# Towards a Pólya–Carlson dichotomy for algebraic dynamics

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**Abstract**

We present results and background rationale in support of a Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behaviour of dynamical zeta functions of compact group automorphisms.

## 1. Introduction

Let $\theta : X \to X$ be a continuous map on a compact metric space with the property that

$$\mathsf{F}_\theta(n) = |\{x \in X \mid \theta^n x = x\}|$$

is finite for all $n \geqslant 1$. The associated dynamical zeta function

$$\zeta_\theta(z) = \exp \sum_{n \geqslant 1} \frac{\mathsf{F}_\theta(n)}{n} z^n$$

is an invariant of topological conjugacy for the map $\theta$, and the analytic properties of the zeta function and weighted versions of it may be used to study orbit-growth and other properties of $\theta$. In particular, for situations in which the zeta function has a finite positive radius of convergence and a meromorphic extension beyond the radius of convergence Tauberian methods may be used to relate analytic properties of singularities of the zeta function to orbit-growth properties of the map. For smooth maps with sufficiently uniform hyperbolic behaviour, the zeta function is rational (see Manning [10]), and in particular hyperbolic toral automorphisms have this property. On the other hand, for some natural families of dynamical systems the arithmetic and analytic properties of the zeta function are known to be very different. The third author [25], for a family of isometric extensions of the full shift on $p$ symbols ($p$ a prime) parametrised by a probability space, shows that with the possible exception of two values of $p$ the dynamical zeta function is not an algebraic function almost surely.

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Everest, Stangoe and the third author [6] studied the specific automorphism of a compact group dual to the automorphism $r \mapsto 2r$ on $\mathbb{Z}[\frac{1}{6}]$, showing it to have a natural boundary on the circle $|z| = \frac{1}{2}$ (we refer to Segal [23, Ch. 6] for a convenient introduction to the theory of complex functions with natural boundary). Buzzi [2] shows that a certain weighted random zeta function has a natural boundary. In a different direction, for dynamical systems with a polynomial growth bound on the number of periodic orbits, a more natural complex function that captures all the periodic point data is given by an orbit Dirichlet series, and many natural examples are known to have infinitely many singularities on the critical line with no lower bound on their separation by work of Everest, Stevens *et al.* [5], or even to have a natural boundary in calculations of Pakapongpun and the third author [16].

One of the fundamental links between the arithmetic properties of the coefficients of a complex power series and its analytic behaviour is given by the Pólya–Carlson theorem [3], [17].

**Pólya–Carlson Theorem.** *A power series with integer coefficients and radius of convergence* 1 *is either rational or has the unit circle as a natural boundary.*

Unfortunately, while there are some natural group automorphisms whose zeta function has radius of convergence 1, many do not – and for most (in cardinality) it is not at all clear how to compute the radius of convergence without refined information about the arithmetic of linear recurrence sequences.

The suggestion we wish to explore here is that there is a Pólya–Carlson dichotomy for group automorphisms in exactly the same sense: the zeta function of a compact group automorphism is either rational or admits a natural boundary at its radius of convergence.

**Conjecture.** *Let $\theta : X \to X$ be an automorphism of a compact metric abelian group, with the property that $\mathsf{F}_\theta(n) < \infty$ for all $n \geqslant 1$. Then $\zeta_\theta$ is either a rational function or admits a natural boundary.*

We cannot prove this, but will show it for a large class of automorphisms of connected finite-dimensional abelian groups (these groups are called solenoids). In addition, the arguments here do we hope make this suggestion plausible, and clarify what sort of issues would arise in attempting to prove the full statement. The conjecture – if true – may be seen as a rigidity phenomenon in algebraic dynamics, preventing small changes to the analytic properties of the dynamical zeta function for these systems. This contrasts with other orbit-growth properties for compact group automorphisms which can take all values along a continuum (we refer to the work of Baier, Jaidee, Stevens *et al.* [1] for examples of dynamical properties that can vary continuously in this setting).

In addition to the Pólya–Carlson theorem, we will make essential use of the Hadamard quotient theorem (see van der Poorten [18] and Rumely [21]).

**Hadamard Quotient Theorem.** *Let $\mathbb{K}$ be a field of characteristic zero, and suppose that $\sum_{n\geqslant 0} b_n z^n$ and $\sum_{n\geqslant 1} c_n z^n$ in $\mathbb{K}[[z]]$ are expansions of rational functions. If there is a finitely-generated ring $R$ over $\mathbb{Z}$ with $a_n = \frac{b_n}{c_n} \in R$ for all $n \geqslant 1$, then $\sum_{n\geqslant 0} a_n z^n$ is also the expansion of a rational function.*

Given the fact that there are well-known arithmetical constraints on the possible sequences $(\mathsf{F}_\theta(n))$ of periodic point counts for any map (see Puri and the third author [19]), and additional (less well-known) constraints in the case of group automorphisms (see the thesis of Moss [14] for more details, or the survey of Staines *et al.* [13] for an example of a linear recurrent divisibility sequence that counts periodic points for some map but that cannot be the periodic point count for a group automorphism), we should point out that the suggested dichotomy certainly cannot hold for all maps. To see this, notice for example that there is a continuous map $\theta$ with $\mathsf{F}_\theta(n) = \binom{2n}{n}$ for all $n \geqslant 1$ (by work of Puri and the third author [19]) and so

$$\sum_{n\geqslant 1} \mathsf{F}_\theta(n) z^n = \frac{1}{\sqrt{1-4z}} - 1.$$

In the parameter space of all compact metric abelian group automorphisms over which we are suggesting the dichotomy holds, the results below are restricted in three different ways. Removing the assumption of connectedness — or at least dealing with the totally disconnected case — is likely to be relatively straightforward because of the additional information available about the arithmetic properties of linear recurrence sequences over finite fields, and it is expected that the most interesting arithmetic questions occur in the connected setting considered here. The bound on the topological dimension of the compact group is needed to avoid Salem numbers, the familiar bane of several investigations in arithmetic and dynamics. Removing this bound on the face of it would involve subtle Diophantine problems involving linear forms in logarithms. The third way in which we restrict the cases we deal with concerns a subset $S$ of a countable collection $P$ of places of a number field. We are able to handle the situations in which $S$ is finite or infinite but extremely thin, and the case in which $P \setminus S$ is finite. Understanding the general case seems to require different techniques.

## 2. One-dimensional solenoids

A one-dimensional solenoid $X$ has a Pontryagin dual group isomorphic to a subgroup of $\mathbb{Q}$. Given an automorphism $\theta : X \to X$ of a one-dimensional solenoid, the dual group naturally carries the structure of a module over a ring of the form $\mathbb{Z}[r^{\pm 1}]$, where $r \in \mathbb{Q}^\times$ and multiplication by $r$ corresponds to application of the dual automorphism $\widehat{\theta}$. To avoid trivial (from a dynamical point of view, non-ergodic) automorphisms, we assume $r \neq \pm 1$ throughout. For a more detailed account of these systems, we refer to the papers of Chothi, Everest and the third author [4] and the recent survey by Staines *et al.* [13]. Relevant background and references for all the results we will need on linear recurrence sequences may be found in the monograph of Everest, van der Poorten, Shparlinski and the third author [7].

There is a convenient formula for $\mathsf{F}_\theta(n)$, which we shall use as the basis for our discussion. Since there may be uncountably many non-conjugate automorphisms of one-dimensional solenoids that share the same zeta function (we refer to Miles [12, Ex. 1] for an example), this not only avoids complex classification problems for these systems but has the added advantage that no background in algebraic dynamics is needed to access the main results of this section; instead the formula (1) below serves as a starting point.

Let $\mathcal{P}(\mathbb{Q})$ denote the set of rational primes. For any $x \in \mathbb{Q}$ and $S \subset \mathcal{P}(\mathbb{Q})$, write $|x|_S = \prod_{p \in S} |x|_p$. Miles [12, Th. 3.1] shows that there is a distinguished set of primes $T \subset \mathcal{P}(\mathbb{Q})$ such that $\mathsf{F}_\theta(n) = |r^n - 1|_T^{-1}$, for all $n \geqslant 1$, and that $|r|_p = 1$ for all $p \in T$ (see Miles [12, Rmk. 1]). Therefore, by the Artin product formula (see Weil [26, Sec. IV.4] for a complete treatment of the valuation theory of number fields used here) we have

$$\mathsf{F}_\theta(n) = |r^n - 1| \cdot |r^n - 1|_S, \tag{1}$$

where $S = \mathcal{P}(\mathbb{Q}) \setminus T$. Furthermore, $|r|_p \neq 1$ necessarily implies that $p \in S$.

If $T = \varnothing$, then we obtain the trivial sequence $(1, 1, \dots)$, so we assume that $T \neq \varnothing$ throughout. In this section, we do not discuss further the problem of determining which dynamical systems give rise to the same formula (1), it is sufficient to note that for any $r \neq \pm 1$ and any $S \subset \mathcal{P}(\mathbb{Q})$ with $|r|_p = 1$ for all $p \in \mathcal{P}(\mathbb{Q}) \setminus S$, the automorphism $x \mapsto rx$ on the ring $R = \mathbb{Z}[\frac{1}{p} : p \in S]$ dualizes to an automorphism $\theta$ of the solenoid $X = \widehat{R}$, and its sequence of periodic point counts is given by (1). We refer to [4] for further details of this construction.

From now on, we assume that $r \in \mathbb{Q} \setminus \{\pm 1\}$ is fixed and consider the sequences $(f_S)$ defined by $f_S(n) = |r^n - 1| \cdot |r^n - 1|_S$ for $n \geqslant 1$ that arise by varying the set $S$. We also write $F_S(z) = \sum_{n \geqslant 1} f_S(n) z^n$ for the associated ordinary generating function. We are able to concentrate on the ordinary generating function rather than the zeta function (and hence have more ready access to the theory of linear recurrence sequences) because of the following fundamental relationship.

**Lemma 1.** *Let $F(z) = \sum_{n \geqslant 1} \mathsf{F}_\theta(n) z^n$. If $\zeta_\theta$ is rational then $F$ is rational. If $\zeta_\theta$ has analytic continuation beyond its circle of convergence, then so too does $F$. In particular, the existence of a natural boundary at the circle of convergence for $F$ implies the existence of a natural boundary for $\zeta_\theta$.*

PROOF. This follows from the fact that $F(z) = z\zeta_\theta'(z)/\zeta_\theta(z)$. $\square$

In order to handle the sequence $f_S = (f_S(n))$ more easily, we need a way to evaluate expressions of the form $|r^n - 1|_p$ when $|r|_p = 1$. To this end, the following lemma is useful, and we state this in the more general setting of number fields, since this will be needed in the next section. Moreover, it will be necessary to deal with number fields of unknown degree for the putative linear recurrence sequences that arise inside arguments by contradiction. Let $\mathbb{K}$ be an algebraic number field, and let $\mathcal{P}(\mathbb{K})$ denote the set of places of $\mathbb{K}$. For a place $v \in \mathcal{P}(\mathbb{K})$ with $|\xi|_v = 1$, let $\mathfrak{K}_v$ denote the residue class field, let $m_v$ denote the multiplicative order of the image of $\xi$ in $\mathfrak{K}_v^\times$, and let $\varrho_v$ denote the residue degree. We assume that $|\cdot|_v$ is normalized so that the Artin product formula holds (we refer to Ramakrishnan and Valenza [20] or Weil [26, Sec. IV.4] for the details).

**Lemma 2.** *Let $p$ be the characteristic of $\mathfrak{K}_v$. There exists a non-negative integer constant $D_v \geqslant 0$ and a rational constant $C_v > 0$, such that for any $n \in \mathbb{N}$,*

$$|\xi^n - 1|_v = \begin{cases} 1 & \text{if } m_v \nmid n, \\ C_v |n|_p^{\varrho_v} & \text{if } m_v \mid n \text{ and } \operatorname{ord}_p(n) > D_v, \end{cases}$$

*and $|\xi^n - 1|_v$ assumes at most finitely many values otherwise.*

Proof. See [11, Lem. 4.9], for example. □

**Remark 3.** The proof of [11, Lem. 4.9] shows in fact that $D_v$ is the least positive integer $D$ such that

$$|\xi^{m_p p^{D+1}} - 1|_v = |p|_v |\xi^{m_p p^D} - 1|_v,$$

and also that

$$C_v = |p|_v^{-D_v} |\xi^{m_p p^{D_v}} - 1|_v.$$

Furthermore, in the particular case $\mathbb{K} = \mathbb{Q}$, $D_p = 0$ when $p > 2$, and $D_p \in \{0, 1\}$ when $p = 2$.

We begin by establishing precisely when $F_S$ is rational.

**Theorem 4.** *The function $F_S$ is rational if and only if $|r|_p \neq 1$ for all $p \in S$.*

Proof. Let $S' = \{p \in S : |r|_p = 1\}$ and $S'' = \{p \in S : |r|_p > 1\}$. Then

$$f_S(n) = |r^n - 1| |r|_{S''}^n f(n),$$

where $f(n) = |r^n - 1|_{S'}$. If $r = a/b$, then without loss of generality we can assume that $a > |b|$ because $\zeta_\theta = \zeta_{\theta^{-1}}$. Then

$$|r^n - 1| |r|_{S''}^n = a^n - b^n \Rightarrow f_S(n) = (a^n - b^n) f(n), \tag{2}$$

and this shows immediately that $F_S$ is rational if $S' = \varnothing$.

To prove the converse, we assume that $S' \neq \varnothing$ and aim to show that the sequence $f_S$ does not satisfy a linear recurrence over $\mathbb{Q}$, which will in turn imply that $F_S$ is irrational.

To begin with, assume that $S$ is finite, so the sequence $f = (f(n))$ lies in a finitely generated extension of $\mathbb{Z}$. For a contradiction, assume that $f_S$ is given by a linear recurrence relation. Then, using (2) and the Hadamard quotient theorem, it follows that $f$ also satisfies a linear recurrence relation. Let $q$ be a rational prime not in $S$, and define

$$n(e) = q^e \prod_{p \in S'} m_p p^{D_p},$$

where $D_p$ is as in Lemma 2 and $e \geqslant 1$. Applying Lemma 2, we see that

$$f(kn(e)) = f(n(e))$$

whenever $k$ is coprime to $n(e)$. Hence the sequence $f$ assumes infinitely many values infinitely often, and so it cannot satisfy a linear recurrence by a result of Myerson and van der Poorten [15, Prop. 2], giving a contradiction.

Now assume that $S$ is infinite and recall that $S \neq \mathcal{P}(\mathbb{Q})$. Using the Artin product formula and (2), if $T = \mathcal{P}(\mathbb{Q}) \setminus S'$, then

$$f_S(n) f_T(n) = |r^n - 1||r^n - 1|_{S''} = |r^n - 1||r|^n_{S''} = a^n - b^n.$$

Note that both $f_S$ and $f_T$ are positive integer sequences and the product sequence $f_S f_T$ satisfies a linear recurrence over $\mathbb{Q}$. Moreover, by the Hadamard quotient theorem, $f_S$ satisfies a linear recurrence over $\mathbb{Q}$ if and only if $f_T$ satisfies a linear recurrence over $\mathbb{Q}$.

Hence, if either $f_S$ or $f_T$ satisfies a linear recurrence, it follows that for $n$ sufficiently large we have

$$f_S(n) \;=\; \sum_{i=1}^{d} P_i(n)\alpha_i^n \tag{3}$$

and

$$f_T(n) \;=\; \sum_{j=1}^{e} Q_j(n)\beta_j^n, \tag{4}$$

for some natural numbers $d$ and $e$ and rational polynomials

$$P_1, \ldots, P_d, Q_1, \ldots, Q_e,$$

and algebraic numbers $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_e$. Let $N$ be a nonzero natural number with the property that $NP_i$ and $NQ_j$ are integer polynomials and $N\alpha_i$ and $N\beta_j$ are algebraic integers for $i, j \in \{1, \ldots, d\} \times \{1, \ldots, e\}$. We let $N'$ denote the product of the norms of $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_e$.

Let $p$ be a prime number that does not divide $abNN'$. We may then regard $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_e$ as elements of the algebraic closure $\overline{\mathbb{Q}}_p$, and since $p$ does not divide $N$, we have that

$$|\alpha_i|_p, |\beta_j| \leqslant 1$$

for $(i, j) \in \{1, \ldots, d\} \times \{1, \ldots, e\}$. Moreover, since $p$ does not divide $N'$, we have

$$|\alpha_i|_p = |\beta_j|_p = 1$$

for $(i, j) \in \{1, \ldots, d\} \times \{1, \ldots, e\}$. Let $G \in \mathbb{Q}[x]$ be the monic polynomial of smallest degree that has $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_e$ as zeros. Then, by construction, $G$ has no coefficients with denominators divisible by $p$ when written in lowest terms. We note that $G$ modulo $p$ splits in some extension $\mathbb{F}_q$ of $\mathbb{F}_p$, so there is some $k \geqslant 0$ such that

$$|\alpha_i^{p^k} - \alpha_i|_p < 1$$

and

$$|\beta_j^{p^k} - \beta_j|_p < 1$$

for $(i,j) \in \{1,\dots,d\} \times \{1,\dots,e\}$. In particular, we have $|\alpha_i^{p^k-1} - 1|_p < 1$ for $i \in \{1,\dots,d\}$ and $|\beta_j^{p^k-1} - 1|_p < 1$ for $j \in \{1,\dots,e\}$. We also have

$$|P_i(p^k-1) - P_i(-1)|_p < 1$$

and

$$|Q_j(p^k-1) - Q_j(-1)|_p < 1$$

for $(i,j) \in \{1,\dots,d\} \times \{1,\dots,e\}$ (simply because $P_i$ and $Q_j$ are integral polynomials). It follows from (3) and (4) that $p$ divides $f_S(p^k-1)$ if and only if $p$ divides $\sum_{i=1}^d P_i(-1)$ and $p$ divides $f_T(n)$ if and only if $p$ divides $\sum_{j=1}^e Q_i(-1)$. Since $f_S(p^k-1)f_T(p^k-1) = a^{p^k-1} - b^{p^k-1} \equiv 0 \pmod p$, we see that

$$\left(\sum_{i=1}^d P_i(-1)\right)\left(\sum_{j=1}^e Q_i(-1)\right) \equiv 0 \pmod p$$

for all sufficiently large primes $p$. Hence one of $\sum_{i=1}^d P_i(-1)$ and $\sum_{j=1}^e Q_j(-1)$ must be zero. Without loss of generality, assume that $\sum_{i=1}^d P_i(-1) = 0$. This means that $p$ divides $f_S(p^k-1)$ for every prime $p$ that does not divide $abNN'$. In particular, $p \notin S$ for any such prime $p$.

Thus we may assume that $S$ is finite and non-empty, and this case has been handled already. □

We record two curious consequences for the special case of one-dimensional solenoids.

**Corollary 5.** *The function $F_S$ is rational if and only if the associated zeta function is rational.*

PROOF. The proof above shows that if $F_S$ is rational then $f_S(n) = a^n - b^n$ with $a > |b| \geqslant 1$, in which case $\zeta_\theta(z) = \frac{1-bz}{1-az}$. The other implication is covered by Lemma 1. □

**Corollary 6.** *If $F_S$ has radius of convergence 1, then $F_S$ has the unit circle as a natural boundary.*

PROOF. The theorem above shows that if $F_S$ is rational, then $f_S(n) = a^n - b^n$, where $a > |b| \geqslant 1$, so $F_S$ has radius of convergence $\frac{1}{a} < 1$. Thus, $F_S$ having radius of convergence 1 implies that $F_S$ is irrational, and the result follows from the Pólya–Carlson Theorem itself. □

As an example of Corollary 6, consider the case where $\mathcal{P}(\mathbb{Q}) \setminus S$ is finite and non-empty, so $f_S(n)$ grows polynomially in $n$ (as $f_S$ corresponds to a sequence of periodic point counts for a system of finite combinatorial rank [5]), and $F_S$ has radius of convergence 1. Then the corollary shows that $F_S$ has a natural boundary. Corollary 6 also applies to many of the cases where $S$ is infinite, but it is not straightforward to exibit examples.

We now turn our attention to finite $S$, for which the radius of convergence of $F_S$ is always strictly less than 1. It will be useful to consider certain rational functions that arise from congruence conditions, especially for our subsequent work involving Lambert series. The following (a generating function analogue of the Euler product construction for certain Dirichlet series) is readily established using a simple application of the inclusion-exclusion principle.

**Lemma 7.** *For a finite set of rational primes $S$, the function $H_S(z) = \sum_{n\geqslant 1} z^n$, where $n$ runs over all positive integers with $p \nmid n$ for all $p \in S$, is a rational function of the form*

$$\sum_{I\in\mathcal{I}} \frac{d_I z^{k_I}}{1 - z^{k_I}},$$

*where $\mathcal{I} = \mathcal{I}(S)$ is a finite indexing set, $d_I \in \{-1, 1\}$, and each $k_I \in \mathbb{N}$ is divisible only by primes appearing in $S$.*

For a singleton set, we write more briefly $H_{\{p\}} = H_p$. Thus, for example,

$$H_3(z) = \frac{z}{1-z} - \frac{z^3}{1-z^3}.$$

The next example is a simplified account of the case considered by Everest, Stangoe and the third author [6]. It gives a simple illustration of the irrational case in Theorem 4, and the functional equation found may be used to show the existence of a natural boundary in this case.

**Example 8.** For $r = 2$, consider $F_{\{2,3\}}$, which is the ordinary generating function for the periodic point sequence for the map dual to $x \mapsto 2x$ on $\mathbb{Z}[\frac{1}{6}]$. For this simple example, we can establish a functional equation to show that $F_{\{2,3\}}$ has a natural boundary. Let

$$F(z) = \sum_{n\geqslant 1} |2^n - 1|_3 z^n,$$

so that $F_{\{2,3\}}(z) = F(2z) - F(z)$. Since $F$ has radius of convergence 1, showing that the unit circle is a natural boundary for $F$ is enough to prove that the circle $|z| = \frac{1}{2}$ is a natural boundary for $F_{\{2,3\}}$. Since

$$F(z) = \frac{1}{3} \sum_{2|n} |n|_3 z^n + \sum_{2\nmid n} z^n,$$

we have $F(z) = \frac{1}{3}G(z^2) + H_2(z)$, where $G(z) = \sum_{n\geqslant 1} |n|_3 z^n$. Furthermore, since $H_2$ is rational, it is enough to show that $G$ has the natural boundary $|z| = 1$ to establish this for $F$. Writing $n = 3^e k$, where $e \geqslant 0$ and $3 \nmid k$, gives

$$G(z) = \sum_{e\geqslant 0} \frac{1}{3^e} \sum_{3\nmid k} z^{3^e k} = \sum_{e\geqslant 0} \frac{1}{3^e} H_3(z^{3^e}) = H_3(z) + \frac{1}{3} \sum_{e\geqslant 0} \frac{1}{3^e} H_3(z^{3^{e+1}}).$$

It follows that $G(z) = H_3(z) + \frac{1}{3}G(z^3)$. Using this functional equation inductively, we deduce that there are dense singularities of $G$ on the unit circle, occurring at $3^e$-th roots of unity, $e \in \mathbb{N}$.

In general, it is difficult to establish functional equations of the same sort to demonstrate a natural boundary. However, for finite sets $S$, we are nonetheless able to identify distinguished singularites on the circle of convergence for $F_S$ that lead to a natural boundary, by means of the following calculation.

**Theorem 9.** *Let $S$ be a finite set of rational primes such that $|r|_p = 1$ for all $p \in S$, and let $F(z) = \sum_{n\geqslant 1} |r^n - 1|_S z^n$. Then there is a constant $E(S) > 0$ such that for any $q \in S$ and any $\delta \in \mathbb{Z}[1/q]$, with the possible exception of finitely many values of $\delta$,*

$$|F(\lambda \exp(2\pi\delta i))| \to \infty$$

*as $\lambda \to 1^-$ whenever $|\delta|_q > q^{E(S)}$.*

Proof. Let $m_p$ denote the multiplicative order of $r$ modulo $p$ for any prime $p$ in $S$, and note that $|r^n - 1|_p \neq 1$ if and only if $m_p \mid n$ by Lemma 2. Let $T$ comprise the set of primes in $S$ together with all those that divide $m_p$ for some $p \in S$. Choose $\delta \in \mathbb{Z}[1/q]$ with

$$-\operatorname{ord}_q(\delta) > E(S) = 1 + \max\{\operatorname{ord}_t(m_p) : t \in T, p \in S\}$$

and set $E = -\operatorname{ord}_q(\delta) - 1$. We wish to consider the behaviour of $F(z)$ when

$$z = \lambda \exp(2\pi\delta i)$$

and $\lambda \to 1^-$. To do this, we will split up the sum defining $F$ as follows.

Let

$$J = \left\{ \prod_{p\in T} p^{e_p} : 0 \leqslant e_p < E \text{ for all } p \in T \right\}$$

and for any $j \in J$, let $N(j)$ denote the set of positive integers $n$ such that

$$\operatorname{ord}_p(n) = \operatorname{ord}_p(j)$$

for all $p \mid j$ and $\operatorname{ord}_p(n) \geqslant E$ for all $p \in T$ with $p \nmid j$, so that $\{N(j)\}_{j\in J}$ forms a partition of $\mathbb{N}$. Notice that

$$m_p \mid n \text{ for some } n \in N(j) \Longleftrightarrow m_p \mid n \text{ for all } n \in N(j)$$

for any $p \in S$. Furthermore, if we define

$$S(j) = \{p \in S : p \nmid j \text{ and } m_p \mid n \text{ for some } n \in N(j)\},$$

then by Lemma 2, for all $n \in N(j)$,

$$|r^n - 1|_S = c_j |n|_{S(j)}$$

for some non-negative rational constant $c_j$. Hence, we can write

$$\begin{aligned} F(z) &= \sum_{j \in J} \sum_{n \in N(j)} c_j |n|_{S(j)} z^n \\ &= \underbrace{\sum_{j : S(j) \neq \varnothing} c_j \sum_{n \in N(j)} |n|_{S(j)} z^n}_{G(z)} + \sum_{j : S(j) = \varnothing} c_j \sum_{n \in N(j)} z^n. \end{aligned}$$

The second series on the right-hand side (given by summands for which $S(j)$ is empty) is a rational function with radius of convergence 1 by Lemma 7, so has only finitely many singularities on the circle of convergence $|z| = 1$. Therefore, for all but finitely many choices of $\delta \in \mathbb{Z}[1/q]$, this second series is bounded as $\lambda \to 1^-$. From now on assume that $\delta$ has been chosen so as to avoid these singularities. To demonstrate the required conclusion of the theorem, it is sufficient to show that $|G(z)| \to \infty$ as $\lambda \to 1^-$. For ease of notation, we now omit the clause $S(j) \neq \varnothing$ when writing $G(z)$.

For any $j \in J$, let

$$U(j) = \left\{ \prod_{p \in S(j)} p^{e_p} : e_p \geqslant E \text{ for all } p \in S(j) \right\},$$

and

$$j' = j \prod_{p \in T \setminus S(j) : p \nmid j} p^E.$$

Note that any $n \in N(j)$ can be written uniquely in the form $n = uj'k$ with $u$ in $U(j)$ and $p \nmid k$ for all $p \in S'(j) = S(j) \cup \{p : p \mid j\}$. Hence,

$$G(z) = \sum_j \sum_{u \in U(j)} \frac{c_j}{u} \sum_k z^{uj'k}, \tag{5}$$

where $k$ runs through all positive integers with $p \nmid k$ for all $p \in S'(j)$. Consider first the terms in the series for which $\operatorname{ord}_q(uj) > E$. In this case, $z^{uj'k} = \lambda^{uj'k}$ and the inner sum comprises strictly positive real terms, and diverges as $\lambda \to 1^-$. To complete the proof, we will show that the sum of the remaining terms in the series for $G(z)$ is bounded. For these terms, we have $\operatorname{ord}_q(uj) \leqslant E$.

First suppose $q \nmid uj$, so $q \notin S(j)$ and $q \nmid j$, which implies that $q \notin S'(j)$. Using Lemma 7 for the corresponding terms of $G(z)$, the inner sum in (5) may be written as $H_{S'(j)}(z^{uj'})$. Moreover, the rational expression given in Lemma 7 shows that

$$|H_{S'(j)}(z^{uj'})| \leqslant \sum_{I \in \mathcal{I}(S'(j))} \frac{1}{|1 - z^{uj'k_I}|}, \tag{6}$$

and since $q \notin S'(j)$, $\mathrm{ord}_q(uj'k_I) = \mathrm{ord}_q(uj') = E$. Thus there is a constant $M(\delta, j) > 0$ such that

$$|1 - z^{uj'k_I}| > M(\delta, j).$$

Therefore, using (5) and (6), the sum of terms of $G(z)$ for which $q \nmid uj$ is bounded in absolute value by

$$\sum_j \sum_{u \in U(j)} \frac{c_j |\mathcal{I}(S'(j))|}{M(\delta, j) u} = \sum_j \frac{c_j |\mathcal{I}(S'(j))|}{M(\delta, j)} \prod_{p \in S(j)} \frac{1}{p^E(1 - 1/p)}.$$

It remains to consider the terms of $G(z)$ for which $0 < \mathrm{ord}_q(uj) < E$. In this case, $\mathrm{ord}_q(uj) = \mathrm{ord}_q(j) > 0$, so $q \in S'(j)$. Let $S''(j) = S'(j) \setminus \{q\}$, and notice that we may write

$$\begin{aligned} H_{S'(j)}(w) &= H_{S''(j)}(w) - H_{S''(j)}(w^q) \\ &= \sum_{I \in \mathcal{I}(S''(j))} d_I \left( \frac{w^{k_I}}{1 - w^{k_I}} - \frac{w^{qk_I}}{1 - w^{qk_I}} \right) \\ &= \sum_{I \in \mathcal{I}(S''(j))} d_I \left( \frac{w^{k_I} + w^{2k_I} + \cdots + w^{(q-1)k_I}}{1 - w^{qk_I}} \right), \end{aligned} \tag{7}$$

where, $q \nmid k_I$ for all $I \in \mathcal{I}(S''(j))$.

Once again, for the terms of $G(z)$ with $0 < \mathrm{ord}_q(uj) < E$, the inner sum in (5) is $H_{S'(j)}(z^{uj'})$, and using (7) we obtain the bound

$$|H_{S'(j)}(z^{uj'})| \leqslant \sum_{I \in \mathcal{I}(S''(j))} \frac{q - 1}{|1 - z^{uj'qk_I}|}. \tag{8}$$

Furthermore, since

$$\mathrm{ord}_q(uj'qk_I) = 1 + \mathrm{ord}_q(uj) \leqslant E,$$

there is a constant $M(\delta, j) > 0$ such that $|1 - z^{uj'qk_I}| > M(\delta, j)$. Just as before we can use (5) and (8), to obtain a bound for the remaining terms of $G(z)$. □

**Corollary 10.** *If $S$ is finite and $F_S$ is irrational, then the circle of convergence of $F_S$ is a natural boundary for the function.*

Proof. Exactly as in the proof of Theorem 4, and without loss of generality, we can assume that $f_S(n)$ is given by (2). Then, as in Example 8, write

$$F_S(z) = F(az) - F(bz),$$

where $F$ is given by Theorem 9. Since $a > |b|$ and since $F$ has the unit circle as a natural boundary, this shows that $F_S$ has the circle $|z| = \frac{1}{a}$ as a natural boundary. □

In the next section we will give an alternative proof of Corollary 10 for higher-dimensional solenoids based on the Pólya–Carlson Theorem. However, this method does not explicitly reveal why the natural boundary occurs in the way that Theorem 9 does.

To conclude this section, we consider the occurrence of natural boundaries for certain infinite sets of primes $S$. We begin with the following application of Fabry's gap Theorem (see Segal [8, Sec. 6.4]).

**Lemma 11.** *Let $a > 1$ and let $g(n)$ be an integer-valued sequence satisfying $g(n) \leqslant a^n$ for all $n \geqslant 1$. Suppose that $g(n)$ does not satisfy a linear recurrence and suppose that for every real number $s > 1$ there is a sequence of natural numbers $n_1 < n_2 < \cdots$ with $n_j/j \to \infty$ such that $g(n) < s^n$ for $n \notin \{n_1, n_2, \ldots\}$. Then there is some $R \in [\frac{1}{a}, 1]$ such that $\sum_{n\geqslant 1} g(n)z^n$ admits the circle $|z| = R$ as its natural boundary.*

Proof. Let

$$R^{-1} = \limsup_{n\to\infty} g(n)^{1/n}$$

and notice that $\frac{1}{a} \leqslant R \leqslant 1$. If $R = 1$, then the Pólya–Carlson theorem immediately gives us the result. Let $s \in (1, \frac{1}{R})$, let $N$ denote the set of natural numbers $n$ such that $g(n) < s^n$, and write $\{n_1, n_2, \ldots\}$ for the complement of $N$ in the natural numbers. By assumption $n_j/j \to \infty$. Let

$$G(z) = \sum_{n\geqslant 1} g(n)z^n$$

and

$$G_N(z) = \sum_{n\in N} g(n)z^n.$$

Then $G_N$ has radius of convergence at least $\frac{1}{s}$, which is strictly greater than $R$. Hence the set of singularities of $G(z) - G_N(z)$ that lie on the circle of radius $R$ is identical to the set of singularities of $G(z)$ on the circle of radius $R$. However

$$G(z) - G_N(z) = \sum_{j} g(n_j)z^{n_j}$$

has the circle of radius $R$ as its natural boundary by Fabry's gap Theorem. □

**Lemma 12.** *Let $p$ be a rational prime such that $|r|_p = 1$. Then there is an integer constant $A$ depending only on $r$ such that*

$$|r^n - 1|_p^{-1} < \left(\frac{\log(n)}{n} + \frac{\ell_p \log(A)}{n}\right)^n,$$

*where $\ell_p$ denotes the smallest natural number $n$ for which $|r^n - 1|_p < \frac{1}{2}$.*

Proof. First note that if $p > 2$, $|r^n - 1|_p < 1/2 \Longleftrightarrow |r^n - 1|_p < 1$, so $\ell_p = m_p$, where $m_p$ is as in Lemma 2, but this is not the case if $p = 2$. If $n$ is not a multiple of $\ell_p$, then $|r^n - 1|_p^{-1} \leqslant 2$, by Lemma 2 and Remark 3, so the result is immediate. For $n$ a multiple of $\ell_p$, write $n = \ell_p p^e k$ where $p \nmid k$ and $e \geqslant 0$. Then, by Lemma 2 and Remark 3,

$$|r^n - 1|_p^{-1} = |r^{\ell_p p^e k} - 1|_p^{-1} \leqslant \ell_p |r^{\ell_p} - 1|_p^{-1} p^e \leqslant A^{\ell_p} p^e,$$

where $A$ is the height of $r$ (so $A$ is an integer constant depending only on $r$ and is independent of $p$). Furthermore,

$$A^{\ell_p} p^e = \left( \frac{e \log(p) + \ell_p \log(A)}{n} \right)^n,$$

giving the desired result since $e \log(p) \leqslant \log(n)$. $\square$

**Theorem 13.** *Let $S$ be a set of primes such that $\mathcal{P}(\mathbb{Q}) \setminus S = \{p_1 < p_2 < \cdots\}$ is infinite and satisfies*

$$\frac{\log(p_{n+1})}{p_n} \longrightarrow \infty$$

*as $n \to \infty$. Then $F_S$ admits some circle of radius $R \leqslant 1$ as its natural boundary.*

Proof. Let $T = \mathcal{P}(\mathbb{Q}) \setminus S = \{p_1 < p_2 < \ldots\}$. By our earlier assumptions on $S$, $|r|_p = 1$ for all $p \in T$. Furthermore $f_S(n) = |r^n - 1|_T^{-1}$ by the Artin product formula.

Let $s > 1$ and let $C$ be a natural number with $2\log(A)/C < \log(s)/2$, where $A$ is the constant appearing in Lemma 12. For each $k$, we let $\ell_k$ denote the smallest natural number such that $|r^{\ell_k} - 1|_{p_j} < 1/2$. Note that $A^{\ell_k} > p_k$ and so $\ell_k > \log(p_k)$. On the other hand, $\ell_k \leqslant p_k - 1$, when $p_k \neq 2$. Since

$$p_k = \mathrm{o}(\log(p_{k+1}))$$

and $\ell_{k+1} > \log(p_{k+1})$, we see that $\ell_{k+1}/\ell_k \to \infty$.

Assume that $k$ is large enough to ensure that $\ell_i > C\ell_{i-1}$ for $i \geqslant k$ and

$$\ell_{k-1} > \max\{C\ell_1, \ldots, C\ell_{k-2}\}.$$

Let $n$ be a natural number in $\{\ell_{k-1}, \ldots, \ell_k - 1\}$ that is not in $\{\ell_{k-1}j : j \leqslant C\}$. Then we have $f_S(n) = \prod_{i<k} |r^n - 1|_{p_i}^{-1}$, since $p_i$ cannot divide $f_S(n)$ for $i \geqslant k$. Note that if $n$ is not a multiple of $\ell_i$ then $|r^n - 1|_{p_i}^{-1} \geqslant 1/2$; if $n$ is a multiple of $\ell_i$ then $|r^n - 1|_{p_i}^{-1} \leqslant s_i^n$, where $s_i \leqslant \log(n)/n + \ell_i \log(A)/n$, by Lemma 12. Hence

$$\frac{\log(f_S(n))}{n} \leqslant \sum_{i<k} \left( \frac{\log(n)}{n} + \frac{\ell_i \log(A)}{n} \right) \leqslant \frac{k \log(n)}{n} + \sum_{i<k} \frac{\ell_i \log(A)}{n}.$$

Since $\ell_i/\ell_{i-1} \to \infty$, we see that for $n$ sufficiently large we must have

$$\frac{\log(f_S(n))}{n} < \begin{cases} k\log(n)/n + 2\ell_{k-1}\log(a)/n & \text{if } \ell_{k-1} \mid n, \\ k\log(n)/n + \mathrm{o}(1) & \text{if } \ell_{k-1} \nmid n. \end{cases}$$

If $n$ is a multiple of $\ell_{k-1}$ then we must have $n > C\ell_{k-1}$ by construction, and hence in either case we have

$$\frac{\log(f_S(n))}{n} < \frac{k\log(n)}{n} + \frac{\log(s)}{2}$$

for $k$ sufficiently large. We claim that $k = \mathrm{o}(n/\log(n))$. To see this, it is sufficient to show that $n = \mathrm{o}(k\log(k))$. But $n \geqslant \ell_{k-1}$ and $\ell_{k-1} > \log(p_{k-1})$; moreover, by assumption, $\log(\log(p_{k-1}))) > p_{k-3}$ for $k$ sufficiently large and since $p_{k-3}$ is necessarily greater than $k$ for $k$ large, we have

$$n \geqslant \ell_{k-1} \geqslant \log(p_{k-1}) > \exp(k)$$

for $k$ large, giving the claim. It follows that for $k$ sufficiently large there are at most $C$ values of $n \in \{\ell_{k-1}, \ldots, \ell_k - 1\}$ for which we have $f_S(n) > s^n$. Hence there is some constant $B$ such that

$$|\{n < \ell_{k+1} : f_S(n) > s^n\}| \leqslant Ck + B.$$

Let $n_1 < n_2 < \cdots$ be those natural numbers with the property that $f_S(n) > s^n$; that is, $f_S(n) > s^n$ if and only if $n = n_i$ for some $i$. Then, given a natural number $i$, there is some $k$ such that $Ck+B < i \leqslant C(k+1)+B$. By the remarks above we have $n_i > \ell_{k+1}$. Thus

$$\frac{n_i}{i} > \frac{\ell_{k+1}}{Ck+B+C} \to \infty$$

as $k \to \infty$, since we have shown that $\ell_k \geqslant \exp(k)$ for $k$ sufficiently large. The result now follows from Lemma 11, taking $g(n) = f_S(n)$ and noting that $g(n)$ does not satisfy a non-trivial linear recurrence by Theorem 4. □

## 3. Higher-dimensional groups

A compact connected abelian group $X$ of dimension $d \geqslant 1$ has a Pontryagin dual group $\widehat{X}$ that is a subgroup of $\mathbb{Q}^d$. For an automorphism $\theta : X \to X$, we use the following periodic point counting formula, taken from [12, Th. 1.1]. As before, $\mathsf{F}_\theta(n)$ denotes the number of points fixed by the automorphism $\theta^n$, $\mathcal{P}(\mathbb{K})$ denotes the set of places of the number field $\mathbb{K}$, and for any set of places $S$ in $\mathcal{P}(\mathbb{K})$, we write $|x|_S = \prod_{v\in S}|x|_v$.

**Proposition 14.** *If $\theta : X \to X$ is an ergodic automorphism of a finite dimensional compact connected abelian group, then there exist algebraic number fields $\mathbb{K}_1, \ldots, \mathbb{K}_k$, sets of finite places $T_j \subset \mathcal{P}(\mathbb{K}_j)$ and elements $\xi_j \in \mathbb{K}_j$, no one of which is a root of unity for $j = 1, \ldots, k$, such that*

$$\mathsf{F}_\theta(n) = \prod_{j=1}^{k} |\xi_j^n - 1|_{T_j}^{-1}. \tag{9}$$

The fields $\mathbb{K}_j$ and sets of places $T_j$ appearing above depend only on $X$ and $\theta$, and are obtained by considering $\widehat{X}$ as a module over the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$, where the module structure is given by identifying multiplication by $t$ with the application of the dual map $\widehat{\theta}$ (a standard procedure for the study of automorphisms of compact abelian groups, see Schmidt [22] for an overview). The precise method for obtaining the formula is constructive and is described in [12, Sec. 4]; it is useful to note from this that $\mathbb{K}_j = \mathbb{Q}(\xi_j)$, $j = 1, \ldots, k$. As in the one-dimensional case, applying the Artin product formula to (9) gives

$$\mathsf{F}_\theta(n) = \prod_{j=1}^{k} |\xi_j^n - 1|_{P_j^\infty \cup S_j}, \tag{10}$$

where $P_j^\infty$ denotes the set of infinite places of $\mathbb{K}_j$ and $S_j = \mathcal{P}(\mathbb{K}_j) \setminus T_j$. It is also worth noting that [12, Rmk. 1] implies that $|\xi_j|_v = 1$ for all $v \in T_j$, $j = 1, \ldots, k$, as $\theta$ is an automorphism. The main result of this section is the following.

**Theorem 15.** *Under the assumptions of Proposition 14, suppose that the product in* (10) *only involves finitely many places and that* $|\xi_j|_v \neq 1$ *for all* $v$ *in* $P_j^\infty$ *and* $j = 1, \ldots, k$. *Then* $\zeta_\theta$ *is either rational or has a natural boundary at its circle of convergence, and the latter occurs if and only if* $|\xi_j|_v = 1$ *for some* $v \in S_j$, $1 \leqslant j \leqslant k$.

The condition that $|\xi_j|_v \neq 1$ for all $v \in P_j^\infty$, $j = 1, \ldots, k$, is equivalent to the statement that none of the $\xi_j$ have algebraic conjugates on the unit circle. Hence, for example, the theorem applies if each $\xi_j$ is a Pisot number. We remark that we do not believe the avoidance of conjugates on the unit circle is necessary for the dichotomy stated in the theorem, but it is essential in the proof we provide. Nonetheless, we obtain the following generalization of Corollary 10.

**Corollary 16.** *If the dimension of* $X$ *is at most* 3 *and the product in* (10) *comprises finitely many places, then* $\zeta_\theta$ *is rational or has a natural boundary at the circle of convergence.*

Proof. If the dimension of $X$ is at most 3 then each of the field extensions $\mathbb{K}_j|\mathbb{Q}$ has degree at most 3. Therefore, if any algebraic conjugate of any $\xi_j$ lies on the unit circle then $\xi_j$ must be a root of unity. However, this is precluded by the hypothesis of Proposition 14. $\square$

For the proof of Theorem 15 we use the following.

**Lemma 17.** *Let* $S$ *be a finite list of places of algebraic number fields and, for each* $v \in S$, *let* $\xi_v$ *be a non-unit root in the appropriate number field such that* $|\xi_v|_v = 1$. *Then the function*

$$F(z) = \sum_{n \geqslant 1} f(n) z^n,$$

*where* $f(n) = \prod_{v \in S} |\xi_v^n - 1|_v$ *for* $n \geqslant 1$, *has the unit circle as a natural boundary.*

Proof. We will use the notation of Lemma 2. First note that $F$ has radius of convergence 1. For any function $G$ that is analytic inside the unit circle and which can be analytically continued beyond it, both $G'$ and $z \longmapsto zG'(z)$ can also be analytically continued beyond it.

Assume, for the purposes of a contradiction, that $F$ has analytic continuation beyond the unit circle. Let $\varrho = \sum_{v\in S} \varrho_v$ be the sum of all the residue degrees, and repeat the process of differentiating then multiplying by $z$ precisely $\varrho$ times, beginning with the function $F$ and finally obtaining the function $\sum_{n\geqslant 1} n^{\varrho} f(n) z^n$, which by construction is analytic in a region strictly containing the open unit disc (by our initial observation). So too then is the function $G$ defined by

$$G(z) = C \sum_{n\geqslant 1} n^{\varrho} f(n) z^n,$$

where $C = \prod_{v\in S} C_v^{-1}$, the constant $C_v$ being given by Remark 3. Note that

$$\limsup_{n\to\infty} (n^{\varrho} f(n))^{1/n} = 1,$$

so $G$ also has radius of convergence 1.

We claim that $Cn^{\varrho} f(n)$ is a positive integer for all $n \geqslant 1$. To see this, it is enough to notice that $C_v^{-1} n^{\varrho_v} |\xi^n - 1|_v$ is a positive integer for all $n \geqslant 1$, which is shown by Lemma 2. Since $G$ has integer coefficients, radius of convergence 1 and analytic continuation beyond the unit circle, it is a rational function by the Pólya–Carlson Theorem. Furthermore, $H(z) = C \sum_{n\geqslant 1} n^{\varrho} z^n$ defines a rational function, and $F$ is the Hadamard quotient of $G$ by $H$. Since the sequence of coefficients $f = (f(n))$ is drawn from a finitely generated ring over $\mathbb{Z}$, the Hadamard quotient theorem implies that $F$ is also a rational function. Therefore, $f$ is given by a linear recurrence sequence.

Let $p_v$ denote the characteristic of the residue field $\mathfrak{K}_v$ and let $q$ be a rational prime coprime to all $p_v$ for each $v \in S$. Proceeding as in the proof of Theorem 4, define $n(e) = q^e \prod_{v\in S} m_v p_v^{D_v}$, where $D_v$ is given by Lemma 2 and $e \in \mathbb{N}$. Then $f(kn(e)) = f(n(e))$ whenever $k$ is coprime to $n(e)$. Hence, the sequence $f$ takes on infinitely many values infinitely often, which is not possible for a linear recurrence sequence by the result of Myerson and van der Poorten [15]. This contradiction means that the assumption that $F$ has analytic continuation beyond the unit circle is untenable. $\square$

Proof (of Theorem 15). Let $S_j' = \{v \in S_j : |\xi_j|_v \neq 1\}$, let

$$f(n) = \prod_{j=1}^{k} |\xi_j^n - 1|_{S_j \setminus S_j'},$$

and let

$$g(n) = \prod_{j=1}^{k} |\xi_j^n - 1|_{P_j^{\infty} \cup S_j'}.$$

So, $\mathsf{F}_\theta(n) = f(n)g(n)$ by (10). By the ultrametric property

$$g(n) = \prod_{j=1}^{k} |\xi_j|_{S_j''}^n |\xi_j^n - 1|_{P_j^\infty},$$

where $S_j'' = \{v \in S_j : |\xi_j|_v > 1\}$. Extending the method of Smale [24] for toral automorphisms, we can expand the product over infinite places using an appropriate symmetric polynomial (see for example [11, Lem. 4.1]) to obtain an expression of the form

$$g(n) = \sum_{I \in \mathcal{I}} d_I w_I^n, \tag{11}$$

where $\mathcal{I}$ is a finite indexing set, $d_I \in \{-1, 1\}$ and $w_I \in \mathbb{C}$.

Moreover, since $|\xi_j|_v \neq 1$ for all $v \in P_j^\infty$, $j = 1, \ldots, k$, there is a dominant term $w_J$ in the expansion (11), for which

$$|w_J| = \prod_{j=1}^{k} |\xi_j|_{S_j''} \prod_{v \in P_j^\infty} \max\{|\xi_j|_v, 1\} = \prod_{j=1}^{k} \prod_{v \in P_j^\infty \cup \mathcal{P}(\mathbb{K}_j)} \max\{|\xi_j|_v, 1\},$$

and $|w_J| > |w_I|$ for all $I \neq J$ (note that $\log |w_J|$ is the topological entropy, as given by [9]). Furthermore, by (11),

$$\zeta_\theta(z) = \exp\left( \sum_{I \in \mathcal{I}} d_I \sum_{n \geqslant 1} \frac{f(n)(w_I z)^n}{n} \right).$$

If $S_j \setminus S_j' = \varnothing$ for all $j = 1, \ldots, k$, then $f(n) \equiv 1$, and it follows immediately that $\zeta_\theta$ is rational.

Now suppose that $S_j \setminus S_j' \neq \varnothing$ for some $j$. As noted in Lemma 1, we need only exhibit a natural boundary at the circle of convergence for

$$\sum_{I \in \mathcal{I}} d_I \sum_{n \geqslant 1} f(n)(w_I z)^n$$

to exibit one for $\zeta_\theta(z)$. Moreover, $\limsup_{n \to \infty} f(n)^{1/n} = 1$, so for each $I \in \mathcal{I}$, the series

$$\sum_{n \geqslant 1} f(n)(w_I z)^n$$

has radius of convergence $|w_I|^{-1}$, and since $|w_J|^{-1} < |w_I|^{-1}$ for all $I \neq J$, this means that it suffices to show that the circle of convergence $|z| = |w_J|^{-1}$ is a natural boundary for $\sum_{n \geqslant 1} f(n)(w_J z)^n$. But this is the case precisely when $\sum_{n \geqslant 1} f(n) z^n$ has the unit circle as a natural boundary, and this has already been dealt with by Lemma 17. □

We conclude with the following example.

**Example 18.** Suppose $X$ is a two dimensional solenoid, so $\widehat{X} \hookrightarrow \mathbb{Q}^2$. Consider an automorphism $\theta : X \to X$ dual to multiplication by the matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on $\widehat{X}$. In the case $X = \mathbb{T}^2$, $\theta$ is the well-known cat map which has dynamical zeta function

$$\zeta_\theta(z) = \frac{(1-z)^2}{(1-\xi z)(1-\eta z)}, \tag{12}$$

where $\xi = (3+\sqrt{5})/2$ and $\eta = \xi^{-1} = (3-\sqrt{5})/2$. More generally, the method of [12, Sec. 4] shows that there are natural $\mathbb{Z}[\xi^{\pm 1}]$-module embeddings $\mathbb{Z}[\xi^{\pm 1}\ ] \hookrightarrow \widehat{X} \hookrightarrow \mathbb{Q}(\xi)$, where $\widehat{X}$ is considered as a $\mathbb{Z}[\xi^{\pm 1}]$-module by identifying multiplication by $A$ with multiplication by $\xi$. Dynamically, this means that the cat map on $\mathbb{T}^2$ is always an algebraic factor of $(X, \theta)$. Furthermore, the finite places appearing in (10) are a subset of $\mathcal{P}(\mathbb{Q}(\xi))$ and, by expanding the product over the two infinite places of $\mathbb{Q}(\xi)$, this formula simplifies to

$$\mathsf{F}_\theta(n) = (\xi^n + \eta^n - 2)|\xi^n - 1|_S,$$

where $S$ is the set of places $v \in \mathcal{P}(\mathbb{Q}(\xi))$ for which $|\cdot|_v$ is unbounded on $\widehat{X}$ under the natural embedding $\widehat{X} \hookrightarrow \mathbb{Q}(\xi)$. In the case $S = \varnothing$, $\zeta_\theta$ is given by (12). In any other case when $S$ is finite, $\zeta_\theta$ is shown to have a natural boundary on the circle $|z| = \eta$ by Theorem 15.


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