THE KANENOBU KNOTS AND KHOVANOV-ROZANSKY HOMOLOGY

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Abstract. Kanenobu has given infinite families of knots with the same HOMFLYPT polynomials. We show that these knots also have the same $sl(n)$ and HOMFLYPT homologies, thus giving the first example of an infinite family of knots undistinguishable by these invariants. This is a consequence of a structure theorem about the homologies of knots obtained by twisting up the ribbon of a ribbon knot with one ribbon.

1. Context and results

This paper has three sections. In the first section we shall give our results and some context for them, postponing proofs for the second section. In the final section we shall indicate a way to generalize our result.

1.1. **The Kanenobu knots.** Kanenobu has described knots $K(p,q)$ given by a pair of integers p, q [2]. We draw these knots in Figure 1. Kanenobu showed that these knots satisfy certain properties:

Theorem 1.1 (Kanenobu [2])**.** *Suppose p and q are even. Then we have*

(1) $K(p,q) = K(r,s) \iff \{p,q\} = \{r,s\} \subset \mathbb{Z}$,
(2) $\overline{K(p,q)} = K(-p,-q)$, *and*
(3) $P(K(p,q)) = P(K(r,s))$ *whenever* $p+q = r+s$,

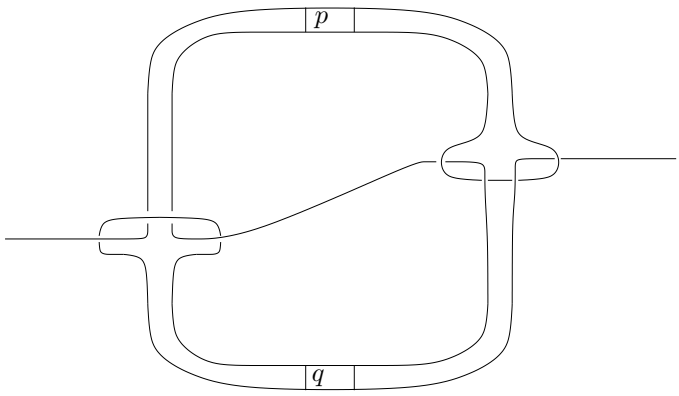

Figure 1. We give a diagram of the Kanenobu knots $K(p,q)$, passing through the point at infinity. In this picture $|p|$ and $|q|$ are integers representing the number of half-twists added in the two strands, the sign of the half-twists depending on the signs of p and q.

where we write $\overline{K}$ for the mirror image of K, and $P(K)$ for the HOMFLYPT polynomial of K. From Kanenobu's arguments, although he did not state it explicitly, we also have (2) and the $\Longleftarrow$ direction of (1) with no parity restriction on p and q.

These families of knots $K(p,q)$ with $p+q$ constant and p,q even are therefore infinite families of knots with the same HOMFLYPT and hence the same specializations of HOMFLYPT (the same Alexander polynomials, Jones polynomials, $sl(n)$ polynomials *et cetera*).

Given a knot, Khovanov and Rozansky have defined bigraded vector spaces which recover the $sl(n)$ polynomials of the knot as the graded Euler characteristic [3] and also trigraded vector spaces which recover the HOMFLYPT polynomial as the bigraded Euler characteristic [4]. Jacob Rasmussen [8] has given spectral sequences which start from the HOMFLYPT homology of a knot and converge to the $sl(n)$ homology. As a consequence of the existence of these spectral sequences, one can think of the HOMFLYPT homology as being the limit as $n \to \infty$ of the $sl(n)$ homologies.

In this paper we shall assume a working familiarity with the definitions and basic properties of the $sl(n)$ and HOMFLYPT homologies as found in [3, 4].

It is natural to ask whether the HOMFLYPT homology or, more generally, the $sl(n)$ homologies can detect the difference between $K(p,q)$ and $K(r,s)$ when $p+q = r+s$ and p,q,r,s are all even. In this paper we show that they cannot detect this difference. As a consequence, the Kanenobu knots provide the first examples of an infinite collection of knots with the same HOMFLYPT and $sl(n)$ homologies.

Theorem 1.2. *We write $H_n(K)$ for the $sl(n)$ homology of a knot K. We may mean the unreduced, reduced, or the equivariant homology with potential $x^{n+1}-(n+1)ax$. Then for the Kanenobu knots $K(p,q)$ where $p,q \in \mathbb{Z}$ we have*

$$H_n(K(p,q)) = H_n(K(r,s)) \text{ whenever } p+q = r+s \text{ and } pq \equiv rs \pmod 2.$$

Corollary 1.3. *We write $\overline{H}(K)$ for the reduced HOMFLYPT homology of a knot K. Then for the Kanenobu knots $K(p,q)$ where $p,q \in \mathbb{Z}$ we have*

$$\overline{H}(K(p,q)) = \overline{H}(K(r,s)) \text{ whenever } p+q = r+s \text{ and } pq \equiv rs \pmod 2.$$

Liam Watson [9] has an analogue of Theorem 1.2 for standard Khovanov homology over $\mathbb{Z}$, and Greene-Watson are working on analogues for odd Khovanov homology and for Heegaard-Floer knot homology.

1.2. **Ribbon knots with one ribbon.** The important point in our proof of Theorem 1.2 is that the knots $K(p,q)$ are ribbon knots with one ribbon. If $K = K_0$ is a ribbon knot with one ribbon , then by twisting along the ribbon we obtain a sequence of knots K_p for $p \in \mathbb{Z}$; this process is illustrated in Figure 2. For these knots we have the following result.

Theorem 1.4. *We write $H_n(K)$ for the $sl(n)$ homology of a knot K. We may mean the unreduced, reduced, or the equivariant homology with potential $x^{n+1} - (n+1)ax$. When working with equivariant homology every module in this statement should be read as a finitely-generated bigraded $\mathbb{C}[a]$-module, and otherwise as a finite dimensional bigraded $\mathbb{C}$-vector space. Let U be the unknot. Then for any $p \in \mathbb{Z}$ there exists a module $M_n(K_p)$ such that*

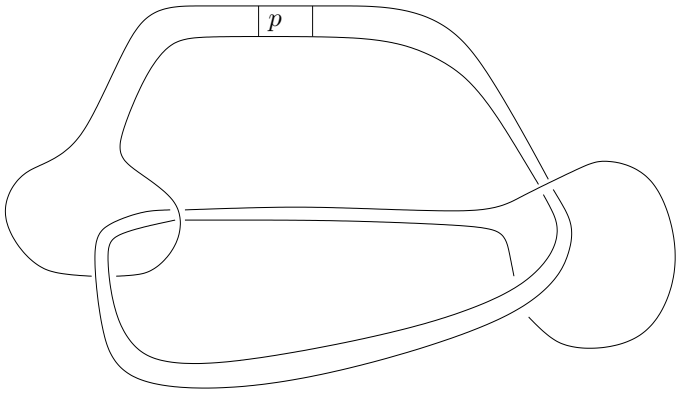

FIGURE 2. Here is an example of a class of ribbon knots K_p with one ribbon. On the ribbon we have inserted $|p|$ half-twists, positively or negatively depending on the sign on p.

$$H_n(K_p) = H_n(U) \oplus M_n(K_p)$$

and

$$M_n(K_{p+2}) = M_n(K_p)[2]\{-2n\} \text{ for all } p,$$

where the square brackets indicate a shift in the homological grading and the curly brackets indicate a shift in the quantum grading.

Corollary 1.5. *We write $\overline{H}^{i,j,k}(K)$ for the reduced HOMFLYPT homology of a knot K, using the grading conventions of* [8]*. We use square brackets to denote a shift in the j-grading, and curly brackets to denote a shift in the k-grading. Let U be the unknot. Then for any $p \in \mathbb{Z}$ there exists a trigraded $\mathbb{C}$-vector space $M(K_p)$ such that*

$$\overline{H}(K_p) = \overline{H}(U) \oplus M(K_p)$$

and

$$M(K_{p+2}) = M(K_p)[-2]\{2\} \text{ for all } p.$$

Remark. *In* [6]*, we indicated how the class of objects with well-defined Khovanov-Rozansky homologies could be enlarged to include knots with* infinite twist sites*, which are sites where we add an infinite number of twists to two oppositely-oriented strands. In the case of the ribbon knots K_p considered in this subsection, Theorem 1.4 and Corollary 1.5 imply that the Khovanov-Rozansky homologies of K_∞ are the same as those of the unknot.*

1.3. **Acknowledgements.** We thank Liam Watson for suggesting this problem and we thank Josh Greene for suggesting this problem later when we were better able to answer it. We also thank the anonymous referee for suggesting changes that greatly improved our exposition.

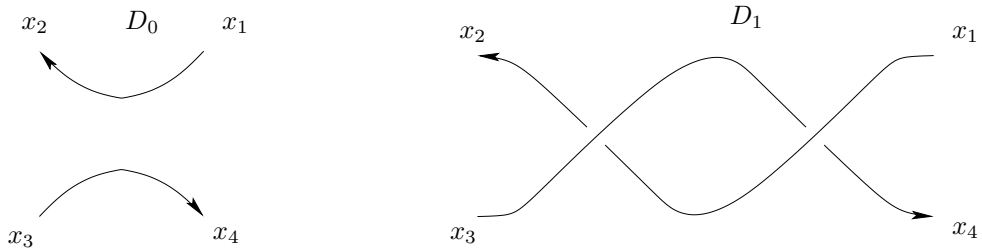

FIGURE 3. We have drawn two tangle diagrams, D_0 and D_1. Associated in the $sl(n)$ Khovanov-Rozansky theory to each tangle diagrams is a complex of (vectors of) matrix factorizations. We shall denote these up to chain homotopy equivalence by $C_n(D_0)$ and $C_n(D_1)$.

2. Proofs

We organize this section somewhat in reverse, ending with our proof of Theorem 1.4.

Proof of Corollaries 1.3 and 1.5. These Corollaries follow immediately from Theorems 1.2 and 1.4 and Rasmussen's Theorem 1 from [8] which realizes the reduced HOMFLYPT homology as the limit of the reduced $sl(n)$ homologies as $n \to \infty$. □

Proof of Theorem 1.2 assuming Theorem 1.4. There are two places in Figure 1 where adding a ribbon to the knot $K(p,q)$ will result in the 2-component unlink: we could add a ribbon at one side of the p-twist region or at one side of the q-twist region. Hence we observe that $K(p,q)$ is a ribbon knot with one ribbon in two possibly distinct ways. This puts us in the situation of Theorem 1.4.

Applying Theorem 1.4 at the two different sites tells us that

$$H_n(K(p,q)) = H_n(K(p+2,q-2))$$

for all p, q.

Iterating this, and using the fact that $K(p,q) = K(q,p)$, we obtain the statement of Theorem 1.2. □

Before we begin the proof of Theorem 1.4 we collect some results, many of which appeared in [6].

Proposition 2.1. *Let $H_n(K)$ stand for the equivariant $sl(n)$ knot homology of K with potential $x^{n+1} - (n+1)ax$ over the ring $\mathbb{C}[a]$. Here a is graded of degree $2n$ and x of degree 2 so that the potential is of homogeneous degree $2n+2$.*

Then $H_n(K)$ has the structure of a $\mathbb{C}[a]$-module and the free part F of $H_n(K)$ is of dimension n, supported entirely in homological degree 0 and we have

$$F = H_n(U)\{s_n(K)\},$$

where we write U for the unknot, $s_n(K)$ for the analogue of the Lee-Rasmussen invariant $s(K)$ coming from Khovanov-Rozansky homology, and the curly brackets mean a shift in quantum degree.

Proof. Setting $a = 1$ we recover Gornik's perturbation of Khovanov-Rozansky homology [1], which he showed was of dimension n and supported in homological degree 0. This implies that the free part of $H_n(K)$ must be supported in homological degree 0 and be of dimension n, since the free part is the only part that survives under setting $a = 1$. In [7] we have shown the dependence of the quantum degrees of Gornik's perturbation on a single even integer $s_n(K)$. The result follows. □

Next consider the complexes $C_n(D_0)$ and $C_n(D_1)$ of matrix factorizations given in Figure 3. Krasner has given a representative of the chain homotopy equivalence class of $C_n(D_1)$ which has a particularly simple form, the chain complex being made up of maps between matrix factorizations V and Z corresponding the diagrams given in Figure 4.

Proposition 2.2 (Krasner [5])**.** *Up to chain homotopy equivalence*

$$C_n(D_1) = V[0]\{1-n\} \stackrel{x_2-x_4}{\to} V[1]\{-1-n\} \stackrel{S}{\to} Z[2]\{-2n\},$$

where square brackets indicate a shift in homological degree, curly brackets indicate a shift in quantum degree, and S is the map induced by saddle cobordism.

We argued in [6] that this theorem holds even equivariantly, a flavour of Khovanov-Rozansky homology that did not exist when Krasner first formulated Proposition 2.2. Since the matrix factorization Z is equal, up to degree shifts, to the only matrix factorization appearing in the complex $C(D_0)$, there is an obvious chain map induced by the identity map on Z:

$$G : C_n(D_0) \to C_n(D_1)[-2]\{2n\}.$$

We have shifted $C_n(D_1)$ here so that G is graded of degree 0 both in the homological and in the quantum gradings.

Proposition 2.3. *The cone of the chain map G, working either equivariantly or over $\mathbb{C}$, is the chain complex*

$$Co(G) = V[-2]\{1+n\} \stackrel{x_2-x_4}{\to} V[-1]\{-1+n\}.$$

Proof. This is a straightforward application of Gaussian elimination. □

The map G appears in [6] as the map $G_{0,1}$. In that paper, maps on knot homologies induced by maps such as G were fitted together into large commutative diagrams with exact sequences for the rows. For our application in the current paper we only need one of the rows, not the whole commutative diagram.

Proposition 2.4. *Consider the knots K_p as in the statement of Theorem 1.4. Then there is a long exact sequence in homology*

$$\begin{aligned} \cdots \quad &\to \quad N^{-1} \to H_n^0(K_p) \to H_n^2(K_{p+2})\{2n\} \to \\ &\to \quad N^0 \to H_n^1(K_p) \to H_n^3(K_{p+2})\{2n\} \to \cdots \end{aligned}$$

where N is a bigraded module given by

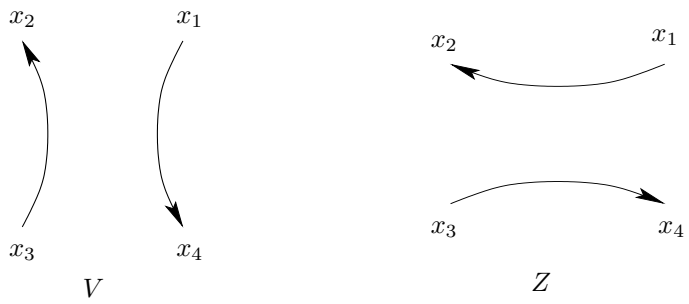

FIGURE 4. We have drawn here the diagrams which correspond to the matrix factorizations V and Z.

$$N = H_n(U)[-1]\{0\} \oplus H_n(U)[-2]\{2n\},$$

and we use superscripts to denote homological degree and all maps are of quantum-degree 0.

This long exact sequence exists for equivariant, unreduced, or reduced flavours of homology. In the first case, all modules are $\mathbb{C}[a]$*-modules, in the other cases all modules are* $\mathbb{C}$*-vector spaces.*

We note that this long exact sequence appears as the top line of the commutative diagram in Proposition 2.7 of [6].

Proof. Since the knots K_p and K_{p+2} differ locally by replacing an occurrence of the tangle D_0 by the tangle D_1, the chain map G induces a chain map

$$G : C_n(K_p) \to C_n(K_{p+2})[-2]\{2n\},$$

where we have written C_n to denote the $sl(n)$ Khovanov-Rozansky chain complex. If we let $\tilde{N}$ be the cone of this chain map then setting N to be the homology of $\tilde{N}$ gives us the desired long exact sequence. It remains to identify the module structure of N.

Note that Proposition 2.3 realizes the cone $\tilde{N}$ as an explicit map between two chain complexes, each associated (up to some degree shifts) to a diagram of the 2-component unlink.

Taking account of these degree shifts we see that N is supported in homological degrees -2 and -1 and sits in a long exact sequence whose support is

$$0 \to N^{-2} \to H_n(U \cup U)\{1+n\} \stackrel{x-y}{\to} H_n(U \cup U)\{-1+n\} \to N^{-1} \to 0.$$

Here we are writing the homology of the 2-component unlink as

$$H_n(U \cup U) = \mathbb{C}[x,y]/(x^n = y^n = 0)\{2-2n\}$$

in the standard case and as

$$H_n(U \cup U) = \mathbb{C}[a,x,y]/(x^n = y^n = a)\{2-2n\}$$

in the equivariant case.

Computing the kernel and cokernel of the map $x - y$ determines N as in the statement of the Proposition. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Proposition 2.4 gives a long exact sequence relating the $sl(n)$ homologies of K_p and K_{p+2} with the bigraded module N, which is supported in only homological degrees -2 and -1 and whose structure has been explicitly computed. If we can compute the maps in the long exact sequence

$$\varphi : H_n^0(K_{p+2})\{2n\} \to N^{-2}$$

and

$$\psi : N^{-1} \to H_n^0(K_p)$$

then we shall be able to describe $H_n(K_{p+2})$ completely in terms of $H_n(K_p)$.

Let us start by considering the equivariant case. Since K_p and K_{p+2} are both smoothly slice, we have $s_n(K_p) = s_n(K_{p+2}) = 0$. So Proposition 2.1 tells us everything about the free parts of $H_n(K_p)$ and $H_n(K_{p+2})$. The module $M_n(K_r)$ for $r = p, p+2$ appearing in the statement of the theorem is thus the a-torsion part of the homology $H_n(K_r)$ and we have

$$H_n(K_r) = H_n(U) \oplus M_n(K_r).$$

The module N^{-2} is a free $\mathbb{C}[a]$-module. If the map φ is not an injection on the free part of $H_n^0(K_{p+2})\{2n\}$ then there must be some non-zero free part of $H_n^{-1}(K_p)$, but this cannot happen by Proposition 2.1. Since φ preserves the quantum grading, we see that φ is therefore an isomorphism on the free part of $H_n^0(K_{p+2})\{2n\}$. Similarly, we see that ψ must map N^{-1} isomorphically onto the free part of $H_n^0(K_p)$.

Hence the long exact sequence gives isomorphisms (with some degree shifts) between the a-torsion parts of the homologies. This establishes Theorem 1.4 in the equivariant case.

Specializing the equivariant case to $a = 0$ we obtain the unreduced standard $sl(n)$ homology. It is clear that φ descends to a surjective map when we specialize and that ψ descends to an injective map. This establishes the unreduced case.

For the reduced case we recall the definition of the reduced homology groups from the end of Section 7 of [3]. Recall that the homology of each matrix factorization is a free $\mathbb{C}[x]/x^n$-module where x is a marked point. This gives the unreduced Khovanov-Rozansky complex of a knot diagram the structure of a complex of free $\mathbb{C}[x]/x^n$-modules. One obtains the *reduced* Khovanov-Rozansky complex (a complex purely of $\mathbb{C}$-vector spaces) by applying the functor $* \otimes_{\mathbb{C}[x]/x^n} \mathbb{C}$.

We note that in the unreduced case N has the structure of a rank$=2$ free module over $\mathbb{C}[x]/x^n$ and the maps φ and ψ (since they are surjective and injective respectively) are $\mathbb{C}[x]/x^n$-isomorphisms of the two summands of N with the preimage and image of the maps respectively.

When we reduce, N becomes a rank$=2$ $\mathbb{C}$-vector space and φ and ψ are again isomorphisms of the two summands of N with the preimage and image of the maps respectively. This is because of the naturality of the short exact sequence in the universal coefficients theorem for principal ideal domains and the vanishing of the Tor term when computing the reduced homology N in terms of the unreduced version. This establishes the reduced case. □

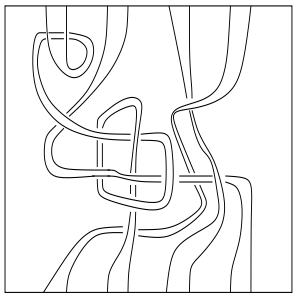

FIGURE 5. Here we show an example of a *ribbon tangle* on 4 ribbons.

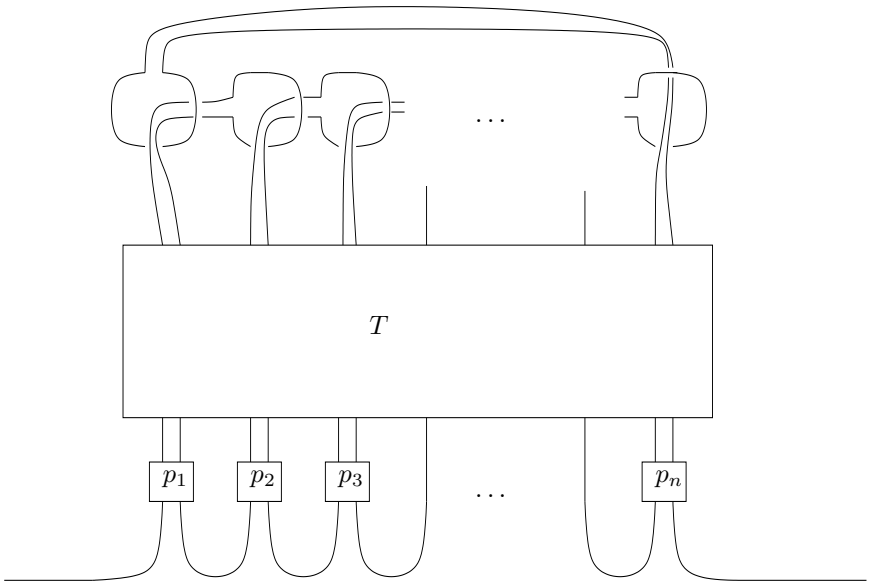

FIGURE 6. We have drawn a knot $K_T(p_1, p_2, \ldots, p_n)$ passing through the point ∞. The knot depends on a ribbon tangle T and n integer parameters $p_1, p_2, \ldots, p_n$ describing the half-twists added to each ribbon.

3. Extension of results

The salient point in our proof of Theorem 1.2 was that the Kanenobu knots are ribbon knots on one ribbon in two different ways. There are many ways to generate families of knots in which one can hope to retain this property. As an example, we present one such way.

Definition 3.1. *In Figure 5 we draw a* ribbon tangle *on* 4 *ribbons. A ribbon tangle on n ribbons is a tangle with n inputs at the bottom and n outputs at the top, where we replace each tangle strand by two strands using the blackboard framing.*

We draw a knot $K_T(p_1, p_2, \ldots, p_n)$ in Figure 6 depending on n integers $p_i \in \mathbb{Z}$, and a ribbon tangle T. By adding a ribbon to any of the twist regions we obtain

the 2-component unlink. Hence we can apply Theorem 1.4 in this situation and so obtain the following result.

Theorem 3.2. *Let H be either reduced HOMFLYPT homology or $sl(n)$ homology (reduced, unreduced, or equivariant with potential $x^{n+1} - (n+1)ax$) and T be any ribbon tangle. Then we have that*

$$H(K_T(p_1, p_2, \ldots, p_n)) = H(K_T(q_1, q_2, \ldots, q_n))$$

whenever

$$p_1 + p_2 + \cdots + p_n = q_1 + q_2 + \cdots + q_n \text{ and } p_i \equiv q_i \pmod 2 \text{ for all } i.$$

□

Of course, to show that in such examples you are generating infinitely many *distinct* knots with the same Khovanov-Rozansky homologies, you need another invariant with which to distinguish between them.

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