ON SINGULAR FOLIATIONS ON THE SOLID TORUS

JOSÉ L. ARRAUT, LUCIANA F. MARTINS, AND DIRK SCHÜTZ

Abstract. We study smooth foliations on the solid torus $S^1 \times D^2$ having $S^1 \times \{\mathbf{0}\}$ and $S^1 \times \partial D^2$ as the only compact leaves and $S^1 \times \{\mathbf{0}\}$ as singular set. We show that all other leaves can only be cylinders or planes, and give necessary conditions for the foliation to be a suspension of a diffeomorphism of the disc.

1. Introduction

In this paper we consider codimension 1 foliations allowing singular leaves. Singular foliations can be defined in different ways and have been studied by several authors (see [6, 17, 18]). For a recent account of the theory see for example [1, 3, 11, 12, 15, 16]. We use as definition the one given in [14]: a C^r-*foliation* $\mathscr{F}$, $r \geq 1$, of a m-manifold M is a partition of M in connected immersed C^r submanifolds, called *leaves*, such that the module $\mathfrak{X}^r(\mathscr{F})$ of the C^r vector fields of M tangent to the leaves is transitive, that is, given $p \in M$ and $v \in T_pL$, where L is the leaf by p, there exists $X \in \mathfrak{X}^r(\mathscr{F})$ such that $X(p) = v$. This definition is equivalent to those stated in [17] and [18]. There may be leaves of different dimensions, the *dimension* d of the foliation is the greatest of these numbers, and $c = m - d$ its *codimension*. If $\dim L < d$, then L is called a *singular leaf* and $\mathrm{Sing}(\mathscr{F}) = \{p \in L; L \text{ is singular}\}$ the *singular set* of $\mathscr{F}$. A leaf of dimension d is called a *regular leaf* and the subset $\mathrm{Reg}(\mathscr{F})$ of points of M which belong to regular leaves is open.

In the case of a codimension 1 foliation on a 3-manifold, the singular leaves are therefore isolated points or 1-dimensional manifolds. Camacho and Scárdua considered in [3] codimension one foliations with isolated singularities of Morse type. Scárdua and Seade studied in [15] and [16] codimension one transversally oriented foliation on oriented closed manifolds having non-empty compact singular set which is locally defined by Bott-Morse functions. We restrict our investigation to a seemingly basic situation, namely C^2 foliations on the solid torus $S^1 \times D^2$ which have $L_0 = S^1 \times \{\mathbf{0}\}$ as their only singular leaf, and $L_1 = S^1 \times \partial D^2$ as their only compact regular leaf. The family of such foliations will be denoted by $\mathcal{A}$. To quote L. Conlon from the MathSciNet review of [16]: "Foliations with singularities are a real zoo, but they do arise in nature (e.g., the orbit foliation of a Lie group action). In order

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to get any reasonable structure theory, it is necessary to severely restrict the types of singularities." We hope that our study can contribute for this theory.

Foliations in $\mathcal{A}$ defined by orbits of an action of $\mathbb{R}^2$ are studied in [11] by Maquera and Martins. In this case the possible leaves in $S^1 \times D^2 \setminus (L_0 \cup L_1)$ are homeomorphic to a cylinder or a plane. So we can ask if the same is true for a general foliation in $\mathcal{A}$. This is the main motivation of this work.

In Section 2 we give various examples, usually starting from a vector field X on D^2 such that the intersection of $\mathscr{F}$ with $D_\theta = \{\theta\} \times D^2 \subset S^1 \times D^2$ gives exactly the phase portrait of X. A typical situation is where X has no other singularities than the origin of D^2. Using a diffeomorphism $h : D^2 \to D^2$ preserving the orbits of X we obtain a foliation on the mapping torus of h, called a *suspension*. If h preserves orientation, the mapping torus is $S^1 \times D^2$ and we get a foliation with singular set L_0.

We also give examples where X has singularities, and the resulting foliation does not come from a suspension.

In Section 3 we deal with properties of foliations in $\mathcal{A}$. For a general foliation $\mathscr{F} \in \mathcal{A}$ the discs D_θ need not intersect $\mathscr{F}$ in a phase portrait as in our examples, for any $\theta \in S^1$. However, it is possible to perturbe D_θ into general position so that we obtain a singular foliation on this disc given by the intersections of the leaves of $\mathscr{F}$ with this new disc, what clearly yield good information about the geometric behavior of the leaves of $\mathscr{F}$. It also turns out (Theorem 3.2) that this foliation is the phase portrait of a vector field, a fact which enables us to use Poincaré-Bendixson theory to study the elements of $\mathcal{A}$.

In Theorem 3.6 we describe the possible phase portraits for the case when $\mathscr{F} \in \mathcal{A}$ came from a suspension; Figure 1 lists the possibilities.

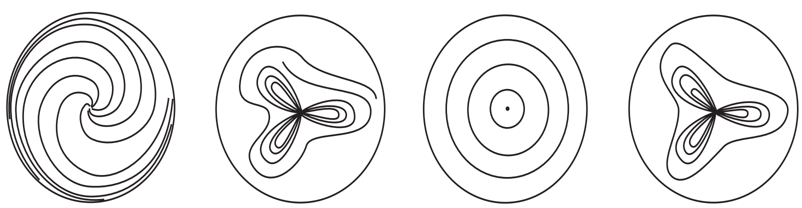

FIGURE 1. Topological models for the traces in D_θ, $\theta \in S^1$, of a foliation in $\mathcal{A}$ which is given by a suspension of a diffeomorphism of D^2.

If we allow compact regular leaves other than L_1 to exist, we show that these necessarily have to be tori (Corollary 3.12). This is then used to prove our main result:

Theorem 1.1. *Let $\mathscr{F} \in \mathcal{A}$ and L a non-compact leaf of $\mathscr{F}$. Then the inclusion $L \subset S^1 \times (D^2 - \{\mathbf{0}\})$ induces an injection on fundamental group and, consequently, L is diffeomorphic to $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$.*

This was known for foliations coming from actions of $\mathbb{R}^2$ on $S^1 \times D^2$, see [11], and Corollary 3.7 gives a simplified proof for suspensions.

We expect to be able to use the results of our investigation to study more complicated situations, such as singular foliations on 3-manifolds which admit a Heegard splitting of genus 1.

2. Examples of foliations in $\mathcal{A}$

In this section we give examples of foliations in $\mathcal{A}$. We construct elements $\mathscr{F} \in \mathcal{A}$ starting from a phase portrait of a vector field X on D^2 and such that the intersection of the leaves of $\mathscr{F}$ with $D_\theta = \{\theta\} \times D^2$ (the *traces* of $\mathscr{F}$ on D_θ) gives precisely the phase portrait of X, for all $\theta \in S^1$.

Remark 2.1. In what follows $\mathfrak{X}^r(D^2)$, $r \geq 1$, denotes the set of C^r vector fields on D^2. Let $X \in \mathfrak{X}^r(D^2)$ with a finite number of singularities and assume that ∂D^2 is an orbit. Suppose that $h \in \mathrm{Diff}^r(D^2)$ preserves the orbits of X and $h(\partial D^2) = \partial D^2$. Let M be the manifold obtained from $\mathbb{R} \times D^2$ by identifying (t,p) with $(t-1, h(p))$. The suspension of h defines a C^r foliation $\mathscr{F}(X,h)$ of M, which is the image of the foliation of $\mathbb{R} \times D^2$ whose leaves are $\mathbb{R} \times \mathcal{O}_X(p)$ by the quotient map, with $\mathcal{O}_X(p)$ the orbit of X by the point p. The foliation $\mathscr{F}(X,h)$ is called the *suspension of X by h*. Notice that if h is isotopic to the identity, then $\mathscr{F}(X,h)$ is a foliation of $S^1 \times D^2$. Its singular set consists of compact leaves of dimension one, each one associated to a singularity of X.

Example 2.2. *(Suspension)* Consider vector fields $X, Y, Z \in \mathfrak{X}(D^2)$ whose phase portraits are given by the first three pictures in Figure 1, respectively. Suppose that $X(x,y) = (-y + (1-(x^2+y^2)^{1/2})x, x + (1+(x^2+y^2)^{1/2})y)$ (note that X is invariant under a rotation R_ϕ of D^2 centered at the origin) and that $Z(x,y) = (-y, x)$. Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a C^∞-map increasing on $[0,1]$ with $\lambda(t) = 0$, if $t \leq 0$, and $\lambda(t) = 1$, if $t \geq 1$. Writing the coordinates of a point $p \in D^2$ as $\|p\|(\cos\alpha, \sin\alpha)$, we have $h \in \mathrm{Diff}^\infty(D^2)$ the map given by $h(p) = \lambda(\|p\|)(\cos\alpha, \sin\alpha)$. So we have foliations $\mathscr{F}(X, R_\phi), \mathscr{F}(Y, id), \mathscr{F}(Z, h)$ in $\mathcal{A}$, with id meaning the identity map of D^2. Every non-compact leaf of $\mathscr{F}(X, R_\phi)$ is a plane leaf if θ is irrational, or a cylinder leaf, otherwise. For $\mathscr{F}(Y, id)$ and $\mathscr{F}(Z, h)$, every non-compact leaf is a cylinder, with the first one having the property that each homoclinic orbit of Y gives a cylinder leaf L such that $L \cup L_0$ bounds a tube which is topologically a torus.

Example 2.3. *(From a phase portrait)* Let X be the smooth vector field on D^2 such that its phase portrait is given in Figure 2. The singularities of X are $\mathbf{0} = (0,0)$ and a saddle $p_0 = (a, 0)$, for some $0 < a < 1$. We suppose that X is symmetric with respect to the x-axis and the eigenvalues $\alpha < 0 < \beta$ of $DX(p_0)$ satisfy the non-resonance relation. We shall construct now a C^{k-1} foliation $\mathscr{F} \in \mathcal{A}$ such that the traces of $\mathscr{F}$ in D_θ are orbits of X, for all $\theta \in S^1$. Note that the non-resonance condition implies that $|\alpha| \neq \beta$ and so $-\frac{\alpha}{\beta} > k$ or $-\frac{\beta}{\alpha} > k$, with $k \geq 1$. We suppose $k \geq 3$ in order to have $\mathscr{F}$ at least of C^2 class. By Sternberg's Theorem (see [9, p. 126]), the non-resonance condition also implies that X is C^∞-conjugate to $DX(p_0)$ in some neighborhood of p_0.

Let ξ denote the homoclinic orbit of X indicated in bold in Figure 2, $\overline{\xi} = \xi \cup \mathbf{0}$ and η^s and η^u the stable and unstable manifolds of the saddle point p_0, respectively.

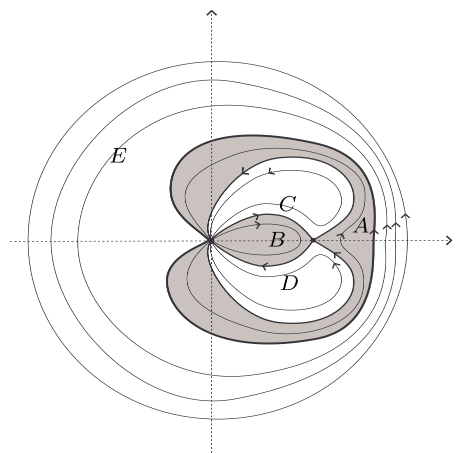

FIGURE 2. Phase-portrait of the vector field of the Example 2.3.

Let A, B, C, D and E be the open regions in D^2 indicated in the figure, which are invariant under X. Note that E is the region between $\overline{\xi}$ and ∂D^2 and every orbit of X in E is periodic.

We shall exhibit a C^k-function $g : D^2 - (\overline{\xi} \cup \partial D^2) \to \mathbb{R}$, $k \geq 3$, such that its level curves are orbits of X. The partition $\mathscr{G}$ of $\mathbb{R} \times D^2$ given by the two dimensional manifolds $graph(g+c)$, $c \in \mathbb{R}$, $\mathbb{R} \times \xi$ and $\mathbb{R} \times \partial D^2$, and by the one dimensional manifold $\mathbb{R} \times \{\mathbf{0}\}$ gives a C^{k-1}-foliation of $\mathbb{R} \times D^2$, i.e. $\mathfrak{X}^{k-1}(\mathscr{G})$ is transitive. Since $\mathscr{G}$ is invariant by translations $(q,z) \mapsto (q, z+cte)$, it induces a C^{k-1}-foliation $\mathscr{F}$ on $S^1 \times D^2$ with the property that $\mathscr{F} \in \mathcal{A}$.

Let take $\delta > 0$ and $B_\delta(p_0) \subset D^2$ the open ball with radius δ centered at p_0 contained at the interior of closed curve $\overline{\xi}$ and such that there exist a C^∞-conjugation $\varphi : B_\delta(p_0) \to B_\delta(p_0)$ between X and $DX(p_0)$ and a linear conjugation $\psi : B_\delta(p_0) \to B_\delta(p_0)$ between $DX(p_0)$ and the vector field $Y(x,y) = (\alpha x, \beta y)$. Since $-\frac{\alpha}{\beta} > k$ or $-\frac{\beta}{\alpha} > k$, it is easy to find a first integral $\varrho : \mathbb{R}^2 \to \mathbb{R}$ of Y of class C^k such that ϱ is identically null at x and y-axis, $\varrho(x,y) > 0$ if $xy > 0$ and $\varrho(x,y) < 0$ if $xy < 0$. So we get a C^k-function $f = \varrho \circ \psi \circ \varphi : B_\delta(p_0) \to \mathbb{R}$ which is a first integral of X in $B_\delta(p_0)$, f vanishes on $(\eta^s \cup \eta^u) \cup D_\delta(p_0)$, $f > 0$ in $(A \cup B) \cap D_\delta(p_0)$, and $f < 0$ in $(C \cup D) \cap D_\delta(p_0)$.

Consider now the vector field $X_\perp$ orthogonal to X at every point, that is, if $X(x,y) = (a_1(x,y), a_2(x,y))$ then $X_\perp(x,y) = (-a_2(x,y), a_1(x,y))$. So p_0 is also a hyperbolic saddle of $X_\perp$. Denote by ζ^s and ζ^u its invariant stable and unstable manifolds, respectively. Notice that $\zeta^s = \{(x,0) \in D^2\,;\, 0 < x \leq 1\}$. Let $(m,0) = \xi \cap \zeta^s$, $\zeta_0^s = \zeta^s - \{(m,0),(1,0)\}$ (see Figure 3).

Let $f_0 = f|_{(\zeta^u \cup \zeta_0^s) \cap B_\delta}$. Thus f_0 is a C^k-function and we can extend it to a C^k-function $g_0 : \zeta_0^s \cup \zeta^u \to \mathbb{R}$ that satisfies the following properties:

(a) g_0 is symmetric with respect to the x-axis;

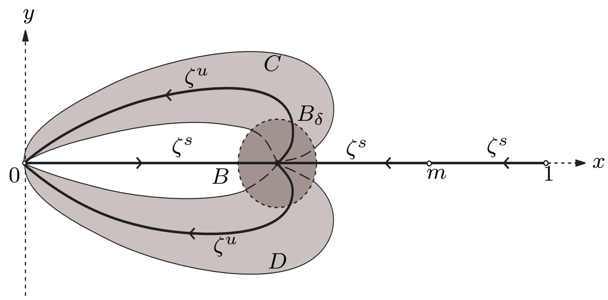

FIGURE 3. Stable and unstable manifolds of $X_\perp$.

(b) $\lim\limits_{(x,y)\to(0,0)} g_0(x,y) = -\infty$ if $(x,y)\in\zeta^u$, $\lim\limits_{x\to 0} g_0(x,0) = +\infty$,
$\lim\limits_{x\to m^-} g_0(x,0) = +\infty$, $\lim\limits_{x\to m^+} g_0(x,0) = +\infty$ and $\lim\limits_{x\to 1} g_0(x,0) = -\infty$.

(c) $\left(\dfrac{1}{g_0'}\right)^{(j)}(x,y)\to 0$ when $(x,y)\to(0,0),(m,0),(1,0)$, for all integers $j\geq 0$, and $\dfrac{1}{g_0'}$ is C^{k-1} at a neighborhood of $(0,0),(m,0)$ and $(1,0)$ (except in these points).

Note that except in $\eta^u\cup\eta^s$ and in $\overline{\xi}\cup\partial D^2$, each orbit of X intercepts $\zeta^u\cup\zeta_0^s$ in a unique point. Finally, define $g : D^2-(\overline{\xi}\cup\partial D^2)\to\mathbb{R}$ by $g(q) = g_0(p)$ if $q\notin\eta^s\cup\eta^u$, where $p=\mathcal{O}_X(q)\cap(\zeta^u\cup\zeta_0^s)$, and $g(q) = 0$ if $q\in\eta^s\cup\eta^u$. It is easy to check that g is a C^k-function satisfying:

(i) The level curves of g are orbits of X;
(ii) g vanishes on $\eta^s\cup\eta^u$;
(iii) $g(x,y)>0$ if $(x,y)\in A\cup B$ and $g(x,y)<0$ if $(x,y)\in C\cup D$;
(iv) $\lim\limits_{(x,y)\to\overline{\xi}} g(x,y) = +\infty$ when $(x,y)\in A\cup B$,
$\lim\limits_{(x,y)\to(0,0)} g(x,y) = -\infty$ when $(x,y)\in C\cup D$,
$\lim\limits_{(x,y)\to\overline{\xi}} g(x,y) = +\infty$ when $(x,y)\in E$ and $\lim\limits_{(x,y)\to\partial D^2} g(x,y) = -\infty$;
(v) $\dfrac{g_x}{g_x^2+g_y^2}$ and $\dfrac{g_y}{g_x^2+g_y^2}$ are C^{k-1} in a neighborhood of $\overline{\xi}\cup\partial D^2$ (except for points in this set).

Therefore g is the required function which induces the partition $\mathscr{G}$ of $\mathbb{R}\times D^2$ as we wanted. Now we just need to verify that $\mathfrak{X}^{k-1}(\mathscr{G})$ is transitive, i.e., given $(t,p)\in\mathbb{R}\times D^2$ and $v\in T_{(t,p)}L$, where L is the manifold of $\mathscr{G}$ by (t,p), we need to find $Z\in\mathfrak{X}^{k-1}(\mathscr{G})$ with $Z(t,p)=v$. It is easy to verify this property for a vector v which is tangent to a leaf $graph(g+c)$, or if it is the lifting to $\mathscr{G}$ of a non-zero vector of the form $X(x,y)$. To verify it for a non-horizontal vector tangent to a vertical leaf, we will construct a vector field $Y\in\mathfrak{X}^{k-1}(\mathscr{G})$ such that $Y=\frac{\partial}{\partial z}$ in a neighborhood of these leaves. Let K_1 and K_2 be compact neighborhoods of $\overline{\xi}$ and ∂D^2, respectively, where the properties (v) of g and (c) of g_0 are satisfied and such that $K_1 \cap K_2 = \emptyset$. Put $K = K_1 \cup K_2$ and define $Y_0 \in \mathfrak{X}^{k-1}(D^2)$ by:

$$Y_0|_{K-(\bar{\xi}\cup\partial D^2)} = \frac{1}{\langle grad(g), grad(g)\rangle} \cdot grad(g), \quad Y_0|_{\bar{\xi}\cup\partial D^2} = (0,0)$$

and Y_0 on $D^2 - K$ is given by any C^{k-1}-extension. Now we define $Y \in \mathfrak{X}^{k-1}(\mathscr{G})$ by:

$$(1) \qquad Y(s,q) \;=\; \begin{cases} (Y_0(q), grad(g)(q)\cdot Y_0(q)) & \text{if} \quad q \in D^2 - (\bar{\xi}\cup\partial D^2) \\ \dfrac{\partial}{\partial z} & \text{if} \quad q \in \bar{\xi}\cup\partial D^2 \end{cases}.$$

Note that $Y(s,q) = (Y_0(q),1)$, for all $q \in K - (\bar{\xi}\cup\partial D^2)$. Now it is sufficient to take $Z = \alpha Y + \beta X$, with $\alpha,\beta \in \mathbb{R}$ satisfying $v = \alpha\, Y(s,q) + \beta\, X(q)$.
Therefore $\mathscr{G}$ is a C^{k-1}-foliation of $\mathbb{R}\times D^2$ and consequently induces a C^{k-1}-foliation $\mathscr{F}$ on $S^1\times D^2$ such that $\mathscr{F} \in \mathcal{A}$. We notice that $S^1\times\bar{\xi}$ is the union of L_0 with the cylinder-leaf $S^1\times\xi$, which is topologically a torus bounding a tube T invariant by $\mathscr{F}$. All leaves in the interior of T are planes and the other leaves, except L_1, are cylinders.

Remark 2.4. The foliation $\mathscr{F}$ constructed in Example 2.3 cannot be given by an action φ of $\mathbb{R}^2$ on $S^1\times D^2$. In fact, if $\mathscr{F}$ was given by φ, since there are traces of $\mathscr{F}$ on D_θ, for some $\theta\in S^1$, which are closed curves near ∂D_θ, it was shown in [11] that then all other regular traces on D_θ are closed.

3. Properties of foliations in $\mathcal{A}$

In the above section we constructed foliations in $\mathcal{A}$ from a vector field X defined in D^2 (or its phase portrait) such that the traces of $\mathscr{F}$ in D_θ are orbits of X. Now we shall consider the inverse situation. More precisely, given a foliation $\mathscr{F}\in\mathcal{A}$ and a disk D_θ, perturbing it slightly we obtain a 2-disk Σ embedded in $S^1\times D^2$, in general position with respect to $\mathscr{F}$ and such that the foliation $\mathscr{F}^*$ in Σ induced by leaves of $\mathscr{F}$ is orientable (Theorem 3.2). This can be done such that Σ agrees with D_θ in neighborhoods of ∂D_θ and $L_0\cap D_\theta$. The foliation $\mathscr{F}^*$ clearly yields good information about the geometric behavior of the leaves of $\mathscr{F}$ in $S^1\times D^2$. We also get information about the foliation $\mathscr{F}^*$. When $\mathscr{F}$ is given by a suspension of a vector field X in D^2 by a diffeomorphism, we shall characterize X and, therefore, $\mathscr{F}^*$ (Theorem 3.6). In Subsection 3.3 we deal with the leaf structure of foliations in $\mathcal{A}$.

For the rest of the paper, given $\theta\in S^1$, we shall denote $D_\theta = \{\theta\}\times D^2 \subset S^1\times D^2$ and $q_\theta = (\theta,0)\in L_0$.

3.1. **Orientability of the traces of $\mathscr{F}\in\mathcal{A}$ in Σ.** Let $\mathscr{F}\in\mathcal{A}$ and $\mathscr{F}_0$ be the restriction of $\mathscr{F}$ to $S^1\times D^2\setminus L_0$. An embedding $g: D^2\to S^1\times D^2$ with $g(\mathbf{0})\in L_0$ is said to be in *general position* with respect to $\mathscr{F}$ if g is transverse to $\mathscr{F}$ at $g(\mathbf{0})$ and, for every distinguished map f of $\mathscr{F}_0$, the map $(f\circ g)|_{D^2\setminus\{\mathbf{0}\}}$ is locally of Morse type. The submanifold $g(D^2)$ is said to be in *general position* with respect to $\mathscr{F}$. In the examples of Section 2 the inclusion $j: D^2\to S^1\times D^2$ given by $j(D^2) = \{\theta\} \times D^2 = D_\theta$ is in general position with respect to the foliations constructed there.

Remark 3.1. If $g : D^2 \to S^1 \times D^2$ is in general position with respect to $\mathscr{F} \in \mathcal{A}$, then g induces a foliation $\mathscr{F}^*$ in $g(D^2)$ whose leaves are the connected components of the intersection of the leaves of $\mathscr{F}$ with $g(D^2)$ (the *traces* of $\mathscr{F}$ in $g(D^2)$). Furthermore, the singularities of $\mathscr{F}^*$ are the points where $g(D^2)$ is tangent to a leaf of $\mathscr{F}$.

Theorem 3.2. *If $\mathscr{F} \in \mathcal{A}$ and $\theta \in S^1$, then there exists Σ, a closed 2-disk embedded in $S^1 \times D^2$, in general position with respect to $\mathscr{F}$, such that $\partial\Sigma = \partial D_\theta$, $\Sigma \cap L_0 = q_\theta$ and the foliation $\mathscr{F}^*$ in Σ induced by $\mathscr{F}$ is given by a vector field $X_{\mathscr{F}} \in \mathfrak{X}^1(\Sigma)$ which satisfies:*

(a) *the singularities of $\mathscr{F}^*$ are isolated, hence finite in number, and are saddles or centers, except possibly the singularity q_θ;*

(b) *no two singularities of $\mathscr{F}^*$ in $\Sigma\backslash\{q_\theta\}$ are on the same leaf of $\mathscr{F}$ and so there is no connection between two different saddles in $\Sigma\backslash\{q_\theta\}$.*

Proof. Let $j : D^2 \to S^1 \times D^2$ be the inclusion such that $j(D^2) = D_\theta$. Since j is transverse to leaves L_0 and L_1, j is transverse to D_θ in neighborhoods of ∂D^2 and $\mathbf{0}$. Given $\varepsilon > 0$ and an integer $r \geq 2$, with an adaptation of Haefliger's techniques (see [7]), we obtain a C^∞-embedding $g : D^2 \to S^1 \times D^2$ in general position with respect to $\mathscr{F}$ such that g is ε-close to j in the C^r-topology and coincides with j in neighborhoods of ∂D^2 and $\mathbf{0}$. So g induces a C^r-foliation $g^*(\mathscr{F})$ on D^2 whose leaves are the connected components of the sets $g^{-1}(F)$, with F a leaf of $\mathscr{F}$. Set $\Sigma = g(D^2)$. So we have the foliation $\mathscr{F}^*$ in Σ satisfying part (a) of the theorem. By a small isotopy of $g(D^2)$ in a neighborhood of each singularity different from $\mathbf{0}$, we obtain part (b).

We shall prove now that $\mathscr{F}^*$ is orientable. For this, it is enough to prove it for the foliation $\mathscr{G} = g^*(\mathscr{F})$ in D^2. We need to show that there exists a vector field $X \in \mathfrak{X}^1(D^2)$ such that if $\mathrm{Sing}(\mathscr{G}) = S = \{p_0, p_1, \ldots, p_l\} \subset \mathrm{int}(D^2)$ with $p_0 = (0,0)$ and $p_1, \ldots, p_l$ being saddles or centers, then $X(q) = \mathbf{0}$, if $q \in S$, and $X(q)$ is a non-null vector tangent to the leaf of $\mathscr{G}$, if $q \notin S$.

The foliation $\mathscr{G}$ induces a line bundle L on $D^2 - S$ so that for $x \in D^2 - S$ we have $L_x = T_xF$, where F is the leaf of $\mathscr{G}$ by x. Note that this line bundle is a subbundle of the tangent bundle of $D^2 - S$. If we can show that L is orientable as a vector bundle, we obtain a non-zero vector field Y on $D^2 - S$ which is tangent to $\mathscr{G}$ and with $\|Y(x)\| = 1$ for all $x \in D^2 - S$, as a map $Y : D^2 - S \to \mathbb{R}^2$. Let $h : D^2 \to [0, \infty)$ be a C^∞-map with $h^{-1}(\{0\}) = S$ and such that all partial derivatives vanish at these points. Then $X : D^2 \to \mathbb{R}^2$ given by $X(x) = h(x)Y(x)$ is the desired vector field on D^2.

It remains to show that L is orientable. Since each $L_x = T_xF$ is a subspace of $\mathbb{R}^2$, we get a map $f : D^2 - S \to \mathbb{R}P^1$ sending x to L_x, which is the classifying map of the vector bundle L. Let $w \in H^1(\mathbb{R}P^1; \mathbb{Z}_2)$ be the generator, then $f^*(w) = w_1(L)$, the first Stiefel-Whitney class, whose vanishing is equivalent to orientability of L, see Milnor-Stasheff [13].

Let $D_0, \dots, D_l \subset D^2$ be small disks centered at $p_0, \dots, p_l$, respectively, such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Also let $S_j = \partial D_j$ for $j = 0, \dots, l$.
Then $H^1(D^2 - S; \mathbb{Z}_2) = \bigoplus_{j=0}^{l} H^1(S_j; \mathbb{Z}_2)$, and the j-th component of $w_1(L)$ is $w_1(L|_{S_j})$. For $j = 1, \dots, l$ we get that p_j is either a saddle or a center, hence $L|_{S_j}$ is orientable, so $w_1(L|_{S_j}) = 0$. It follows that $w_1(L) = w_1(L|_{S_0})$.
Let $i : S^1 \to D^2 - S$ be inclusion of the boundary $S^1 = \partial D^2$. Then $i^* : H^1(D^2 - S; \mathbb{Z}_2) \to H^1(S^1; \mathbb{Z}_2)$ sends each summand $H^1(S_j; \mathbb{Z}_2)$, $j = 0, \dots, l$, isomorphically onto $H^1(S^1; \mathbb{Z}_2)$. But we also get $i^*(w_1(L)) = w_1(L|_{S^1})$ and the latter is 0, as $L|_{S^1}$ is just the tangent bundle of S^1 (recall that S^1 is a leaf of $\mathscr{G}$). Therefore $i^*(w_1(L)) = 0$, and since all components except the 0-th one have been shown to be zero, we get $w_1(L) = 0$, which implies that L is orientable, which concludes the proof. □

Note that if $\mathscr{F}$ is a regular foliation on $S^1 \times D^2$, then *all* singularities of $\mathscr{F}^*$ are saddles or centres and its orientability follows from its local orientability (see [2, p. 127]). Of course Theorem 3.2 remains true if $\mathscr{F} \in \mathcal{A}$ is given by an action of $\mathbb{R}^2$ on $S^1 \times D^2$. This result is proved in [11, Prop. 3.2], but in this case we have the local orientability for all singularities of $\mathscr{F}^*$, so the proof follows with the same argument as in the regular case.

3.2. **Foliations in $\mathcal{A}$ given by suspension.** We will give necessary conditions on a vector field $X \in \mathfrak{X}^r(D^2)$, $r \geq 2$, so that a foliation $\mathscr{F} \in \mathcal{A}$ is obtained from a suspension of X by a diffeomorphism of D^2.

Remark 3.3. Let $\mathscr{F} \in \mathcal{A}$, $\theta \in S^1$, Σ be the 2-disk in general position with respect to $\mathscr{F}$ and $X_{\mathscr{F}} \in \mathfrak{X}^2(\Sigma)$ be the vector field induced by $\mathscr{F}$, as in Theorem 3.2. Since D_θ is transverse to $S^1 \times D^2$ in neighborhoods of ∂D_θ and q_θ, then there exist neighborhoods V_0 and V_1 of q_θ and ∂D_θ in D_θ, respectively, such that $D_\theta|_{V_0} = \Sigma|_{V_0}$ and $D_\theta|_{V_1} = \Sigma|_{V_1}$ (so $V_0 \cup V_1$ is transversal to $\mathscr{F}$). Let $q_1 = (\theta, x) \in \partial\Sigma \cap S^1 \times D^2$, where $x \in \partial D^2$ (note that $\partial\Sigma = \mathcal{O}_{X_{\mathscr{F}}}(q_1)$) and $\ell \subset V_1$ be a cross section to $X_{\mathscr{F}}$ at q_1 such that $P_1 : \ell' \subset \ell \to \ell$, the Poincaré map of $\partial\Sigma$, is well defined.

With the notation of the above remark, the next result follows. We note that the condition of C^2-differentiability is necessary. This has to be taken into account in Theorem 3.6 below, as the differentiability class of X is usually one less than that of $\mathscr{F}$, compare Theorem 3.2.

Lemma 3.4. *Let $\mathscr{F} \in \mathcal{A}$ and $X_{\mathscr{F}} \in \mathfrak{X}^2(\Sigma)$ be a vector field induced by $\mathscr{F}$. Then one of the following cases holds:*

(a) $P_1 = \mathrm{id}$, *i.e. every $X_{\mathscr{F}}$-orbit near $\partial\Sigma$ is periodic;*
(b) *either P_1 or $(P_1)^{-1}$ is a topological contraction, i.e. every $X_{\mathscr{F}}$-orbit near $\partial\Sigma$ spirals towards $\partial\Sigma$.*

Proof. Given $q_1 = (\theta, x) \in \{\theta\} \times \partial D^2$, let $S_1 = \{\theta\} \times \partial D^2 = \partial D_\theta$, $S_2 = S^1 \times \{x\}$ and $\alpha_i : [0, 1] \to S^1 \times D^2$ be parametrization of S_i, $i = 1, 2$ (recall that $\partial D_\theta = \partial\Sigma$). As $\ell \subset V_1$ is transverse to $X_{\mathscr{F}}$ and V_1 is transverse to $\mathscr{F}$, then ℓ is a cross section to $\mathscr{F}$ at q_1 and so we can consider

$$\mathrm{Hol} : \pi_1(L_1, q_1) \cong \mathbb{Z}^2 \to \mathrm{Diff}^2(\ell', q_1),$$

the holonomy of L_1 . Thus $P_1 = \mathrm{Hol}([\alpha_1])$. Let $P_2 = \mathrm{Hol}([\alpha_2])$. So P_1 and P_2 are commuting germs of diffeomorphisms.

Assume that P_1 does not satisfy (b), i.e. P_1 has fixed points arbitrarily close to q_1. We claim that P_2 (or P_2^{-1}) is a contraction. Otherwise, there would exist a point $q \in \ell'$, $q \neq q_1$, such that $P_2(q) = q$ and, consequently, $P_1(q) \neq q$ since L_1 is the only regular compact leaf of $\mathscr{F}$. As $X_{\mathscr{F}}$ has no singularities in V_1 we get for $q_n = P_1^n(q)$ that $P_2(q_n) = q_n$ and $\lim_{n\to\infty} q_n = q'$ (after possibly replacing P_1 with P_1^{-1}), for some $q' \neq q_1$ with $P_1(q') = q'$. Then $P_2(q') = q'$. So the leaf of $\mathscr{F}$ by q' is a regular compact leaf other than L_1, which is a contradiction. By Kopell's Lemma [10], $P_1 = id$. This proves that P_1 satisfies (a) or (b). □

We conclude from the above lemma that, in a neighborhood of $\partial\Sigma$, the orbits of $X_{\mathscr{F}}$ either spiral towards the boundary or form circles.

Definition 3.5. Let Z be a vector field on $\mathbb{R}^2$, p a singularity of Z, $k \geq 0$ and $\Gamma = \cup_{i=0}^{k} \gamma_i$, where $\gamma_0 = p$ and γ_i , $i = 1, \ldots, k$, is a regular orbit of Z. We then say that Γ is a *k-petal* of Z at p if $\mathrm{cl}(\gamma_i) \setminus \gamma_i = \{p\}$ and $\mathrm{cl}(\gamma_i)$ is the frontier of a open 2-disk D_i such that $D_i \cap D_j = \emptyset$ for $j = 1, \ldots, k$ with $j \neq i$.

Note that a 0-petal of Z at p is only the point p. See Figure 1(b) or (d) for a 3-petal at p. It is possible that $k = \infty$, in that case the diameters of the discs D_i have to converge to 0. If $\Gamma = \bigcup_{i=0}^{k} \gamma_i$ is a k-petal of X at p, then $\mathrm{int}(\Gamma) = \bigcup_{i=0}^{k} \mathrm{int}(\overline{\gamma_i})$, with $\mathrm{int}(\overline{\gamma_i})$ denoting the interior of the open 2-disk in $\mathbb{R}^2$ which γ bounds. We also note that when $\mathscr{F} \in \mathcal{A}$ is given by a suspension of $X \in \mathfrak{X}^r(D^2)$, $r \geq 2$, by some $h \in \mathrm{Diff}^r(D^2)$, then $\Sigma = \{\theta\} \times D^2$, where Σ is the 2-disk given in Theorem 3.2, and the foliation $\mathscr{F}^*$ in Σ induced by $\mathscr{F}$ coincides with the phase portrait of X (i.e. $X \equiv X_{\mathscr{F}}$).

Theorem 3.6. *If $\mathscr{F}(X,h) \in \mathcal{A}$ is a suspension of $X \in \mathfrak{X}^r(D^2)$, $r \geq 2$, for some $h \in \mathrm{Diff}^r(D^2)$, then $\mathbf{0} \in D^2$ is the unique singular point of X and there exists an X-invariant neighborhood V of ∂D^2 in D^2, homeomorphic to $S^1 \times (0,1]$, such that $\Gamma = \mathrm{Front(V)}$, the frontier of V in D^2, is a k-petal of X at $\mathbf{0}$, for some $k \geq 0$, and all orbits inside $V \setminus \partial D^2$ have the same topological type. Furthermore, if $p \in V \setminus \partial D^2$ and $\mathcal{O}_p(X)$ is homeomorphic to $\mathbb{R}$ then $\alpha(p) \cup \omega(p) = \partial D^2 \cup \Gamma$. If $k \geq 1$ then $\mathrm{int}(\Gamma)$ is a union of homoclinic orbits of X at $\mathbf{0}$.*

Proof. Since L_0 is the only singular leaf of $\mathscr{F}$, then $\mathbf{0}$ is the only singular point of X and therefore $\mathbf{0} \in \mathrm{int}(\gamma)$, for all periodic orbit γ of X. Furthermore, by the Poincare-Bendixson Theorem, if $q \in D^2 - \{\mathbf{0}\}$ then the ω-limit set of q, $\omega(q)$, either is a periodic orbit of X or there exists Γ, a k-petal of X at 0, $k \geq 0$, such that $\omega(q) = \Gamma$. If $k > 0$, since $\mathbf{0}$ is the only singularity of X, it follows that $\mathrm{int}(\Gamma)$ is a union of homoclinic orbits of X at $\mathbf{0}$. The same is true for the α-limit set of q, $\alpha(q)$.

Let γ be a periodic orbit of X in $\mathrm{int}(D^2)$. Since $h(\gamma)$ is also a period orbit of X, it follows that either $h(\gamma) \subset \mathrm{int}(\gamma)$ or $h(\gamma) \subset \mathrm{ext}(\gamma)$, where $\mathrm{ext}(\gamma)$ is the connected component of $D^2 - \gamma$ which contain ∂D^2. Let us suppose that $h(\gamma) \subset \mathrm{ext}(\gamma)$. Consequently there exists a sequence $(h^n(\gamma))_{n\geq 0}$ of period orbit of X satisfying $h^{n+1}(\gamma) \subset \mathrm{ext}(h^n(\gamma))$, for all $n \geq 0$. Let γ' the periodic orbit of X such that $\gamma' = \lim_{n\to\infty} h^n(\gamma)$. So, $\gamma' = h(\gamma')$ and therefore $\gamma' = \partial D^2$. (If $h(\gamma) \subset \mathrm{int}(\gamma)$ then to get the same sequence, it is enough to take h^{-1} instead h.)
By Lemma 3.4, there exists a maximal X-invariant neighborhood V of ∂D^2 in D^2, homeomorphic to $S^1 \times (0, 1]$, where every orbit of X is periodic. Let $\Gamma = \mathrm{Front(V)}$. We claim that Γ is a k-petal of X at $\mathbf{0}$, for some $k \geq 0$. Since Γ is a connected set, it is enough to show that Γ is not a periodic orbit of X. So, suppose that Γ is periodic. Hence $h(\Gamma)$ is a periodic orbit of X. Since V is maximal then $h(\Gamma)$ can not be a subset of $\mathrm{int}(\Gamma)$. But if $h(\Gamma) \subset \mathrm{ext}(\Gamma)$ then there exist a periodic orbit of X, $\sigma \subset \mathrm{int}(\Gamma)$, such that $h(\sigma) = \Gamma$. But this again contradicts the fact that V is maximal. This contraction proves that Γ is not a periodic orbit of X. Therefore $\mathrm{Front}(V) = \Gamma$, where Γ os a k-petal at $\mathbf{0}$, $k \geq 0$, and this completes the proof. □

As an immediate consequence of the above theorem, we have:

Corollary 3.7. *If $\mathscr{F} \in \mathcal{A}$ is a suspension of $X \in \mathfrak{X}^r(D^2)$, $r \geq 2$, then all leaves other than L_0 and L_1 are diffeormorphic to $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$.*

3.3. **The leaf structure of foliations in $\mathcal{A}$.**

Definition 3.8. A singular foliation is called *transversely orientable* if the restriction to the regular foliation is transversely orientable.

Lemma 3.9. *Let $\mathscr{F} \in \mathcal{A}$. Then $\mathscr{F}$ is transversely orientable.*

Proof. Let $T\mathscr{F}$ be the subbundle of the tangent bundle of $S^1 \times (D^2 - \{\mathbf{0}\})$ induced by the restriction $\mathscr{F}_0$ of $\mathscr{F}$ to the regular foliation, and $N\mathscr{F}$ the corresponding quotient bundle. We need to show that $N\mathscr{F}$ is an orientable bundle, that is, that the first Stiefel-Whitney class $w_1(N\mathscr{F})$ vanishes.
We have $H^1(S^1 \times (D^2 - \{\mathbf{0}\}); \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ corresponding to the product structure of $S^1 \times (D^2 - \{\mathbf{0}\})$, so we write $w_1(N\mathscr{F}) = (a, b)$ with $a, b \in \mathbb{Z}_2$.
Let $\mathcal{R}$ be the Reeb foliation on $S^1 \times D^2$, then $\mathcal{R}$ and $\mathscr{F}$ agree on $S^1 \times S^1$, and we can glue them to a singular foliation $\mathscr{G}$ on $S^1 \times S^2$, which is a regular $C^{r,0+}$-foliation on $S^1 \times S^2 - \{\mathbf{0}\}$, in the sense of [4, §3.4]. In particular, we still have a C^0-subbundle $T\mathscr{G}$ of $TS^1 \times (S^2 - \{\mathbf{0}\})$, and the corresponding quotient bundle $N\mathscr{G}$. If $i : S^1 \times (D^2 - \{\mathbf{0}\}) \to S^1 \times (S^2 - \{\mathbf{0}\})$ denotes inclusion, we get $w_1(N\mathscr{F}) = i^*(w_1(N\mathscr{G}))$ which implies $b = 0$, as $i^* : H^1(S^1 \times (S^2 - \{\mathbf{0}\}); \mathbb{Z}_2) \to H^1(S^1 \times (D^2 - \{\mathbf{0}\}); \mathbb{Z}_2)$ is just inclusion of the first factor.
Now let $\mathcal{R}'$ be the Reeb foliation on $D^2 \times S^1$, which we can glue to $\mathscr{F}_0$ to get a foliation $\mathscr{G}'$ on $S^3 - S^1 = D^2 \times S^1 \cup S^1 \times (D^2 - \{\mathbf{0}\})$. Now $j^* : H^1(S^3 - S^1; \mathbb{Z}_2) \to H^1(S^1 \times (D^2 - \{\mathbf{0}\}); \mathbb{Z}_2)$ is induced by inclusion of the second factor, so the analogous argument for $\mathscr{G}'$ gives $a = 0$, which means that $w_1(N\mathscr{F}) = 0$. □

Let $\mathcal{B}$ be the set of singular C^r-foliations, $r \geq 2$, of $S^1 \times D^2$, which have $L_0 = S^1 \times \{\mathbf{0}\}$ and $L_1 = S^1 \times \partial D^2$ as compact leaves, and such that L_0 is the only singular leaf. Note that $\mathcal{A} \subset \mathcal{B}$ consist of those foliations which do not have other compact leaves.

Lemma 3.10. *Let $\mathscr{F} \in \mathcal{B}$. There exists a tangential vector field on $S^1 \times D^2$ which is non-zero in a neighborhood of L_0.*

Proof. Since L_0 is a singular leaf, for every $\theta \in S^1$ there exists a tangential vector field which is non-zero near $(\theta, 0) \in L_0$. By choosing the same direction for all θ, we can glue these vector fields together to give a tangential vector field which has L_0 as a closed orbit. This is the required vector field. $\square$

Theorem 3.11. *Assume we have a C^1-embedded closed surface L in $S^1 \times \operatorname{int} D^2$ and there exists a non-zero vector field X on $S^1 \times D^2$ such that X is transverse to $L \cup S^1 \times \partial D^2$. Then L is a torus.*

Proof. We can embed $S^1 \times D^2$ into S^3, and it follows by Alexander duality that $S^3 - L$ consists of two connected components. By an easy Mayer-Vietoris argument we get that $(S^1 \times D^2) - L$ also has two connected components. Denote by M_1 and M_2 these two components.
Since L is C^1-embedded in $S^1 \times D^2$, we get that both M_1 and M_2 are compact oriented manifolds with boundaries $\partial M_1 = L$ and $\partial M_2 = L \cup (S^1 \times S^1)$. By the Poincaré-Hopf Theorem, we get $\chi(M_i) = 0$ for $i = 1, 2$. Here we may have to change the vector field near $S^1 \times S^1$ so that it points outward (or inward) on ∂M_2. It follows that $\chi(L) = 0$, so L is the torus, as it is the boundary of an orientable manifold. $\square$

Corollary 3.12. *Let $\mathcal{F} \in \mathcal{B}$ and $L \subset S^1 \times (\operatorname{int} D^2 - \{\mathbf{0}\})$ be a compact leaf. Then L is diffeomorphic to $S^1 \times S^1$.*

Proof. By Lemma 3.9 we know that $\mathcal{F}$ is transversely orientable, so there exists a vector field on $S^1 \times (D^2 - \{\mathbf{0}\})$ which is transverse to the restriction of $\mathcal{F}$. We can extend this to a vector field Y on $S^1 \times D^2$ which is zero on $S^1 \times \{\mathbf{0}\}$ and transverse to $\mathcal{F}$ everywhere else. Let X be the tangential vector field from Lemma 3.10. Then $X + Y$ is a nonzero vector field on $S^1 \times D^2$ which is transverse to L. $\square$

We now want to begin the proof of Theorem 1.1. It is roughly along the lines of the proof in [2, Ch.VII] that every smooth foliation of S^3 has a compact leaf. The crucial steps in [2] are that non-compact leaves L admit transverse loops going through L while leaves containing a simple vanishing cycle do not admit such loops. Since we are dealing with a regular foliation on the non-compact manifold $S^1 \times (D^2 - \{\mathbf{0}\})$, we have to be more careful with these arguments, and in particular we have to deal with certain special cases that arise.
First of all we will study the leaf structure of $\mathscr{F} \in \mathcal{A}$ in neighborhood of singular leaf L_0 .
Let $\mathscr{F} \in \mathcal{B}$ and $\theta \in S^1$. Using Theorem 3.2 we can perturbe D_θ away from q_θ and its boundary, getting a 2-disk Σ so that it is in general position with respect to $\mathscr{F}$.

This gives rise to a singular foliation on Σ which we denote by $\mathscr{F}^*$ and is given by the phase portrait of a vector field $X_{\mathscr{F}}$. Notice that $q_\theta \in \Sigma$ is a singularity of $\mathscr{F}^*$.

Proposition 3.13. *If $\mathscr{F} \in \mathcal{B}$, then there exist a neighborhood W of L_0, a C^2 diffeormorphism $h : A \to U$ and $X \in \mathfrak{X}^2(A)$, where A and U are neighborhoods in D_θ of q_θ, such that h preserves the orbits of X and $\mathscr{F}|_W$ is topologically equivalent to the suspension of X by h.*

Proof. Let $Z \in \mathfrak{X}^2(\mathscr{F})$ be the tangential vector field given in Lemma 3.10 and U be a neighborhood of q_θ in $D_\theta \cap \Sigma$ such that q_θ is the only singularity of $X_{\mathscr{F}}$ and the Poincaré diffeomorphism of Z at q_θ, $h : A \to U$, is well defined, where $A = \{(\theta, x) \in S^1 \times D^2; |x| < \varepsilon\}$, for some $\varepsilon > 0$. Note that h is of class C^2 and preserves the orbits of $X = X_{\mathscr{F}}|_A$. So the desired neighborhood exists. $\square$

Recall that a *simple vanishing cycle* on a codimension 1 foliation $\mathscr{F}$ on a 3-manifold M consists of a smooth loop $\sigma_0 : S^1 \to L$ to a leaf L of $\mathscr{F}$ which is not homotopic to a constant loop in L, and there is a smooth map $\sigma : S^1 \times [0, \varepsilon] \to M$ with $\sigma(\cdot, 0) = \sigma_0$ and

1. (1) Each $\sigma_t : S^1 \to M$ has image in a leaf L_t of $\mathscr{F}$ for all $t \in [0, \varepsilon]$.
2. (2) For each $z \in S^1$, the path $t \mapsto \sigma_t(z)$ is transverse to $\mathscr{F}$.
3. (3) For $t > 0$, σ_t is null-homotopic in L_t, and the lift to the universal cover of L_t, which is assumed to be $\mathbb{R}^2$, $\hat{\sigma}_t : S^1 \to \mathbb{R}^2$, is a simple closed curve.

Lemma 3.14. *Let $\mathscr{F} \in \mathcal{A}$. If the leaf $L \in \mathscr{F}$ admits a simple vanishing cycle, then there does not exist a loop transverse to $\mathscr{F}$ going through L.*

Proof. As in [2, Prop.VII.5] or [5, §9.3] we get a smooth immersion $H : D^2 \times (0, \varepsilon] \to S^1 \times (D^2 - \{\mathbf{0}\})$ such that

1. (1) H extends smoothly to $S^1 \times [0, \varepsilon] \to S^1 \times (D^2 - \{\mathbf{0}\})$, and this is a simple vanishing cycle.
2. (2) For each $t > 0$ the image of $H_t(D^2)$, denoted by D_t, is in a leaf L_t.

Notice that the compactness of the ambient manifold is not necessary, indeed, the arguments in [5, §9.3] work in a manifold which has part of the boundary removed. We claim there is a monotone decreasing sequence $t_n \to 0$ such that L_{t_n} and L_{t_m} are the same leaf and $D_{t_n} \subset D_{t_m}$ for $m > n$. As in the proof of [2, Prop.VII.6], let

$$U \quad = \quad \{x \in D^2 \mid \lim_{t \to 0} H(x, t) \text{ exists } \},$$

which is an open set and is not all of D^2. If $x_0 \in D^2 - U$, we can find a sequence $t_n \to 0$ with $H(x_0, t_n) \to p_0 \in S^1 \times D^2$. If p_0 is contained in a regular leaf, the claim follows as in the proof of [2, Prop.VII.6].

So assume that for all $x \in D^2 - U$ the sets $\{H(x, t) \mid t \in (0, \varepsilon)\}$ have all accumulation points in $L_0 = S^1 \times \{\mathbf{0}\}$, the singular leaf.

For $\delta > 0$ let $D(0, \delta) = \{x \in D^2 \mid \|x\| \leq \delta\}$ and $S(\delta) = \partial D(0, \delta)$. If $\delta > 0$ is small enough, we have $H_t(S^1) \cap S^1 \times D(0, \delta) = \emptyset$ for all $t \in [0, \varepsilon]$.

Let $q : S^1 \times D^2 \to D^2$ be the projection to the second factor. Then for each $x_0 \in D^2 - U$ we have $\lim_{t \to 0} q(H(x_0, t)) = \mathbf{0} \in D^2$, for otherwise there is a $\delta > 0$

and a sequence $t_n \to 0$ with $q(H(x_0, t_n)) \in S(\delta)$, in which case we would get accumulation points not in L_0.

Using Theorem 3.2 choose a disc Σ which intersects $\mathscr{F}$ transversely, and let $\mathscr{G}$ be the induced singular foliation of Σ. We can assume that $D(0,\delta) \subset \Sigma$ is the disc around $q_\theta \equiv \mathbf{0}$. Then $S_t \cap D(0,\delta) = \emptyset$, where $S_t = \partial D_t$, so $D_t \cap D(0,\delta)$ is a union of intervals whose boundaries are on $S(0,\delta)$. Hence these intervals lie in the hyperbolic parts of hyperbolic or parabolic sectors in $S(0,\delta)$, compare [8, §VII.8]. As the D_t approach q_θ, they are in the finitely many hyperbolic sectors for small $t > 0$.

Therefore there exists a hyperbolic sector S_h such that $D_t \cap S_h$ contains points x_t with $x_t \to q_\theta$ as $t \to 0$. Denote by $[x_t]$ the component of $D_t \cap S_h$. As the suspension, which we denote by h, sends hyperbolic sectors to hyperbolic sectors, some positive iteration h^n fixes S_h. After replacing n with $-n$, if necessary, we get that $h^n([x_t]) = [x_{s(t)}]$ with $s(t) \leq t$ for all sufficiently small t. Note that $L_t = L_{s(t)}$, and the suspension gives a path $\alpha : [0,1] \to L_t$ from x_t to $h^n(x_t)$ which has empty intersection with S_t.

Repeating this, we get a path $\alpha : [0,\infty) \to L_t$ starting in x_t and never crossing S_t. After lifting α to $\tilde{\alpha} : [0,\infty) \to \tilde{L}_t$ we see that $\tilde{\alpha}$ has to stay in a compact piece of $\tilde{L}_t \cong \mathbb{R}^2$ bounded by a circle lifting S_t. However, after composing with $\tilde{\imath} : \tilde{L}_t \to \mathbb{R} \times D^2$, which is the lift of the inclusion $L_t \subset S^1 \times D^2$ to covering spaces, we see that the image does not stay in a compact subset of $\mathbb{R} \times D^2$, as we can assume that $p \circ \tilde{\imath} \circ \tilde{\alpha}(s) = s \in \mathbb{R}$ for $s \in [0,\infty)$, where $p : \mathbb{R}^2 \times D^2 \to \mathbb{R}^2$ is projection. This follows from the way α was defined using the suspension. But this is a contradiction. Therefore, there have to be other accumulation points away from L_0.

The claim then follows as in [2, Prop.VII.6]. The rest of the proof is as in the proof of [2, Thm.VII.2], where a mapping torus is used to show the non-existence of a transverse loop through the leaf containing the simple vanishing cycle. $\square$

Lemma 3.15. *Let $\mathscr{F} \in \mathcal{A}$ and $L \in \mathscr{F}$ be a regular leaf. If L does not intersect any transverse loop of $\mathscr{F}$, then the map $\pi_1(L) \to \pi_1(S^1 \times (D^2 - \{\mathbf{0}\}))$, induced by inclusion, is injective.*

Proof. Note that the compact leaf $L_1 = S^1 \times S^1$ induces an injection on fundamental group. We may therefore assume that L is non-compact. It then follows that L has accumulation points in the compact set $S^1 \times D^2$. If there is an accumulation point on a regular leaf L', we can use the argument of [2, Ch.VII, Prop.4] to construct a transverse loop intersecting L. Therefore, all accumulation points are contained in the singular leaf L_0.

Now choose a disc Σ which is transverse to the foliation using Theorem 3.2, and such that $L \cap \Sigma$ does not contain singularities. Then $L \cap \Sigma$ consists of copies of $\mathbb{R}$ and S^1. We have to consider the following three cases:

(1) There do not exist copies of $\mathbb{R}$ in $L \cap \Sigma$.
(2) There are finitely many copies of $\mathbb{R}$ in $L \cap \Sigma$.
(3) There are infinitely many copies of $\mathbb{R}$ in $L \cap \Sigma$.

The idea in all three cases is basically the same. As in the proof of Theorem 3.12, there is a vector field Y on $S^1 \times D^2$ which is zero on L_0 and transverse to every regular leaf. Furthermore, there is a tangential vector field X which is non-zero only in a small neighborhood of L_0, and for which L_0 is a closed orbit. Thus the vector field $X+Y$ is non-zero on $S^1 \times D^2$ and transverse to the regular foliation on $S^1 \times (D^2 - \{\mathbf{0}\})$. We then modify L to a closed surface L' which is still transverse to $X+Y$, possibly after slightly modifying X and Y. By Theorem 3.11 L' is a torus, and because of the way that L' is obtained from L we get that L is either diffeomorphic to $S^1 \times \mathbb{R}$ or $\mathbb{R}^2$.

Case 1. Since the foliation is given by a suspension near L_0, there have to be infinitely many circles in $L \cap \Sigma$. Of these, only finitely many cannot contain q_θ. This follows as every circle in $L \cap \Sigma$ has to contain a singularity of which there are only finitely many, and no further accumulation points away from q_θ exist.
Let $U \subset \Sigma$ be a small neighborhood of q_θ such that the Poincaré return map of X is defined on U. Then only finitely many of the circles in $L \cap \Sigma$ are not contained in U. Furthermore, if $i : L \to S^1 \times D^2$ is the inclusion, then $K = i^{-1}(S^1 \times D^2 - (S^1 \times U))$ is compact, for otherwise there would be more accumulation points. Also, by slightly changing the neighborhood $S^1 \times U$ of L_0, we can assume that K is a compact surface with boundary. Because of the suspension property of U, the remaining circles in $L \cap \Sigma$ are pushed closer to L_0 in one direction, and closer to K in the other. This means that each such circle is part of a tame end of L, and since K is compact, we get that L has finitely many ends. Note that K need not be connected. However, we now replace K with the compact surface obtained from L by removing the ends. Each end can be thought of as a $S^1 \times [0,\infty)$, embedded into $S^1 \times D^2 - \{\mathbf{0}\}$, with each $S^1 \times \{i\}$ a circle in Σ for i a non-negative integer. Each such circle represents the same generator in $H_1(\Sigma - \{q_\theta\}) \cong \mathbb{Z}$, where the orientation comes from the transverse orientability of $\mathscr{F}$. For each end e of L denote by S_e the circle $S^1 \times \{\mathbf{0}\} \subset \Sigma - \{q_\theta\} \subset S^1 \times (D^2 - \{\mathbf{0}\})$, and its homology class by $[S_e] \in H_1(S^1 \times (D^2 - \{\mathbf{0}\}))$. Now notice that $K \subset S^1 \times (D^2 - \{\mathbf{0}\})$ bounds these finitely many circles, so

$$\sum [S_e] \quad = \quad 0 \quad \in \quad H_1(S^1 \times (D^2 - \{\mathbf{0}\})),$$

which implies that there is an even number of ends, and the induced orientations cancel out in pairs. In terms of the normal direction in Σ this means that for half of the circles the transverse vector field in Σ points outward, and for the other half it points inward.
Assume for the moment that we only have two ends, one pointing inward and one pointing outward. It is clear that the two ends have to alternate when we look at all circles near $q_\theta \in \Sigma$. We can therefore assume that the circle S_e corresponding to the end which points inward has smaller diameter than the circle $S_{e'}$ corresponding to the other end, and there are no circles between the two.
Now add a cylinder $S^1 \times [-1,1]$ between the two circles (viewed in $S^1 \times D^2$) as in Figure 4. Note that the vector field X is pointing upward in Figure 4, while the vector field Y is indicated as pointing horizontally. Also, we can think of X

as being large compared to Y, as we can assume to be very close to L_0, where Y vanishes.

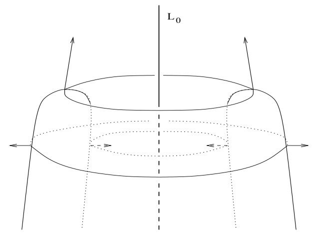

FIGURE 4. Two ends connected by a cylinder.

The vector field $X+Y$ remains transverse to the resulting surface L'.
Note that the circles need not be perfectly round, but we have an annulus between them, which gives a piecewise smooth surface. Rounding the corners then gives the surface L'.
If we have more than two ends, we can still arrange the circles S_e in Σ so that all ends are represented exactly once in an annulus around q_θ, with the innermost circle pointing inward and the outermost circlepointing outward. By induction we can put cylinders between adjacent circles with opposite directions as in the above case, and so that the resulting surface remains transverse with respect to $X+Y$.
By Theorem 3.11 we get that L' is a torus. As L' was obtained from K by adding handles to boundary circles, this is only possible if K is a sphere with two open discs removed. This implies that L is diffeomorphic to $S^1\times\mathbb{R}$. Since the circle S_{e_1} represents a non-zero element in $H_1(\Sigma-\{q_\theta\}))$, we get that $\pi_1(L)\to\pi_1(S^1\times(D^2-\{\mathbf{0}\}))$ is injective.

Case 2. Since we assume that L only has accumulation points in L_0, each copy of $\mathbb{R}$ is compactified by $q_\theta\in\Sigma$ by Poincaré-Bendixson theory. We call such a compactified copy of $\mathbb{R}$ a petal. There can only be finitely many circles in $\Sigma\cap L$: Each circle has to have a singularity on the inside, and as there are only finitely many in Σ, the circles would have to accumulate at one. This can only be $q_\theta\in\Sigma$, but since we now assume the existence of at least one petal, the circles would contain a whole petal in their closure.
Choose a small neighborhood U of q_θ in Σ on which the suspension property for $\mathscr{F}$ holds, and so that U intersected with each petal consists of two lines. These finitely many lines are permuted by the suspension, so after passing to a finite cover we can assume that the suspension acts on these lines as the identity. It follows that there is a neighborhood W_0 of L_0 diffeomorphic to $S^1\times\operatorname{int}D^2$ such that $W_0\cap L$ consists of an even number of copies of $S^1\times\mathbb{R}$.
Two of these copies form a $S^1\times\vee$ and we can form a C^1-surface L' from L by replacing this with $S^1\times\smile$, compare Figure 5. We can also change the vector field Y near L_0 so that it is transverse to L'. This can produce new singularities of Y,

but by making changes only close to L_0, we can assume that $X+Y$ is still non-zero, and furthermore $X+Y$ is transverse to L'.
We have that $L \cup L_0$ is compact, and only a compact part of L stays away from L_0, so we get that L' is a closed surface. By Theorem 3.11 it follows that L' is a torus. Changing L' back to L, we see that L is diffeomorphic to $S^1 \times \mathbb{R}$ and so is every space finitely covered by L. The statement on the fundamental group is clear from the above.

Case 3. As in the second case there are only finitely many circles in $L \cap \Sigma$, and each copy of $\mathbb{R}$ will compactify to a petal with q_θ. Only finitely many of these petals can not be contained in a given neighborhood of q_θ, for we would otherwise get accumulation points away from L_0. In particular, for a neighborhood U of q_θ where the Poincaré return map h of X is defined, only finitely many petals are not contained in U.
Each petal has an induced orientation, and the ends of the petal are permuted by h with orientations preserved. Of course, petals are mapped to petals if they are contained in U, but if a petal is not contained in U, the two ends need not be mapped to ends of the same petal under h. Each petal bounds a disc by the Schoenflies theorem, and we will refer to the interior of this disc as the interior of the petal.
If a petal is contained in U, there can be no singularity of the induced foliation of Σ in the interior of the petal. Now assume that two petals P and P' are contained in U such that the interior of P is contained in P'. We claim that there is a transverse path in Σ from P to P'. This path can be extended to a transverse loop through L, contradicting our assumption on L.
To see this, note that by Poincaré-Bendixson theory every leaf of the foliation on Σ in the interior of P' is also a petal. We look at all transverse paths starting on P going to the outside. If we can find a transverse path starting at P and going to another leaf P^* in the interior of P', we can find a transverse path starting at P and ending at any given point of P^* using typical arguments as in [2, Prop.VII.4]. Therefore, we can always extend such a path until we reach P'.
Therefore we can assume that the petals coming from L contained in U have disjoint interiors.
Each petal P has an orientation by Theorem 3.2. If we think of P as a copy of $\mathbb{R}$, it has two ends, one oriented by $+$ and one by $-$. We write e_P^+ and e_P^- for these ends. We can act on this set of oriented ends using h.
If for some end e_P^+ each $h^i(e_P^+)$ is the positive end of a petal $P_i \subset U$ for all $i \in \mathbb{Z}$, these petals can be connected by the flow of X and we get a connected component of L homeomorphic to $\mathbb{R}^2$ (if $P_i \neq P$ for all $i \neq 0$) or $S^1 \times \mathbb{R}$ (if $P_i = P$ for some $i > 0$). As L is connected we would be finished.
Therefore we can assume that every petal contained in U has their ends equivalent under the action of h to the ends of a petal not contained in U. Since there are only finitely many such petals, we can only have finitely many orbits under the action of h. This also implies that L can only have finitely generated homology and is obtained from a compact surface by removing finitely many points.

Let $P \subset U$ be a petal such that $h^i(P) \subset U$ for all $i \leq 0$. Since the interior of P contains no other petals of L, the ends $h^i(e_P^+)$ and $h^i(e_P^-)$ are always next to each other for all $i \in \mathbb{Z}$. Note that they need not be ends of the same petal, but they are ends of adjacent petals. Since the petals $h^i(P)$ for $i \leq 0$ are all different, there is an $i_0 > 0$ such that for $i \geq i_0$ also the ends $h^i(e_P^+)$ and $h^i(e_P^-)$ belong to petals contained in U.

We claim that for $i \geq i_0$ we actually get that $h^i(e_P^+)$ and $h^i(e_P^-)$ belong to the same petal P_i.

To see this, note that we can think of P as an interval connecting the two ends e_P^+ and e_P^-. We can think of this interval as a path in an end $S^1 \times [0, \infty)$, which can be pushed away from every compact subset of $S^1 \times [0, \infty)$ using h^i with $i < 0$. Furthermore, such a path can be extended to a loop in this end, away from any compact subset of $S^1 \times [0, \infty)$. Here we think of $h^i(P) \subset S^1 \times \{-i\}$ for $i < 0$, and we can extend this path at both endpoints very close to L_0, passing through $h^j(e_P^+)$ for $j \geq i$ on one end, and through $h^j(e_P^-)$ for $j \geq i$ on the other end. To close the loop for large $j \geq i_0$ we need the ends $h^j(e_P^+)$ and $h^j(e_P^-)$ to be on the same petal.

Let $P_1, \dots, P_k$ be the finitely many petals not contained in U and let $Q_1, \dots, Q_m$ be petals in U with the property that $h^i(Q_j) \subset U$ for all $i \leq 0$, with all of their ends in different orbits and so that every infinite orbit of an end has a unique representation among the ends of the Q_j. By the previous claim there exist petals $Q_1', \dots, Q_m'$ with $h^i(Q_j') \subset U$ for all $i \geq 0$ and $h^{i_0}(e_{Q_j}^\pm) = e_{Q_j'}^\pm$.

Furthermore, let $P_{k+1}, \dots, P_n \subset U$ be the petals which are not of the form $h^{-i}(Q_j)$ or $h^i(Q_j')$ for $i \geq 0$. Then any petal P of L is either one of the $P_1, \dots, P_n$, or $P = h^i(Q_j)$ for some j and $i \leq 0$, or $P = h^i(Q_j')$ for some j and $i \geq 0$.

Note that there may be orbits of ends which are finite, but each one would have to contain a petal not contained in U, as otherwise the leaf L would just be $S^1 \times \mathbb{R}$ and we would be in Case 2. After passing to an appropriate finite cover of $S^1 \times D^2$ we can assume that h acts trivially on such ends, that is, each such orbit only contains one element. The petals $Q_1, \dots, Q_m$ and $Q_1', \dots, Q_m'$ are each the boundary of submanifolds $\mathbb{R} \times [0, \infty)$ contained in L. Let us denote by $\tilde{K}$ the submanifold obtained from L by removing these copies of $\mathbb{R} \times (0, \infty)$, so that the boundary of $\tilde{K}$ is the disjoint union of the $Q_1, \dots, Q_m$ and $Q_1', \dots, Q_m'$. The ends $e^\pm$ of each such petal together with q_θ forms a '$\vee$' figure in Σ, which we simply denote as $\vee$. The flow of X can then be used to immerse $\vee \times [0, i_0]$ into $S^1 \times D^2$ with boundary $\vee \times \{0\}$ contained in $Q_j \cup \{q_\theta\}$ and $\vee \times \{i_0\}$ contained in $Q_j' \cup \{q_\theta\}$.

We now smoothen $\vee \times [0, i_0]$ to a $\smile \times [0, i_0]$ to change $\tilde{K}$ to $\tilde{K}'$ so that the boundary of $\tilde{K}'$ consists of $2m$ circles instead of the petals, compare Figure 5.

This is done in a similar fashion to Case 2. Note that $\tilde{K}'$ might not yet be compact, as there can be ends from finite orbits. But as there can only be finitely many of them, and we have that h acts trivially on them, we can smoothen the remaining $\vee \times [0, 1]$ as before to turn $\tilde{K}'$ into a compact surface K', whose boundary circles correspond to the petals $Q_1, \dots, Q_m, Q_1', \dots, Q_m'$. This smoothing has to take place further close to L_0 than in the previous cases, as some of the boundary circles of $\tilde{K}'$ may have been inside the petals.

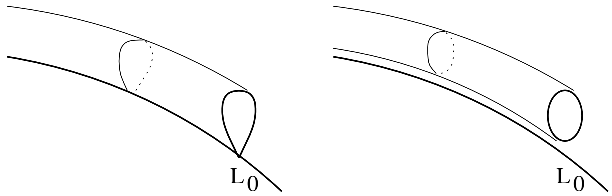

FIGURE 5. Smoothing $\tilde{K}$ to $\tilde{K}'$.

We also change X and Y slightly so that X is tangential to K' and Y is transverse to K'. This can be done so that $X + Y$ is still nonzero, as we only have to make changes very close to L_0, where X is large compared to Y and nonzero.
Note that X points inward at Q_j and outward at Q'_j. For each petal Q_j we now glue a cylinder $S^1 \times [0, i_0]$ between the boundaries corresponding to Q_j and Q'_j close to L_0 such that the cylinder travels once around the circle of $S^1 \times D^2$, compare Figure 6, although notice that K' has to travel more times around S^1 for the handles to exist (and the handles cannot exist, see below). It is possible the circles corresponding to Q_j and Q'_j are contained in a petal not contained in U so that h acted trivially on its ends, but because of the action is coming from a flow, it would have to be the same petal for both Q_j and Q'_j. Therefore the cylinder can be chosen to not intersect K' in interior points.

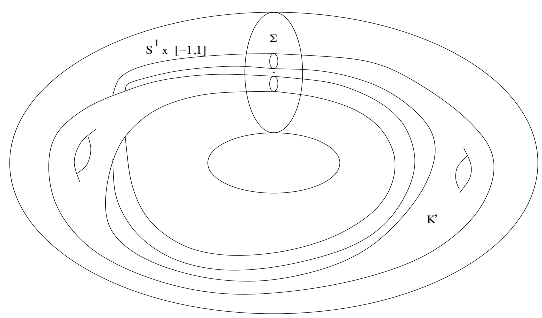

FIGURE 6. The surface K.

The resulting surface K is a closed surface. We also change the vector fields X and Y slightly so that X is tangential to K and Y is transverse to K. This can again be done without introducing singularities on $X + Y$. By Theorem 3.11 K is a torus, hence K' a cylinder and L a plane. □

Proof of Theorem 1.1. Assume we have a non-compact leaf L such that the map $\pi_1(L) \to \pi_1(S^1 \times (D^2 - \{\mathbf{0}\}))$, induced by inclusion, is not injective. By [5, Prop.9.2.5] and [5, Cor.9.3.7] there exists a simple vanishing cycle on some leaf $\tilde{L}$. Notice that the proof of this in [5] does not require the ambient manifold to be compact.

By Lemma 3.14 the leaf $\tilde{L}$ does not intersect a transverse loop. Now by Lemma 3.15 the map $\pi_1(\tilde{L}) \to \pi_1(S^1 \times (D^2 - \{\mathbf{0}\}))$ is injective. This is a contradiction, as the vanishing cycle in $\tilde{L}$ is clearly a non-zero element of the kernel. Hence every non-compact leaf induces an injective map on fundamental group.
If such a leaf L were different from $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$, its fundamental group is free and non-commutative, as L is orientable by Lemma 3.9. But such a group cannot be mapped injectively into $\mathbb{Z} \times \mathbb{Z} \cong \pi_1(S^1 \times (D^2 - \{\mathbf{0}\}))$. $\square$

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