Nonparametric predictive inference with combined data under different right-censoring schemes

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Abstract

This paper presents nonparametric predictive inference (NPI) for meta-analysis in which multiple independent samples of lifetime data are combined, where different censoring schemes may apply to the different samples. NPI is a frequentist statistical approach based on few assumptions and with uncertainty quantified via lower and upper probabilities. NPI has the flexibility to deal with a mixture of different types of censoring, mainly because the inferences do not depend on counterfactuals, which affect several inferences for more established frequentist approaches. We show that the combined sample, consisting of differently censored independent samples, can be represented as one sample of progressively censored data. This allows explicit formulae for the NPI lower and upper survival functions to be presented which are generally applicable. The approach is illustrated through an example using a small data set from the literature, for which several scenarios are presented.

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1 Introduction

In reliability and lifetime testing, censored data can occur due to different circumstances. For example, an experiment may be terminated early, at a prefixed time or as soon as a particular number of units have failed, in order to save time and cost. This leads to the well-known Type-I and Type-II censoring schemes, respectively. In progressive censoring schemes, units which have not failed are removed during the experiment at several stages, which may enable expensive units to be used for other purposes (Balakrishnan and Aggarwala, 2000; Burke, 2011) or detailed investigations of both failed and unfailed units in order to get more insight into the failure process. The units can be removed from the experiment at specified times, which leads to progressive Type-I censoring, or at times when prefixed numbers of failures have occurred, leading to progressive Type-II censoring. A variety of inferences is possible based on such censored samples, e.g. Burke (2011) and Bordes (2004) proposed nonparametric estimation of the survival function under progressive Type-I and Type-II censoring, respectively. Progressive censoring has been the topic of many research papers over recent decades, for an overview and applications we refer to Balakrishnan and Aggarwala (2000) and Balakrishnan (2007).

There are scenarios where multiple independent samples are obtained from the population of interest. For example, when the number of units that can be placed in one lifetime experiment is limited per run, then the experiment may need to be run several times in order to collect the required data. If such runs involve censoring then one may consider it logical that the same censoring scheme is used for each run, but this is not necessarily the case and may not be possible in practice, for example if available test facilities differ per run. The combination of information from several samples, resulting from similar yet not identical experiments, leads to many statistical challenges which are generally indicated as 'meta-analysis' and for which very many methods have been presented in the literature (Borenstein et al., 2009), reflecting the huge practical importance of combining information from different sources. Combining multiple samples has several objectives, including increasing the accuracy of estimation and enhancing the coverage of confidence and prediction intervals (Volterman and Balakrishnan, 2010; Balakrishnan et al., 2010). Balakrishnan et al. (2010) combined two ordinary Type-II and progressively Type-II right censored samples to derive exact nonparametric confidence, prediction, and tolerance intervals. They found that in the case of combining two ordinary Type-II censored samples, the distribution of the order statistics from the combined sample is closely related to the distribution of progressively Type-II censored order statistics. This has been extended to the situation where multiple independent Type-II right censored samples (Volterman and Balakrishnan, 2010), doubly Type-II censored samples (Volterman et al., 2012) and progressively Type-II censored samples (Volterman et al., 2013) are pooled together. A major problem for several established frequentist statistical methods, including hypothesis testing and corresponding confidence intervals, it that counterfactuals, so outcomes of an experiment that could have occurred but did not, affect the outcomes of analyses. Different censoring schemes tend to lead to differences in possible counterfactuals, which therefore leads to problems when one wishes to combine information from samples obtained under different censoring schemes.

In this paper we present how nonparametric predictive inference (NPI) can deal with combined information from multiple independent samples under different right-censoring schemes. NPI explicitly focusses on prediction for a future random quantity, based on observations of random quantities that are exchangeable with the future one. It should be emphasized that NPI does not make use of an assumed underlying population, hence no joint probability distribution for the combined (pooled) sample is required for the NPI approach. Such inferences do not include counterfactuals, hence the major difficulty for established frequentist statistical methods in dealing with data from different samples with different censoring schemes is avoided. We present explicit formulae for the lower and upper survival functions for progressive Type-I and Type-II censoring schemes, which enable a variety of censoring schemes to be dealt with. Throughout this paper, it is explicitly assumed that the right-censoring mechanisms are all non-informative, which will be explained further shortly.

Nonparametric predictive inference (NPI) is a statistical method based on Hill's assumption $A_{(n)}$ (Hill, 1968, 1988, 1993), which gives a direct conditional probability for a future real-valued random quantity, conditional on observed values of n related random quantities (Augustin and Coolen, 2004; Coolen, 2006). Effectively, it assumes that the rank of the future observation among the observed values is equally likely to have each possible value $1, \ldots, n+1$. We assume here, for ease of presentation, that there are no tied observations (these can be dealt with by assuming that such observations differ by a very small amount, a common method to break ties in statistics). The assumption $A_{(n)}$ is not sufficient to derive precise probabilities for many events of interest, but optimal bounds for probabilities for all events of interest can be derived via the 'fundamental theorem of probability' (De Finetti, 1974). These optimal bounds are lower and upper probabilities in interval probability theory (Augustin and Coolen, 2004; Walley, 1991; Weichselberger, 2001).

In this paper we consider the combination of data from samples under different right-censoring schemes. Such data often occur in reliability and survival analysis, where right-censoring of event times means that, for a specific unit or individual, it is only known that the event has not yet taken place at the time of observation. Coolen and Yan (2004) presented a generalization of $A_{(n)}$, called rc-$A_{(n)}$, which is suitable for right-censored data. In comparison to $A_{(n)}$, rc-$A_{(n)}$ uses the additional assumption that, at the moment of censoring, the residual lifetime of a right-censored unit is exchangeable with the residual lifetimes of all other units that have not yet failed or been censored, see Coolen and Yan (2004) and Yan (2002) for further details of rc-$A_{(n)}$. This exchangeability assumption is a natural way to formulate the usual non-informative censoring assumption that underlies most established statistical methods, including the Kaplan-Meier estimator (Kaplan and Meier, 1958), and is actually slightly weaker than assuming full independence of the failure and censoring processes as is commonly done. The main difference is that the censoring process is allowed to depend on the failure process upto the censoring time, as long as the remaining time to the event for a censored unit is exchangeable with the remaining times to the event of other non-censored units (which have been in the experiment equally long).

Coolen et al. (2002) introduced NPI to some reliability applications, including lower and upper survival functions for the next observation, illustrated with an application with competing risks data. They illustrated the lower and upper marginal survival functions, so each restricted to a single failure mode. While predictive inference, as considered in this approach, is different to estimation, as it explicitly considers a single future unit instead of estimating characteristics of a population distribution, it is interesting to mention that these NPI lower and upper survival functions (Coolen et al., 2002; Maturi et al., 2010b) bound the well-known Kaplan-Meier estimator (Kaplan and Meier, 1958), which is the nonparametric maximum likelihood estimator for the population survival function in case of lifetime data with right-censored observations (Coolen and Yan, 2004; Coolen-Maturi et al., 2012c).

NPI has been presented for comparison of two or more groups of right-censored data (Coolen-Schrijner et al., 2009; Coolen-Maturi et al., 2011, 2012c), for pairwise comparison with competing risks (Coolen-Maturi, 2014), and for comparison of two groups under several types of progressive censoring schemes (Maturi et al., 2010a). The main results in these papers are closed-form expressions for the NPI lower and upper survival functions and corresponding lower and upper probabilities for events comparing future observations from different groups. For comparison of more than two groups under progressive censoring schemes the expressions become quite cumbersome, but such methods can be applied using the `R` commands provided by Maturi (2010).

This paper is organized as follows. In Section 2 we briefly review NPI for progressive censoring schemes. Section 3 demonstrates how the NPI approach with combined data from samples under different censoring schemes. Finally, an example is given in Section 4 to illustrate the method presented in this paper, followed by some concluding remarks in Section 5.

2 NPI for progressive censoring

In order to propose the NPI approach for combing multiple independent samples under different censoring schemes, we first introduce some definitions and notation. Then we present new formulae for the NPI lower and upper survival functions based on event time data under Type-I and Type-II progressive censoring schemes.

2.1 Notation and setting

Maturi et al. (2010a) presented NPI for different progressive censoring schemes, the main results of which we use in this paper and introduce as definitions below. For further details and justification we refer to Maturi et al. (2010a). Suppose that n_z units were placed on a lifetime experiment. Of these n_z units, r_z failed during the experiment. For simplicity of presentation, we assume throughout this paper that they failed at r_z different failure times $z_1 < z_2 < \ldots < z_{r_z}$, and we set $z_0 = 0$ and $z_{r_z+1} = \infty$. For details on the use of NPI if data contain tied observations we refer to Coolen and Yan (2004) and Maturi et al. (2010a). NPI for Type-II and Type-I progressive censoring schemes is achieved according to the following definitions (Maturi et al., 2010a). These contain M-functions, which allow partial specification of a probability distribution and are closely related to Shafer's basic probability assignments (Shafer, 1976)[1]. M-functions assign non-negative probability masses, summing to 1, to intervals which may overlap (multiple values may even be assigned to the same interval). If interest is in the event that the random quantity of interest is in a specific interval A, the lower probability for this event is derived by summing the M-function values assigned to intervals that are fully contained within A (so all probability mass that *must* be in A) and the corresponding upper probability by summing the M function values assigned to intervals that have non-empty intersection with A (all probability mass that *can* be in A).

[1] It is important to emphasize that NPI does not use the corresponding Dempster-Shafer framework for statistical inference.

Definition 2.1 *(Maturi et al., 2010a)*
An experiment with a progressive Type-II censoring scheme is characterized by $\breve{R} = (R_1, R_2, \ldots, R_{r_z})$, where R_l non-failing units are withdrawn from the experiment at failure time z_l (in addition to the failing unit), for $l = 1, \ldots, r_z$. It is assumed that all non-failing units still in the experiment at the final observed failure time z_{r_z} are removed at that moment, at which the experiment ends. NPI for data from such an experiment is based on the assumption rc-$A_{(n_z)}$ (Coolen and Yan, 2004), which implies that the probability distribution for a nonnegative random quantity Z_{n_z+1} on the basis of such data, including r_z observations of the actual event of interest and $(n_z - r_z)$ progressively censored observations, is partially specified by the following M-function values, for $i = 0, 1, \ldots, r_z$,

$$M_{Z_{n_z+1}}(z_i, z_{i+1}) = \frac{1}{n_z+1} \prod_{k=1}^{i-1} \frac{n_z - k - \sum_{l=1}^{k-1} R_l + 1}{n_z - k - \sum_{l=1}^{k} R_l + 1} \tag{1}$$

$$M_{Z_{n_z+1}}(z_i^+, z_{i+1}) = \left[\frac{R_i}{n_z - i - \sum_{l=1}^{i} R_l + 1}\right] M_{Z_{n_z+1}}(z_i, z_{i+1}) \tag{2}$$

where z_i^+ is used to indicate a value infinitessimally greater than z_i, which can be interpreted as representing the lower bound for the interval that would contain the actual lifetimes for all units censored at z_i. Then the total probability mass assigned to the interval (z_i, z_{i+1}) is the sum of the two M-functions corresponding to (z_i, z_{i+1}) and (z_i^+, z_{i+1}) (for $i = 0, 1, \ldots, r_z$), and is given by

$$P(Z_{n+1} \in (z_i, z_{i+1})) = \frac{1}{n_z+1} \prod_{k=1}^{i} \frac{n_z - k - \sum_{l=1}^{k-1} R_l + 1}{n_z - k - \sum_{l=1}^{k} R_l + 1} \tag{3}$$

Definition 2.2 *(Maturi et al., 2010a)*
In a progressive Type-I censoring scheme for n_z units on a lifetime experiment, R_q units are withdrawn from the experiment at time T_q ($q = 1, \ldots, Q$), and define $T_0 = 0$ while it is assumed that T_Q is greater than the largest observed failure time (typically T_Q is the end of the experiment, of course $R_Q = 0$ is possible). Let s_q denote the number of observed failure times between T_{q-1} and T_q, with in total $r_z = \sum_{q=1}^{Q} s_q$ observed failures. For ease of presentation, we assume no ties among the observed times (both failure and right-censoring times) in the data, any ties can be broken in the usual way (where right-censoring is normally assumed to happen just later than a failure event if their event times are tied). The data can be represented as in Figure 1, where $z_{i_q}^q$ denotes the i_qth observed failure time between T_{q-1} and T_q ($i_q = 1, \ldots, s_q$,

$$
\begin{array}{ccccc}
& R_{q-1} & & R_q & & R_{q+1} \\
& \nearrow & & \nearrow & & \nearrow \\
\cdots < T_{q-1} < z_1^q < \cdots < z_{s_q}^q < T_q < z_1^{q+1} < \ldots < z_{s_{q+1}}^{q+1} < T_{q+1} < \ldots
\end{array}
$$

Figure 1: Data representation for progressive Type-I censoring

$q = 1, \ldots, Q$). Let

$$B_q = \frac{1}{n_z + 1} \prod_{k=1}^{q} \frac{n_z - \sum_{l=1}^{k} s_l - \sum_{l=1}^{k-1} R_l + 1}{n_z - \sum_{l=1}^{k} s_l - \sum_{l=1}^{k} R_l + 1}$$

then the M-functions for the NPI approach based on data corresponding to a progressive Type-I censoring scheme, are (for $q = 1, \ldots, Q$ and $i_q = 1, \ldots, s_q$)

$$
\begin{aligned}
& M_{Z_{n_z+1}}(0, z_1^1) = B_0 \ , \quad M_{Z_{n_z+1}}(z_{i_q}^q, z_{i_q+1}^q) = B_{q-1} \ , \\
& M_{Z_{n_z+1}}(T_q, z_1^{q+1}) = \left[\frac{R_q}{n_z - \sum_{l=1}^{q} s_l - \sum_{l=1}^{q} R_l + 1} \right] B_{q-1} \ , \\
& P(Z_{n_z+1} \in (z_{i_q}^q, z_{i_q+1}^q)) = B_q
\end{aligned}
$$

where z_1^{q+1} ($z_{s_q}^q$) is the first (last) failure time observed after (before) we removed R_q units at time T_q, and where $z_{s_q+1}^q = z_1^{q+1}$ and $z_1^{Q+1} = \infty$.

NPI for ordinary Type-II and Type-I right-censored data can be obtained from the above definitions as these are special cases of Type-II and Type-I progressive censoring schemes, respectively. For ordinary Type-II right-censoring we have $R_i = 0$ for all $i = 1, 2, \ldots, r_z - 1$ and $R_{r_z} = n_z - r_z$ in Definition 2.1, while ordinary Type-I right-censoring corresponds to $Q = 1$ in Definition 2.2.

2.2 NPI lower and upper survival functions under progressive censoring

We now present explicit formulae for the NPI lower and upper survival functions under Type-II and Type-I progressive censoring, we denote these by $\underline{S}_{Z_{n_z+1}}(t)$ and $\overline{S}_{Z_{n_z+1}}(t)$, respectively. The proofs of these results are presented in the appendix.

We first consider the Type-II progressive censoring scheme (Definition 2.1). For $t \in [z_i, z_{i+1})$, with $i = 0, 1, \ldots, r_z$, the NPI upper survival function is

$$\overline{S}_{Z_{n_z+1}}(t) = \tilde{n}_{z_i} M_{Z_{n_z+1}}(z_i, z_{i+1}) \tag{4}$$

where $\tilde{n}_{z_i} = n_z - i - \sum_{l=1}^{i-1} R_l + 1$ and $M_{Z_{n_z+1}}(z_i, z_{i+1})$ is given by (1).

The corresponding NPI lower survival function under the Type-II progressive censoring scheme is as follows. For $t \in (t^i_{l_i}, t^i_{l_i+1}]$, with $i = 0, 1, \ldots, r_z$ and $l_i = 0, 1, \ldots, R_i$, and further notation $t^i_{R_i+1} = t^{i+1}_0 = z_{i+1}$ for $i = 0, 1, \ldots, r_z - 1$,

$$\underline{S}_{Z_{n_z+1}}(t) = \tilde{n}_{t^i_{l_i+1}} \left[\frac{\tilde{n}_{z_i}}{\tilde{n}_{t^i_{l_i+1}} + 1} \right]^{1-a} M_{Z_{n_z+1}}(z_i, z_{i+1}) \tag{5}$$

where $\tilde{n}_{t^i_{l_i+1}} = n_z - i - \sum_{l=1}^{i-1} R_l - l_i$ and $a = 1$ if $\{j : z_i < c_j < t^i_{l_i+1}\} = \emptyset$ and $a = 0$ if $\{j : z_i < c_j < t^i_{l_i+1}\} \neq \emptyset$. Notice that at observed failure times, so $t = z_i$, these NPI lower and upper survival functions are equal, i.e. $\underline{S}_{Z_{n_z+1}}(z_i) = \overline{S}_{Z_{n_z+1}}(z_i)$ for all $i = 0, \ldots, r_z$.

Next we consider the Type-I progressive censoring scheme (Definition 2.2). For $t \in [z^q_{i_q}, z^q_{i_q+1})$, with $i_q = 1, \ldots, s_q$ and $q = 1, \ldots, Q$, the NPI upper survival function is

$$\overline{S}_{Z_{n_z+1}}(t) = \tilde{n}_{z^q_{i_q}} M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1}) \tag{6}$$

where $\tilde{n}_{z^q_{i_q}} = n_z - (i_q - 1) - \sum_{l=1}^{q-1} R_l - \sum_{l=1}^{q-1} s_l$ and $M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1})$ is given by Definition 2.2. Furthermore, for $t \in [0, z^1_1)$ we have $\overline{S}_{Z_{n_z+1}}(t) = 1$.

The corresponding NPI lower survival function under the Type-I progressive censoring scheme for Z_{n_z+1} is

$$\underline{S}_{Z_{n_z+1}}(t) = M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1}) \left[\tilde{n}_{t^q_{i_q+1}} \right]^{1-\delta} \left[\frac{\tilde{n}_{z^q_{s_q}} \tilde{n}_{t^q_{l_q+1}}}{\tilde{n}_{t^q_{l_q+1}} + 1} \right]^{\delta} \tag{7}$$

where $\delta = 0$ for $t \in (t^q_{i_q}, t^q_{i_q+1}]$ with $t^q_{s_q+1} = c^q_1$, and $\delta = 1$ for $t \in (t^q_{l_q}, t^q_{l_q+1}]$ with $t^q_{R_q+1} = z^{q+1}_1$, for $q = 1, \ldots, Q$, $i_q = 1, \ldots, s_q$ and $l_q = 1, \ldots, R_q$. In this expression, $\tilde{n}_{t^q_{l_q+1}}$ ($\tilde{n}_{t^q_{i_q+1}}$) is the number of units at risk at $t^q_{l_q+1}$ ($t^q_{i_q+1}$), so the number of units that have not failed or been censored before this time, and $M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1})$ is given in Definition 2.2. Furthermore, for $t \in (0, z^1_1]$ we have $\underline{S}_{Z_{n_z+1}}(t) = n_z/(n_z + 1)$.

3 Combining data resulting from different censoring schemes

In this section, we present how information from multiple independent samples under different right-censoring schemes can be combined in NPI, in order to use all combined data for the inference on the next future observation. We should emphasize here that we assume that the lifetimes of all the units from the different samples are exchangeable, which relates to the usual assumption in the more established frequentist statistics framework that all samples are drawn from the same population. For example, one may wish to based inference on the combined information from two or more Type-II (or Type-I) right-censored samples, with different censoring times applying per sample. The combined data in this case can be represented as a single Type-II (or Type-I) progressively censored sample, for which NPI has been introduced by Maturi et al. (2010a). Moreover, if we combine two or more Type-II (or Type-I) progressively censored samples we obtain a single Type-II (or Type-I) progressively censored sample. This means that we can apply the same inferential method as presented by Maturi et al. (2010a) on the resulting combined progressively censored sample, as discuss in detail in the rest of this section. This approach for combining information from samples under different censoring schemes is based on the approach presented by Balakrishnan et al. (2010). However, they study the problem from the order statistics perspective while we apply the NPI method to such scenarios to derive frequentist predictive inference for the next observation. Working with order statistics is complicated as it requires the distributions for the different order statistics. The proposed NPI approach is pretty flexible and implementation of the NPI lower and upper survival functions as presented in Section 2 is quite straightforward. With the usual, rather weak, assumptions underlying NPI together with assumed non-informative right-censoring (assumed as rc-$A_{(n)}$ for each sample and for the combined sample) as discussed in Section 1, any type of right-censored or progressively censored samples can be combined in the NPI approach. We discuss how this is achieved for the main situations of interest below, for each case then the NPI lower and upper survival functions follow from Section 2 with some quite obvious notation introduced for convenience in the general theory in that section, e.g. $z_0 = 0$, not explicitly mentioned again here in each case. These results are illustrated in an example in Section 4.

3.1 Progressive Type-II censored samples

Data from m independent progressive Type-II censored samples, with for sample j ($j = 1, \dots, m$) observations $x_{j,1} < x_{j,2} < \dots < x_{j,r_j}$ and $n_j - r_j$ right-censored observations according to censoring scheme $(R_{j,1}, R_{j,2}, \dots, R_{j,r_j})$, can be combined into a single progressive Type-II censored sample (Definition 2.1) with observed times $z_1 < z_2 < \dots < z_{r_z}$ and censoring scheme $R_i = R_{j,i_j}$ if $z_i = x_{j,i_j}$, $i_j = 1, \dots, r_j$, where $i = 1, \dots, r_z$, $n_z = \sum_{j=1}^{m} n_j$ and $r_z = \sum_{j=1}^{m} r_j$. As a special case one can combine m Type-II right-censored samples where $(R_{j,1}, R_{j,2}, \dots, R_{j,r_j}) = (0, 0, \dots, n_j - r_j)$.

3.2 Type-II right-censored samples and progressive Type-II censored samples

Suppose we have m_1 independent progressive Type-II censored samples, with for sample j ($j = 1, \ldots, m_1$) observations $x_{j,1} < x_{j,2} < \ldots < x_{j,r_j}$ and $n_j - r_j$ right-censored observations according to censoring scheme $(R_{j,1}, \ldots, R_{j,r_j})$. Suppose that, in addition, we have $m - m_1$ independent Type-II right-censored samples, with for sample j ($j = m_1 + 1, \ldots, m$) observations $x_{j,1} < x_{j,2} < \ldots < x_{j,r_j}$ and $n_j - r_j$ right-censored observations. These m samples can be combined into one progressive Type-II censored sample (Definition 2.1) with failure times $z_1 < z_2 < \ldots < z_{r_z}$ and censoring scheme R_i ($i = 1, \ldots, r_z$) defined as follows: for $j = 1, \ldots, m_1$, $R_i = R_{j,i_j}$ if $z_i = x_{j,i_j}$, $i_j = 1, \ldots, r_j$; for $j = m_1 + 1, \ldots, m$, $R_i = n_j - r_j$ if $z_i = x_{j,i_j}$, $i_j = 1, \ldots, r_j$; all other $R_i = 0$.

3.3 Progressive Type-I censored samples

Under a progressive Type-I censoring scheme for n_j ($j = 1, \ldots, m$) units in a lifetime experiment, in case of m such independent experiments ($j = 1, \ldots, m$), R_{j,i_j} units are withdrawn from the experiment at T_{j,i_j} ($i_j = 1, \ldots, p_j$) and for $r_j = \sum_{i_j=1}^{p_j} s_{j,p_j}$ units the failure times are observed, where s_{j,i_j} is the number of observed failure times between T_{j,i_j-1} and T_{j,i_j}. Combining the data from these m samples leads to a single progressive Type-I censored sample (Definition 2.2) where $R_q = R_{j,i_j}$ units are withdrawn from the experiment at $T_q = T_{j,i_j}$ ($j = 1, \ldots, m$, $i_j = 1, \ldots, p_j$) and $r_z = \sum_{j=1}^{m} r_j = \sum_{q=1}^{Q} s_q$ failure times are observed, with s_q the number of observed failure times between T_{q-1} and T_q. Similarly, combining m Type-I right-censored samples can be considered to be a special case of this scenario.

3.4 Type-I right-censored samples and progressive Type-I censored samples

Suppose that we have m_1 independent progressive Type-I censored samples, where R_{j,i_j} units are withdrawn from the j-th experiment at T_{j,i_j} ($j = 1, \ldots, m_1$, $i_j = 1, \ldots, p_j$). Suppose that, in addition, we have $m - m_1$ independent Type-I right-censored samples, with sample j ($j = m_1 + 1, \ldots, m$) consisting of r_j observations before T_j and $R_j = n_j - r_j$ right-censored observations at T_j. Combining these samples leads to one progressive Type-I censored sample (Definition 2.2) with $R_q = R_{j,i_j}$ ($j = 1, \ldots, m_1$) units withdrawn from the experiment at $T_q = T_{j,i_j}$ ($j = 1, \ldots, m_1$, $i_j = 1, \ldots, p_j$, $i_j = 1, \ldots, p_j$) and $R_q = R_j = n_j - r_j$ ($j = m_1 + 1, \ldots, m$) units withdrawn from the experiment at $T_q = T_j$ ($j = m_1 + 1, \ldots, m$). The combined data include $r_z = \sum_{j=1}^{m} r_j = \sum_{q=1}^{Q} s_q$ failure times, with s_q the number of observed failure times between T_{q-1} and T_q.

3.5 General mixtures of right-censored and progressively censored samples

We presented the specific cases above as they have attracted attention in the literature and lead to neatly formulated combined samples which directly fit with the NPI lower and upper survival functions presented in Section 2. Perhaps most interesting is the fact that one can combine the information from any mixture of different types of independent right-censored and progressively censored samples without complications using the NPI method. All such scenarios can be expressed as mixtures of Type-I and Type-II progressively censored samples and the corresponding NPI lower and upper survival functions for the combined sample can be derived following similar steps to those presented in Section 2.2[2]. This opportunity to combine information from any mixture of independent right-censored samples, in a quite straightforward manner, is an advantage of the NPI method, particularly when compared to more established frequentist methods which involve considerations of counterfactuals, which indeed can become very cumbersome for general mixtures of differently censored samples.

4 Example

To illustrate the methods presented in this paper, we use a subset of Nelson's dataset (Nelson, 1982, p. 462) on breakdown times (in minutes) of an insulating fluid that is subject to high voltage stress. The data are given in Table 1 and consist of two samples, which we assume to be independent, each with 10 units. The methods presented in this paper are applicable to any number of samples with any numbers of observations, which indeed can be different per sample. We have opted to illustrate the methods using only two small samples in order to clearly show the differences for the different cases in the plots, with increasing sample sizes the differences between corresponding upper and lower survival functions decreases and, if most observations are not censored, the differences due to different censoring schemes would also become small. As mentioned, implementation of the methods for larger applications is facilitated by the R-functions.

We use these data to illustrate the combination of the information from both samples in NPI under different censoring schemes by presenting the lower and upper survival functions for the failure time Z_{n_z+1} of the next unit based on the combined information, and we compare these with the NPI lower and upper survival functions based

[2]R functions for this are available from *www.npi-statistics.com*.

Sample	Failure times									
X	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	4.75
Y	1.34	1.49	1.56	2.10	2.12	3.83	3.97	5.13	7.21	8.71

Table 1: Insulating fluid failure times of units from two samples

only on the individual samples X and Y, so for the failure time of the next units X_{n_x+1} and Y_{n_y+1}, respectively. For ease of presentation, we use $\underline{S}_X$, $\overline{S}_X$, $\underline{S}_Y$, $\overline{S}_Y$, $\underline{S}_Z$, $\overline{S}_Z$, to refer to these NPI lower and upper survival functions in the figures. We consider 6 cases, mostly related to the scenarios presented in Section 3.

Case A: We now consider progressive Type-II censoring applied independently to both samples X and Y with schemes $\breve{R}^x = (3, 1, 1, 0, 0)$ at observed values (0.49, 0.64, 0.93, 2.06, 2.15) in sample X and $\breve{R}^y = (3, 2, 0, 0, 0)$ at observed times (1.34, 1.56, 2.10, 3.83, 7.21) in sample Y. This can be combined into a single progressive Type-II censored sample (Section 3.1) with $\breve{R}^z = (3, 1, 1, 3, 2, 0, 0, 0, 0, 0)$ at observed times (0.49, 0.64, 0.93, 1.34, 1.56, 2.06, 2.10, 2.15, 3.83, 7.21). The NPI lower and upper survival functions based on this combined sample and based on the individual samples, with the same censorings, are presented in Figure 2.

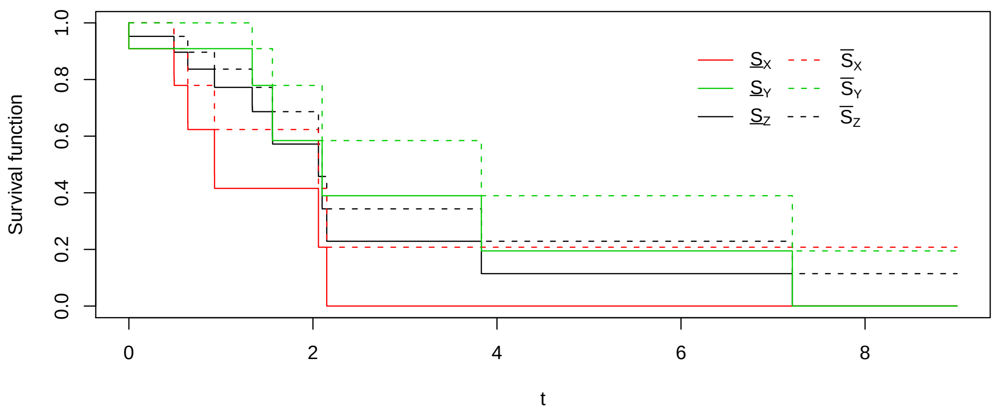

Figure 2: Lower and upper survival functions, Case A

Case B: Suppose that some units of samples X and Y are removed from the experiment before breakdown, at different times. Suppose that two units are removed from each sample, at $T_1 = 1.5$, say the units with actual failure times 2.06 and 2.57 from the X sample and the units with actual failure times 2.10 and 8.71 from the Y sample are removed (hence those failure times are not observed). In addition, one unit from each sample is removed at $T_2 = 3.5$, let this be the X sample unit with actual failure time 4.75 and and the Y sample unit with actual failure time 5.13. The resulting data consist of 7 failure times and 3 censored observations for each sample. This can be combined into one progressive Type-I censored sample (Section 3.3). The NPI lower and upper survival functions based on this combined sample and based on the individual samples, with the same censorings, are presented in Figure 3.

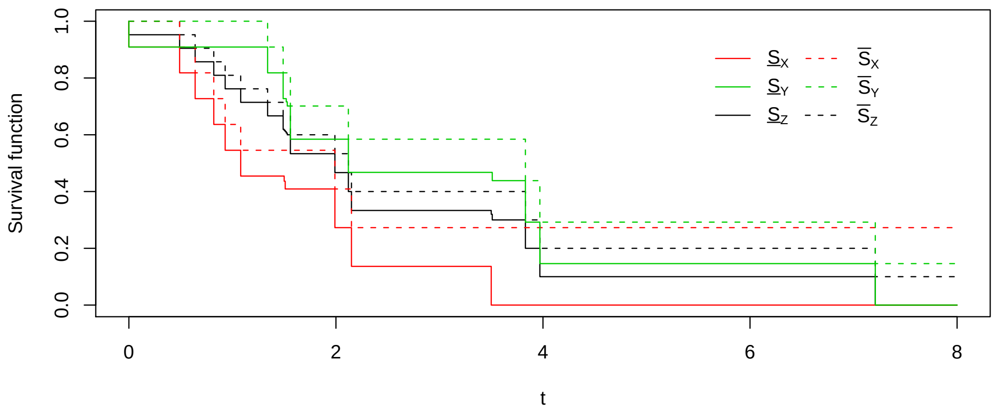

Figure 3: Lower and upper survival functions, Case B

Case C: Assume that both experiments started simultaneously and are terminated at the fourth observed failure time for the Y sample, hence this sample is Type-II right-censored at 2.10. It is non-trivial how this stop-criterion for the experiment translates into a specific censoring scheme for sample X, as it is based on a random event not related to this sample. However, as mentioned in this paper, as NPI is not affected by counterfactuals the actual censoring mechanism is irrelevant as long as we can assume exchangeability at the censoring time of the remaining times till failure of all units which are still at risk. For example, to fit with the theory presented in Section 3, we can treat this X sample, with right-censoring at time 2.10 of all three units still at risk at that time, as a Type-I right-censored sample. These two right-censored samples can be combined into a single Type-II progressive hybrid censored sample with 11 failure times and 9 censored observations; we do not consider this scheme further in this paper, for more detail we refer to (Maturi et al., 2010a). The NPI lower and upper survival functions based on this combined sample and based on the individual samples, with the same censorings, are presented in Figure 4.

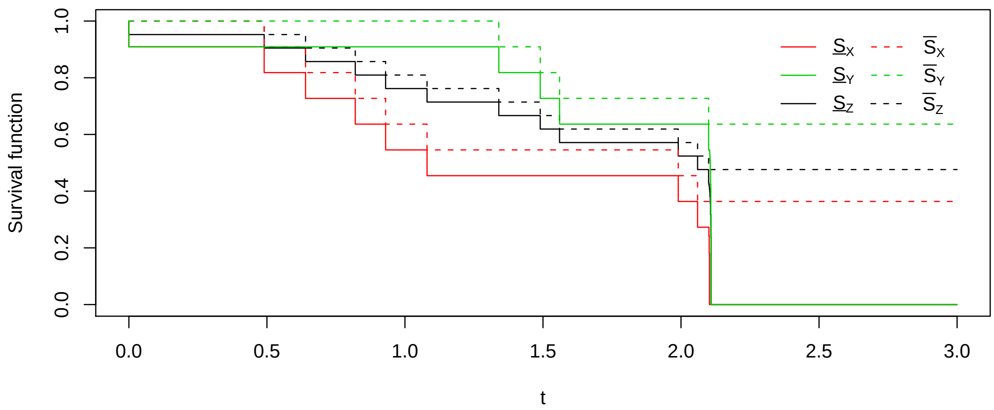

Figure 4: Lower and upper survival functions, Case C

Case D: We now apply the progressive Type-II censoring scheme with $\breve{R}^x = (3, 1, 1, 0, 0)$ to sample X, at the observed times (0.49, 0.64, 0.93, 2.06, 2.15), and we assume that the experiment leading to sample Y is terminated at time 4, leading to a Type-I right-censored sample with 7 observed failure times and 3 right-censored observations at time 4. The NPI lower and upper survival functions based on this combined sample and based on the individual samples, with the same censorings, are presented in Figure 5.

Case E: To illustrate the possible combination of samples with different mixtures of censoring schemes, as discussed in Section 3.5, consider the progressive Type-II censoring scheme being applied to sample X with $\breve{R}^x = (3, 1, 1, 0, 0)$ at the observed times (0.49, 0.64, 0.93, 2.06, 2.15), and the progressive Type-I censoring scheme applied to sample Y, with the two units with failure times 2.10 and 8.71 instead being removed at $T_1 = 1.5$ and the unit with failure time 5.13 instead being removed at $T_2 = 3.5$. The NPI lower and upper survival functions based on this combined sample and based on the individual samples, with the same censorings, are presented in Figure 6.

Case F: Let us end this example by combining the two samples without applying any censoring schemes, so effectively assuming that both samples came from the same single experiment without any censoring of observations. The NPI lower and upper survival functions based on this combined sample and based on the individual samples

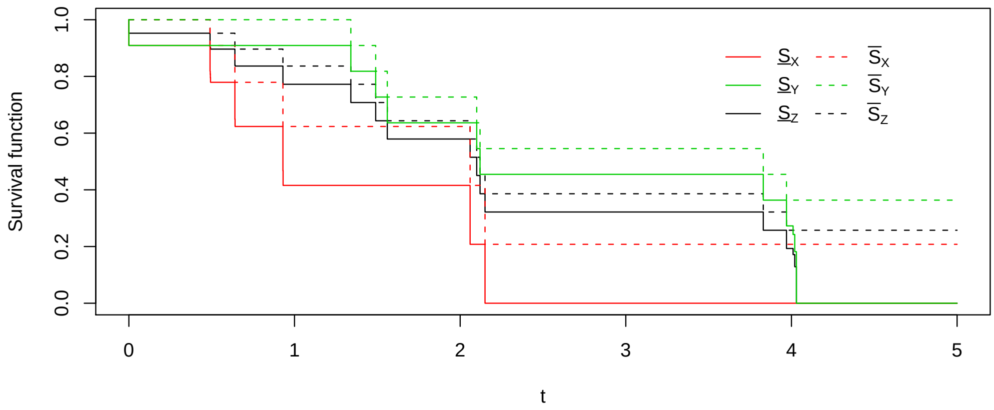

Figure 5: Lower and upper survival functions, Case D

are presented in Figure 7.

Figures 2 to 7 illustrate the combination of information from the two samples in this example, with differing censoring schemes applied. The lower (upper) survival function based on a combined sample is always (pointwise) bounded by the two lower (upper) survival functions based on the individual samples with the same censoring, so $\min\{\underline{S}_X(t), \underline{S}_Y(t)\} \leq \underline{S}_Z(t) \leq \max\{\underline{S}_X(t), \underline{S}_Y(t)\}$ and $\max\{\overline{S}_X(t), \overline{S}_Y(t)\} \leq \overline{S}_Z(t) \leq \max\{\overline{S}_X(t), \overline{S}_Y(t)\}$ at every $t > 0$. In most cases the combination of information from different samples has also led to reduced imprecision (the difference between the corresponding upper and lower survival functions) when compared to the imprecision for both corresponding cases based on a single sample. Intuitively it is quite logical that combining information from different samples leads to more total information being taken into account, reflected through less imprecision in the resulting inferences. However, due to some specific censoring schemes, it can occur that imprecision in case of the combined sample is larger, at some times t, than when only one of the samples is used. This happens, for example, in Case C, where the imprecision in case the combined sample is used is smaller than when either one of the individual samples is used up to $t = 2.10$, but beyond that value of t the upper survival function based only on the data from sample X is less than the upper survival function based on the combined sample. This occurs because there are 10 more units at risks beyond $t = 2.06$ (the last observed failure time for sample X in this case) for the combined sample, which affects the upper survival function. It should be remarked

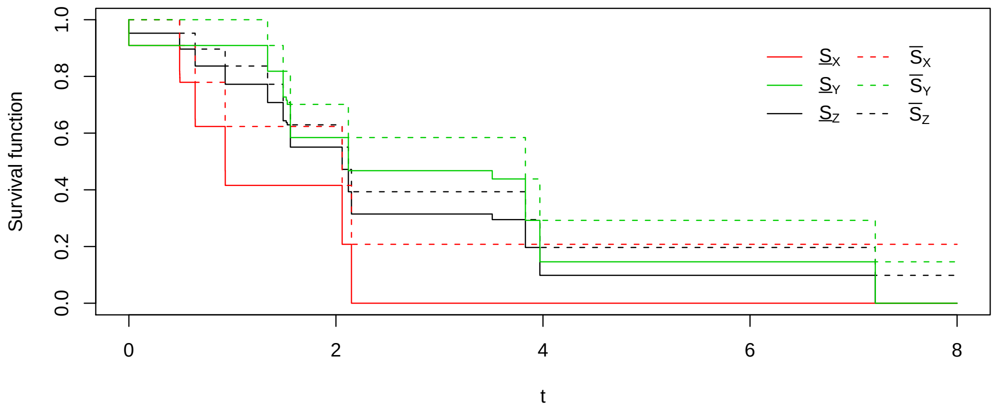

Figure 6: Lower and upper survival functions, Case E

that the NPI lower survival function, based on any data set, is equal to zero beyond the largest observation in the data set, both if this observation is an actual failure time or a right-censored observation. Of course, differences between the NPI lower and upper survival functions in these six figures are caused by the different censoring schemes, where more censoring typically leads to more imprecision, particularly for larger values of t. At any right-censoring time, the NPI lower survival function decreases in value, reflecting that beyond such a time point (compared to just before it) there is less evidence in the data in favour of the next unit surviving past that point. However, as a right-censored observation does not actually provide evidence against such survival, NPI upper survival functions do not decrease at right-censoring times, they do so only at observed failure times (at which of course also the NPI lower survival functions decrease).

5 Concluding remarks

This paper has introduced NPI based on multiple independent samples under different censoring schemes, which fits in general theory of meta-analysis. We used the fact that the resulting combined sample in such cases can be represented as a single NPI progressive censoring sample, as defined in Section 2. Explicit formulae for the NPI lower and upper survival functions for progressive Type-I and Type-II censoring schemes are also presented for the first time. In addition to using the NPI lower and upper survival

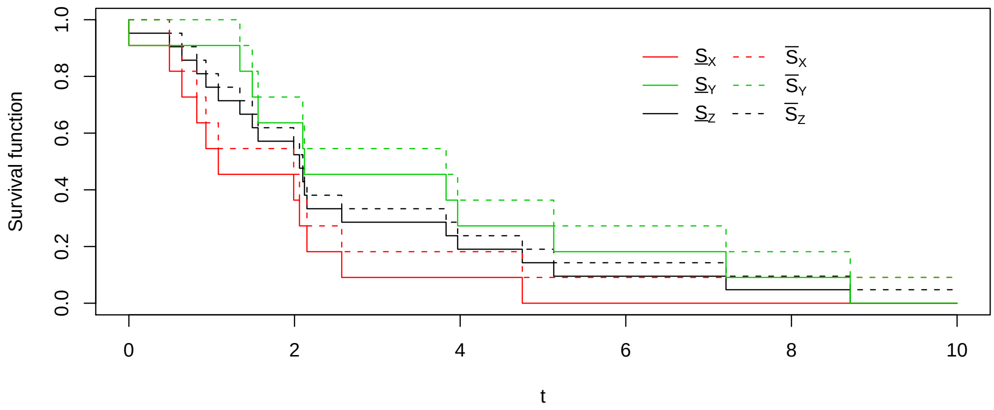

Figure 7: Lower and upper survival functions, Case F

functions for inference about the next observation based on the combined sample, several other inferences can be based on the NPI approach in such cases. For example, one can consider the event that the next observation from the combined sample will be in a specific interval, NPI lower and upper probabilities for such an event are easily derived using the M-functions presented in Section 2. One may also want to compare the next observations related to different sets of combined samples, NPI for such problems is possible along the same lines as the methods presented by Maturi (2010), who developed NPI methods for multiple comparisons for a range of right-censored and progressively censored data (but not with combination of data from different samples).

Appendix

We present the proofs of the results in Section 2.2. First, consider Type-II progressive censoring (Definition 2.1). We need notation for multiple right-censorings at the same time, which for the NPI approach we actually assume all to differ infinitesimally and, if such censorings occur simultaneously with an actual failure event, then we assume the censoring times to be infinitesimally greater than the corresponding failure time. To achieve this, let $c^i_{l_i}$ denote the right-censoring time of the l_i-th unit censored at z_i, for $i = 1, \ldots, r_z$ and $l_i = 1, \ldots, R_i$. We mean here that $c^i_{l_i}$ is infinitessimally greater than z_i. So, for any $k \in \{1, \ldots, r_z\}$, we have $z_k < c^k_1 < \ldots < c^k_{l_k} < \ldots < c^k_{R_k} < z_{k+1}$ where the actual differences for all but the final inequalities are very small, the limiting case for these differences reducing to 0 leads to the NPI results. Furthermore, let $\tilde{n}_{c^k_{l_k}}$ be the number of units at risk at $c^k_{l_k}$, that is $\tilde{n}_{c^k_{l_k}} = n_z - k - (l_k - 1) - \sum_{l=1}^{k-1} R_l$. The NPI upper and lower survival functions for Z_{n_z+1}, given in Equations (4) and (5), are derived as follows.

For $t \in [z_i, z_{i+1})$ with $i = 0, 1, \ldots, r_z$, the definition of the NPI upper survival function by Maturi et al. (2010b) leads to

$$\begin{aligned}
\overline{S}_{Z_{n_z+1}}(t) &= \frac{1}{n_z+1}\,\tilde{n}_{z_i} \prod_{\{j: c_j < z_i\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= \frac{1}{n_z+1}\,\tilde{n}_{z_i} \prod_{k=1}^{i-1} \prod_{l_k=1}^{R_k} \frac{\tilde{n}_{c^k_{l_k}}+1}{\tilde{n}_{c^k_{l_k}}} \\
&= \frac{1}{n_z+1}\,\tilde{n}_{z_i} \prod_{k=1}^{i-1} \frac{n_z - k - \sum_{l=1}^{k-1} R_l + 1}{n_z - k - \sum_{l=1}^{k} R_l + 1} \\
&= \tilde{n}_{z_i} M_{Z_{n_z+1}}(z_i, z_{i+1})
\end{aligned}$$

where $\tilde{n}_{z_i} = n_z - i - \sum_{l=1}^{i-1} R_l + 1$.

To prove the corresponding NPI lower survival function, let $t \in (t^i_{l_i}, t^i_{l_i+1}]$ with $i = 0, 1, \ldots, r_z$ and $l_i = 0, 1, \ldots, R_i$, and in, addition to notation introduced above, let $t^i_{R_i+1} = t^{i+1}_0 = z_{i+1}$, for $i = 0, 1, \ldots, r_z - 1$. The definition of the NPI lower survival function by Maturi et al. (2010b) leads to

$$\begin{aligned}
\underline{S}_{Z_{n_z+1}}(t) &= \frac{1}{n_z+1}\,\tilde{n}_{t^i_{l_i+1}} \prod_{\{j: c_j < t^i_{l_i+1}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= \frac{1}{n_z+1} \prod_{\{j: c_j < z_i\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \times \tilde{n}_{t^i_{l_i+1}} \prod_{\{j: z_i < c_j < t^i_{l_i+1}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= M_{Z_{n_z+1}}(z_i, z_{i+1}) \times \tilde{n}_{t^i_{l_i+1}} \prod_{\{j: z_i < c_j < t^i_{l_i+1}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= M_{Z_{n_z+1}}(z_i, z_{i+1}) \times \tilde{n}_{t^i_{l_i+1}} \left[\frac{\tilde{n}_{z_i}}{\tilde{n}_{t^i_{l_i+1}}+1}\right]^{1-a}
\end{aligned}$$

where $\tilde{n}_{t^i_{l_i+1}} = n_z - i - \sum_{l=1}^{i-1} R_l - l_i$ and $a = 1$ if $\{j : z_i < c_j < t^i_{l_i+1}\} = \emptyset$ and $a = 0$ if $\{j : z_i < c_j < t^i_{l_i+1}\} \neq \emptyset$.

Secondly, consider Type-I progressive censoring (Definition 2.2). Let $c^q_{l_q}$ denote the right-censoring time of the l_q-th unit censored at T_q, for $q = 1, \ldots, Q$ and $l_q =$

$1, \ldots, R_q$, with infinitesimally small differences between coinciding right-censored observations and the corresponding T_q as explained above. For any $k \in \{1, \ldots, Q\}$, $z^k_{s_k} < T_k < c^k_1 < \ldots < c^k_{l_k} < \ldots < c^k_{R_k} < z^{k+1}_1$, and let $\tilde{n}_{c^k_{l_k}}$ be the number of units at risk at $c^k_{l_k}$, so $\tilde{n}_{c^k_{l_k}} = n_z - \sum_{l=1}^{k} s_l - (l_k - 1) - \sum_{l=1}^{k-1} R_l$. Similarly, let $\tilde{n}_{z^k_{i_k}}$ be the number of units at risk at $z^k_{i_k}$, so $\tilde{n}_{z^k_{i_k}} = n_z - (i_k - 1) - \sum_{l=1}^{k-1} R_l - \sum_{l=1}^{k-1} s_l$, $i_k = 1, \ldots, s_k$. The NPI upper and lower survival functions for Z_{n_z+1}, given in Equations (6) and (7), are derived as follows.

For $t \in [z^q_{i_q}, z^q_{i_q+1})$ with $i_q = 1, \ldots, s_q$ and $q = 1, \ldots, Q$, the definition of the NPI upper survival function by Maturi et al. (2010b) leads to

$$
\begin{aligned}
\overline{S}_{Z_{n_z+1}}(t) &= \frac{1}{n_z+1}\,\tilde{n}_{z^q_{i_q}} \prod_{\{j: c_j < z^q_{i_q}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= \frac{1}{n_z+1}\,\tilde{n}_{z^q_{i_q}} \prod_{k=1}^{q-1} \prod_{l_k=1}^{R_k} \frac{\tilde{n}_{c^k_{l_k}}+1}{\tilde{n}_{c^k_{l_k}}} \\
&= \frac{1}{n_z+1}\,\tilde{n}_{z^q_{i_q}} \prod_{k=1}^{q-1} \frac{n_z - \sum_{l=1}^{k} s_l - \sum_{l=1}^{k-1} R_l + 1}{n_z - \sum_{l=1}^{k} s_l - \sum_{l=1}^{k} R_l + 1} \\
&= \tilde{n}_{z^q_{i_q}}\, M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1})
\end{aligned}
$$

with $M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1})$ as given in Definition 2.2. Furthermore, for $t \in [0, z^1_1)$ we have $\overline{S}_{Z_{n_z+1}}(t) = 1$.

The corresponding NPI lower survival function is derived in two steps. First, consider $t \in (t^q_{l_q}, t^q_{l_q+1}]$ with $q = 1, \ldots, Q$ and $l_q = 1, \ldots, R_q$, and we introduce additional notation $t^q_{R_q+1} = z^{q+1}_1$ for $q = 1, \ldots, Q$. The definition of the NPI lower survival function by Maturi et al. (2010b) leads to

$$
\begin{aligned}
\underline{S}_{Z_{n_z+1}}(t) &= \frac{1}{n_z+1}\,\tilde{n}_{t^q_{l_q+1}} \prod_{\{j: c_j < t^q_{l_q+1}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= \frac{1}{n_z+1} \prod_{\{j: c_j < z^q_{i_q}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \times \tilde{n}_{t^q_{l_q+1}} \prod_{\{j: z^q_{i_q} < c_j < t^q_{l_q+1}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1}) \times \tilde{n}_{t^q_{l_q+1}} \prod_{\{j: z^q_{i_q} < c_j < t^q_{l_q+1}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1}) \times \tilde{n}_{t^q_{l_q+1}} \times \frac{\tilde{n}_{z^q_{s_q}}}{\tilde{n}_{t^q_{l_q+1}}+1}
\end{aligned}
$$

where $\tilde{n}_{t^q_{l_q+1}}$ is the number of units at risk at $t^q_{l_q+1}$ and $M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1})$ is given in

Definition 2.2.

Secondly, consider $t \in (t^q_{i_q}, t^q_{i_q+1}]$ with $q = 1, \ldots, Q$ and $i_q = 1, \ldots, s_q$, and introduce additional notation $t^q_{s_q+1} = c^q_1$ for $q = 1, \ldots, Q$. The definition of the NPI lower survival function by Maturi et al. (2010b) leads to

$$\begin{aligned}
\underline{S}_{Z_{n_z+1}}(t) &= \frac{1}{n_z+1}\,\tilde{n}_{t^q_{i_q+1}} \prod_{\{j: c_j < t^q_{i_q+1}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= \frac{1}{n_z+1} \prod_{\{j: c_j < z^q_{i_q}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \times \tilde{n}_{t^q_{i_q+1}} \prod_{\{j: z^q_{i_q} < c_j < t^q_{i_q+1}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1}) \times \tilde{n}_{t^q_{i_q+1}} \prod_{\{j: z^q_{i_q} < c_j < t^q_{i_q+1}\}} \frac{\tilde{n}_{c_j}+1}{\tilde{n}_{c_j}} \\
&= M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1}) \times \tilde{n}_{t^q_{i_q+1}} \times 1
\end{aligned}$$

where $\tilde{n}_{t^q_{i_q+1}}$ is the number of units at risk at $t^q_{i_q+1}$ and $M_{Z_{n_z+1}}(z^q_{i_q}, z^q_{i_q+1})$ is given in Definition 2.2. Furthermore, for $t \in (0, z^1_1]$ we have $\underline{S}_{Z_{n_z+1}}(t) = n_z/(n_z+1)$.

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