

# Should lotteries offer discounts on multiple tickets?

Damian S. Damianov<sup>\*1</sup>

## Abstract

In a complete information setting we show that the standard lottery – in which each lottery ticket is offered for the same price – is an optimal fundraising mechanism in the presence of strong asymmetries in the way bettors value the prize and the public good provided with the lottery proceeds. When participants are more homogeneous, it is optimal to offer discounts for the purchase of multiple tickets.

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\* Durham University Business School, Mill Hill Lane, Durham DH1 3LB, United Kingdom.  
Telephone: +44 (0) 191 33 45528. Email: damian.damianov@durham.ac.uk

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## 1. Introduction

Lotteries are popular mechanisms for the provision of public goods and for charitable fundraising. Sales of lottery tickets generate revenue for the funding of a number of good causes including health, education, infrastructure development, environmental protection, sports, arts, and a variety of other programs. A recent study by Bloomberg using 2010 U.S. Census data reports that adults in the U.S. spend on average about \$230 on lottery tickets per year.<sup>1</sup> This suggests that even small changes in the way lottery games are operated might result in substantial changes in funding for charitable causes.

The recent theoretical literature demonstrates that fundraising mechanisms in which a prize is awarded to one of the contributors – such as lotteries – are better at raising funds for public goods than voluntary contribution schemes. In an environment in which a finite number of identical consumers decide how much to contribute to a public good and how much to retain for private consumption, Morgan (2000) shows that a fixed prize raffle generates a higher level of public good contributions (net of the value of the prize) compared to voluntary donations. Goeree et al (2005) study the optimal design of fundraising mechanisms in the symmetric independent private value model. They demonstrate that in a setting in which the proceeds are used for the funding of a public good, the optimal mechanism is an all-pay auction; in this setting all-pay auctions outperform winner-pay auctions. More recently, Bos (2011) studies a complete information model and shows that lotteries generate more revenue than all-pay auctions when participants are sufficiently heterogeneous in the way they value the prize and the contributions to the public good.

In this paper we derive the optimal fundraising mechanism in a setting in which the designer chooses a nonlinear pricing scheme so as to maximize the revenue from lottery ticket sales. We build on the recent theoretical advancements presented in Goeree et al (2005) and Bos (2011) along several dimensions. Similarly to Goeree et al (2005), in our model the designer chooses from a continuum of mechanisms, yet we allow for asymmetries between participants. We analyse a setting in which bettors can have different valuations for the prize and different marginal per capita returns associated with the provision of the public good. In that respect our analysis is conducted in the complete information framework presented in Bos (2011), yet we do not constrain the mechanism designer to the choice between the all-pay auction and the lottery only. Instead, by varying the pricing scheme of lottery tickets, we consider a set of mechanisms in which the lottery and the all-pay auction arise as special cases.

The choice of a pricing scheme determines the relationship between the individual contribution of a participant and the chance of winning the prize as is standard in the mechanism design literature. The main focus of the paper is on determining whether, and in which circumstances, discounts for the purchase of multiple lottery tickets increase lottery proceeds. We operationalize the idea of discounts by considering power functions as a form of nonlinear pricing. That is, the designer specifies the power function parameter thus choosing among a continuum of pricing schemes.

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<sup>1</sup> Out of the 43 states currently operating lotteries, the highest spending per adult is in Massachusetts where residents spend on lottery tickets \$861 per year, or 1.3% of their personal income; see <http://www.bloomberg.com/video/popout/86793284/68.896> for further details.

This representation is useful because, as we show, the so described lottery games are isomorphic to Tullock (1980) contests. This feature facilitates the analysis by allowing us to use the equilibrium existence and characterization results from the Tullock contest literature (see e.g. Nti, 1999). The main point of departure from the contest literature, however, is in viewing the contest parameter as an object of choice by the mechanism designer.

Our main contribution is a closed form expression which links the valuations of participants to the parameter describing the revenue maximizing ticket pricing function. Solving this mechanism design problem, we provide conditions under which the lottery is optimal among a continuum of fundraising mechanisms. In particular we show that, when bettors are sufficiently asymmetric in the way they value the prize and the public good, lottery ticket sales without discounts are optimal. Discounts, however, increase revenue in the presence of smaller asymmetries.<sup>2</sup>

## 2. The model

We consider a scenario in which two participants  $i = 1, 2$  choose the number of tickets  $x_1$  and  $x_2$  that they purchase in a lottery. The value of the prize for participant  $i$  is denoted by  $v_i$ . The revenue from the lottery is used to provide a public good. We denote the per capita return of participant  $i$  from the public good by  $\delta_i$ , where  $0 < \delta_i < 1$ . In this model the lottery designer chooses a function  $c(x_i)$  which determines the total amount that bettor  $i$  has to pay for a number of  $x_i$  tickets. The expected payoff of bettor  $i$  is thus given by

$$U_i(x_i, x_{-i}) = v_i \cdot P_i(x_i, x_{-i}) - c(x_i) + \delta_i [c(x_i) + c(x_{-i})]$$

where

$$P_i(x_i, x_{-i}) = \frac{x_i}{x_i + x_{-i}}$$

is the probability that participant  $i$  wins the prize. The lottery designer seeks to determine the function  $c(x_i)$  in such a way that the lottery revenue  $c(x_1^*) + c(x_2^*)$  is maximized given the number of tickets  $x_1^*$  and  $x_2^*$  purchased in the Cournot-Nash equilibrium of the lottery. We will consider ticket pricing functions of the type  $c(x_i) = x_i^\alpha$ . A value of  $\alpha = 1$  is the default scenario in which each lottery ticket has the same price, and values  $0 < \alpha < 1$  describe cases in which discounts are offered for the purchase of multiple tickets.<sup>3</sup> We first demonstrate that the presented setting is equivalent to a Tullock contest. Let us denote by  $y_i = c(x_i)$  the total expenditures on lottery tickets by bettor  $i$  and by  $V_i = \frac{v_i}{1-\delta_i}$  the “adjusted valuation” of bettor  $i$ . This valuation adjustment takes into account that participants benefit from their own contribution to the public good and transforms the original game into an equivalent game in which participants do not benefit from the public good, but compete for a larger prize which equals the original prize scaled by the factor  $\frac{1}{1-\delta_i}$ .

<sup>2</sup> In the limit when all asymmetries disappear, it is known that the optimal mechanism is an all-pay auction (Goeree et al, 2005; Engers and McManus, 2007; Bos, 2011; Damianov and Peeters, 2012).

<sup>3</sup> Values  $\alpha > 1$ , while theoretically possible, might not be implementable as they imply additional charges for the purchase of multiple tickets.

**Lemma 1.** *The fundraising lottery is isomorphic to a Tullock contest in which players choose their expenditures on lottery tickets  $y_1$  and  $y_2$  and their payoff function is given by*

$$\pi_i(y_i, y_{-i}) = \frac{y_i^r}{y_i^r + y_{-i}^r} \cdot V_i - y_i$$

where  $r = 1/\alpha$  is the Tullock contest parameter.

The proof is provided in the Appendix. The equilibria of the Tullock rent-seeking game have been studied by several authors (see, e.g. Perez-Castrillo and Verdier, 1992; Nti, 1999; and Fang, 2002). Nti (1999) derives a necessary and sufficient condition for the existence of a unique pure strategy Nash equilibrium.<sup>4</sup> In the following analysis we will consider the cases for which this condition is satisfied. For the first order conditions we obtain

$$\frac{\partial \pi_i(y_1, y_2)}{\partial y_i} = \frac{r y_i^{r-1} y_{-i}^r}{(y_i^r + y_{-i}^r)^2} V_i - 1 = 0, \quad i = 1, 2.$$

Solving this system of equations we derive the equilibrium contribution levels

$$y_1^* = \frac{r V_1^{r+1} V_2^r}{(V_1^r + V_2^r)^2}; \quad y_2^* = \frac{r V_1^r V_2^{r+1}}{(V_1^r + V_2^r)^2}$$

The lottery designer chooses  $r$  so that the lottery revenue

$$R(V_1, V_2, r) = y_1^* + y_2^* = \frac{r V_1^r V_2^r (V_1 + V_2)}{(V_1^r + V_2^r)^2}$$

is maximized. The first order condition is given by the equation

$$\frac{\partial R(V_1, V_2, r)}{\partial r} = \frac{V_1^r V_2^r (V_1 + V_2)}{(V_1^r + V_2^r)^3} [V_1^r + V_2^r + r(V_1^r - V_2^r)(\ln(V_2) - \ln(V_1))] = 0 \quad (1)$$

The optimal value of  $r$  thus satisfies the equation

$$r = \frac{V_1^r + V_2^r}{(V_1^r - V_2^r)(\ln(V_1) - \ln(V_2))} \quad (2)$$

The following lemma shows that the lottery revenue function is concave in  $r$  so that the solution to equation (2) indeed represents the global maximum.

**Lemma 2.** *The lottery revenue  $R(V_1, V_2, r)$  is concave in the parameter  $r$ .*

The proof is provided in the Appendix. We can assume now without loss of generality that  $V_1 \geq V_2$ . Following Nti (1999) and Bos (2011) we define the level of asymmetry between the players by the ratio of their adjusted valuations  $a := V_1/V_2$ . The next statement summarizes our main result.

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<sup>4</sup>Nti (1999) shows that the Tullock contest has a unique pure strategy Nash equilibrium if and only if the adjusted valuations satisfy the inequality  $V_1^r + V_2^r > r \cdot V_2^r$  (see Proposition 3).

**Proposition 1.** *The optimal value of  $r$  depends only on the asymmetry level  $a \geq 1$  and is given by the expression*

$$r = \frac{\ln(z^*)}{\ln(a)}$$

where  $z^* \approx 4.6805$  is the unique solution to the equation

$$\ln(z) = \frac{z + 1}{z - 1}.$$

The proof is provided in the Appendix. From Proposition 1 we can see that the optimal value of  $\alpha$  is linear in the level of asymmetry and is given by the equation

$$\alpha = \frac{1}{r} = \frac{\ln(a)}{\ln(z^*)}$$

That is, for high levels of asymmetry, i.e. when  $a > z^*$ , it is actually optimal to impose additional charges for the purchase of multiple tickets. In other words, the optimal mechanism for high asymmetry involves increasing rather than constant marginal price. As such a pricing scheme might not be easily implementable<sup>5</sup> our result implies that discounts are not optimal for high levels of asymmetry. When  $a = z^*$  the standard case of linear pricing is optimal, and when  $a < z^*$ , which occurs when bettors are more homogeneous, it is optimal to offer discounts on multiple tickets.

### 3. Conclusion

This note contributes to the mechanism design literature on charitable fundraising. We show that the standard lottery – in which each ticket is sold for the same price – is an optimal fundraising mechanism in the presence of strong asymmetries in the way participants value the prize and the public good. When participants are less asymmetric, it is optimal to offer discounts for the purchase of multiple tickets. This result might have implications for the potential of charitable organization to raise funds through lotteries in communities with varying degrees of income inequality.

### Appendix

**Proof of Lemma 1.** As we denoted the spending of bettor  $i$  by  $y_i = c(x_i) = x_i^\alpha$ , we can express the utility function of participant  $i$

$$U_i(x_i, x_{-i}) = v_i \frac{x_i}{x_i + x_{-i}} - c(x_i) + \delta_i [c(x_i) + c(x_{-i})]$$

in terms of the lottery expenditures of both players and the parameter  $r = 1/\alpha$  as follows

$$U_i(y_i, y_{-i}) = (1 - \delta_i) \left[ \frac{y_i^r}{y_1^r + y_2^r} \cdot V_i - y_i \right] + \delta_i \cdot y_{-i}$$

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<sup>5</sup> In order to implement an increasing marginal pricing schedule, the lottery organizers must keep track of the participants' identities and the number of tickets each participant has purchased. Otherwise buyers can purchase one ticket at a time effectively buying multiple tickets at a lower price.

$$= (1 - \delta_i) \cdot \pi_i(y_i, y_{-i}) + \delta_i \cdot y_{-i}$$

Note that the lottery payoff function is linearly increasing in the Tullock contest payoff function  $\pi_i(y_i, y_{-i})$ . Because both functions have the same maximizers, the equilibria of the lottery game and the Tullock contest are identical.

□

**Proof of Lemma 2.** Differentiating the expression presented in (1) with respect to  $r$  yields

$$\frac{\partial^2 R(V_1, V_2, r)}{\partial r^2} = -\frac{V_1^r \cdot V_2^r (\ln(V_1) - \ln(V_2))}{(V_1^r + V_2^r)^4} D(V_1, V_2, r) \quad (3)$$

where

$$D(V_1, V_2, r) = 2V_1^{2r} - 2V_2^{2r} - r(V_1^{2r} + V_2^{2r} - 4V_1^r V_2^r)(\ln(V_1) - \ln(V_2))$$

Using the expression for  $r$  in (2) we obtain

$$D(V_1, V_2, r) = 2V_1^{2r} - 2V_2^{2r} - \frac{V_1^r + V_2^r}{V_1^r - V_2^r} (V_1^{2r} + V_2^{2r} - 4V_1^r V_2^r) \Leftrightarrow$$

$$D(V_1, V_2, r) = \frac{1}{V_1^r - V_2^r} [2(V_1^r + V_2^r)(V_1^r - V_2^r)^2 - (V_1^r + V_2^r)(V_1^{2r} + V_2^{2r} - 4V_1^r V_2^r)] \Leftrightarrow$$

$$D(V_1, V_2, r) = \frac{(V_1^r + V_2^r)}{V_1^r - V_2^r} [2(V_1^r - V_2^r)^2 - (V_1^{2r} + V_2^{2r} - 4V_1^r V_2^r)] \Leftrightarrow$$

$$D(V_1, V_2, r) = \frac{(V_1^r + V_2^r)}{V_1^r - V_2^r} [2(V_1^r - V_2^r)^2 - (V_1^{2r} + V_2^{2r} - 4V_1^r V_2^r)] \Leftrightarrow$$

$$D(V_1, V_2, r) = \frac{(V_1^r + V_2^r)}{V_1^r - V_2^r} (V_1^{2r} + V_2^{2r})$$

Substituting this expression for  $D(V_1, V_2, r)$  in equation (3) we obtain

$$\frac{\partial^2 R(V_1, V_2, r)}{\partial r^2} = -\frac{V_1^r \cdot V_2^r (V_1^r + V_2^r)(V_1^{2r} + V_2^{2r})}{(V_1^r + V_2^r)^4} \cdot \frac{(\ln(V_1) - \ln(V_2))}{V_1^r - V_2^r} < 0.$$

□

**Proof of Proposition 1.** Dividing the numerator and denominator of the right hand-side of equation (1) by  $V_2^r$  we obtain

$$r = \frac{a^r + 1}{(a^r - 1) \cdot \ln(a)} \quad (4)$$

Let us define  $z := a^r$ . We can express  $r$  as

$$r = \ln(z)/\ln(a) \quad (5)$$

From equation (4) we obtain

$$\frac{\ln(z)}{\ln(a)} = \frac{z + 1}{(z - 1)} \cdot \frac{1}{\ln(a)}$$

Multiplying both sides by  $\ln(a)$  we obtain the value of  $z$  reported in Proposition 1. Equation (5) yields the desired result.  $\square$

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