Magnetic bags in hyperbolic space

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A magnetic bag is an Abelian approximation to a large number of coincident $SU(2)$ Bogomol'nyi-Prasad-Sommerfield monopoles. In this paper we consider magnetic bags in hyperbolic space and derive their Nahm transform from the large-charge limit of the discrete Nahm equation for hyperbolic monopoles. An advantage of studying magnetic bags in hyperbolic space, rather than Euclidean space, is that a range of exact charge N hyperbolic monopoles can be constructed, for arbitrarily large values of N, and compared with the magnetic bag approximation. We show that a particular magnetic bag (the magnetic disc) provides a good description of the axially symmetric N-monopole. However, an Abelian magnetic bag is not a good approximation to a roughly spherical N-monopole that has more than N zeros of the Higgs field. We introduce an extension of the magnetic bag that does provide a good approximation to such monopoles and involves a spherical non-Abelian interior for the bag, in addition to the conventional Abelian exterior.

I. INTRODUCTION

In three-dimensional Euclidean space there is a $4N$-dimensional moduli space of $SU(2)$ charge N Bogomol'nyi-Prasad-Sommerfield (BPS) magnetic monopole solutions of the Bogomol'nyi equation. If the N monopoles are coincident, it has been proposed that in the large-N limit there is an Abelian description, known as a magnetic bag [1]. This is a solution of the Abelian Bogomol'nyi equation for a real scalar field, that approximates the length of the Higgs field, and a $U(1)$ gauge field that models the component of the non-Abelian gauge field in the Higgs direction. The bag is defined by a surface in $\mathbb{R}^3$ and the Abelian fields are taken to vanish in the interior of the bag. Direct evidence for the magnetic bag description, in terms of a comparison with the non-Abelian fields of a monopole, is limited to low charge [1,2], where a few axial and platonic monopole examples are available [3]. In particular, it has been observed that the magnetic bag provides a reasonable prediction for the size of these monopoles [4]. There is also a monopole wall [5], with infinite magnetic charge, that resembles a local patch of the surface of a large magnetic bag. Supporting evidence for the magnetic bag idea comes from the fact that the Nahm transform [6] for monopoles becomes a transform for magnetic bags in the large-N limit [7]. Rigorous results relating to the size of a magnetic bag have recently been obtained [8], and attempts have been made to numerically compute non-Abelian field configurations with large values of N, with similar properties to a magnetic bag, by gluing together cones of unit charge [9].

The low-charge platonic monopoles may be divided into two types, by the structure of the zeros of the Higgs field [10]. The $N = 4$ cubic monopole and the $N = 7$ dodecahedral monopole have a single zero of the Higgs field, with multiplicity[1] N, at their center. This property is shared by the axially symmetric N-monopole, for all $N > 1$. Turning to the platonic solids with triangular faces, the tetrahedral, octahedral and icosahedral monopoles, with charges $N = 3, 5, 11$, have $N + 1$ zeros of the Higgs field on the vertices of the platonic solid and at their center there is an additional zero with multiplicity -1 (an anti-zero). This led Lee and Weinberg [2] to propose that these low-charge monopoles are embryonic versions of large-charge monopoles that can be described by two extreme types of monopole bag, which they named non-Abelian and Abelian bags respectively. The first type models a monopole with a single zero of the Higgs field (with multiplicity N) at the center of the bag. The second type describes a monopole that has most of the Higgs zeros (in fact $N + 1$ of them) distributed on the surface of the bag. In this paper we shall have something to say about both types of monopole bag, but the terms non-Abelian and Abelian are potentially confusing given our later analysis. We therefore prefer to use the terms cherry and strawberry flavor, to distinguish monopoles that have a large (in terms of multiplicity) zero of the Higgs field at their center from those that have most of the Higgs zeros distributed on a surface. The nomenclature is chosen because the distribution of the Higgs zeros mirrors the distribution of the seeds in a cherry or a strawberry.

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[1]The multiplicity is the winding number of the normalized Higgs field on a small sphere surrounding the Higgs zero.

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BPS monopoles in Euclidean space have a natural generalization to hyperbolic space, although a Nahm transform is known only if there is a discrete relationship between the curvature of hyperbolic space and the asymptotic length of the Higgs field. In this case hyperbolic monopoles correspond to circle-invariant Yang-Mills instantons in $\mathbb{R}^4$ [11] and are related to solutions of a discrete Nahm equation [12]. In this paper we study magnetic bags in hyperbolic space and investigate their properties. We describe a transform that maps hyperbolic magnetic bags to solutions of a $u(\infty)$ Nahm equation and show how to derive this equation as the large-N limit of the discrete Nahm equation. If the asymptotic length of the Higgs field is suitably tuned then exact charge N hyperbolic monopole solutions can be obtained in terms of free data specifying $N + 1$ points on the sphere (together with a set of positive weights) [13]. By taking large values of N (we shall consider values of several hundred) this provides large-charge hyperbolic monopoles that can be used for comparison with the magnetic bag approximation. This is a significant advantage over the Euclidean situation.

Taking the points to be at the vertices of a regular $(N + 1)$-gon, in an equatorial circle on the sphere, yields the axially symmetric charge N hyperbolic monopole. This monopole is cherry flavor, having a single zero of the Higgs field of multiplicity N at its center. In the large-N limit the associated magnetic bag is squashed into a circular disc—a magnetic disc. We compute an exact solution for the magnetic disc and show that it provides a good approximation to the axial N-monopole in the large-N limit. If the $N + 1$ points are sufficiently distributed over the sphere, at the vertices of a deltahedron, then the hyperbolic monopole is roughly spherical. This monopole is strawberry flavor, with $N + 1$ zeros of the Higgs field on the vertices of the deltahedron and an anti-zero at the origin. This is the large-N generalization of the tetrahedral, octahedral and icosahedral hyperbolic monopoles that arise from this construction with $N = 3, 5, 11$ [13]. However, we find that the spherical Abelian magnetic bag is not a good approximation to such hyperbolic N-monopoles, because the Higgs field does not remain small inside the bag and also has a significant spatial structure. We introduce an extension of the magnetic bag that applies when there are extra zeros of the Higgs field and show that this new bag does provide a good approximation to these large-charge exact hyperbolic monopole solutions. This sheds new light on the mysterious nature of monopole anti-zeros.

II. HYPERBOLIC MONOPOLES AND MAGNETIC BAGS

In this section, we consider $SU(2)$ magnetic monopoles and bags on three-dimensional hyperbolic space, $\mathbb{H}^3_\kappa$, with constant curvature $-\kappa^2$. The discussion is a straightforward generalization of the Euclidean case $\mathbb{H}^3_0 = \mathbb{R}^3$, and

includes this flat-space limit. We denote the metric on $\mathbb{H}^3_\kappa$ by

$$ds^2(\mathbb{H}^3_\kappa) = g_{ij}dx^i dx^j, \qquad (2.1)$$

and its boundary by $\partial\mathbb{H}^3_\kappa$.

The static energy of the $SU(2)$ Yang-Mills-Higgs theory is

$$E = \int_{\mathbb{H}^3_\kappa} \left(-\frac{1}{8}\text{Tr}(F_{ij}F^{ij}) - \frac{1}{4}\text{Tr}(D_i\Phi D^i\Phi) \right)\sqrt{g}d^3x, \qquad (2.2)$$

where Φ, A_i, are the $\mathfrak{su}(2)$-valued Higgs field and the components of the gauge potential, with $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ and $D_i\Phi = \partial_i\Phi + [A_i, \Phi]$, for $i = 1, 2, 3$.

The boundary condition on the Higgs field is that it has constant positive magnitude v at spatial infinity, that is

$$|\Phi|^2 = -\frac{1}{2}\text{Tr}(\Phi^2) = v^2 \quad \text{on} \quad \partial\mathbb{H}^3_\kappa. \qquad (2.3)$$

The monopole charge, $N \in \mathbb{Z}$, is given by the magnetic flux through the boundary at infinity

$$N = -\frac{1}{4\pi v} \int_{\partial\mathbb{H}^3_\kappa} \text{Tr}(F\Phi), \qquad (2.4)$$

where F is the field strength two-form $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$. To simplify the presentation, we shall restrict to the case $N > 0$. A standard Bogomol'nyi argument yields the energy bound

$$E \geq 2\pi vN, \qquad (2.5)$$

which is attained by solutions of the first-order Bogomol'nyi equation

$$F_{ij} = \sqrt{g}\epsilon_{ijk}D^k\Phi. \qquad (2.6)$$

As in flat space, there is a $4N$-dimensional moduli space of solutions to Eq. (2.6), corresponding to arbitrary positions and $U(1)$ phases for each of the N individual monopoles.

For monopoles in hyperbolic space there are two length scales, namely, the curvature scale of hyperbolic space $1/\kappa$, and the core size $1/v$ of a single monopole. The relevant quantity is the ratio of these length scales, v/κ. As first pointed out by Atiyah [11], if $2v/\kappa \in \mathbb{Z}$ then a charge N hyperbolic monopole is equivalent to a circle-invariant self-dual Yang-Mills instanton in $\mathbb{R}^4$, with instanton number $2Nv/\kappa$. As a result, the study of hyperbolic monopoles simplifies for discrete values of the asymptotic length of the Higgs field, relative to the curvature of hyperbolic space. As only the ratio is important, without loss of generality we may choose to fix either κ or v. We shall choose to fix the former, by setting $\kappa = 1$ from now on, which means that

the flat-space limit is equivalent to the limit $v \to \infty$ and the special tuned values for the length scale are given by $2v \in \mathbb{Z}$. For notational convenience we denote $\mathbb{H}_1^3$ by $\mathbb{H}^3$.

A magnetic bag [1] is an Abelian approximation to a monopole solution in the large-N limit, where all N monopoles are coincident. It involves a real scalar field ϕ and a $U(1)$ gauge field $f_{ij} = \partial_i a_j - \partial_j a_i$, that are to be interpreted as approximations to the length of the Higgs field and the projection of the non-Abelian gauge field onto the Higgs direction respectively

$$\phi \approx |\Phi| \quad \text{and} \quad f_{ij} \approx -\frac{\text{Tr}(F_{ij}\Phi)}{2|\Phi|}. \qquad (2.7)$$

These Abelian fields are required to satisfy the Abelian Bogomol'nyi equation

$$f_{ij} = \sqrt{g}\varepsilon_{ijk}\partial^k\phi, \qquad (2.8)$$

which implies that ϕ satisfies the Laplace-Beltrami equation

$$\partial_i(\sqrt{g}g^{ij}\partial_j\phi) = 0. \qquad (2.9)$$

The magnetic bag is defined by specifying the surface of the bag Σ, that divides $\mathbb{H}^3$ into an interior and exterior part. In the interior of the bag the Abelian fields ϕ and f_{ij} are taken to vanish. The scalar field is required to vanish on the surface of the bag and to have the correct asymptotic value at spatial infinity

$$\phi = 0 \quad \text{on} \quad \Sigma \quad \text{and} \quad \phi = v \quad \text{on} \quad \partial\mathbb{H}^3. \qquad (2.10)$$

Finally, the magnetic charge is identified with the Abelian magnetic flux through the surface of the bag

$$N = \frac{1}{2\pi}\int_\Sigma f, \qquad (2.11)$$

where $f = \frac{1}{2}f_{ij}dx^i \wedge dx^j$ is the Abelian two-form field strength.

The idea is that the magnetic bag approximation improves with increasing N and becomes exact in the limit $N \to \infty$, if accompanied by the limit $v \to \infty$, with N/v nonzero and finite. This double scaling limit is required to keep the size of the bag (and the hyperbolic N-monopole) finite as $N \to \infty$. Note that this limit does not correspond to the Euclidean limit, which is $v \to \infty$ with $N/v \to 0$.

The freedom in choosing the surface Σ reflects the fact that the dimension of the N-monopole moduli space tends to infinity as $N \to \infty$. The simplest example is the spherical bag, as follows. We work with the ball model of hyperbolic space, given by the metric

$$ds^2(\mathbb{H}^3) = \frac{4}{(1-R^2)^2}((dX^1)^2 + (dX^2)^2 + (dX^3)^2), \qquad (2.12)$$

with radial coordinate $R = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2} < 1$. For a spherical bag, $\phi(R)$, the Laplace-Beltrami equation (2.9) reduces to

$$\partial_R\left(\frac{R^2}{1-R^2}\partial_R\phi\right) = 0. \qquad (2.13)$$

Denoting the radius of the bag by $R_\star$, then $\phi(R) = 0$ for $0 \le R < R_\star$ and for $R \ge R_\star$ we require the solution of Eq. (2.13) that satisfies the boundary conditions $\phi(R_\star) = 0$ and $\phi(1) = v$. This solution is easily found to be

$$\phi = \frac{v}{(1-R_\star)^2}\left(R_\star^2 + 1 - \frac{R_\star}{R}(R^2 + 1)\right). \qquad (2.14)$$

Substituting this solution into the Abelian Bogomol'nyi equation (2.8) yields the Abelian field strength, from which the magnetic charge N can be calculated using Eq. (2.11). This provides the following relation between the radius of the bag and the magnetic charge:

$$\frac{N}{v} = \frac{4R_\star}{(1-R_\star)^2}. \qquad (2.15)$$

This relation can be used to rewrite Eq. (2.14) as

$$\phi = v - \frac{N}{4R}(1-R)^2. \qquad (2.16)$$

This explicit example, and in particular the formula (2.15), illustrates the above discussion regarding the double scaling limit, required to keep the size of the bag finite as $N \to \infty$.

It is helpful to rewrite the bag radius formula (2.15) in terms of the geodesic distance from the origin $\rho = 2\tanh^{-1}R$, to give

$$\rho_\star = \frac{1}{2}\log\left(\frac{N}{v} + 1\right). \qquad (2.17)$$

From this we see that if the radius of the bag is much smaller than the curvature length scale, $\rho_\star \ll 1$, then we recover the flat-space result $\rho_\star \approx N/(2v)$, that the bag radius grows linearly with the magnetic charge. In contrast, for large bags $\rho_\star \gg 1$, the radius has a logarithmic growth with the magnetic charge.

For later use, we note that in terms of the geodesic distance from the origin, the expression (2.16) for the scalar field of the spherical magnetic bag is

$$\phi = v(N + 1 - N\coth\rho). \qquad (2.18)$$

A hyperbolic monopole is determined by the fields on $\partial\mathbb{H}^3$ [12], in contrast to Euclidean monopoles, where the fields on the sphere at infinity only fix the charge N. This

distinction is also reflected in the magnetic bag description, because the surface of the bag Σ is encoded in the Abelian field strength on the boundary. To show this property, we introduce spherical coordinates R, θ, χ in the ball model of $\mathbb{H}^3$,

$$X^1 = R \sin\theta \cos\chi, \qquad X^2 = R \sin\theta \sin\chi,$$
$$X^3 = R \cos\theta. \tag{2.19}$$

As the scalar field ϕ of a magnetic bag is a harmonic function, it can be written as an expansion in terms of spherical harmonics $Y_{l,m}(\theta, \chi)$ as

$$\phi = v - \frac{N}{4R}(1-R)^2 + \sum_{l=1}^{\infty} \psi_l(R) \sum_{m=-l}^{l} c_{l,m} Y_{l,m}(\theta, \chi), \tag{2.20}$$

where $\psi_l(R)$ is the solution of the ordinary differential equation

$$\partial_R\left(\frac{R^2}{1-R^2}\partial_R\psi_l\right) - \frac{l(l+1)}{1-R^2}\psi_l = 0, \tag{2.21}$$

satisfying the boundary condition

$$\frac{\psi_l(R)}{(1-R)^2} \to 1 \quad \text{as} \quad R \to 1. \tag{2.22}$$

ψ_l can be expressed in terms of an associated Legendre function of the first kind

$$\psi_l(R) = \frac{(-1)^l}{(l+1)!}\sqrt{\frac{\pi(1-R^2)}{R}}P_{\frac{1}{2}}^{l+\frac{1}{2}}\left(\frac{1+R^2}{1-R^2}\right), \tag{2.23}$$

but we shall not need this explicit representation.

It is clear from Eq. (2.20) that all the expansion coefficients $c_{l,m}$ contribute to the computation of the surface of the bag Σ, given by $\phi = 0$. Substituting the expansion (2.20) into the Abelian Bogomol'nyi equation (2.8) and taking the limit $R \to 1$ yields the Abelian field strength on the boundary sphere

$$f = \left(\frac{N}{2} - 2\sum_{l=1}^{\infty}\sum_{m=-l}^{l} c_{l,m} Y_{l,m}(\theta, \chi)\right)\sin\theta d\theta \wedge d\chi. \tag{2.24}$$

This shows that all the expansion coefficients $c_{l,m}$ contribute to the Abelian field strength on $\partial\mathbb{H}^3$ and hence this contains the information required to reconstruct Σ. Note that all the coefficients $c_{l,m}$ vanish for a spherical bag, and hence these coefficients provide a measure of the deviation of the bag from a spherical shape.

III. THE HYPERBOLIC $\mathfrak{u}(\infty)$ NAHM EQUATION

In Euclidean space there is a Nahm transform that relates magnetic bags to solutions of a $\mathfrak{u}(\infty)$ Nahm equation [7]. In this section, we describe a natural generalization of this transform to hyperbolic space.

$\mathfrak{u}(\infty)$ is the Lie algebra of smooth real functions on S^2, with Lie bracket given by the Poisson bracket, and it may be regarded as the large-N limit of the Lie algebra $\mathfrak{u}(N)$ of Hermitian $N \times N$ matrices [14]. To be explicit, consider S^2 as the unit sphere in $\mathbb{R}^3$ with Cartesian coordinates $\mathbf{u} = (u^1, u^2, u^3)$. The standard area two-form on the sphere is given by $\omega = \frac{1}{2}\varepsilon_{ijk}u^i du^j \wedge du^k$ and the associated Poisson bracket is

$$\{P, Q\} = \varepsilon_{ijk}u^i \frac{\partial P}{\partial u^j}\frac{\partial Q}{\partial u^k} \tag{3.1}$$

for functions $P(\mathbf{u})$, $Q(\mathbf{u})$ on S^2. The algebra of functions on S^2 is generated by the Cartesian coordinates, which clearly satisfy

$$[u^i, u^j] = 0 \quad \text{and} \quad (u^1)^2 + (u^2)^2 + (u^3)^2 = 1, \tag{3.2}$$

together with the Poisson bracket relation

$$\{u^i, u^j\} = \varepsilon_{ijk}u^k. \tag{3.3}$$

To reveal the connection to the large-N limit of $\mathfrak{u}(N)$, let J^1, J^2, J^3 denote the generators of the N-dimensional irreducible representation of $\mathfrak{su}(2)$, satisfying $[J^i, J^j] = \varepsilon_{ijk}J^k$. The algebra of Hermitian $N \times N$ matrices is generated by $U^j = \frac{2i}{N}J^j$, satisfying

$$[U^i, U^j] = \frac{2i}{N}\varepsilon_{ijk}U^k \quad \text{and}$$
$$(U^1)^2 + (U^2)^2 + (U^3)^2 = 1 - \frac{1}{N^2}. \tag{3.4}$$

The relations (3.4) converge to the relations (3.2) in the limit as $N \to \infty$, if we make the identification $U^j \to u^j$. Furthermore, in this limit the Poisson bracket relation (3.3) gives

$$N[U^i, U^j] = 2i\varepsilon_{ijk}U^k \to 2i\varepsilon_{ijk}u^k = 2i\{u^i, u^j\}, \tag{3.5}$$

providing the prescription for replacing commutators by Poisson brackets.

To define the hyperbolic $\mathfrak{u}(\infty)$ Nahm equation, let $\mathbf{x} = (x^1, x^2, x^3)$ be coordinates in $\mathbb{H}^3$ with metric (2.1) and consider the mapping

$$\mathbf{x}: S^2 \times [0, v) \mapsto \mathbb{H}^3, \tag{3.6}$$

defined by a solution of the equation

$$\frac{dx^i}{ds} = \frac{\sqrt{g}}{N} g^{il} \varepsilon_{jkl}\{x^j, x^k\}, \qquad (3.7)$$

where s is the independent variable in the interval $[0, v)$. The boundary condition is that $\mathbf{x}$ is a coordinate on $\partial \mathbb{H}^3$ as $s \to v$.

The Nahm transform for magnetic bags is simply an exchange of the independent and dependent variables in Eq. (3.7). The scalar field ϕ is identified with the variable s and the Abelian two-form f is proportional to the area two-form ω on S^2,

$$\phi = s, \qquad f = \frac{N}{2}\omega. \qquad (3.8)$$

In particular, this identification means that $\mathbf{x}$ evaluated at $s = 0$ is a coordinate on Σ, the surface of the bag.

In the Euclidean case, the proof that the $\mathbf{u}(\infty)$ Nahm equation is equivalent to the Abelian Bogomol'nyi equation can be found in Ref. [7]. As Eq. (3.7) is simply the covariant version of the Euclidean equation, the proof follows from a simple covariant version of the Euclidean proof. The main step is to multiply Eq. (3.7) by $\omega \wedge ds$ and to use the property of the Poisson bracket $\{x^j, x^k\}\omega \wedge ds = dx^j \wedge dx^k \wedge ds$ to see that $\frac{N}{2}\omega = *ds$, where $*$ denotes the Hodge dual on $\mathbb{H}^3$. This is the Abelian Bogomol'nyi equation (2.8), given the identification (3.8).

As the Nahm transform linearizes the hyperbolic $\mathbf{u}(\infty)$ Nahm equation (3.7), then we expect this to be an integrable system with an infinite number of conserved quantities. In fact it is easy to show that

$$\int_{\{\phi=s\}} \Psi f, \qquad (3.9)$$

is independent of s, for any harmonic function Ψ on $\mathbb{H}^3$ with no singularities. The proof is a simple application of Stokes' theorem, as follows:

$$\int_{\{\phi=s_2\}} \Psi f - \int_{\{\phi=s_1\}} \Psi f = \int_{\{s_1 \le \phi \le s_2\}} d\Psi \wedge f = \int_{\{s_1 \le \phi \le s_2\}} d\Psi \wedge *d\phi$$

$$= \int_{\{s_1 \le \phi \le s_2\}} d\phi \wedge *d\Psi = \int_{\{\phi=s_2\}} \phi * d\Psi - \int_{\{\phi=s_1\}} \phi * d\Psi$$

$$= s_2 \int_{\{0 \le \phi \le s_2\}} d * d\Psi - s_1 \int_{\{0 \le \phi \le s_1\}} d * d\Psi = 0. \qquad (3.10)$$

In terms of spherical coordinates (2.19), we may take $\Psi = \tilde{\psi}_l(R) Y_{l,m}(\theta, \chi)$, where

$$\tilde{\psi}_l(R) = \frac{(l+1)!}{2} \sqrt{\frac{\pi(1-R^2)}{R}} P_{\frac{1}{2}}^{-l-\frac{1}{2}}\left(\frac{1+R^2}{1-R^2}\right) \qquad (3.11)$$

solves the radial equation (2.21) and is normalized so that $\tilde{\psi}_l(1) = 1$. The conserved quantities (3.9) are then proportional to the constants $c_{l,m}$ that appear in the expansion (2.20) of ϕ.

To illustrate the Nahm transform for hyperbolic magnetic bags, we consider the example of the spherical bag, introduced in the previous section. Using the ball model metric (2.12) the hyperbolic $\mathbf{u}(\infty)$ Nahm equation (3.7) becomes

$$\frac{dX^i}{ds} = \frac{2}{N(1-R^2)} \varepsilon_{ijk}\{X^j, X^k\}. \qquad (3.12)$$

In terms of the Cartesian coordinates u^i on S^2, the spherically symmetric ansatz is given by

$$X^i = u^i R(s). \qquad (3.13)$$

Using the Poisson bracket relation (3.3) reduces Eq. (3.12) to the ordinary differential equation

$$\frac{dR}{ds} = \frac{4R^2}{N(1-R^2)}. \qquad (3.14)$$

The solution satisfying the required boundary condition, $R(v) = 1$, is

$$R(s) = 1 - \frac{2\sqrt{v-s}}{\sqrt{v-s} + \sqrt{N+v-s}}. \qquad (3.15)$$

Setting $s = \phi$ in Eq. (3.15) indeed reproduces the spherical bag solution (2.16), in inverse function form. The bag radius is

$$R_\star = R(0) = 1 - \frac{2}{1 + \sqrt{1 + N/v}}, \qquad (3.16)$$

which agrees with Eq. (2.15).

IV. THE LARGE-N LIMIT OF THE DISCRETE NAHM EQUATION

In Euclidean space, the $\mathfrak{u}(\infty)$ Nahm equation can be derived as a large-N limit of the Nahm equation for $N \times N$ matrices [7]. This approach is not an option in hyperbolic space, as there is no known Nahm transform for generic values of v. However, for the tuned values $2v \in \mathbb{Z}$, there is a transform between hyperbolic monopoles and solutions of a discrete Nahm equation [12]. This lattice system is obtained by identifying hyperbolic monopoles with circle-invariant instantons and imposing circle symmetry within the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [15]. The lattice is indexed by the weight under the circle action and the construction yields a hyperbolic monopole within the upper half-space model of $\mathbb{H}^3$, with metric

$$ds^2(\mathbb{H}^3) = \frac{(dy^1)^2 + (dy^2)^2 + (dy^3)^2}{(y^3)^2}, \qquad (4.1)$$

where $y^3 > 0$. The relation between the upper half-space coordinates and the ball coordinates is

$$y^3 = \frac{1 - R^2}{1 + R^2 - 2X^3}, \quad y^1 + iy^2 = \frac{2(X^1 + iX^2)}{1 + R^2 - 2X^3}, \qquad (4.2)$$

with the plane $y^3 = 0$ mapping to the boundary of hyperbolic space, $R = 1$.

As a brief aside, we note that this is the most convenient coordinate system in which to write down an Abelian magnetic wall, namely a solution of the Abelian Bogomol'nyi equation that has translational symmetry in a plane. Take (y^1, y^2) to be the symmetry plane of the wall, located at the position $y^3 = y_\star^3$. The Abelian fields vanish above the wall $(y^3 > y_\star^3)$, whereas below the wall $(0 < y^3 \leq y_\star^3)$ they are given by

$$\phi = v - v\left(\frac{y^3}{y_\star^3}\right)^2, \qquad f = \frac{2v}{(y_\star^3)^2} dy^2 \wedge dy^1. \qquad (4.3)$$

We see that the magnetic flux is constant, and hence the total magnetic flux through the wall is infinite, as expected from the translational symmetry. However, a finite piece of this wall provides a good description of a local patch of the surface of a large magnetic bag.

We now derive the hyperbolic $\mathfrak{u}(\infty)$ Nahm equation from the large-N limit of the discrete Nahm equation. This discrete system is defined on a one-dimensional lattice consisting of $2v$ lattice points, $k = 0, \ldots, 2v - 1$ with complex $N \times N$ matrices B_{2j} and W_{2j+1} defined on even and odd lattice sites respectively. For ease of presentation, we assume that $2v$ is an odd integer. The matrices are propagated along the lattice by applying the relations [12]

$$B_{2j+2} = W_{2j+1}^{-1} B_{2j} W_{2j+1} \quad \text{and}$$

$$W_{2j+1} W_{2j+1}^\dagger = W_{2j-1} W_{2j-1}^\dagger + [B_{2j}^\dagger, B_{2j}]. \qquad (4.4)$$

Boundary conditions are introduced by extending the lattice to negative values and defining $B_{-2j} = B_{2j}^t$ and $W_{-(2j+1)} = W_{2j+1}^t$, plus adding an extra lattice site and demanding that W_{2v} has rank one, so that $W_{2v} W_{2v}^\dagger = L^t L^\dagger$ for some N-component row vector L.

The Nahm equation is obtained in the Euclidean flat-space limit, $v \to \infty$, as follows [12]. Define the scaled lattice variable $\sigma = k/(2v)$ and write

$$B_{2j} = -iT^1(\sigma) - T^2(\sigma) \quad \text{and} \quad W_{2j+1} = v + T^3\left(\sigma + \frac{1}{2}v^{-1}\right). \qquad (4.5)$$

There is a gauge symmetry of this system that allows W_{2j+1} to be chosen to be Hermitian. σ becomes a continuous variable in the limit as $v \to \infty$ and the lattice system (4.4) becomes the Nahm equation [6]

$$\frac{dT^i}{d\sigma} = -\frac{i}{2}\varepsilon_{ijk}[T^j, T^k], \qquad (4.6)$$

for the triplet of Hermitian matrices T^1, T^2, T^3.

The starting point to derive the large-N limit of the discrete Nahm equation is similar to the above. We introduce the same scaled lattice variable σ but we modify Eq. (4.5) by dropping the explicit v-dependent term proportional to the identity matrix, to give

$$B_{2j} = -iT^1(\sigma) - T^2(\sigma) \quad \text{and} \quad W_{2j+1} = T^3\left(\sigma + \frac{1}{2}v^{-1}\right). \qquad (4.7)$$

Substituting this form into the discrete Nahm equation (4.4), taking the large-v continuum limit and neglecting terms of order v^{-1} yields

$$2\frac{dT^1}{d\sigma}T^3 - \left[\frac{dT^1}{d\sigma}, T^3\right] + iv\left[2T^2 + \frac{1}{v}\frac{dT^2}{d\sigma}, T^3 + \frac{1}{2v}\frac{dT^3}{d\sigma}\right] = 0, \qquad (4.8)$$

$$2\frac{dT^2}{d\sigma}T^3 - \left[\frac{dT^2}{d\sigma}, T^3\right] - iv\left[2T^1 + \frac{1}{v}\frac{dT^1}{d\sigma}, T^3 + \frac{1}{2v}\frac{dT^3}{d\sigma}\right] = 0, \qquad (4.9)$$

$$2\frac{dT^3}{d\sigma}T^3 - \left[\frac{dT^3}{d\sigma}, T^3\right] + 2iv[T^1, T^2] = 0. \qquad (4.10)$$

Apply the large-N limit by replacing matrices by functions on the sphere $T^j(\sigma) \to y^j(\sigma, \mathbf{u})$, and using Eq. (3.5) to replace commutators by Poisson brackets, $[T^i, T^j] \to \frac{2i}{N}\{y^i, y^j\}$. This gives

$$\frac{dy^1}{d\sigma}y^3 - \frac{i}{N}\left\{\frac{dy^1}{d\sigma}, y^3\right\} - \frac{v}{N}\left\{2y^2 + \frac{1}{v}\frac{dy^2}{d\sigma}, y^3 + \frac{1}{2v}\frac{dy^3}{d\sigma}\right\} = 0,$$

(4.11)

$$\frac{dy^2}{d\sigma}y^3 - \frac{i}{N}\left\{\frac{dy^2}{d\sigma}, y^3\right\} + \frac{v}{N}\left\{2y^1 + \frac{1}{v}\frac{dy^1}{d\sigma}, y^3 + \frac{1}{2v}\frac{dy^3}{d\sigma}\right\} = 0,$$

(4.12)

$$\frac{dy^3}{d\sigma}y^3 - \frac{i}{N}\left\{\frac{dy^3}{d\sigma}, y^3\right\} - \frac{2v}{N}\{y^1, y^2\} = 0.$$

(4.13)

Finally, we take the limit $v \to \infty$ and $N \to \infty$ with v/N finite, to get

$$\frac{dy^i}{d\sigma}y^3 = \frac{v}{N}\varepsilon_{ijk}\{y^j, y^k\},$$

(4.14)

where $\sigma \in [0, 1)$.

To apply Eq. (4.14) in the large-N limit, with N and v finite, we introduce the independent variable $s = v\sigma \in [0, v)$ to get the final form

$$\frac{dy^i}{ds} = \frac{1}{Ny^3}\varepsilon_{ijk}\{y^j, y^k\}.$$

(4.15)

This is the hyperbolic $\mathfrak{u}(\infty)$ Nahm equation (3.7) in upper half-space coordinates with the metric (4.1). The boundary condition on the discrete Nahm equation, that the rank of W_k drops by a factor $1/N$ when $k = 2v$, translates to the boundary condition that as $s \to v$ then $y^3 \to 0$, which is indeed the boundary of hyperbolic space, in upper half-space coordinates.

V. EXACT HYPERBOLIC MONOPOLES WITH LARGE CHARGE

By restricting to the simplest tuned value, $v = \frac{1}{2}$, explicit exact charge N hyperbolic monopole solutions can be obtained from free data specifying $N + 1$ points on the sphere (together with a positive weight for each point) [13]. At the heart of this construction is the identification of a hyperbolic N-monopole with a circle-invariant N-instanton in $\mathbb{R}^4$ obtained using the Jackiw-Nohl-Rebbi (JNR) ansatz [16] for instantons, with JNR poles restricted to the fixed-point set of the circle action. An alternative view of the same solution is via the discrete Nahm equation discussed in the previous section, where the restriction $v = \frac{1}{2}$ reduces the lattice to a single point. All that remains of the discrete

Nahm equation is then a boundary condition for the complex $N \times N$ symmetric matrix B_0 and the complex row vector L that gives W_1. The solution associated with the free data is essentially obtained by taking B_0 to be diagonal, with the remaining data providing the components of L in a simple way that automatically satisfies the boundary condition [17].

An explicit formula for the Higgs field is most naturally written using the upper half-space coordinates (4.1), no matter whether the JNR or discrete Nahm route is taken to obtain the solution. To present this formula, let $\{\gamma_j \in \mathbb{CP}^1, j = 0, ..., N\}$ be a set of $N + 1$ distinct points on the Riemann sphere and use these points to define the following real function:

$$\Xi = \sum_{j=0}^{N} \frac{1 + |\gamma_j^2|}{|y^1 + iy^2 - \gamma_j|^2 + (y^3)^2}.$$

(5.1)

The square of the length of the Higgs field is then given by [13,17]

$$|\Phi|^2 = \left(\frac{y^3}{2\Xi}\right)^2 \left(\left(\frac{\partial\Xi}{\partial y^1}\right)^2 + \left(\frac{\partial\Xi}{\partial y^2}\right)^2 + \left(\frac{\Xi}{y^3} + \frac{\partial\Xi}{\partial y^3}\right)^2\right).$$

(5.2)

Although this formula for the Higgs field is most readily obtained in upper half-space coordinates, the symmetry of the solution is most apparent by converting to the ball model using the relations (4.2) between the two coordinate systems. This reveals that the points γ_j on the Riemann sphere should be regarded as points on the sphere $R = 1$, that is the boundary of $\mathbb{H}^3$ in the ball model. Furthermore, the monopole inherits the symmetry of this set of points on the sphere, due to the choice of weights in Eq. (5.1). Replacing the weights $1 + |\gamma_j|^2$ in Eq. (5.1) with arbitrary real and positive weights also yields a hyperbolic monopole solution, but generally this will not share the symmetry of the set of points on the sphere.

The axially symmetric hyperbolic N-monopole (positioned at the origin, with X^3 the axis of symmetry) is obtained in this formalism by the choice $\gamma_j = e^{2\pi ij/(N+1)}$. Naively, it might be expected that placing $N + 1$ points on the vertices of a regular $(N + 1)$-gon in an equatorial circle would produce a monopole with a discrete cyclic symmetry, but the fact that all the points lie on a circle enhances the cyclic symmetry to an axial symmetry. For later reference, in the plane $X^3 = 0$ the length of the Higgs field has the simple expression [18]

$$|\Phi| = \frac{(N + 1)R^N(1 - R^2)}{2(1 - R^{2N+2})}.$$

(5.3)

From this formula we see that the axial N-monopole indeed has a zero of the Higgs field at the origin, with multiplicity N. This means that the axial N-monopole is cherry flavor.

The energy density of a monopole solution can be obtained directly from the length of the Higgs field by acting with the Laplace-Beltrami operator on $|\Phi|^2$. In the left image in Fig. 1 we display an energy density isosurface, using the ball model of $\mathbb{H}^3$, for the axial monopole with $N = 371$ (the reason for this particular choice of N will be revealed shortly). The blue sphere in this image represents the boundary of hyperbolic space, $R = 1$. We see that, for a large value of N, the energy density isosurface of the axial N-monopole takes the form of a thin disc. In the next section we study the magnetic bag approximation to this type of solution, namely the magnetic disc, and show that it provides a good description.

Applying the above construction with $N = 3, 5, 11$ and placing the $N + 1$ points on the sphere at the vertices of a tetrahedron, octahedron and icosahedron, respectively, yields a tetrahedral 3-monopole, an octahedral 5-monopole and an icosahedral 11-monopole [13]. All these monopoles are strawberry flavor, with an anti-zero at the origin and $N + 1$ zeros of the Higgs field on the vertices of a Platonic solid. We can continue this family to large N, by placing the $N + 1$ points on the vertices of a suitable deltahedron, so that the points are in some sense evenly distributed. Although there are no spherically symmetric monopoles with $N > 1$, within the moduli space of monopoles obtained from the free data of points on a sphere, this family generates an N-monopole that is the best candidate to have a spherical Abelian bag description.

To generate $N + 1$ evenly distributed points on the sphere we turn to the following well-known physical problem. Given a positive integer M, the Thomson problem is to find the positions of M unit-charge point particles on the sphere that attain the global minimum of their total electrostatic Coulomb energy (for a review see Ref. [19]). For $M = 4, 6, 12$ the solution of the Thomson problem is to place the point particles at the vertices of a tetrahedron, octahedron and icosahedron respectively. By taking our points on the sphere to be the positions of the particles that solve the Thomson problem we can generate a family of hyperbolic monopoles with charge $N = M - 1$ that includes and extends our platonic strawberry flavor examples.

Computing solutions of the Thomson problem for large M is a difficult computational task, due to the large number of local minima that exist. However, this is a well-studied optimization problem, that is often used to benchmark new algorithms, so there is a wealth of data available. In particular, magic numbers have been found at which icosahedrally symmetric local energy minima have been obtained that are believed to be the global minima. Icosahedral symmetry is the best approximation to spherical symmetry that can be obtained with a finite number of points, and hence this is the closest that we can come to a spherical configuration. As an example, it is believed that $M = 372$ is a magic number with icosahedral symmetry [20]. Taking this configuration of points yields the icosahedrally symmetric hyperbolic monopole with charge 371 displayed in the right image in Fig. 1. This explains our earlier nonobvious choice of $N = 371$ for the axial monopole, as we want to display the energy density isosurfaces of the two different kinds of monopole with the same charge, to aid the comparison.

An examination of the Higgs field of the icosahedrally symmetric 371-monopole displayed in the right image in Fig. 1 confirms that this is indeed strawberry flavor, with an anti-zero at the origin and 372 zeros on a shell. In Sec. VII we shall discuss the Higgs field of this monopole in detail and explain why an Abelian magnetic bag is not a good description. We then introduce a new magnetic bag with a non-Abelian interior that does provide a good approximation to strawberry-flavor monopoles.

Finally in this section, we stress that we expect there to be a family of charge N cherry-flavor hyperbolic monopoles that approach the spherical Abelian magnetic bag in
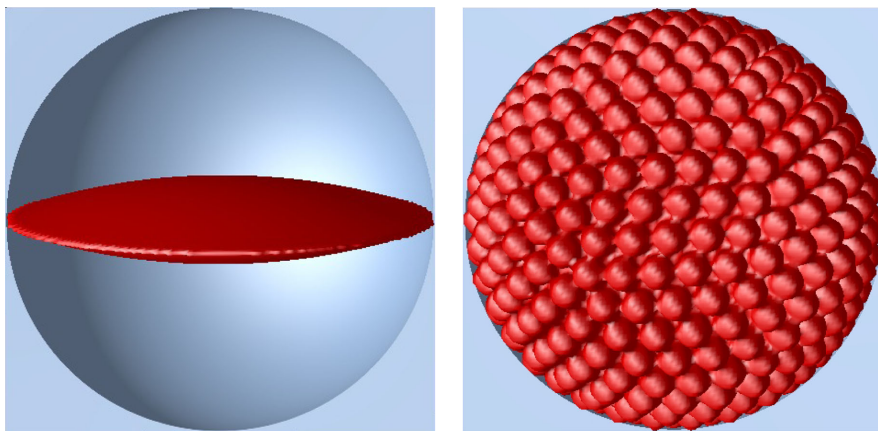

FIG. 1 (color online).   Energy density isosurfaces, in the ball model of $\mathbb{H}^3$, for hyperbolic monopoles with $N = 371$. The hyperbolic monopole in the left image has axial symmetry and is cherry flavor, whereas the one in the right image has icosahedral symmetry and is strawberry flavor. The blue sphere represents the boundary of hyperbolic space.

the large-N limit. This is a family that extends the cubic 4-monopole and the dodecahedral 7-monopole, obtained by imposing constraints on the ADHM construction that ensures a circle symmetry of the instanton [13]. These examples are not within the scheme of specifying free data as points on a sphere and hence we are currently unable to extend the family to large values of N because of the technical difficulty in imposing the required constraints.

VI. THE MAGNETIC DISC

A magnetic disc is the degenerate limit in which the surface Σ of the magnetic bag becomes a disc. Therefore, to obtain a magnetic disc we require a harmonic function that vanishes on a disc. It is possible to obtain the required solution explicitly by introducing an appropriate coordinate system, in terms of Jacobi elliptic functions, with the property that the Laplace-Beltrami equation has solutions that can be obtained via a separation of variables [21].

Consider the disc, $\mathcal{D}_S$, of geodesic radius S, given in ball coordinates by $X^3 = 0$ and $\sqrt{(X^1)^2 + (X^2)^2} \leq \tanh(S/2)$. Let sn denote the Jacobi elliptic function with elliptic modulus $\tanh S$ and $\widetilde{\mathrm{sn}}$ the Jacobi elliptic function with elliptic modulus $\mathrm{sech}S$. We extend the same notation to the other Jacobi elliptic functions and to the complete elliptic integral of the first kind, so that K denotes this elliptic integral with elliptic modulus $\tanh S$ and $\tilde{K}$ is the complete elliptic integral of the first kind with elliptic modulus $\mathrm{sech}S$.

We introduce the coordinates r, Θ, χ on $\mathbb{H}^3$, where $0 \leq r < \tilde{K}$ and the angular coordinates have the ranges $-K < \Theta < K$ and $0 \leq \chi \leq 2\pi$. The relation to the ball coordinates is given by

$$X^1 = \frac{\sinh S \, \widetilde{\mathrm{nc}}(r)\mathrm{cn}(\Theta)\cos\chi}{1 + \cosh S \, \widetilde{\mathrm{dc}}(r)\mathrm{dn}(\Theta)}, \qquad (6.1)$$

$$X^2 = \frac{\sinh S \, \widetilde{\mathrm{nc}}(r)\mathrm{cn}(\Theta)\sin\chi}{1 + \cosh S \, \widetilde{\mathrm{dc}}(r)\mathrm{dn}(\Theta)}, \qquad (6.2)$$

$$X^3 = \frac{\tanh S \, \widetilde{\mathrm{sc}}(r)\mathrm{sn}(\Theta)}{1 + \cosh S \, \widetilde{\mathrm{dc}}(r)\mathrm{dn}(\Theta)}, \qquad (6.3)$$

and yields the metric

$$ds^2(\mathbb{H}^3) = (\widetilde{\mathrm{dc}}^2(r) - \mathrm{dn}^2(\Theta))(dr^2 + d\Theta^2) \\ + \sinh^2 S \widetilde{\mathrm{nc}}^2(r)\mathrm{cn}^2(\Theta)d\chi^2. \qquad (6.4)$$

The first reason for using this coordinate system is that the disc $\mathcal{D}_S$ is simply given by $r = 0$. The second reason is that this allows a separable solution of the Laplace-Beltrami equation (2.9) with ϕ a function of r only. The ansatz $\phi(r)$ reduces Eq. (2.9) to the ordinary differential equation

$$\frac{d}{dr}\left(\widetilde{\mathrm{nc}}(r)\frac{d\phi}{dr}\right) = 0. \qquad (6.5)$$

We require the solution that vanishes on the disc $\mathcal{D}_S$, and hence $\phi(0) = 0$, and has the correct asymptotic value, $\phi(r) \to v$ as $r \to \tilde{K}$. As we wish to compare the magnetic disc with the axial exact solution in the previous section we take $v = \frac{1}{2}$, so the required solution is

$$\phi(r) = \frac{\cos^{-1}(\widetilde{\mathrm{dn}}(r))}{2\cos^{-1}(\tanh S)}. \qquad (6.6)$$

The relation between the magnetic charge and the geodesic radius of the disc is given by

$$N = \frac{1}{2\pi}\int_{\Sigma_r} *d\phi = \frac{\sin^{-1}(\tanh S)}{\cos^{-1}(\tanh S)}, \qquad (6.7)$$

where Σ_r is any surface of constant r. Inverting this formula provides the geodesic radius of the disc

$$S = \tanh^{-1}\left(\sin\left(\frac{\pi N}{2(N+1)}\right)\right) = \log N + \log\left(\frac{4}{\pi}\right) + \mathcal{O}\left(\frac{1}{N}\right). \qquad (6.8)$$

Along the positive X^1 axis, in the exterior of the disc, the relation between $X^1 = R$ and the coordinate r is

$$R = \frac{\sinh S \, \widetilde{\mathrm{nc}}(r)}{1 + \cosh S \, \widetilde{\mathrm{dc}}(r)}. \qquad (6.9)$$

Using this formula, in Fig. 2 we plot the solution (6.6) as a function of the geodesic distance from the origin $\rho = 2\tanh^{-1} R$, for the charges $N = 100$ and $N = 10000$ (blue curves). For comparison, the red curves in Fig. 2 display the corresponding exact solution (5.3) along the same axis, again as a function of ρ. We see that the magnetic disc provides a reasonable approximation to the exact axial monopole and that the error appears to have very little dependence on N for these large values. As we now explain, this is exactly the result expected of a magnetic bag approximation.

To compare the disc radius (6.8) with the exact axial monopole solution, we use Eq. (5.3) to define the value $\hat{R}$ at which the Higgs field attains half the asymptotic value, $|\Phi| = \frac{1}{4}$. This provides a sufficient definition of the size of the axial monopole. The geodesic radius of the axial monopole is then given by

$$\hat{S} = 2\tanh^{-1}(\hat{R}) = \log N + \log\left(\frac{2}{\sqrt{3}}\right) + \mathcal{O}\left(\frac{1}{N}\right). \qquad (6.10)$$

Comparing Eqs. (6.8) and (6.10) shows that the two agree up to terms that are $\mathcal{O}(1)$. Recall that the magnetic bag is
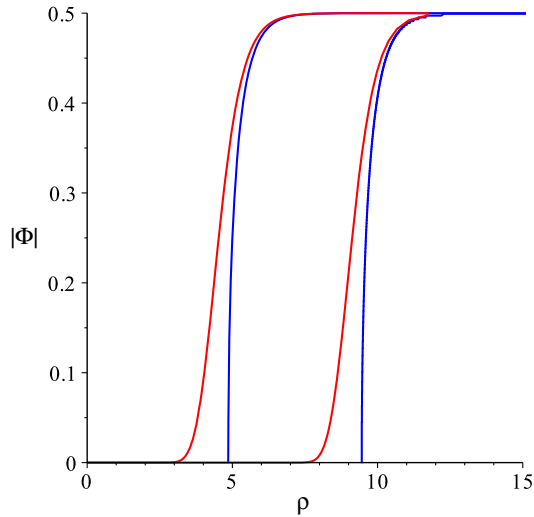
FIG. 2 (color online). The red curves display the length of the Higgs field $|\Phi|$ as a function of geodesic distance from the origin ρ, along an axis that is perpendicular to the symmetry axis of the axial hyperbolic N-monopole with $N = 100$ and $N = 10000$. The blue curves show the corresponding magnetic disc approximation.
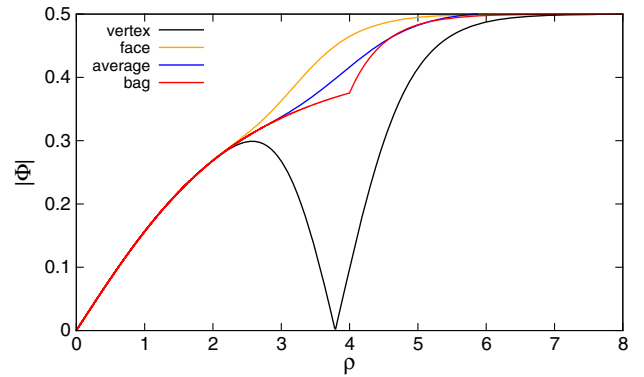

FIG. 3 (color online). The length of the Higgs field, $|\Phi|$, for an icosahedrally symmetric strawberry-flavor monopole with charge $N = 371$. The plot shows $|\Phi|$ as a function of geodesic distance from the origin ρ along a radial half line that passes through a vertex (black curve) and a face center (yellow curve) of the associated polyhedron. The blue curve is the spherical average of $|\Phi|$, obtained by integrating over the angular coordinates. The red curve is the new magnetic bag approximation.

expected to become exact in the limit $N \to \infty$ and $v \to \infty$ with N/v finite. The $v \to \infty$ limit is required to keep the size of the magnetic bag finite. However, our exact solutions are only available for $v = \frac{1}{2}$, so we are unable to take the $v \to \infty$ limit to keep the size finite as $N \to \infty$. An alternative is to measure geodesic distance in units of $\log N$, so that, by Eq. (6.8), the magnetic disc has geodesic radius one in these units as $N \to \infty$. In these units, terms that are $\mathcal{O}(1)$ tend to zero as $N \to \infty$, and hence the exact axial monopole converges to the magnetic disc.

Note that Eq. (2.17) shows that in the large-N limit, with $v = \frac{1}{2}$, the leading-order term for the geodesic radius of the spherical bag is $\log \sqrt{N}$, in comparison to the geodesic radius of the magnetic disc, $\log N$. Thus the spherical bag is a substantially more compact object than the magnetic disc.

VII. A MAGNETIC BAG FOR STRAWBERRY-FLAVOR MONOPOLES

In the previous section we considered a particular type of cherry-flavor monopole, the axial monopole, and demonstrated that the Abelian magnetic bag indeed provides a good description in the large-charge limit. In this section we turn our attention to strawberry-flavor monopoles and find that the Abelian magnetic bag is no longer a good approximation.

A typical example of a large-charge strawberry-flavor hyperbolic monopole is the icosahedrally symmetric charge 371 monopole displayed in the right image in Fig. 1. This has an anti-zero at the origin and 372 zeros of the Higgs field on the vertices of a polyhedron with icosahedral symmetry. A more detailed picture of the Higgs field is

provided in Fig. 3, where we plot the length of the Higgs field $|\Phi|$ as a function of geodesic distance from the origin ρ along a radial half line that passes through a vertex of the polyhedron (black curve) and a face center of the polyhedron (yellow curve). The blue curve is the spherical average of $|\Phi|$, obtained by integrating over the angular coordinates.

It is immediately clear from Fig. 3 that an Abelian magnetic bag does not provide a good description of this large-charge hyperbolic monopole, because the length of the Higgs field does not remain close to zero in a region that could be associated with the interior of an Abelian bag. Furthermore, suggested generalizations [2,4], in which the length of the Higgs field is assumed to be a nonzero constant in the interior of the bag, are also not appropriate here, as $|\Phi|$ has a significant ρ dependence.

Figure 3 reveals that the best that any spherical bag description could hope to achieve is an approximation to the spherical average of $|\Phi|$. This is because there is a substantial angular variation of $|\Phi|$ on the sphere that contains most of the zeros of the Higgs field. As $v = \frac{1}{2}$, the spherical Abelian magnetic bag (2.18) is given by

$$\phi = \frac{1}{2}(N + 1 - N \coth \rho). \qquad (7.1)$$

As we shall see, this does provide a good description of the spherical average of $|\Phi|$ in the exterior of a suitable bag, but clearly it fails in the interior.

A key observation from Fig. 3 is that the monopole appears to be spherically symmetric in a large region around the origin, that we identify as the interior of our new bag. Although a spherical Abelian description is not valid in the interior, it turns out that a spherical non-Abelian

solution of the Bogomol'nyi equation is an excellent approximation in this region.

Let θ, χ be the usual angular coordinates on the sphere, as in Eq. (2.19). The standard spherical hedgehog ansatz, in radial gauge $A_\rho = 0$, is given by

$$\Phi = ih(\sin\theta(\tau_1\cos\chi + \tau_2\sin\chi) + \tau_3\cos\theta), \qquad (7.2)$$

$$A_\theta = \frac{i}{2}(k-1)(\tau_1\sin\chi - \tau_2\cos\chi), \qquad (7.3)$$

$$A_\chi = \frac{i}{2}(k-1)\sin\theta((\tau_1\cos\chi + \tau_2\sin\chi)\cos\theta - \tau_3\sin\theta), \qquad (7.4)$$

where τ_i are the Pauli matrices and h, k are radial profile functions that depend only on ρ. Substituting this hedgehog ansatz into the Bogomol'nyi equation (2.6) yields the following ordinary differential equations for $h(\rho)$ and $k(\rho)$:

$$\frac{dh}{d\rho} = \frac{1-k^2}{2\sinh^2\rho}, \qquad \frac{dk}{d\rho} = -2hk. \qquad (7.5)$$

Regularity at the origin imposes the boundary conditions $h(0) = 0$ and $k(0) = 1$. Requiring the correct asymptotic value for the length of the Higgs field imposes the condition

$$|\Phi| = |h| \to v = \frac{1}{2} \quad \text{as} \quad \rho \to \infty. \qquad (7.6)$$

The standard 1-monopole solution of Eq. (7.5) is given by

$$h = \coth(2\rho) - \frac{1}{2}\coth\rho, \qquad k = \text{sech}\rho. \qquad (7.7)$$

This solution has the small-ρ expansion $h = \frac{\rho}{2} + \mathcal{O}(\rho^3)$, and the fact that the coefficient of the linear term is positive corresponds to a zero of the Higgs field at the origin with multiplicity $+1$. Note that $k \to 0$ as $\rho \to \infty$, which is a finite energy requirement.

There is another solution of Eq. (7.5) that satisfies the regularity conditions at the origin and the boundary condition (7.6). It is given by

$$h = \frac{1}{2\rho} - \frac{1}{2}\coth\rho, \qquad k = \frac{\sinh\rho}{\rho}. \qquad (7.8)$$

This solution does not have a finite charge N because $k \nrightarrow 0$ as $\rho \to \infty$, but rather it grows without bound. However, it is a perfectly regular solution for any finite value of ρ. The small-ρ expansion of this solution gives $h = -\frac{\rho}{6} + \mathcal{O}(\rho^3)$ and hence there is an anti-zero of the Higgs field at the origin, because the coefficient of the linear term is negative.

The scalar field ϕ, that approximates the spherical average of $|\Phi|$, is obtained for our new magnetic bag by taking the non-Abelian solution (7.8) in the interior of the

bag and the Abelian solution (7.1) in the exterior of the bag. Explicitly,

$$\phi = \begin{cases} \frac{1}{2}\coth\rho - \frac{1}{2\rho} & \text{for } 0 \le \rho \le \rho_\star, \\ \frac{1}{2}(N+1-N\coth\rho) & \text{for } \rho > \rho_\star, \end{cases} \qquad (7.9)$$

where the bag radius $\rho_\star$ is determined in terms of the magnetic charge N by requiring that ϕ is continuous at $\rho = \rho_\star$. The result is

$$N = \frac{e^{2\rho_\star} - 2\rho_\star - 1}{2\rho_\star}. \qquad (7.10)$$

For $N = 371$ this gives $\rho_\star \approx 4$ and the associated new magnetic bag (7.9) is shown as the red curve in Fig. 3. This plot demonstrates that the new magnetic bag provides an excellent approximation to the spherical average of this hyperbolic monopole.

We have performed a similar comparison for a range of large-charge strawberry-flavor hyperbolic monopoles obtained from solutions of the Thomson problem, with the result that the same level of excellent agreement is found. Not only does this demonstrate the success of our new magnetic bag approximation, but it elucidates the nature of monopole anti-zeros. Until now, this has been somewhat of a mysterious issue, but now we see that a monopole with an anti-zero is simply making use of a previously overlooked spherically symmetric solution of the Bogomol'nyi equation. There is a similar spherically symmetric solution of the Bogomol'nyi equation in $\mathbb{R}^3$, satisfying the regularity conditions at the origin but not the finite energy condition at infinity, so this new understanding of monopole anti-zeros extends to the Euclidean setting too.

The observant reader may wonder why we chose to impose the $\rho \to \infty$ boundary condition (7.6) on the solution used for the interior of the bag, given that the bag approximation (7.9) only utilizes this solution in the finite range $[0, \rho_*]$. Our justification is that the solution (7.8) fits the exact monopole fields. We note however that the system (7.5) has many solutions with an anti-zero at $\rho = 0$ other than Eq. (7.8); the fact that the particular solution (7.8) fits all available strawberry-flavor monopoles may be a consequence of working within the JNR ansatz.

VIII. CONCLUSION

The Abelian magnetic bag, describing a large number of coincident non-Abelian BPS monopoles, has been extended to hyperbolic space and its properties investigated in detail. In particular, we have made comparisons with exact solutions of the Bogomol'nyi equation containing hundreds of monopoles. This is the main reason for moving to the hyperbolic setting, as such exact solutions are not available for comparison in Euclidean space. Our results show a good agreement for charge N monopoles with a single zero of the

Higgs field (of multiplicity N) and we have derived a Nahm transform for the associated Abelian magnetic bag from the large-N limit of the discrete Nahm equation for hyperbolic monopoles. However, for monopoles with more than N zeros of the Higgs field we found that the Abelian magnetic bag is not a good description, but must be supplemented by a non-Abelian interior for the bag, which we were able to describe in detail. This provides a new understanding of the structure of monopole anti-zeros.

ACKNOWLEDGMENTS

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