) f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)$$
$$= \int d^2\mathbf{k}\,\frac{1}{\pi\mathbf{k}^2}\xi(\mathbf{k}^2)\theta(\mathbf{k}^2-\lambda^2) f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b). \tag{15}$$

With these conventions, the forward BFKL equation in the NLLA can finally be written as

$$\big(\omega - \omega_0(\mathbf{k}_a^2,\lambda^2)\big) f_\omega(\mathbf{k}_a,\mathbf{k}_b)$$
$$= \delta^{(2)}(\mathbf{k}_a-\mathbf{k}_b) + \int d^2\mathbf{k}\left(\frac{1}{\pi\mathbf{k}^2}\xi(\mathbf{k}^2)\theta(\mathbf{k}^2-\lambda^2) + \widetilde{\mathcal{K}}_r(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k})\right) f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b), \tag{16}$$

where [3]

$$\widetilde{\mathcal{K}}_r(\mathbf{q},\mathbf{q}') = \frac{\bar{\alpha}_s^2}{4\pi}\Bigg\{-\frac{1}{(\mathbf{q}-\mathbf{q}')^2}\ln^2\frac{\mathbf{q}^2}{\mathbf{q}'^2} + \left(1+\frac{n_f}{N_c^3}\right)\left(\frac{3(\mathbf{q}\cdot\mathbf{q}')^2 - 2\mathbf{q}^2\mathbf{q}'^2}{16\mathbf{q}^2\mathbf{q}'^2}\right)$$
$$\times\left(\frac{2}{\mathbf{q}^2} + \frac{2}{\mathbf{q}'^2} + \left(\frac{1}{\mathbf{q}'^2} - \frac{1}{\mathbf{q}^2}\right)\ln\frac{\mathbf{q}^2}{\mathbf{q}'^2}\right)$$
$$-\left(3 + \left(1+\frac{n_f}{N_c^3}\right)\left(1 - \frac{(\mathbf{q}^2+\mathbf{q}'^2)^2}{8\mathbf{q}^2\mathbf{q}'^2} - \frac{(2\mathbf{q}^2\mathbf{q}'^2 - 3\mathbf{q}^4 - 3\mathbf{q}'^4)}{16\mathbf{q}^4\mathbf{q}'^4}(\mathbf{q}\cdot\mathbf{q}')^2\right)\right)$$
$$\times\int_0^\infty dx\,\frac{1}{\mathbf{q}^2 + x^2\mathbf{q}'^2}\ln\left|\frac{1+x}{1-x}\right|$$
$$+\frac{2(\mathbf{q}^2-\mathbf{q}'^2)}{(\mathbf{q}-\mathbf{q}')^2(\mathbf{q}+\mathbf{q}')^2}\left(\frac{1}{2}\ln\frac{\mathbf{q}^2}{\mathbf{q}'^2}\ln\frac{\mathbf{q}^2\mathbf{q}'^2(\mathbf{q}-\mathbf{q}')^4}{(\mathbf{q}^2+\mathbf{q}'^2)^4} + \left(\int_0^{-\mathbf{q}^2/\mathbf{q}'^2} - \int_0^{-\mathbf{q}'^2/\mathbf{q}^2}\right)dt\,\frac{\ln(1-t)}{t}\right)$$
$$-\left(1 - \frac{(\mathbf{q}^2-\mathbf{q}'^2)^2}{(\mathbf{q}-\mathbf{q}')^2(\mathbf{q}+\mathbf{q}')^2}\right)\left(\left(\int_0^1 - \int_1^\infty\right)dz\,\frac{1}{(\mathbf{q}'-z\mathbf{q})^2}\ln\frac{(z\mathbf{q})^2}{\mathbf{q}'^2}\right)\Bigg\}. \tag{17}$$

It is important to note that this kernel contains the full angular information in the BFKL evolution. Writing the equation as in (16) is very natural in the sense that there is a clear separation between the virtual contributions on the left-hand side, and the real emissions, integrated over phase space, on the right-hand side.

To study the dependence on λ of Eq. (16) we take the derivative with respect to λ^2 of the λ-dependent terms, i.e.,

$$\frac{\partial}{\partial\lambda^2}\left\{\omega_0(\mathbf{k}_a^2,\lambda^2) f_\omega(\mathbf{k}_a,\mathbf{k}_b) + \int d^2\mathbf{k}\,\frac{1}{\pi\mathbf{k}^2}\xi(\mathbf{k}^2)\theta(\mathbf{k}^2-\lambda^2) f_\omega(\mathbf{k}_a+\mathbf{k},\mathbf{k}_b)\right\}$$
$$= \frac{1}{\lambda^2}\xi(\lambda^2) f_\omega(\mathbf{k}_a,\mathbf{k}_b) - \frac{1}{2\pi\lambda^2}\int_0^{2\pi} d\theta\,\xi(\lambda^2) f_\omega(\mathbf{k}_a+\underline{\lambda},\mathbf{k}_b). \tag{18}$$

For a sufficiently smooth $f_\omega(\mathbf{k}_a,\mathbf{k}_b)$ this expression is small in the $\lambda \to 0$ limit. In fact, the approximation made in Eq. (7) is only good for a smooth $f_\omega(\mathbf{k}_a,\mathbf{k}_b)$, and the λ-dependence can ultimately be studied numerically.

3. Iterative solution in the NLLA

The BFKL equation in the NLLA for the forward case as written in Eq. (16) can be solved using an iterative procedure similar to the one applied in [8] for the LLA. We will, in the following, show how this works in the NLLA. In the NLLA it becomes meaningful to study the dependence on the renormalisation scale. Since there are in principle many different physical scales in the problem, there are many possible choices also for the renormalisation scale. Here we will assume that the renormalisation scale is chosen to depend on the arguments of the Green's function only

$$\mu = \mu\big(\mathbf{k}_a^2, \mathbf{k}_b^2\big). \tag{19}$$

Other choices are possible and can be studied in this formalism. The μ-dependence of the trajectory and the kernel will be explicitly shown in all of our expressions from now on. It is also convenient to use the following notation for the kernel

$$\widehat{\mathcal{K}}_r\big(\mathbf{k}_a, \mathbf{k}_a+\mathbf{k}, \lambda^2, \mu\big(\mathbf{k}_a^2, \mathbf{k}_b^2\big)\big) \equiv \frac{1}{\pi \mathbf{k}^2}\xi(\mathbf{k}^2)\theta(\mathbf{k}^2-\lambda^2) + \widetilde{\mathcal{K}}_r(\mathbf{k}_a, \mathbf{k}_a+\mathbf{k}). \tag{20}$$

Using this notation we can take the ω-dependence to the right-hand side of Eq. (16), i.e.,

$$f_\omega(\mathbf{k}_a,\mathbf{k}_b) = \frac{1}{\omega - \omega_0(\mathbf{k}_a^2, \lambda^2, \mu(\mathbf{k}_a^2, \mathbf{k}_b^2))} \times \left\{\delta^{(2)}(\mathbf{k}_a-\mathbf{k}_b) + \int d^2\mathbf{k}\, \widehat{\mathcal{K}}_r\big(\mathbf{k}_a, \mathbf{k}_a+\mathbf{k}, \lambda^2, \mu\big(\mathbf{k}_a^2, \mathbf{k}_b^2\big)\big) f_\omega(\mathbf{k}_a+\mathbf{k}, \mathbf{k}_b)\right\}, \tag{21}$$

and iterate this expression. This procedure leads to

$$\begin{aligned} f_\omega(\mathbf{k}_a,\mathbf{k}_b) = {} & \frac{\delta^{(2)}(\mathbf{k}_a-\mathbf{k}_b)}{\omega - \omega_0(\mathbf{k}_a^2, \lambda^2, \mu(\mathbf{k}_a^2, \mathbf{k}_b^2))} \\ & + \int d^2\mathbf{k}_1 \frac{\widehat{\mathcal{K}}_r(\mathbf{k}_a, \mathbf{k}_a+\mathbf{k}_1, \lambda^2, \mu(\mathbf{k}_a^2, \mathbf{k}_b^2))}{\omega - \omega_0(\mathbf{k}_a^2, \lambda^2, \mu(\mathbf{k}_a^2, \mathbf{k}_b^2))} \frac{\delta^{(2)}(\mathbf{k}_a+\mathbf{k}_1-\mathbf{k}_b)}{\omega - \omega_0((\mathbf{k}_a+\mathbf{k}_1)^2, \lambda^2, \mu((\mathbf{k}_a+\mathbf{k}_1)^2, \mathbf{k}_b^2))} \\ & + \int d^2\mathbf{k}_1 \frac{\widehat{\mathcal{K}}_r(\mathbf{k}_a, \mathbf{k}_a+\mathbf{k}_1, \lambda^2, \mu(\mathbf{k}_a^2, \mathbf{k}_b^2))}{\omega - \omega_0(\mathbf{k}_a^2, \lambda^2, \mu(\mathbf{k}_a^2, \mathbf{k}_b^2))} \\ & \times \int d^2\mathbf{k}_2 \frac{\widehat{\mathcal{K}}_r(\mathbf{k}_a+\mathbf{k}_1, \mathbf{k}_a+\mathbf{k}_1+\mathbf{k}_2, \lambda^2, \mu((\mathbf{k}_a+\mathbf{k}_1)^2, \mathbf{k}_b^2))}{\omega - \omega_0((\mathbf{k}_a+\mathbf{k}_1)^2, \lambda^2, \mu((\mathbf{k}_a+\mathbf{k}_1)^2, \mathbf{k}_b^2))} \\ & \quad \times \frac{\delta^{(2)}(\mathbf{k}_a+\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_b)}{\omega - \omega_0((\mathbf{k}_a+\mathbf{k}_1+\mathbf{k}_2)^2, \lambda^2, \mu((\mathbf{k}_a+\mathbf{k}_1+\mathbf{k}_2)^2, \mathbf{k}_b^2))} + \cdots. \end{aligned} \tag{22}$$

As we want to obtain the final solution in energy space we perform the inverse Mellin transform of Eq. (22) as defined in Eq. (2). In this way it is possible to obtain the following expression for the solution in the NLLA:[1]

[1] Using the notation $y_0 \equiv \Delta$.

$$
\begin{aligned}
f(\mathbf{k}_a,\mathbf{k}_b,\Delta) ={}& \exp\bigl(\omega_0\bigl(\mathbf{k}_a^2,\lambda^2,\mu\bigl(\mathbf{k}_a^2,\mathbf{k}_b^2\bigr)\bigr)\Delta\bigr)\Biggl\{\delta^{(2)}(\mathbf{k}_a-\mathbf{k}_b) \\
&+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\int d^2\mathbf{k}_i\Biggl[\frac{\theta(\mathbf{k}_i^2-\lambda^2)}{\pi\mathbf{k}_i^2}\xi\Biggl(\mathbf{k}_i^2,\mu\Biggl(\Biggl(\mathbf{k}_a+\sum_{l=0}^{i-1}\mathbf{k}_l\Biggr)^2,\mathbf{k}_b^2\Biggr)\Biggr) \\
&\qquad+\widetilde{\mathcal{K}}_r\Biggl(\mathbf{k}_a+\sum_{l=0}^{i-1}\mathbf{k}_l,\mathbf{k}_a+\sum_{l=1}^{i}\mathbf{k}_l,\mu\Biggl(\Biggl(\mathbf{k}_a+\sum_{l=0}^{i-1}\mathbf{k}_l\Biggr)^2,\mathbf{k}_b^2\Biggr)\Biggr)\Biggr] \\
&\quad\times\int_0^{y_{i-1}}dy_i\exp\Biggl[\Biggl(\omega_0\Biggl(\Biggl(\mathbf{k}_a+\sum_{l=1}^{i}\mathbf{k}_l\Biggr)^2,\lambda^2,\mu\Biggl(\Biggl(\mathbf{k}_a+\sum_{l=0}^{i}\mathbf{k}_l\Biggr)^2,\mathbf{k}_b^2\Biggr)\Biggr) \\
&\qquad-\omega_0\Biggl(\Biggl(\mathbf{k}_a+\sum_{l=1}^{i-1}\mathbf{k}_l\Biggr)^2,\lambda^2,\mu\Biggl(\Biggl(\mathbf{k}_a+\sum_{l=0}^{i-1}\mathbf{k}_l\Biggr)^2,\mathbf{k}_b^2\Biggr)\Biggr)\Biggr)y_i\Biggr] \\
&\quad\times\delta^{(2)}\Biggl(\sum_{l=1}^{n}\mathbf{k}_l+\mathbf{k}_a-\mathbf{k}_b\Biggr)\Biggr\}.
\end{aligned}
\tag{23}
$$

The expression (23) is only weakly λ-dependent in the $\lambda \to 0$ limit. We can show this point by applying it to a given smooth function, $\Phi(\mathbf{k}_a,\mathbf{k}_b)$, and selecting those terms proportional to Δ, i.e.,

$$
\begin{aligned}
&\int d^2\mathbf{k}_b\, f(\mathbf{k}_a,\mathbf{k}_b,\Delta)\Phi(\mathbf{k}_a,\mathbf{k}_b) \\
&\quad=\Phi(\mathbf{k}_a,\mathbf{k}_a)+\Delta\Biggl\{\omega_0\bigl(\mathbf{k}_a^2,\lambda^2\bigr)\Phi(\mathbf{k}_a,\mathbf{k}_a) \\
&\qquad+\int d^2\mathbf{k}_1\Biggl(\frac{\theta(\mathbf{k}_1^2-\lambda^2)}{\pi\mathbf{k}_1^2}\xi\bigl(\mathbf{k}_1^2\bigr)+\widetilde{\mathcal{K}}_r(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}_1)\Biggr)\Phi(\mathbf{k}_a,\mathbf{k}_a+\mathbf{k}_1)\Biggr\}+\cdots.
\end{aligned}
\tag{24}
$$

The λ-dependence of this expression can be studied in a similar manner to the one applied for Eq. (18).

As a final remark we would like to point out that our solution in the LLA limit is consistent with similar results in the LLA in the literature [8]. If we take the leading-logarithmic limit of our expressions we have

$$
\omega_0(\mathbf{q}^2,\lambda^2)=-\bar{\alpha}_s\ln\frac{\mathbf{q}^2}{\lambda^2},\qquad \xi=\bar{\alpha}_s,\qquad \eta=0,\qquad \widetilde{\mathcal{K}}_r(\mathbf{q},\mathbf{q}')=0.
\tag{25}
$$

In this case the solution takes the simple form

$$
\begin{aligned}
f(\mathbf{k}_a,\mathbf{k}_b,\Delta)={}&\left(\frac{\lambda^2}{k_a^2}\right)^{\bar{\alpha}_s\Delta}\Biggl\{\delta^{(2)}(\mathbf{k}_a-\mathbf{k}_b) \\
&+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\bar{\alpha}_s\int d^2\mathbf{k}_i\,\frac{\theta(\mathbf{k}_i^2-\lambda^2)}{\pi\mathbf{k}_i^2}\int_0^{y_{i-1}}dy_i\left(\frac{(\mathbf{k}_a+\sum_{l=1}^{i-1}\mathbf{k}_l)^2}{(\mathbf{k}_a+\sum_{l=1}^{i}\mathbf{k}_l)^2}\right)^{\bar{\alpha}_s y_i}\delta^{(2)}\Biggl(\sum_{l=1}^{n}\mathbf{k}_l+\mathbf{k}_a-\mathbf{k}_b\Biggr)\Biggr\}.
\end{aligned}
\tag{26}
$$

This expression is equivalent to that obtained in [8] in the LLA where the regularisation of the infrared divergences was performed in a different manner. The main result of this Letter is the extension of this iterative method to the NLLA. This has been possible by using the NLLA BFKL equation in $4+2\epsilon$ dimensions to introduce a cut-off in the phase space. This renders the $\epsilon \to 0$ limit finite and thus allows us to iterate the BFKL equation to obtain the solution in Eq. (23).

4. Conclusions

We have presented a procedure to solve the BFKL equation for forward scattering in the next-to-leading logarithmic approximation. We have shown how it is important to use the kernel in dimensional regularisation and introduce a cut-off, λ, in the phase space, Eq. (16). This allows us to write the solution in a compact form, Eq. (23), suitable for numerical studies, which will be presented in a future work. We have also shown the mechanism by which the solution is weakly dependent on the cut-off for small values of λ. We would like to point out that we keep the full angular information in our solution by solving the equation for any conformal spin, i.e., we do not use the angular-averaged kernel (see Eq. (17)). This will allow the study of spin-dependent observables in the NLLA.

Work is in progress to understand the BFKL resummed gluon Green's function using this approach, and to quantify the effect of those terms related to the running of the coupling [9] compared to the scale invariant ones. A study of the solution in the NLLA of the $N = 4$ supersymmetric case [10], where the coupling does not run, is also under consideration.

The ultimate goal in the application of our solution is the calculation of cross sections in the NLLA for those processes where the BFKL resummation should be relevant. The presented solution has the advantage of not operating in the Mellin space, i.e., we can use the NLLA impact factors [11] directly in transverse momentum space in order to make phenomenological predictions. It will, in principle, be possible to disentangle the structure of the final state allowing the study of, e.g., multiplicities, extending the work of [12] to the NLLA. As a final point we would like to indicate that this approach is also suitable for studying the solution of the BFKL equation in the NLLA for the non-forward case when the kernel for this case has been completed.

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