

DIRECTIONAL UNIFORMITIES, PERIODIC POINTS, AND ENTROPY

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ABSTRACT. Dynamical systems generated by $d \geq 2$ commuting homeomorphisms (topological \mathbb{Z}^d -actions) contain within them structures on many scales, and in particular contain many actions of \mathbb{Z}^k for $1 \leq k \leq d$. Familiar dynamical invariants for homeomorphisms, like entropy and periodic point data, become more complex and permit multiple definitions. We briefly survey some of these and other related invariants in the setting of algebraic \mathbb{Z}^d -actions, showing how, even in settings where the natural entropy as a \mathbb{Z}^d -action vanishes, a powerful theory of directional entropy and periodic points can be built. An underlying theme is uniformity in dynamical invariants as the direction changes, and the connection between this theory and problems in number theory; we explore this for several invariants. We also highlight Fried's notion of average entropy and its connection to uniformities in growth properties, and prove a new relationship between this entropy and periodic point growth in this setting.

1. Introduction. A topological dynamical system general by $d \geq 2$ commuting homeomorphisms may be studied on multiple scales. At one end it may be viewed as a single action of \mathbb{Z}^d , with global invariants for the action like the joint topological entropy playing a central role, and with well-known rigidity phenomena arising. At the opposite extreme it may be studied one direction at a time, which essentially reduces to asking for the relationship between properties of one homeomorphism and properties of other commuting ones. In attempting to navigate between these two extremes, the first of which can lose the rich information and structures present inside an abelian group action, and the second of which loses touch with the novel phenomena that can only arise for higher-dimensional acting groups, two thematic approaches arise naturally.

The first is *subdynamics*, introduced by Boyle and Lind, in which dynamical properties are formulated for k -dimensional subspaces of \mathbb{R}^d , and the whole action is studied via properties of its subactions. Two particular features of this approach are:

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- (i) the natural emergence of critical dimensions for properties like entropy and expansiveness, leading to notions of entropy rank and expansive rank;
- (ii) a ‘subdynamics philosophy’, that dynamical properties of k -dimensional subactions should be constant, or should vary in a smooth or predictable fashion, while the k -space moves within connected components of the subset of the Grassmanian manifold of k -spaces in \mathbb{R}^d comprising the expansive k -spaces.

We refer to the work of Boyle and Lind [5] for a detailed account of subdynamics, and that of Einsiedler, Lind and the authors [11] for an account in the algebraic setting.

The second is *directional dynamics and uniformities*, in which the philosophy of subdynamics is brought to bear on specific questions in single directions as they vary. Here two particular focal points arise:

- (i) uniformities, in which the parameters describing dynamical properties are found to have no ‘singularities’ as the direction is changed, even across non-expansive directions;
- (ii) in contrast, inverse subdynamics results, where non-expansive directions are detected by changes in directional dynamical properties.

Our main focus is on algebraic \mathbb{Z}^d -actions where every element of the action has finite entropy, though essentially all the questions we discuss have natural formulations for k -dimensional subactions in \mathbb{Z}^d -actions with k -dimensional subactions of finite positive entropy. Roughly speaking, the questions then become notationally and technically a little more difficult, but the real obstacle to progress in this direction is that the Diophantine results underpinning the methods are no longer available. The actions we will be discussing in the main are called *entropy rank one actions* in the language of Einsiedler and Lind [10], whose work provides a structure theorem for such actions. Where we refer to actions of higher rank, the reader should consult the papers of Boyle and Lind [5] or Einsiedler, Lind and the authors [11] for more information.

To help orient the reader, we briefly list some standard examples of the type of systems we will be discussing.

- (i) The commuting endomorphisms $x \mapsto 2x$ and $x \mapsto 3x$ modulo 1 on the additive circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ together generate an action of \mathbb{N}^2 . The natural extension of this system is then a \mathbb{Z}^2 action by continuous automorphisms of a compact metrizable group. The action is mixing, essentially because of the arithmetic independence condition

$$2^i 3^j = 1 \implies i = j = 0.$$

The action has entropy rank one because each individual map (corresponding to multiplication by the integer $2^a 3^b$ if $a, b > 0$) has finite entropy, positive if $(a, b) \neq (0, 0)$. A property that lies much deeper is that the S -unit theorem implies that the system is also mixing of all orders, as shown by Schmidt and the second author [36].

- (ii) If $A, B \in \text{GL}_3(\mathbb{Z})$ are commuting matrices with the property that

$$A^i B^j = I \implies i = j = 0$$

then they together generate a \mathbb{Z}^2 -action α by automorphisms of the torus \mathbb{T}^3 , with the action of $\alpha^{(a,b)}$ being multiplication by $A^a B^b$. This will be a mixing action of entropy rank one, and once again will be mixing of all orders for the same Diophantine reason.

(iii) Higher entropy rank may be seen in the following example. Let

$$X = \{x = (x_n) \mid x \in \mathbb{T}^3, n \in \mathbb{Z}\} = (\mathbb{T}^3)^{\mathbb{Z}}$$

and choose A, B as above. Then we may define an action α of \mathbb{Z}^3 on X by defining

$$\left(\alpha^{(a,b,c)}(x)\right)_k = A^a B^b x_{k+c}.$$

That is, the first two coordinates (a, b) act on each coordinate in the shift space as a toral automorphism, and the third coordinate c acts as the shift on the sequence space X with alphabet \mathbb{T}^3 . Any single direction (represented by a point $(a, b, c) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$) has infinite entropy (indeed, any automorphism of positive entropy commuting with a \mathbb{Z}^2 action of completely positive entropy is forced to have infinite entropy by a result of Morris and the second author [33]), so the entropy rank is strictly greater than one. This system is mixing, and has entropy rank two.

(iv) The natural shift action of \mathbb{Z}^2 on the compact metrizable group

$$\{x = (x_{n,m}) \in \{0, 1\}^{\mathbb{Z}^2} \mid x_{n,m} + x_{n+1,m} + x_{n,m+1} = 0 \pmod{2} \forall m, n \in \mathbb{Z}\}$$

is a \mathbb{Z}^2 -action by automorphisms of a compact totally disconnected group.

This also is mixing and has entropy rank one, but is not mixing of all orders.

For a general overview of systems of this sort, we refer to the monograph of Schmidt [35].

2. Entropies and Lyapunov lists. For a \mathbb{Z}^d -action α by automorphisms of a compact abelian group X , the notion of entropy developed by Milnor [32] may be used to define a directional entropy for any $\mathbf{t} \in \mathbb{R}^d$. We denote this simply as $h(\mathbf{t})$, as dependence on the action will always be clear. Whenever $\mathbf{t} \in \mathbb{Z}^d$, the value $h(\mathbf{t})$ coincides with the topological entropy of the single homeomorphism $\alpha^{\mathbf{t}}$. For other values of \mathbf{t} , the definition is a little technical, corresponding to the more general concept of the entropy of a k -frame in the work of Milnor [32] (see also Boyle and Lind [5]), with $k = 1$. Instead of going back to the definition, we refer to a convenient formula for $h(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^d$, which is analogous to the directional entropy formula for a smooth action of $\mathbb{Z}^d \times \mathbb{R}^n$ on a compact manifold in terms of Lyapunov exponents. When α has entropy rank one, the formula reveals the *directional entropy function* $h : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}$ to be a continuous piecewise linear function, and also a seminorm on \mathbb{R}^d . Although we are only concerned with entropy rank one actions here, we note that Milnor's general framework provides a subtle insight into the structure of algebraic \mathbb{Z}^d -actions where individual elements of the action may have infinite entropy, described in the work of Einsiedler, Lind and the authors [11].

To state the directional entropy formula, some preliminaries are required. The *descending chain condition* on closed α -invariant subgroups of X is the condition that any chain of such subgroups

$$X = X_0 \supset X_1 \supset X_2 \supset \dots$$

stabilizes. When α has entropy rank one, this condition ensures that the dynamical system (X, α) has a finite Lyapunov list, in the sense of Einsiedler and Lind [10], which we will proceed to describe. Following Kitchens and Schmidt [18], we recall the standard correspondence between (X, α) and a module M over the ring of Laurent polynomials $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$, obtained by identifying the application of each dual automorphism $\hat{\alpha}^{\mathbf{n}}$ with multiplication by $u^{\mathbf{n}} = u_1^{n_1} \dots u_d^{n_d}$ and

extending this in a natural way to polynomials. The descending chain condition is equivalent to the assumption that M is a Noetherian module, in which case it will have a finite set of associated prime ideals $\text{Ass}(M)$. Furthermore, under the assumption that α has entropy rank one, Einsiedler and Lind [10, Prop. 6.1] showed that for each $\mathfrak{p} \in \text{Ass}(M)$ that is not a maximal ideal, the domain R_d/\mathfrak{p} has Krull dimension 1, meaning that the field of fractions of R_d/\mathfrak{p} is a global field, which we denote $\mathbb{K}(\mathfrak{p})$. Further background on global fields may be found in the monograph of Cohn [7], for example. There is a set of places $\mathcal{P}(\mathbb{K}(\mathfrak{p}))$ corresponding to inequivalent absolute values defined on $\mathbb{K}(\mathfrak{p})$, and we let $|\cdot|_v$ denote the normalised absolute value corresponding to the place v . Set

$$\mathcal{S}(\mathfrak{p}) = \{v \in \mathcal{P}(\mathbb{K}(\mathfrak{p})) \mid |R_d/\mathfrak{p}|_v \text{ is an unbounded subset of } \mathbb{R}\}.$$

and note that this is a finite set because R_d/\mathfrak{p} is finitely generated. Define the *Lyapunov list* for the prime \mathfrak{p} by

$$\mathcal{L}(\mathfrak{p}) = \{(\log |\bar{u}_1|_v, \dots, \log |\bar{u}_d|_v) \mid v \in \mathcal{S}(\mathfrak{p})\},$$

where \bar{u}_i denotes the image of u_i in the domain R_d/\mathfrak{p} . To define the *Lyapunov list* $\mathcal{L}(\alpha)$ for (X, α) , we form the union of Lyapunov lists over all those associated primes $\mathfrak{p} \in \text{Ass}(M)$ that are not maximal ideals, with each $\mathcal{L}(\mathfrak{p})$ repeated the same number of times as the dimension of the $\mathbb{K}(\mathfrak{p})$ -vector space $M \otimes \mathbb{K}(\mathfrak{p})$. In this way, the Lyapunov list reflects exactly the list of domains of the form R_d/\mathfrak{p} with Krull dimension 1 that appear as factors in any prime filtration of the module M by Einsiedler and Lind [10, Lem. 8.2], and the only other possible factors in such a filtration are of the form R_d/\mathfrak{m} , where \mathfrak{m} is a maximal ideal. Furthermore, if α is also assumed to be mixing, then if any such maximal ideals appear then they cannot be associated primes by work of the first author [23, Lem. 3]. For any $\mathbf{n} \in \mathbb{R}^d$, the directional entropy is given by Abramov's formula

$$h(\mathbf{n}) = \sum_{\ell \in \mathcal{L}(\alpha)} \max\{\ell \cdot \mathbf{n}, 0\}. \quad (1)$$

For further discussion and the proof of (1) in this context see Einsiedler and Lind [10, Sec. 8]; the original calculation is essentially due to Abramov [1] and a modern geometrical proof may be found in a treatment by Lind and the second author [21].

Einsiedler and Lind applied their methods to give a similar formula for fibre entropies related to skew product transformations, and subsequently to give a formula for the relational entropy introduced by Friedland [14], which may be defined for any closed subset $\mathcal{R} \subset X \times X$, as follows. Set

$$\mathcal{X}(\mathcal{R}) = \{x \in X^{\mathbb{N}} \mid (x_i, x_{i+1}) \in \mathcal{R} \text{ for all } i \in \mathbb{N}\},$$

and let $\sigma_{\mathcal{X}(\mathcal{R})}$ denote the one-sided shift on $\mathcal{X}(\mathcal{R})$. The *relational entropy* $h_{\text{rel}}(\mathcal{R})$ of \mathcal{R} is the topological entropy of $\sigma_{\mathcal{X}(\mathcal{R})}$. For example, if \mathcal{R}_A is the graph of a single automorphism $A : X \rightarrow X$, then $h_{\text{rel}}(\mathcal{R}_A)$ is the topological entropy of A . Geller and Pollicott [15] considered relational entropy for the union of the graphs of two commuting transformations, and they conjectured a formula for $h_{\text{rel}}(\mathcal{R}_A \cup \mathcal{R}_B)$ when A and B are commuting automorphisms of a torus \mathbb{T}^m . This was corrected by Einsiedler and Lind, who show [10, Th. 2.4] that, under the independence assumption that $\{x \in X \mid Ax = Bx\}$ has Haar measure zero,

$$h_{\text{rel}}(\mathcal{R}_A \cup \mathcal{R}_B) = \max_{E \subset \{1, \dots, m\}} \log \left(\prod_{j \in E} e^{s_j} + \prod_{j \in E} e^{t_j} \right),$$

where $(s_1, t_1), \dots, (s_m, t_m)$ are the Lyapunov vectors of the \mathbb{Z}^2 -action generated by A and B . This is also in fact the topological entropy of the skew product of A and B over the two shift, and Einsiedler and Lind provide more general formulæ for relations using skew products and topological pressures.

Another notion of entropy, that we will show to be closely related to the uniform growth of periodic points for the systems we study, was introduced by Fried [13] for an action of a compactly generated Lie group on a probability space. This entropy is defined in our setting with Haar measure as the measure on X , so that the corresponding measure theoretic entropy coincides with topological entropy. Firstly, note that the directional entropy function $h : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}$ is clearly measurable by construction. Fried's original definition includes the volume of the generalized octahedron $\{(t_i) \in \mathbb{R}^d \mid \sum_{i=1}^d |t_i| < 1\}$, which is $\frac{2^d}{d!}$. For ease, we accommodate this preliminary calculation into the following.

Definition 2.1. Suppose that α is an entropy rank one action with directional entropy function $h : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}$ and let $U(\alpha) = \{\mathbf{t} \in \mathbb{R}^d \mid h(\mathbf{t}) \leq 1\}$. The *Fried average entropy* of α is

$$h^*(\alpha) = \frac{2^d}{d! \operatorname{vol}(U(\alpha))}.$$

In Section 4, we will see that if α is assumed to be mixing, then h is bounded away from zero on the unit sphere

$$\mathbb{S}^{d-1} = \{\mathbf{t} \in \mathbb{R}^d \mid \|\mathbf{t}\| = 1\},$$

and it therefore defines a norm on \mathbb{R}^d (throughout, $\|\cdot\|$ will denote the Euclidean norm, and we continue to use the notation $h(\cdot)$ when the directional entropy is a norm). It then follows that $U(\alpha)$ is the closed unit ball in this norm, and $U(\alpha)$ is a convex polytope in \mathbb{R}^d . In the setting of smooth $\mathbb{Z}^d \times \mathbb{R}^n$ -actions on compact manifolds, A. Katok, S. Katok and Hertz [17] describe such details, together with numerous other aspects of Fried entropy, including consequences of the directional entropy formula (or *Pesin entropy formula*, as it is a special case of the more general results for diffeomorphisms in the case that X is a manifold), lower bounds for h^* , and a relationship with slow entropy and regulators in algebraic number fields.

3. Directional entropy and Fried average entropy examples. This section is devoted to visualising the unit ball $U(\alpha)$ in the directional entropy norm and calculating the Fried average entropy for some well-known algebraic examples.

Example 1 (The $\times 2, \times 3$ example). The compact group dual to the ring $\mathbb{Z}[\frac{1}{6}]$ is a one dimensional solenoid X . Via duality, both of the maps $x \mapsto 2x$ and $x \mapsto 3x$ are invertible and define automorphisms of X that generate a mixing \mathbb{Z}^2 -action α . Furthermore, the group X is locally isometric to a product of an interval in \mathbb{R} , an open set in the 2-adic numbers \mathbb{Q}_2 , and an open set in the 3-adic numbers \mathbb{Q}_3 . As described by Einsiedler and Lind [10, Ex. 6.7], the Lyapunov list for this example is given by the prime ideal $\mathfrak{p} = (u_1 - 2, u_2 - 3) \subset R_2$ and the set of places $\{2, 3, \infty\}$ in $\mathcal{P}(\mathbb{Q})$, so that

$$\mathcal{L}(\alpha) = \langle (\log |2|_2, \log |3|_2), (\log |2|_3, \log |3|_3), (\log |2|_\infty, \log |3|_\infty) \rangle.$$

The x - and y -axes, together with the line $x \log 2 + y \log 3 = 0$ divide the plane into 6 regions, as shown in Figure 1. Starting in the first quadrant and moving

anticlockwise, label these $\mathcal{C}_1, \dots, \mathcal{C}_6$. Then, for any $(x, y) \in \mathbb{R}^2$,

$$h((x, y)) = \begin{cases} |x| \log 2 + |y| \log 3 & \text{for } (x, y) \in \mathcal{C}_1 \cup \mathcal{C}_4; \\ |y| \log 3 & \text{for } (x, y) \in \mathcal{C}_2 \cup \mathcal{C}_5; \\ |x| \log 2 & \text{for } (x, y) \in \mathcal{C}_3 \cup \mathcal{C}_6. \end{cases}$$

This formula is related to the expansive subdynamics of this system described by Boyle and Lind [5, Ex. 6.5]. The directional entropy unit ball $U(\alpha)$ is shown in Figure 1, and we find

$$\text{vol}(U(\alpha)) = \frac{3}{(\log 2)(\log 3)}$$

so

$$h^*(\alpha) = \frac{6}{(\log 2)(\log 3)}.$$

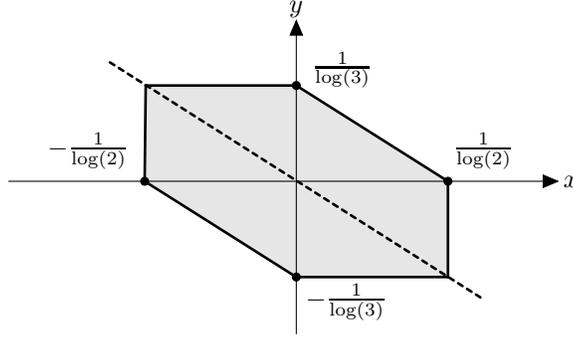


FIGURE 1. The directional entropy unit ball for $\times 2 \times 3$.

Example 2 (Ledrappier's example). The following example is one of a type introduced by Ledrappier [19] to show that mixing does not imply higher-order mixing for \mathbb{Z}^2 -actions with countable Lebesgue spectrum. The restriction of the full \mathbb{Z}^2 -shift to the compact abelian group

$$\{x = (x_{i,j}) \in \mathbb{F}_2^{\mathbb{Z}^2} \mid x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \text{ for all } (i,j) \in \mathbb{Z}^2\}$$

is mixing, yet is not mixing of higher orders. Just as in the work of Boyle and Lind [10, Exs. 6.8, 8.1], Ledrappier's example corresponds to the R_2 -module R_2/\mathfrak{p} , where $\mathfrak{p} = (2, 1 + u_1 + u_2)$, and

$$R_2/\mathfrak{p} \cong \mathbb{F}_2[t^{\pm 1}, (1+t)^{\pm 1}],$$

for an indeterminate t , where the isomorphism is given by $u_1 \mapsto t$ and $u_2 \mapsto 1+t$. The set of places $\mathcal{S}(\mathfrak{p})$ comprises the usual infinite place of $\mathbb{F}_2(t)$ corresponding to the degree of a polynomial, together with the places induced by the prime ideals (t) and $(1+t)$ in $\mathbb{F}_2[t^{\pm 1}, (1+t)^{\pm 1}]$. These give rise to the Lyapunov list

$$\begin{aligned} \mathcal{L}(\alpha) &= \langle (\log |t|_v, \log |t+1|_v) \mid v = \infty, t, t+1 \rangle \\ &= \langle (\log 2, \log 2), (-\log 2, 0), (0, -\log 2) \rangle. \end{aligned}$$

This time, the x - and y -axes together with the line $x + y = 0$ divide the plane into 6 regions, which we label as before $\mathcal{C}_1, \dots, \mathcal{C}_6$. In this case, for any $(x, y) \in \mathbb{R}^2$,

$$h((x, y)) = \begin{cases} (|x| + |y|) \log 2 & \text{for } (x, y) \in \mathcal{C}_1 \cup \mathcal{C}_4; \\ |y| \log 2 & \text{for } (x, y) \in \mathcal{C}_2 \cup \mathcal{C}_5; \\ |x| \log 2 & \text{for } (x, y) \in \mathcal{C}_3 \cup \mathcal{C}_6. \end{cases}$$

This results in the directional entropy unit ball $U(\alpha)$ in Figure 2, and we find

$$\text{vol}(U(\alpha)) = \frac{3}{(\log 2)^2}$$

so

$$h^*(\alpha) = \frac{6}{(\log 2)^2}.$$

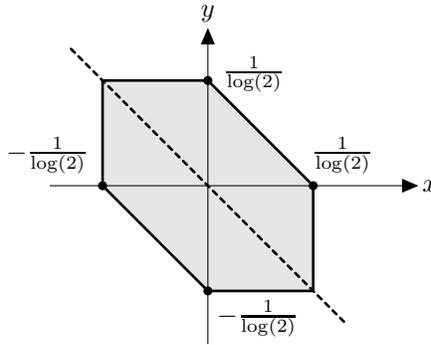


FIGURE 2. The directional entropy unit ball for Ledrappier's example.

Example 3 (Commuting toral automorphisms). The algebraic integers

$$\xi_1 = 1 + \sqrt{2},$$

$$\xi_2 = 2 + \sqrt{5},$$

and

$$\xi_3 = 3 + \sqrt{2}\sqrt{5}$$

are units in the ring $\mathbb{Z}[\sqrt{2}, \sqrt{5}]$ (which is isomorphic to \mathbb{Z}^4 as an additive abelian group). The automorphisms dual to multiplication by each of ξ_1 , ξ_2 and ξ_3 are therefore commuting hyperbolic automorphisms of the torus \mathbb{T}^4 . The Lyapunov list for this example is given by the prime ideal

$$\mathfrak{p} = (u_1^2 - 2u_1 - 1, u_2^2 - 4u_2 - 1, u_3^2 - 6u_3 - 1) \subset R_3,$$

and the set of places $\mathcal{S}(\mathfrak{p})$, which in this case correspond to the archimedean absolute values defined by $|\cdot|_\tau = |\tau(\cdot)|$, where $\tau \in \text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{5})|\mathbb{Q})$ and $|\cdot|$ is the usual absolute value on \mathbb{R} . Therefore,

$$\mathcal{L}(\alpha) = \langle (\log |\tau(\xi_1)|, \log |\tau(\xi_2)|, \log |\tau(\xi_3)|) \mid \tau \in \text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{5})|\mathbb{Q}) \rangle$$

and the directional entropy function is given by

$$h(\mathbf{t}) = \sum_{\tau \in \text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{5})|\mathbb{Q})} \max\{(\log |\tau(\xi_1)|, \log |\tau(\xi_2)|, \log |\tau(\xi_3)|) \cdot \mathbf{t}, 0\}.$$

This results in the directional entropy unit ball $U(\alpha)$ shown in Figure 3, which is a 14-faced polyhedron (a cuboid with 8 tetrahedral corners removed). For this polyhedron,

$$\text{vol}(U(\alpha)) = \frac{5}{6(\log \xi_1)(\log \xi_3)(\log \xi_3)}$$

and so

$$h^*(\alpha) = \frac{10}{9(\log \xi_1)(\log \xi_3)(\log \xi_3)}.$$

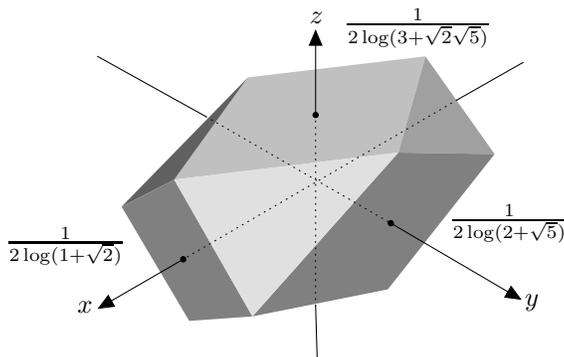


FIGURE 3. The directional entropy unit ball for 3 commuting toral automorphisms.

4. Directional entropy and periodic points.

If the directional entropy function h restricted to the unit sphere \mathbb{S}^{d-1} has a positive lower bound, this may be interpreted as an indication of uniform growth in orbit complexity.

Before stating and proving a non-trivial result in this direction, we recall that the entropy of any mixing automorphism of a compact group lies in the closure of the set

$$\{\mathfrak{m}(f) \mid \mathfrak{m}(f) > 0, f \in \mathbb{Z}[x]\},$$

and the classical *Lehmer problem* asks if

$$\inf\{\mathfrak{m}(f) \mid \mathfrak{m}(f) > 0, f \in \mathbb{Z}[x]\} > 0.$$

Moreover, if the topological dimension of X is finite, then the entropy of any automorphism of X is given by an expression of the form $\mathfrak{m}(f)$ with $f \in \mathbb{Z}[x]$ of degree no more than $\dim(X)$. We refer to the monograph of Everest and the second author [12] for the background and references on the properties of Mahler measure we will need, and to the survey by the authors and Staines [26] for references on how this relates to the dynamical properties of group automorphisms. What is known is that there is a lower bound for the logarithmic Mahler measure in the following sense. Blanskby and Montgomery [4] and Dobrowolski [9] give explicit functions $C : \mathbb{N} \rightarrow \mathbb{R}_{>0}$ for which

$$\inf\{\mathfrak{m}(f) \mid \mathfrak{m}(f) > 0, \deg(f) \leq k\} > C(k) > 0.$$

This means that there is a positive function H defined on the space of all compact groups that admit mixing automorphisms with the property that any mixing

automorphism $\alpha : X \rightarrow X$ has

$$h(\alpha) \geq H(X).$$

The following result appears in part in the proof of Theorem 1.1 in the authors' paper [28], and shows how this relates to the property of mixing in our setting.

Theorem 4.1. *Let α be an entropy rank one \mathbb{Z}^d -action by automorphisms of a compact abelian group X . If (X, α) satisfies the descending chain condition and is mixing, then there exist positive constants C_1, C_2 such that $C_1 \leq h(\mathbf{t}) \leq C_2$ for all $\mathbf{t} \in \mathbb{S}^{d-1}$.*

Proof. Firstly note that since the restriction of h to \mathbb{S}^{d-1} is a continuous function on a compact set, it is bounded and attains its bounds. Hence, we need only show that $C_1 > 0$ if α is mixing.

Suppose there exists $\mathbf{t} \in \mathbb{S}^d$ with $h(\mathbf{t}) = 0$. In the simplest case, the line through \mathbf{t} intersects a non-zero point $\mathbf{n} \in \mathbb{Z}^d$ giving $h(\mathbf{n}) = 0$. It immediately follows that $\alpha^{\mathbf{n}}$ is not ergodic and α is not mixing. If the line does not intersect any such lattice point, then note that for any $\mathbf{n} \in \mathbb{Z}^d$, using (1) we have

$$|h(\mathbf{n}) - h(\mathbf{t})| \leq \max_{(\ell_1, \dots, \ell_d) \in \mathcal{L}(\alpha)} \max_{1 \leq i \leq d} \{\ell_i\} \|\mathbf{n} - \mathbf{t}\|. \quad (2)$$

Furthermore, since $h(\lambda \mathbf{t}) = 0$ for all $\lambda > 0$, we may select a sequence of vectors \mathbf{n}_j in \mathbb{Z}^d and a sequence of scalars λ_j with $\lim_{j \rightarrow \infty} \|\mathbf{n}_j - \lambda_j \mathbf{t}\| = 0$, thus giving

$$\lim_{j \rightarrow \infty} h(\mathbf{n}_j) = 0, \quad (3)$$

by (2).

If $\text{Ass}(M)$ contains maximal ideal \mathfrak{p} , then as noted in Section 2, α cannot be mixing. Hence, for a non-trivial case assume that $\text{Ass}(M)$ contains no maximal ideals, so for each $\mathfrak{p} \in \text{Ass}(M)$, R_d/\mathfrak{p} has Krull dimension 1. If $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ for some rational prime p , then the entropy addition formula shows that for each non-zero $\mathbf{n} \in \mathbb{Z}^d$, either $h(\mathbf{n}) = 0$ or $h(\mathbf{n}) \geq \log p$. It follows from (3) that $h(\mathbf{n}_j) = 0$ for some j , and so $\alpha^{\mathbf{n}_j}$ is not ergodic and α is not mixing. The only remaining case is that $\text{Ass}(M)$ contains a prime \mathfrak{p} for which $\mathbb{K}(\mathfrak{p})$ is an algebraic number field. In this case, the addition formula for entropy shows that for each non-zero $\mathbf{n} \in \mathbb{Z}^d$, either $h(\mathbf{n}) = 0$ or $h(\mathbf{n}) \geq m(f) > 0$, where $m(f)$ denotes the logarithmic Mahler measure of some fixed polynomial f , depending only on \mathfrak{p} , of degree no greater than the degree of the field extension $\mathbb{K}(\mathfrak{p})/\mathbb{Q}$. Reasoning identical to the previous case can now be applied. As noted at the start of this section, there are explicit lower bounds for the non-zero logarithmic Mahler measure of polynomials of bounded degree. \square

Remark 1. If α is an entropy rank one \mathbb{Z}^d -action by automorphisms of a compact abelian group X , and α is both non-mixing and *irreducible* (that is, the only closed α -invariant subgroups of X are finite), then there exists a non-zero $\mathbf{n} \in \mathbb{Z}^d$ such that $\alpha^{\mathbf{n}}$ is not ergodic, so $h(\mathbf{n}) = 0$, by irreducibility. Since (1) gives

$$h(\mathbf{n}) = \|\mathbf{n}\| h(\hat{\mathbf{n}}), \quad (4)$$

it follows that $h(\hat{\mathbf{n}}) = 0$, where $\hat{\mathbf{n}}$ is a unit vector in the direction of \mathbf{n} . Thus, the directional entropy function cannot be bounded away from zero in this case.

For any closed α -invariant subgroup $Y \subset X$, let α_Y and $\alpha_{X/Y}$ denote the induced actions of α on Y and X/Y respectively. A standard dimension argument shows

that for any mixing \mathbb{Z}^d -action α by automorphisms of a finite-dimensional compact connected abelian group X , there exists a closed α -invariant subgroup $Y \subset X$ such that $(X/Y, \alpha_{X/Y})$ is mixing, satisfies the descending chain condition and, for all $\mathbf{n} \in \mathbb{Z}^d$, $h(\alpha_X^{\mathbf{n}}) = h(\alpha_{X/Y}^{\mathbf{n}})$, where h denotes the usual topological entropy. This follows from the Yuzvinskii entropy addition formula and the fact that the subgroup Y may be selected so that $h(\alpha_Y^{\mathbf{n}}) = 0$ for all $\mathbf{n} \in \mathbb{Z}^d$. Subsequently, the directional entropy functions are identical for α_X and $\alpha_{X/Y}$. Hence, we also have the following.

Corollary 1. *If α is a mixing \mathbb{Z}^d -action by automorphisms of a compact connected finite-dimensional abelian group X , then there exists a constant $C > 0$ such that $\mathbf{h}(\mathbf{t}) > C$ for all $\mathbf{t} \in \mathbb{S}^{d-1}$.*

Corollary 2. *Let α be a mixing \mathbb{Z}^d -action by automorphisms of a compact abelian group X . If X is connected and finite-dimensional, or if (X, α) has entropy rank one and satisfies the descending chain condition, then the directional entropy function for α defines a norm on \mathbb{R}^d .*

Proof. The formula (1) shows that \mathbf{h} defines a semi-norm on \mathbb{R}^d and Theorem 4.1 and Corollary 1 show that $\mathbf{h}(\mathbf{t}) = 0$ if and only if $\mathbf{t} = 0$. \square

The following theorem, taken from work of the authors [28], shows that the uniform growth just described is also evident in the periodic point data for elements of the action. Interestingly, Pollicott [34] later used a different method to achieve a similar result for commuting toral automorphisms. In the next section, we will obtain a more precise result, relating the uniform growth in the number of periodic points directly to the Fried average entropy. For a \mathbb{Z}^d -action α by automorphisms of a compact abelian group X and for any $\mathbf{n} \in \mathbb{Z}^d$, let

$$\text{Fix}(\alpha^{\mathbf{n}}) = \{x \in X \mid \alpha^{\mathbf{n}}(x) = x\}.$$

The fixed points for individual elements of the action are called *periodic points*. Although there is another notion of periodic point for a lattice in a \mathbb{Z}^d -action used in work of Lind, Schmidt and the second author [20], the periodic points for lattices are more sparse when α is an entropy rank one action and they do not reveal the rates of growth that are our primary interest here. For a more precise statement and for further details, see the work of the first author [22]. There is a convenient formula for $|\text{Fix}(\alpha^{\mathbf{n}})|$ using a product over places of global fields due to the first author [23, Sec. 3], originating in earlier work of Chothi, Everest and the second author [6], that is an essential starting point in obtaining results such as the following. We will not use this formula explicitly until Section 5, and reserve its statement to the beginning of the proof of Theorem 5.1.

For a sequence (\mathbf{n}_j) in \mathbb{Z}^d , write $\mathbf{n}_j \rightarrow \infty$ to mean that $\|\mathbf{n}_j\| \rightarrow \infty$ as $j \rightarrow \infty$. This can also be thought of as the sequence leaving finite sets.

Theorem 4.2 (Miles and Ward [28]). *Let α be a mixing \mathbb{Z}^d -action by automorphisms of a compact abelian group X . If (X, α) has entropy rank one and satisfies the descending chain condition, then the quantities*

$$\limsup_{\mathbf{n} \rightarrow \infty} \frac{1}{\|\mathbf{n}\|} \log |\text{Fix}(\alpha^{\mathbf{n}})|$$

and

$$\liminf_{\mathbf{n} \rightarrow \infty} \frac{1}{\|\mathbf{n}\|} \log |\text{Fix}(\alpha^{\mathbf{n}})|$$

211	227	235	239	241	121	485	971	1943	3887	7775
49	65	73	77	79	5	161	323	647	1295	2591
<i>5</i>	<i>11</i>	<i>19</i>	<i>23</i>	<i>25</i>	<i>13</i>	<i>53</i>	<i>107</i>	<i>215</i>	<i>431</i>	<i>863</i>
23	7	1	5	7	1	17	35	71	143	287
29	13	5	<i>1</i>	<i>1</i>	1	5	11	23	47	95
31	5	7	1	1	∞	1	1	7	5	31

FIGURE 4. Periodic point counts in the $\times 2, \times 3$ -action.

are finite and strictly positive.

The uniform growth in the number of periodic points demonstrated by Theorem 4.2 may not be immediately evident in a given example. The array in Figure 4 is taken from [28] and shows $|\text{Fix}(\alpha^{(n_1, n_2)})|$ for the $\times 2, \times 3$ -action in Example 1, with $-5 \leq n_1 \leq 5$ and $0 \leq n_2 \leq 5$. Note that for lattice points close to the line $n_1 \log 2 + n_2 \log 3 = 0$ (indicated in italics), it is not obvious that we have an exponential growth in the number of periodic points. The fact that we do have this exponential growth independently of the direction chosen (or indeed, the path chosen) is a Diophantine property of Baker type: it is in essence a statement about the possible size of expressions like $|2^a 3^b - 1|$ for integer points (a, b) chosen to lie close to the line $2^x 3^y = 1$ in terms of $\|(a, b)\|$.

The periodic point data for elements of an entropy rank one action, such as that shown in Figure 4, featured in another role in the earlier work of the authors [27], where it was used to detect expansive subdynamics. Using results from work of the first author [23], the poles and zeros of associated dynamical zeta functions were used to reconstruct non-expansive directions and expansive components (analogous to Weyl chambers for a smooth $\mathbb{Z}^d \times \mathbb{R}^n$ -action on a compact manifold). We note the connection here with the directional entropy function, restricted to \mathbb{S}^{d-1} , which is hiding behind the scenes. Figure 5 is taken from [27] and gives a brief illustration of the approach used there via Example 1. For each region \mathcal{C}_i described in the example, let \mathcal{C}'_i denote the set of integral points in the open cone formed by the removal of the axes and the irrational line shown in Figure 1. If $(n_1, n_2) \in \mathcal{C}_i$, $1 \leq i \leq 6$, then the automorphism $\alpha^{(n_1, n_2)}$ is expansive and has a rational zeta function given by

$$\zeta_{(n_1, n_2)}(z) = \begin{cases} (1-z)/(1-2^{|n_1|}3^{|n_2|}z) & \text{if } (n_1, n_2) \in \mathcal{C}'_1 \cup \mathcal{C}'_4; \\ (1-2^{|n_1|}z)/(1-3^{|n_2|}z) & \text{if } (n_1, n_2) \in \mathcal{C}'_2 \cup \mathcal{C}'_5; \\ (1-3^{|n_2|}z)/(1-2^{|n_1|}z) & \text{if } (n_1, n_2) \in \mathcal{C}'_3 \cup \mathcal{C}'_6. \end{cases}$$

For each $\mathbf{n} \in \mathbb{Z}^2$, let $\Psi_{\mathbf{n}}$ denote the set of poles and zeros of $\zeta_{\mathbf{n}}$. Illustrated in Figure 5 is the closure of the set

$$\Omega_{\times 2 \times 3} = \{(\hat{\mathbf{n}}, |z|^{1/\|\mathbf{n}\|}) \mid \mathbf{n} \in \mathcal{C}'_1 \cup \dots \cup \mathcal{C}'_6, z \in \Psi_{\mathbf{n}}\}$$

where we have parameterized \mathbb{S}^1 by the angles $0 \leq \theta < 2\pi$ along the horizontal axis and shown the second coordinate of $\Omega_{\times 2 \times 3}$ on the vertical axis. Note the appearance of the directional entropy function: the lower curve in Figure 5 is given by $h((\sin \theta, \cos \theta)) = -\log y$.

5. Uniform growth of periodic points and Fried average entropy.

As before, consider a mixing \mathbb{Z}^d -action α by automorphisms of a compact abelian group X . Assume that X is connected and (X, α) satisfies the descending chain condition. We will be interested in the volume of the convex hull of the set of points

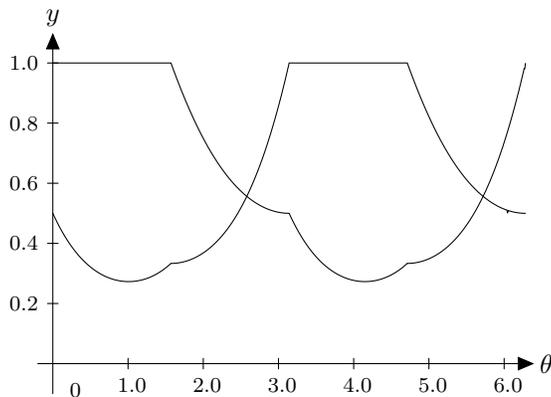


FIGURE 5. The closure of $\Omega_{\times 2 \times 3}$ summarizing directional pole and zero data.

in \mathbb{Z}^d that fix at most $N > 0$ points of X under the action. Hence, define $\mathcal{H}(N)$ to be the convex hull of the set

$$\{\mathbf{n} \in \mathbb{Z}^d : |\text{Fix}(\alpha^{\mathbf{n}})| \leq N\}.$$

The following theorem shows how the asymptotic volume of this convex hull is directly related to the Fried average entropy.

Theorem 5.1. *Let α be a mixing \mathbb{Z}^d -action by automorphisms of a compact connected abelian group X of finite topological dimension such that (X, α) satisfies the descending chain condition. Then*

$$\lim_{N \rightarrow \infty} \frac{\text{vol}(\mathcal{H}(N))}{(\log N)^d} = \text{vol}(U(\alpha)).$$

Hence, dividing the volume of the generalized octahedron in \mathbb{R}^d by this limit gives the Fried average entropy.

Before proving the theorem, we return to the familiar Example 1.

Example 4. Since $|\text{Fix}(\alpha^{-\mathbf{n}})| = |\text{Fix}(\alpha^{\mathbf{n}})|$, for all $\mathbf{n} \in \mathbb{Z}^d$, we need only consider the periodic point counts given in Table 1, where \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_3 refer to the regions in the upper half plane described in Example 1 and shown in Figure 1.

TABLE 1. Periodic point counts for the $\times 2 \times 3$ example.

Location of $(x, y) \in \mathbb{Z}^2$	$ \text{Fix}(\alpha^{(x,y)}) $
Interior of \mathcal{C}_1	$2^x 3^y - 1$
Interior of \mathcal{C}_2	$3^y 1 - 2^x 3^{-y} $
Interior of \mathcal{C}_3	$2^{ x } 1 - 2^x 3^y $
x -axis	$(2^{ x } - 1) 2^{ x } - 1 _3$
y -axis	$(3^y - 1) 3^y - 1 _2$

Now consider, for example, the integral points (x, y) in the interior of \mathcal{C}_3 that satisfy $|\text{Fix}(\alpha^{(x,y)})| \leq N$. Firstly, note that $|1 - 2^x 3^y| < 1$ for all points (x, y) in the interior of \mathcal{C}_3 . On the other hand, by Baker's theorem (see for example Baker's monograph [3] or the monograph of Everest and the second author [12]), there exist positive constants A and B such that when $|x|$ is large,

$$|1 - 2^x 3^y| > A/|x|^B.$$

Suppose also that $|x| > \frac{1}{\log 2}(\log N + (\log N)^\delta)$, where $0 < \delta < 1$ is close to 1. Then, when N is large, in the interior of \mathcal{C}_3

$$|\text{Fix}(\alpha^{(x,y)})| = 2^{|x|} |1 - 2^x 3^y| > N \cdot \frac{A(\log 2)^B \cdot e^{(\log N)^\delta}}{(\log N + (\log N)^\delta)^B} > N$$

and along the negative x -axis

$$|\text{Fix}(\alpha^{(x,y)})| = (2^{|x|} - 1) |2^{|x|} - 1|_3 > (N \cdot e^{(\log N)^\delta} - 1) \frac{\log 2}{3(\log N + (\log N)^\delta)} > N.$$

Similar calculations can be performed in the other five regions described in Example 1 and it follows that $\mathcal{H}(N)$ lies inside a convex region similar to the one shaded in Figure 1, with a boundary that intercepts the axes at $x = \pm \frac{1}{\log 2}(\log N + (\log N)^\delta)$ and $y = \pm \frac{1}{\log 3}(\log N + (\log N)^\delta)$. Also, if $|x| < \frac{\log N}{\log 2}$, then $|\text{Fix}(\alpha^{(x,y)})| < N$ (and this is also true if (x, y) lies on the negative x -axis with $|x| < \frac{\log N}{\log 2}$). Similar reasoning applies for the other five regions, so $\mathcal{H}(N)$ contains a convex region similar to the one shaded in Figure 1 with a boundary that intercepts the axes at $x = \pm \frac{\log N}{\log 2}$ and $y = \pm \frac{\log N}{\log 3}$. Thus,

$$\frac{3(\log N)^2}{(\log 2)(\log 3)} < \text{vol}(\mathcal{H}(N)) < \frac{3(\log N + (\log N)^\delta)^2}{(\log 2)(\log 3)},$$

giving

$$\lim_{N \rightarrow \infty} \frac{\text{vol}(\mathcal{H}(N))}{(\log N)^2} = \frac{3}{(\log 2)(\log 3)} = \text{vol}(U(\alpha)).$$

Proof of Theorem 5.1. We use the notation introduced for entropy rank one actions in Section 2. For any non-zero $\mathbf{n} \in \mathbb{Z}^d$, the periodic point counting formulæ from Miles [23, Sec. 3] give

$$|\text{Fix}(\alpha^{\mathbf{n}})| = \prod_{\mathfrak{p} \in \text{Ass}(M)} \prod_{v \in \mathcal{S}(\mathfrak{p})} |\bar{u}^{\mathbf{n}} - 1|_v^{m(\mathfrak{p})} \quad (5)$$

where $\bar{u}^{\mathbf{n}}$ denotes the image of $u_1^{n_1} \cdots u_d^{n_d}$ in the domain R_d/\mathfrak{p} and $m(\mathfrak{p})$ is the dimension of the $\mathbb{K}(\mathfrak{p})$ -vector space $M \otimes \mathbb{K}(\mathfrak{p})$. Using (1), (5) can be rewritten in the form

$$|\text{Fix}(\alpha^{\mathbf{n}})| = e^{h(\mathbf{n})} g(\mathbf{n}), \quad (6)$$

where $g : \mathbb{Z}^d \rightarrow \mathbb{R}_{\geq 0}$ is given by

$$g(\mathbf{n}) = \prod_{\mathfrak{p} \in \text{Ass}(M)} \prod_{v \in \mathcal{S}(\mathfrak{p})} |1 - \phi_{\mathfrak{p},v}(\mathbf{n})|_v^{m(\mathfrak{p})}$$

and

$$\phi_{\mathfrak{p},v}(\mathbf{n}) = \begin{cases} \bar{u}^{-\mathbf{n}} & \text{if } |\bar{u}^{\mathbf{n}}|_v > 1, \\ \bar{u}^{\mathbf{n}} & \text{if } |\bar{u}^{\mathbf{n}}|_v \leq 1. \end{cases}$$

Note that $g(\mathbf{n}) < 2^D$, where $D = |\text{Ass}(M)| \prod_{\mathfrak{p} \in \text{Ass}(M)} m(\mathfrak{p})$, and that the mixing assumption ensures that $\bar{u}^{\mathbf{n}} \neq 1$, so $g(\mathbf{n}) > 0$. Furthermore, just as in the proof of

Theorem 1.1 in the work of the authors [28], the theorems of Baker [3] and Yu [37] can be used to show that there exist positive constants A and B such that

$$g(\mathbf{n}) > A / \max\{|n_i|\}^B.$$

Denote the closed ball of radius $r > 0$ in the directional entropy norm by

$$B(r) = \{\mathbf{t} \in \mathbb{R}^d \mid \mathbf{h}(\mathbf{t}) \leq r\},$$

so $U(\alpha) = B(1)$. Each such ball is a convex polytope in \mathbb{R}^d . We will show that for large N , $\mathcal{H}(N)$ is trapped between $B(\log(N/2^D))$ and $B(\log N + (\log N)^\delta)$, where $0 < \delta < 1$ is a fixed constant close to 1.

On the one hand, if N is large enough so that $N > 2^D$ and $\mathbf{h}(\mathbf{n}) \leq \log(N/2^D)$, then (6) gives $|\text{Fix}(\alpha^{\mathbf{n}})| \leq N$, so $B(\log(N/2^D)) \subset \mathcal{H}(N)$.

On the other hand, assume that $\mathbf{h}(\mathbf{n}) > \log N + (\log N)^\delta$. Theorem 4.1 shows that there is a constant $C_1 > 0$ such that $\mathbf{h}(\mathbf{t}) > C_1$ for all $\mathbf{t} \in \mathbb{S}^{d-1}$. Therefore, for any non-zero $\mathbf{n} \in \mathbb{Z}^d$,

$$\mathbf{h}(\mathbf{n}) = \|\mathbf{n}\| \sum_{\ell \in \mathcal{L}(\alpha)} \max\{\ell \cdot \widehat{\mathbf{n}}, 0\} > C_1 \|\mathbf{n}\| \geq C_1 \max\{|n_i|\}.$$

Hence, using (6), $|\text{Fix}(\alpha^{\mathbf{n}})|$ is at least as large as

$$e^{\mathbf{h}(\mathbf{n})} g(\mathbf{n}) > \frac{A e^{\mathbf{h}(\mathbf{n})}}{\max\{|n_i|\}^B} > \frac{A C_1^B e^{\mathbf{h}(\mathbf{n})}}{\mathbf{h}(\mathbf{n})^B} > N \cdot \frac{A C_1^B e^{(\log N)^\delta}}{(\log N + (\log N)^\delta)^B} > N,$$

when N is large. That is,

$$\{\mathbf{n} \in \mathbb{Z}^d \mid |\text{Fix}(\alpha^{\mathbf{n}})| \leq N\} \subset B(\log N + (\log N)^\delta)$$

and, since this ball is convex,

$$\mathcal{H}(N) \subset B(\log N + (\log N)^\delta).$$

It follows that

$$\text{vol}(B(\log(N/2^D))) \leq \text{vol}(\mathcal{H}(N)) \leq \text{vol}(B(\log N + (\log N)^\delta)).$$

Thus,

$$\text{vol}(B(1)) \frac{(\log N - D \log 2)^d}{(\log N)^d} \leq \frac{\text{vol}(\mathcal{H}(N))}{(\log N)^d} \leq \text{vol}(B(1)) \frac{(\log N + (\log N)^\delta)^d}{(\log N)^d},$$

so that

$$\lim_{N \rightarrow \infty} \frac{\text{vol}(\mathcal{H}(N))}{(\log N)^d} = \text{vol}(B(1)).$$

□

The assumption of connectedness is essential in Theorem 5.1, as the following example demonstrates.

Example 5. Consider again Ledrappier's example (Example 2). Here, the main obstacle to obtaining a result like Theorem 5.1 is that $\{\mathbf{n} \in \mathbb{Z}^2 \mid |\text{Fix}(\alpha^{\mathbf{n}})| \leq N\}$ is infinite for all $N \geq 1$. To see this, let k be any positive integer and note that (5) gives

$$|\text{Fix}(\alpha^{(2^k, 0)})| = |t^{2^k} - 1|_\infty |t^{2^k} - 1|_t |t^{2^k} - 1|_{t+1} = 2^{2^k} \cdot 1 \cdot 2^{-2^k} = 1,$$

since $t^{2^k} - 1 = (t + 1)^{2^k}$. This shortage of periodic points arises for certain non-expansive elements of the action $\alpha^{(x,y)}$ corresponding to integral points (x, y) on

the axes and on the line $x + y = 0$. Such non-expansive automorphisms of zero-dimensional groups typically display erratic patterns of periodic point growth, as shown in work of Chothi, Everest and the second author [6]. This complication could be avoided by imposing a restriction to only include those elements of the action that are expansive, that is, by replacing $\mathcal{H}(N)$ with the convex hull of

$$\mathcal{H}'(N) = \{\mathbf{n} \in \mathbb{Z}^2 \mid \alpha^{\mathbf{n}} \text{ is expansive and } |\text{Fix}(\alpha^{\mathbf{n}})| \leq N\}.$$

However, in this setting, the simple formula

$$|\text{Fix}(\alpha^{\mathbf{n}})| = e^{h(\mathbf{n})} \tag{7}$$

applies to these expansive elements, so that

$$\lim_{N \rightarrow \infty} \frac{\text{vol}(\mathcal{H}'(N))}{(\log N)^2} = \text{vol}(U(\alpha)) = \frac{3}{(\log 2)^2}$$

follows immediately, and is not particularly surprising.

The reader may have noticed that the last identity obtained in Example 5 is more closely related to the geometry of the acting group than the dynamical properties of the action. This tension between complexity arising from the properties of the acting group and dynamical properties of the action is a constraint on attempts to generalize results to \mathbb{Z}^d -actions. For example, the authors extended the dynamical Mertens' theorem and other orbit growth results to nilpotent group actions [29] but only where the action is a full shift, so that the simple relationship (7) holds for any lattice in the acting group. For more complex actions only partial results could be obtained for \mathbb{Z}^2 -actions, and the authors use examples to show that several new phenomena arise [30].

Example 6. The assumption that the system satisfies the descending chain condition is also essential in Theorem 5.1. To see this, consider the \mathbb{Z}^2 -action generated by the maps $x \mapsto 2x$ and $x \mapsto 3x$ on the solenoid dual to the ring

$$\mathbb{Z}_{(5)} = \{x \in \mathbb{Q} \mid |x|_5 \leq 1\} = \{x = \frac{a}{b} \in \mathbb{Q} \mid a \in \mathbb{Z}, \gcd(5, b) = 1\}.$$

The resulting dynamical system has the well known $\times 2, \times 3$ example (Ex. 1) as an irreducible factor, but has a significantly depleted supply of periodic points. To see this, consider the homomorphism $\chi : \mathbb{Z}^2 \rightarrow \mathbb{F}_5^\times$ given by $\chi(x, y) = 2^x 3^y \pmod{5}$. Since [24] shows that for any $(x, y) \in \mathbb{Z}^2$, $|\text{Fix}(\alpha^{(x,y)})| = |2^x 3^y - 1|_5^{-1}$, we have

$$|\text{Fix}(\alpha^{(x,y)})| \neq 1 \iff (x, y) \in \ker(\chi) = \{(j, j + 4k) : j, k \in \mathbb{Z}\}.$$

In particular, the only points in \mathbb{Z}^2 with non-trivial periodic points lie in a subgroup of index 4. Hence, if (x, y) lies outside this subgroup, then $|\text{Fix}(\alpha^{(x,y)})| = 1$, and so $\mathcal{H}(N)$ is undefined for all $N \in \mathbb{N}$.

6. Synchronization points. There is an essential change in perspective between relational entropy, as considered in the introduction, and the Fried average entropy, in the sense that one is an invariant of a generating set of d commuting automorphisms and the other is an invariant of the \mathbb{Z}^d -action they generate. Although in the previous section it was shown that the Fried average entropy can be obtained naturally using period point counts, there does not appear to be an equally natural way to obtain relational entropy using this data. On the other hand, there is a straightforward way to define non-trivial invariants that capture shared periodic behaviour and reflect related growth rates for a set of generating maps, using a

generalization of the notion of a periodic point, and this is the focus of work of the first author [25].

Let Λ denote any finite set of endomorphisms of a common phase space X and assume $|\Lambda| \geq 2$. Two endomorphisms $\alpha, \beta \in \Lambda$ have a pair of *synchronous orbits* at time $n \in \mathbb{N}$ if there exists $x \in X$ such that $\alpha^n(x) = \beta^n(x)$. Hence, define the set of *weak synchronization points* for Λ at time n by

$$S_n(\Lambda) = \{x \in X : \alpha^n(x) = \beta^n(x) \text{ for some } \alpha, \beta \in \Lambda, \alpha \neq \beta\},$$

and the set of *strong synchronization points* for Λ at time n by

$$S_n^*(\Lambda) = \{x \in X : \alpha^n(x) = \beta^n(x) \text{ for all } \alpha, \beta \in \Lambda, \alpha \neq \beta\}.$$

Clearly, for all $n \in \mathbb{N}$, $S_n^*(\Lambda) = S_n(\Lambda)$ when $|\Lambda| = 2$ and $S_n^*(\Lambda) \subset S_n(\Lambda)$ more generally. If Λ contains the identity map, $S_n^*(\Lambda)$ comprises the points simultaneously fixed by the n th iterates of the non-trivial maps in Λ , so

$$S_n^*(\alpha, \text{Id}) = S_n(\alpha, \text{Id}) = \text{Fix}(\alpha^n).$$

Consequently, synchronization points generalize periodic points. We are thus led to consider the existence and value of the growth rates

$$r_\Lambda = \lim_{n \rightarrow \infty} |S_n(\Lambda)|^{1/n}$$

and

$$r_\Lambda^* = \lim_{n \rightarrow \infty} |S_n^*(\Lambda)|^{1/n}.$$

For example, if $\Lambda = \{\times 2, \times 3\}$ and X is the solenoid dual to $\mathbb{Z}[\frac{1}{6}]$ described in Example 1, then $|S_n(\Lambda)| = |S_n^*(\Lambda)| = |\{x \in X : 2^n x = 3^n x\}| = 3^n - 2^n$ and

$$r_\Lambda = r_\Lambda^* = \lim_{n \rightarrow \infty} (3^n - 2^n)^{1/n} = 3.$$

For two automorphisms α and β of a compact abelian group, the properties of the sequence $(|S_n(\alpha, \beta)|)$ can often be easily described. For example, when α and β are two (not necessarily commuting) automorphisms of the 2-torus, Section 3 in [25] shows that the sequence $(|S_n(\alpha, \beta)|)$ is given by a linear recurrence sequence exhibiting either exponential or polynomial growth (or a mixture of the two), depending on the nature of the defining automorphisms.

If Λ is simply a finite set of two or more commuting homeomorphisms, then it is also possible to give a precise description of r_Λ in terms of topological entropy, under the hypotheses of the following theorem.

Theorem 6.1 (Miles [25]). *Let Λ be a finite set of commuting homeomorphisms of a compact metric space X such that for any two distinct maps $\alpha, \beta \in \Lambda$, the homeomorphism $\beta^{-1}\alpha$ is expansive. Then*

$$r_\Lambda = \max_{\alpha, \beta \in \Lambda, \alpha \neq \beta} \{\exp(h(\beta^{-1}\alpha))\}, \quad (8)$$

where h denotes the topological entropy.

When dealing with automorphisms of a compact abelian group, methods from commutative algebra provide formulæ for counting synchronization points similar to those available for counting periodic points, using products over places of global fields. For example, if Λ is a finite set of automorphisms that generates a mixing \mathbb{Z}^d -action on a solenoid, then a result of the first author [25, Sec. 4] shows there are associated algebraic number fields $\mathbb{K}_1, \dots, \mathbb{K}_r$, associated sets of places P_i of each

field \mathbb{K}_i , and distinguished elements $\lambda_{\alpha,i} \in \mathbb{K}_i$, $\alpha \in \Lambda$, $1 \leq i \leq r$, such that for any $\alpha, \beta \in \Lambda$, $\alpha \neq \beta$,

$$|S_n(\alpha, \beta)| = \prod_{i=1}^r \prod_{v \in P_i} |\lambda_{\alpha,i}^n - \lambda_{\beta,i}^n|_v^{-1}. \quad (9)$$

Amongst several useful consequences, this means that Baker-type estimates become available once more to deal with the erratic behaviour in $|S_n(\alpha, \beta)|$ caused by factors where $|\lambda_{\alpha,i}|_v = |\lambda_{\beta,i}|_v$.

A typical assumption in the context of group automorphisms, that turns out to be somewhat weaker than the expansiveness assumption of the theorem above, is that all but finitely many of the places of the fields \mathbb{K}_i appear in the counting formula (9). Under this assumption, [25, Th. 4.7] shows that (8) holds, and can be expressed as

$$r_\Lambda = \max_{\alpha, \beta \in \Lambda, \alpha \neq \beta} \{H_i(\lambda_{\alpha,i} : \lambda_{\beta,i})\},$$

where H_i denotes the projective height on $\mathbb{P}^1(\mathbb{K}_i)$ (see [25, Sec. 4]).

Turning our attention now to the growth rate of strong synchronization points, a simple non-trivial example is given in [25, Sec. 2] with $X = \mathbb{Z}[\frac{1}{6}]$ and

$$\Lambda = \{\times 1, \times 2, \times 3\},$$

and there it is shown that

$$|S_n^*(\Lambda)| = \gcd(3^n - 1, 2^n - 1).$$

Such sequences feature in a circle of problems of active interest in number theory and are less well understood than linear recurrence sequences. In particular, while it is well-known that $\gcd(3^n - 1, 2^n - 1)$ is unbounded, it is not known whether or not $\gcd(3^n - 1, 2^n - 1) = 1$ infinitely often. Ailon and Rudnick conjecture [2] that for any multiplicatively independent non-zero integers a, b with $\gcd(a - 1, b - 1) = 1$, there are infinitely many $n \in \mathbb{N}$ such that $\gcd(a^n - 1, b^n - 1) = 1$. Nonetheless, using estimates for generalized greatest common divisors due to Bugeaud, Corvaja and Zannier [8], it is shown in [25, Sec. 5] that $r_\Lambda^* = 1$ for this example, and that this minimal growth rate is the case more generally when $|\Lambda| \geq 3$.

Theorem 6.2 (Miles [25]). *Let Λ be a finite set of commuting automorphisms of a connected finite dimensional compact abelian group with $|\Lambda| \geq 3$. If every map of the form $\hat{\alpha}^m \hat{\beta}^n - \hat{\gamma}^{m+n}$ is injective for all distinct $\alpha, \beta, \gamma \in \Lambda$ and m, n not both zero, then $r_\Lambda^* = 1$.*

As well as the estimate provided by [8], a crucial ingredient in the proof of the theorem above is the inequality

$$|S_n^*(\Lambda)| \leq C \prod_{i=1}^r \prod_{v \in P_i} \min\{|\lambda_{\alpha,i}^n - \lambda_{\gamma,i}^n|_v^{-1}, |\lambda_{\beta,i}^n - \lambda_{\gamma,i}^n|_v^{-1}\},$$

where $C > 0$ is a constant determined by Λ , $\alpha, \beta, \gamma \in \Lambda$ are any three distinct automorphisms, and P_i and $\lambda_{\alpha,i}, \lambda_{\beta,i}, \lambda_{\gamma,i}$ are as defined for (9). Notably, the second product on the right may be thought of as a generalized greatest common divisor.

It is also interesting to observe that *synchronization point measures* may be defined in a similar way to periodic point measures, provided the maps in Λ are

invertible. That is, we may define a Λ -invariant measure by

$$\nu_{\Lambda,n}^* = |S_n^*(\Lambda)|^{-1} \sum_{x \in S_n^*(\Lambda)} \delta_x,$$

where δ_x is the Dirac mass at x . The distribution properties of these synchronization point measures may be studied in a similar way to those of periodic point measures. In particular, under reasonable hypotheses, $\nu_{\Lambda,n}^*$ is seen to converge weakly to Haar measure when $|\Lambda| = 2$ [25, Th. 5.5]. In contrast, if the conjecture of Ailon and Rudnick holds, then for the example given with $\Lambda = \{\times 1, \times 2, \times 3\}$, the synchronization point measure $\nu_{\Lambda,n}^*$ is the Dirac mass at zero for infinitely many $n \in \mathbb{N}$.

7. Uniformities in mixing. A map $\alpha : X \rightarrow X$ preserving a measure μ on a space X is mixing if

$$\left| \int f(x)g(\alpha^n x) d\mu(x) - \int f d\mu \int g d\mu \right| \rightarrow 0$$

as $n \rightarrow \infty$ for $f, g \in L_\mu^2(X)$. A well-known consequence of various kinds of desirable dynamical properties for a smooth mixing map on a compact manifold is a *rate of mixing*. This amounts to the description of a subclass of functions \mathcal{C} (characterised by smoothness properties, or in algebraic situations by decay of Fourier coefficients, and so on), an appropriate norm N on \mathcal{C} , and a rate function ϕ with $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$ for which

$$f, g \in \mathcal{C} \implies \left| \int f(x)g(\alpha^n x) d\mu(x) - \int f d\mu \int g d\mu \right| \leq N(f)N(g)\phi(n)$$

for all $n \in \mathbb{N}$. For a mixing \mathbb{Z}^d -action by automorphisms of a compact group several issues arise.

- (i) Describing smooth or Hölder classes of functions is straightforward on the torus (or on nilmanifolds), and in particular there are convenient descriptions in terms of Fourier coefficients. On different groups (an infinite product of solenoids for example) it is less clear how to do this. This already arises for $d = 1$.
- (ii) For $d > 1$ we have potentially very different properties: there may be a rate of mixing in any different direction but what can be said about possible uniformities as the direction changes?
- (iii) The classical Rokhlin problem, asking if mixing forces mixing of all orders for a single measure-preserving transformation, takes on a different shape for measure-preserving \mathbb{Z}^d -actions. On the one hand it is much simpler to answer: Ledrappier's example shows that higher-order mixing is not forced by mixing if $d > 1$. On the other hand, where higher-order mixing does occur it seems to be a by-product of subtle Diophantine phenomena, and extending rate of mixing results to multiple mixing brings in many difficulties.

In this section we describe two types of result that start to address this. There is an inevitable tension between strength of statement and generality of the results, as will become clear. The first result, taken from [31] applies to all finite-dimensional compact abelian groups but gives only a weak description of the class of smooth functions involved.

Theorem 7.1 (Miles and Ward [31]). *Let α be a mixing entropy rank one \mathbb{Z}^d -action by automorphisms of a compact abelian group X satisfying the descending chain*

condition on. Then there is a class of smooth functions $\mathcal{C}(X)$ strictly containing the trigonometric polynomials, and a function $\phi = o(1)$ such that, for any $f, g \in \mathcal{C}(X)$ there is a constant $C(f, g)$ with

$$\left| \int f(x)g(\alpha^{\mathbf{n}}x)d\mu(x) - \int f d\mu \int g d\mu \right| < C(f, g)\phi(\|\mathbf{n}\|),$$

for all $\mathbf{n} \in \mathbb{Z}^d$, where μ denotes the Haar measure on X .

Just as in Theorem 4.2, this is essentially a Diophantine result. In order to see why this is so, we briefly discuss what is involved in the case of the $\times 2, \times 3$ system (X, α) from Example 1. Locally the group X is a product of an open set in $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$, and the action of $\alpha^{(1,1)}$ is to multiply by 6 in each direction. This map is hyperbolic, dilating distances in the real direction by a factor of 6, in the 2-adic direction by a factor of $\frac{1}{2}$, and in the 3-adic direction by a factor of $\frac{1}{3}$. This hyperbolicity leads one to expect an exponential rate of mixing for sufficiently smooth functions under iteration of $\alpha^{(1,1)}$. By contrast, if \mathbf{n} is a large integer vector chosen to lie on, or close to, one of the three non-expansive directions then we lose (or almost lose) hyperbolicity:

- if $\mathbf{n} = (n, 0)$ for some large positive n , then the corresponding automorphism multiplies by 2^n , which dilates in the real direction by 2^n , shrinks in the 2-adic direction by a factor 2^{-n} , and acts as an isometry in the 3-adic direction.
- if $\mathbf{n} = (0, n)$ for some large positive n , then the corresponding automorphism multiplies by 3^n , which dilates in the real direction by 3^n , shrinks in the 3-adic direction by a factor 3^{-n} , and acts as an isometry in the 2-adic direction.
- if $\mathbf{n} = (m, n)$ is chosen to lie very close to the line $2^x 3^y = 1$ for (say) $x > 0$ and $y < 0$, then the corresponding automorphism multiplies by $2^m 3^n$. This shrinks in the 2-adic direction by the factor 2^{-m} , expands in the 3-adic direction by the large factor 3^n , and in the real direction multiplies by $2^m 3^n$, which is close to 1.

Thus in the first two situations we lose hyperbolicity, and in the third we almost do. This slows the rate of mixing in these directions, and part of what is required to prove Theorem 7.1 is checking that the rate of mixing survives at some uniform rate across all these directions.

The second type of result is more specific about the nature of the space but gives much stronger results and in particular much more information about both the class of smooth functions and the rate of mixing. In our setting of \mathbb{Z}^d -actions by automorphisms of a compact group, the next result applies to the case of a torus, but more generally holds on nilmanifolds. What is remarkable about the next result is that it not only gives an explicit rate of mixing, it in fact gives an explicit rate of 3-fold mixing.

In order to state the result, which is due to Gorodnik and Spatzier [16], we recall that a compact nilmanifold is a quotient space $X = G/\Gamma$, where G is a simply connected nilpotent group and Γ is a discrete and co-compact subgroup. Write $\text{Aut}_\Gamma(G)$ for the group of continuous automorphisms of G that fix Γ . These automorphisms then induce maps on X that preserve the measure μ induced on X by the Haar measure on G , and we write $\text{Aut}(X)$ for the group of these maps, which are called automorphisms of the nilmanifold X . Fix a Riemannian metric on X , and finally define $C^\theta(X)$ to be the class of Hölder functions with exponent θ , with corresponding Hölder norm $\|\cdot\|_\theta$. In the toral case, $G = \mathbb{R}^k$, $\Gamma = \mathbb{Z}^k$, μ is Lebesgue measure, and the metric is the usual metric on the torus.

Theorem 7.2 (Gorodnik and Spatzier [16]). *Let $X = G/\Gamma$ be a compact nilmanifold, and let $\alpha : \mathbb{Z}^d \rightarrow \text{Aut}(X)$ be an action of \mathbb{Z}^d by automorphisms of X with the property that for any $\mathbf{n} \in \mathbb{Z}^d \setminus \{0\}$ the map $\alpha^{\mathbf{n}}$ is ergodic with respect to the measure μ induced by Haar measure on G . Then there exists a constant $\eta(\theta)$ such that for any functions $f, g, h \in C^\theta(X)$ we have a constant C with*

$$\left| \int f(\alpha^{\mathbf{l}}x)g(\alpha^{\mathbf{m}}x)h(\alpha^{\mathbf{n}}x)d\mu(x) - \int f d\mu \int g d\mu \int h d\mu \right| \leq C (\exp(\min\{\|\mathbf{l} - \mathbf{m}\|, \|\mathbf{m} - \mathbf{n}\|, \|\mathbf{l} - \mathbf{n}\|\}))^{\eta(\theta)} \|f\|_\theta \|g\|_\theta \|h\|_\theta.$$

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