An Unusual Continued Fraction

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Abstract

We consider the real number σ with continued fraction expansion $[a_0, a_1, a_2, \ldots] = [1, 2, 1, 4, 1, 2, 1, 8, 1, 2, 1, 4, 1, 2, 1, 16, \ldots]$, where a_i is the largest power of 2 dividing $i+1$. We show that the irrationality measure of σ^2 is at least 8/3. We also show that certain partial quotients of σ^2 grow doubly exponentially, thus confirming a conjecture of Hanna and Wilson.

1 Introduction

By a *continued fraction* we mean an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}}$$

or

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cdots}}}$$

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where $a_1, a_2, \ldots,$ are positive integers and a_0 is an integer. To save space, as usual, we write $[a_0, a_1, \ldots, a_n]$ for the first expression and $[a_0, a_1, \ldots,]$ for the second. For properties of continued fractions, see, for example, [14, 9].

It has been known since Euler and Lagrange that a real number has an ultimately periodic continued fraction expansion if and only if it is a quadratic irrational. But the expansions of some other "interesting" numbers are also known explicitly. For example [25],

$$\begin{aligned} e &= [2, (1, 2n, 1)_{n=1}^{\infty}] = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots] \\ e^2 &= [7, (3n-1, 1, 1, 3n, 12n+6)_{n=1}^{\infty}] = [7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, \ldots] \\ \tan 1 &= [(1, 2n-1)_{n=1}^{\infty}] = [1, 1, 1, 3, 1, 5, 1, 7, \ldots] \end{aligned}$$

These three are examples of "Hurwitz continued fractions", where there is a "quasiperiod" of terms that grow linearly (see, for example, [15, 19] and [20, pp. 110–123]). By contrast, no simple pattern is known for the expansions of e^3 or e^4.

Recently there has been some interest in understanding the Diophantine properties of numbers whose continued fraction expansion is generated by a simple computational model, such as a finite automaton. One famous example is the Thue-Morse sequence on the symbols $\{a, b\}$ where $\overline{a} = b$ and $\overline{b} = a$ is given by

$$\mathbf{t} = t_0 t_1 t_2 \cdots = abbabaab \cdots$$

and is defined by

$$t_n = \begin{cases} a, & \text{if } n = 0; \\ t_{n/2}, & \text{if } n \text{ even}; \\ \overline{t_{n-1}}, & \text{if } n \text{ odd}. \end{cases}$$

Queffélec [21] proved that if a, b are distinct positive integers, then the real number $[\mathbf{t}] = [t_0, t_1, t_2, \ldots]$ is transcendental. Later, a simpler proof was found by Adamczewski and Bugeaud [3]. Queffélec [22] also proved the transcendence of a much wider class of automatic continued fractions. More recently, in a series of papers, several authors explored the transcendence properties of automatic, morphic, and Sturmian continued fractions [6, 2, 1, 5, 10, 11].

All automatic sequences (and the more general class of morphic sequences) are necessarily bounded. A more general class, allowing unbounded terms, is the k-regular sequences of integers, for integer $k \geq 2$. These are sequences $(a_n)_{n \geq 0}$ where the k-kernel, defined by

$$\{(a_{k^e n + i})_{n \geq 0} \ : e \geq 0, 0 \leq i < k^e\},$$

is contained in a finitely generated module [7, 8]. We state the following conjecture:

Conjecture 1. Every continued fraction where the terms form a k-regular sequence of positive integers is transcendental or quadratic.

In this paper we study a particular example of a k-regular sequence:

$$\mathbf{s} = s_0 s_1 s_2 \cdots = (1, 2, 1, 4, 1, 2, 1, 8, \ldots)$$

where $s_i = 2^{\nu_2(i+1)}$ and $\nu_p(x)$ is the p-adic valuation of x (the exponent of the largest power of p dividing x). To see that $\mathbf{s}$ is 2-regular, notice that every sequence in the 2-kernel is a linear combination of $\mathbf{s}$ itself and the constant sequence $(1, 1, 1, \ldots)$.

The corresponding real number σ has continued fraction expansion

$$\sigma = [\mathbf{s}] = [s_0, s_1, s_2, \ldots] = [1, 2, 1, 4, 1, 2, 1, 8, \ldots] = 1.35387112842988237438889 \cdots .$$

The sequence $\mathbf{s}$ is sometimes called the "ruler sequence", and is sequence A006519 in Sloane's *Encyclopedia of Integer Sequences* [24]. The decimal expansion of σ is sequence A100338.

Although σ has slowly growing partial quotients (indeed, $s_i \leq i + 1$ for all i), empirical calculation for $\sigma^2 = 1.83296703239600305442721954421041732 4 \cdots$ demonstrates the appearance of some exceptionally large partial quotients. For example, here are the first few terms:

$$\begin{aligned}\sigma^2 = [&1, 1, 4, 1, 74, 1, 8457, 1, 186282390, 1, 1, 1, 2, 1, 430917181166219, \\ &11, 37, 1, 4, 2, 41151315877490090952542206046, 11, 5, 3, 12, 2, 34, 2, 9, 8, 1, 1, 2, 7, \\ &139914688243749673927027521737571169342382939842538070 17, \ldots]\end{aligned}$$

The terms of this continued fraction form sequence A100864 in [24], and were apparently first noticed by Paul D. Hanna and Robert G. Wilson in November 2004. The very large terms form sequence A100865 in [24]. In this note, we explain the appearance of these extremely large partial quotients. The techniques have some similarity with those of Maillet (see [18] and [9, §2.14]).

Throughout the paper we use the following conventions. Given a real irrational number x with partial quotients

$$x = [a_0, a_1, a_2, \ldots]$$

we define the sequence of convergents by

$$\begin{array}{lll} p_{-2} = 0 & p_{-1} = 1 & p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 0) \\ q_{-2} = 1 & q_{-1} = 0 & q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 0) \end{array}$$

and then

$$[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}.$$

The basic idea of this paper is to use the following classical estimate:

$$\frac{1}{(a_{n+1} + 2) q_n^2} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}. \tag{1}$$

Therefore, in order to show that some partial quotients of x are huge, it is sufficient to find convergents p_n/q_n of x such that $|x - p_n/q_n|$ is much smaller than q_n^{-2}. We quantify this idea in Section 3.

Furthermore, we use the Hurwitz-Kolden-Frame representation of continued fractions [16, 17, 13] via 2×2 matrices, as follows:

$$M(a_0, \ldots, a_n) := \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix}. \tag{2}$$

By taking determinants we immediately deduce the classical identity

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1} \tag{3}$$

for $n \geq 0$.

Given a finite sequence $z = (a_0, \ldots, a_n)$ we let z^R denote the reversed sequence $(a_n, \ldots, a_0)$. A sequence is a *palindrome* if $z = z^R$. By taking the transpose of Eq. (2) it easily follows that

$$M(a_n, \ldots, a_0) := \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix}. \tag{4}$$

Hence if

$$[a_0, a_1, \ldots, a_n] = p_n/q_n$$

then

$$[a_n, \ldots, a_1, a_0] = p_n/p_{n-1}.$$

We now briefly mention ultimately periodic continued fractions. By an expression of the form $[x, \overline{w}]$, where x and w are finite strings, we mean the continued fraction $[x, w, w, w, \ldots]$, where the overbar or "vinculum" denotes the repeating portion. Thus, for example,

$$\sqrt{7} = [2, \overline{1, 1, 1, 4}\,] = [2, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \ldots].$$

We now recall a classical result (see, e.g., [20, Satz 3.9, p. 79] or [9, Ex. 2.50, p. 45]).

Lemma 2. *Let a_0 be a positive integer and w denote a finite palindrome of positive integers. Then*

$$[a_0, \overline{w, 2a_0}\,] = \sqrt{\frac{p}{q}},$$

where

$$M(w) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

$p = a_0^2\alpha + 2a_0\beta + \delta,$ *and* $q = \alpha$.

2 Three sequences

We now define three related sequences for $n \geq 2$:

$$
\begin{aligned}
u(n) &= (s_1, s_2, \ldots, s_{2^n-3}) \\
v(n) &= (s_1, s_2, \ldots, s_{2^n-2}) = (u(n), 1) \\
w(n) &= (s_1, s_2, \ldots, s_{2^n-3}, 2) = (u(n), 2).
\end{aligned}
$$

Recall that $s_n = 2^{v_2(n+1)}$ is defined in the introduction. The following table gives the first few values of these quantities:

n	$u(n)$	$v(n)$	$w(n)$
2	2	21	22
3	21412	214121	214122
4	2141218121412	21412181214121	21412181214122

The following proposition, which is easily proved by induction, gives the relationship between these sequences, for $n \geq 2$:

Proposition 3.

(a) $u(n+1) = (v(n), 2^n, v(n)^R)$;

(b) $u(n)$ *is a palindrome;*

(c) $v(n+1) = (v(n), 2^n, 1, v(n))$.

Furthermore, we can define the sequence of associated matrices with $u(n)$ and $v(n)$:

$$
\begin{aligned}
M(u(n)) &:= \begin{bmatrix} c_n & e_n \\ d_n & f_n \end{bmatrix} \\
M(v(n)) &:= \begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix}.
\end{aligned}
$$

The first few values of these arrays are given in the following table. As $d_n = e_n = z_n$ and $c_n = y_n$ for $n \geq 2$, we omit the duplicate values.

n	c_n	d_n	f_n	w_n	x_n
2	2	1	0	3	1
3	48	17	6	65	23
4	40040	14169	5014	54209	19183
5	51358907616	18174434593	6431407678	69533342209	24605842271

If we now define

$$\sigma_n = [1, \overline{w(n)}\,]$$

then Lemma 2 with $a_0 = 1$ and $w = u(n)$ gives

$$\sigma_n = \sqrt{\frac{c_n + 2e_n + f_n}{c_n}}.$$

Write $\sigma = [s_0, s_1, \ldots]$ and $[s_0, s_1, \ldots, s_n] = \frac{p_n}{q_n}$. Furthermore define $\hat{\sigma}_n = [1, u(n)]$. Notice that σ, σ_n, and $\hat{\sigma}_n$ all agree on the first $2^n - 2$ partial quotients. We have

$$|\sigma - \hat{\sigma}_n| < \frac{1}{q_{2^n-3}q_{2^n-2}}$$

by a classical theorem on continued fractions (e.g., [14, Theorem 171]), and furthermore, since $s_{2^n-3} = 2$, $s_{2^n-2} = 1$, we have, for $n \geq 3$, that

$$\sigma < \sigma_n < \hat{\sigma}_n.$$

Hence

$$|\sigma - \sigma_n| < \frac{1}{q_{2^n-3}q_{2^n-2}}.$$

Now by considering

$$M(1)M(u(n))M(1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_n & e_n \\ d_n & f_n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2c_n + d_n & e_n + f_n \\ c_n + d_n & c_n \end{bmatrix},$$

we see that $q_{2^n-3} = c_n$ and $q_{2^n-2} = c_n + d_n$. For simplicity write $g_n = c_n + 2e_n + f_n$. Then

$$|\sigma - \sigma_n| = \left|\sigma - \sqrt{\frac{g_n}{c_n}}\right| < \frac{1}{c_n^2},$$

and so

$$\left|\sigma^2 - \frac{g_n}{c_n}\right| = \left|\sigma - \sqrt{\frac{g_n}{c_n}}\right| \cdot \left|\sigma + \sqrt{\frac{g_n}{c_n}}\right| < \frac{3}{c_n^2}.$$

So we have already found good approximations of σ^2 by rational numbers. In the next section we will show that g_n and c_n have a large common factor, which will improve the quality of the approximation.

3 Irrationality measure of σ^2

From Proposition 3 (a), we get that the matrix

$$\begin{bmatrix} c_{n+1} & e_{n+1} \\ d_{n+1} & f_{n+1} \end{bmatrix}$$

associated with $u(n+1)$ is equal to the matrix associated with $(v(n), 2^n, v(n)^R)$, which is

$$\begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix} \begin{bmatrix} 2^n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_n & x_n \\ y_n & z_n \end{bmatrix} = \begin{bmatrix} 2^n w_n^2 + 2w_n y_n & 2^n w_n x_n + x_n y_n + w_n z_n \\ 2^n w_n x_n + x_n y_n + w_n z_n & 2^n x_n^2 + 2x_n z_n \end{bmatrix}.$$

Notice that

$$c_{n+1} = (2^n w_n + 2y_n) w_n. \tag{5}$$

On the other hand, we have

$$\begin{aligned} g_{n+1} &= c_{n+1} + 2d_{n+1} + f_{n+1} \\ &= c_{n+1} + 2(2^n w_n x_n + x_n y_n + w_n z_n) + 2^n x_n^2 + 2x_n z_n \\ &= c_{n+1} + 2(2^n w_n x_n + x_n y_n + w_n z_n) + 2^n x_n^2 + 2x_n z_n + 2(x_n y_n - w_n z_n + 1) \\ &= c_{n+1} + (2^n w_n + 2y_n) 2x_n + 2^n x_n^2 + 2x_n z_n + 2 \\ &= (2^n w_n + 2y_n)(2x_n + w_n) + 2^n x_n^2 + 2x_n z_n + 2, \end{aligned}$$

where we have used Eq. (3). By Euclidean division, we get

$$\gcd(g_{n+1}, 2^n w_n + 2y_n) = \gcd(2^n w_n + 2y_n, 2^n x_n^2 + 2x_n z_n + 2).$$

Next, we interpret Proposition 3 (c) in terms of matrices. We get that the matrix

$$\begin{bmatrix} w_{n+1} & y_{n+1} \\ x_{n+1} & z_{n+1} \end{bmatrix}$$

associated with $v(n+1)$ is equal to the matrix associated with $(v(n), 2^n, 1, v(n))$, which is

$$\begin{aligned} &\begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix} \begin{bmatrix} 2^n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix} \\ = &\begin{bmatrix} (2^n+1) w_n^2 + 2^n w_n x_n + y_n(w_n + x_n) & (2^n+1) w_n y_n + 2^n w_n z_n + y_n(y_n + z_n) \\ (2^n+1) x_n w_n + 2^n x_n^2 + z_n(w_n + x_n) & (2^n+1) x_n y_n + 2^n x_n z_n + z_n(y_n + z_n) \end{bmatrix}. \end{aligned} \tag{6}$$

Letting $r_n := 2^n(w_n + x_n) + w_n + y_n + z_n$, we see that

$$\begin{aligned} 2^{n+1} w_{n+1} + 2y_{n+1} &= 2^{n+1}((2^n+1) w_n^2 + 2^n w_n x_n + y_n(w_n + x_n)) + \\ &\qquad 2((2^n+1) w_n y_n + 2^n w_n z_n + y_n(y_n + z_n)) \\ &= 2(2^n w_n + y_n)(2^n(w_n + x_n) + w_n + y_n + z_n) \\ &= 2(2^n w_n + y_n) r_n. \end{aligned} \tag{7}$$

Now

$$\begin{aligned} x_{n+1} &= (2^n+1) x_n w_n + 2^n x_n^2 + z_n(w_n + x_n) \\ &= x_n(2^n(w_n + x_n) + w_n + y_n + z_n) + w_n z_n - x_n y_n \\ &= x_n r_n + 1 \end{aligned}$$

and

$$\begin{aligned} z_{n+1} &= (2^n+1)x_ny_n + 2^nx_nz_n + z_n(y_n+z_n) \\ &= z_n(2^n(w_n+x_n) + w_n + y_n + z_n) + (2^n+1)(x_ny_n - w_nz_n) \\ &= z_nr_n - 2^n - 1. \end{aligned}$$

It now follows, from some tedious algebra, that

$$\frac{2^nx_{n+1}^2 + x_{n+1}z_{n+1} + 1}{r_n} = (2^n+1)w_nx_nz_n + 2^n(2^n+1)w_nx_n^2 + z_n + (2^n-1)x_n \\ + 2^nx_n^2y_n + 2^{n+1}x_n^2z_n + x_ny_nz_n + 2^{2n}x_n^3 + x_nz_n^2. \quad (8)$$

From Eq. (5) and reindexing, we get

$$\begin{aligned} c_{n+2} &= w_{n+1}(2^{n+1}w_{n+1} + 2y_{n+1}) \\ &= 2w_{n+1}(2^nw_n + y_n)r_n, \end{aligned}$$

where we used Eq. (7). Also, from the argument above about gcd's and Eq. (8), we see that $2r_n \mid g_{n+2}$. Hence for $n \geq 2$ we have

$$\frac{g_{n+2}}{c_{n+2}} = \frac{P_{n+2}}{Q_{n+2}}$$

for integers $P_{n+2} := \frac{g_{n+2}}{2r_n}$ and $Q_{n+2} := w_{n+1}(2^nw_n + y_n)$. It remains to see that P_{n+2}/Q_{n+2} are particularly good rational approximations to σ^2.

Since w_n/x_n and y_n/z_n denote successive convergents to a continued fraction, we clearly have $w_n \geq x_n$, $w_n \geq y_n$, and $w_n \geq z_n$. It follows that

$$\begin{aligned} Q_{n+2} &= w_{n+1}(2^nw_n + y_n) \\ &= ((2^n+1)w_n^2 + 2^nw_nx_n + y_n(w_n+x_n))(2^nw_n + y_n) \\ &\leq (2^{n+1}+3)w_n^2 \cdot (2^n+1)w_n \\ &= (2^{n+1}+3)(2^n+1)w_n^3. \end{aligned}$$

On the other hand,

$$\begin{aligned} c_{n+2} &= 2Q_{n+2}r_n \\ &> 2(2^n+1)w_n^2 \cdot 2^nw_n \cdot (2^n+1)w_n = 2^{n+1}(2^n+1)^2w_n^4 \\ &\geq Q_{n+2}^{4/3}\frac{2^{n+1}(2^n+1)^2}{((2^{n+1}+3)(2^n+1))^{4/3}} \\ &> Q_{n+2}^{4/3}. \end{aligned}$$

This gives

$$\left|\sigma^2 - \frac{P_{n+2}}{Q_{n+2}}\right| < Q_{n+2}^{-8/3}$$

for all integers $n \geq 2$.

The result we have just shown can be nicely formulated in terms of the irrationality measure. Recall that the *irrationality measure* of a real number x is defined to be the infimum, over all real μ, for which the inequality

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

is satisfied by at most finitely many integer pairs (p, q).

Theorem 4. *The irrationality measure of σ^2 is at least $8/3$.*

It would be very interesting to find the precise value of the irrationality measure of σ_2. The numerical experiments conducted by the second author suggest that the bound $8/3$ in Theorem 4 is sharp; however, we do not currently know how to prove it.

Note that the classical Khintchine theorem (e.g., [12, Chapter VII, Theorem I]) states that for almost all real numbers (in terms of Lebesgue measure), the irrationality exponent equals two. Hence Theorem 4 says that the number σ^2 belongs to a very tiny set of zero Lebesgue measure.

Furthermore, the famous Roth theorem [23] states that the irrationality exponent of every irrational algebraic number is two. Therefore we conclude that σ^2 (and hence σ) are transcendental numbers. However, this result is not new. By [4, Theorem 2.1], if a continued fraction of a real number x starts with arbitrarily long palindromes, then x is transcendental. Clearly σ satisfies this property.

We now provide a lower bound for some very large partial quotients of σ^2. For each $n \geq 2$ we certainly have

$$\left|\sigma - \frac{P_{n+2}}{Q_{n+2}}\right| < Q_{n+2}^{-8/3} < \frac{1}{2Q_{n+2}^2}.$$

In particular this implies that the rational number P_{n+2}/Q_{n+2} is a convergent of σ^2.

Notice that P_{n+2} and Q_{n+2} are not necessarily relatively prime. Let $\tilde{P}_{n+2}/\tilde{Q}_{n+2}$ denote the reduced fraction of P_{n+2}/Q_{n+2}. If $\tilde{P}_{n+2}/\tilde{Q}_{n+2}$ is the m'th convergent of σ^2, then define A_{n+2} to be the $(m+1)$'th partial quotient of σ^2. Then the estimate (1) implies

$$\frac{1}{(A_{n+2}+2)\tilde{Q}_{n+2}^2} < \left|\sigma^2 - \frac{\tilde{P}_{n+2}}{\tilde{Q}_{n+2}}\right| < \frac{3}{c_{n+2}^2} \leq \frac{3}{4r_n^2\tilde{Q}_{n+2}^2},$$

Hence $A_{n+2} \geq 4r_n^2 - 2$.

From the formula for r_n and the inequalities $w_n \geq x_n, w_n \geq y_n, w_n \geq z_n$ one can easily derive

$$(2^n + 1)w_n \leq r_n \leq (2^{n+1} + 3)w_n.$$

This, together with the formula (6) for w_{n+1}, gives the estimate

$$r_{n+1} \geq (2^{n+1} + 1)w_{n+1} \geq (2^{n+1} + 1)(2^n + 1)w_n^2 > r_n^2 + 1.$$

Therefore the sequence $4r_n^2 - 2$, and in turn A_{n+2}, grow doubly exponentially. This phenomenon explains the observation of Hanna and Wilson for the sequence A100864 in [24].

The first few values of the sequences we have been discussing are given below:

n	σ_n^2	$\hat{\sigma}_n$	g_n	r_n	P_n	Q_n	A_n
3	$\frac{11}{6}$	$\frac{65}{48}$	88	834	11	6	74
4	$\frac{834}{455}$	$\frac{54209}{40040}$	73392	1282690	1668	910	8457
5	$\frac{7054795}{3848839}$	$\frac{69533342209}{5135807616}$	94139184480	3151520587778	56438360	30790712	186282390

4 Additional remarks

The same idea can be used to bound the irrationality exponent of an infinite collection of numbers $\sigma = [s_0, s_1, s_2, \ldots,]$. Indeed, there is nothing particularly special about the terms 2^n appearing in Proposition 3. One can check that the same result holds if the strings $u(n)$ and $v(n)$ satisfy the following modified properties from Proposition 3 for infinitely many numbers $n \geq 2$:

(a') $u(n+1) = (v(n), k_n, v(n)^R)$;

(b') $u(n+2)$ is a palindrome;

(c') $v(n+2) = (v(n+1), 2k_n, 1, v(n+1))$.

In particular one can easily check these properties for a string $\mathbf{s} = s_0 s_1 s_2 \cdots$ such that $s_i = f(\nu_2(i+1))$ where $f : \mathbb{N} \to \mathbb{N}$ is a function satisfying the following conditions:

1. $f(0) = 1$;
2. $f(n+1) = 2f(n)$ for infinitely many positive integers n.

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