Yangian Symmetry of Scattering Amplitudes and the Dilatation Operator in $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

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It is known that the Yangian of $PSU(2,2|4)$ is a symmetry of the tree-level S matrix of $\mathcal{N}=4$ super Yang-Mills theory. On the other hand, the complete one-loop dilatation operator in the same theory commutes with the level-one Yangian generators only up to certain boundary terms found by Dolan, Nappi, and Witten. Using a result by Zwiebel, we show how the Yangian symmetry of the tree-level S matrix of $\mathcal{N}=4$ super Yang-Mills theory implies precisely the Yangian invariance, up to boundary terms, of the one-loop dilatation operator.

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Introduction.—The study of $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory has been dominated by two broad strands of research—the first concentrating on the anomalous dimensions of local operators (i.e., the spectral problem) and their correlation functions, and the second investigating the scattering amplitudes of the theory. The successes in these two areas have been considerable in their own right, and at the current time there is vigorous activity focusing on making connections between them in order to deepen our understanding of this fascinating quantum field theory.

In the planar limit the spectral problem is believed to be integrable. This was first shown at one loop in Ref. [1] for a particular sector of the theory. The complete one-loop dilatation operator was later computed in Ref. [2], following earlier results in Ref. [3], and later shown in Ref. [4] to describe a $PSU(2,2|4)$ super spin chain. The one-loop dilatation operator is invariant under the (free) superconformal symmetry, and in fact this condition puts strong constraints on its form.

One of the key features of integrability is the existence of an infinite hierarchy of nonlocal charges Q^A built upon the basic local (or level-zero) $PSU(2,2|4)$ Noether charges J^A of the theory. These nonlocal charges, together with the local ones, obey a Yangian algebra which, in the context of the one-loop dilatation operator H, was described in Ref. [5]. Interestingly, it was found in that paper that H commutes with these additional nonlocal charges up to certain boundary terms,

$$[Q^A, H] \sim J_1^A - J_L^A, \tag{1}$$

where L denotes the length of the chain (or the number of fields in the operator).

The study of scattering amplitudes in $\mathcal{N}=4$ SYM theory started off independent from considerations of integrability, but it has recently begun to be connected to it in various ways. An important discovery was that of dual superconformal symmetry of the $\mathcal{N}=4$ SYM S matrix. This was conjectured in Ref. [6] and tested in several cases, and shortly after proved at tree level in Ref. [7]. At one loop the symmetry is broken because of the presence of infrared divergences in the amplitudes, and the breaking is controlled by a dual conformal Ward identity proposed in Ref. [8] and confirmed with a direct amplitude calculation at one loop in Ref. [9]. Importantly, in Ref. [10] the standard and dual superconformal symmetries were embedded in the Yangian of $PSU(2,2|4)$. Explicit expressions of the level-one generators were constructed and shown to be related to the generators of the dual superconformal algebra. At tree level the symmetry is slightly broken [11] due to collinear singularities of the amplitudes, leading to anomalies that are supported only on special kinematic configurations. As mentioned earlier, at one loop, infrared divergences lead to additional anomalies. Interestingly, these violations can be absorbed into appropriate redefinitions of the Yangian generators at both tree level [11] and one loop [12].

A direct connection between the one-loop nearest-neighbor part of the spin-chain dilatation operator and amplitudes, which will be very relevant for our investigation, was found in Ref. [13] by Zwiebel, working off of an earlier observation of Beisert's. In that paper the one-loop dilatation operator, expressed in the so-called harmonic action form [2], was related to the integration of a four-point superamplitude glued to a tree-level form factor with two external legs over the two-particle phase space; see Fig. 1. In Ref. [14], this connection was explained in terms of one-loop form factors of generic operators. (See also Refs. [15–20] for related work connecting amplitudes, form factors, and the dilatation operator.) Specifically, it was shown there that the result of Ref. [13] is the coefficient of the discontinuity of a bubble integral associated with this

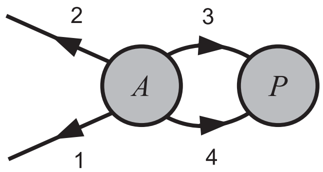

FIG. 1. In Ref. [13] it was shown that the harmonic action (4) is recovered via the sewing together of a tree-level four-point superamplitude A and a tree-level form factor P corresponding to the particular two-site spin-chain state under consideration.

one-loop form factor, and it captures the ultraviolet-divergent part of the calculation.

The presence of a Yangian symmetry on the dilatation operator and the amplitude sides naturally makes one think that these symmetries are the manifestation of a single underlying Yangian symmetry of the theory. However, these two symmetries are seemingly realized in a different manner, given Eq. (1) and the fact that on the amplitude side the symmetry can be realized exactly, with the Yangian generators annihilating the amplitudes [divided by the maximally helicity violating (MHV) part]. The goal of this Letter is that of reconciling these two situations by finding a proof of Eq. (1) which relies on the Yangian symmetry of the tree-level S matrix of $\mathcal{N}=4$ SYM theory, therefore substantiating the connection between the Yangians of the spin chain and the amplitudes.

In the following, we will use Zwiebel's formula to show that the invariance of the amplitudes under the Yangian, and certain special properties of the Yangian of $PSU(2,2|4)$, lead precisely to the expected result (1). One intriguing aspect of Eq. (1) is that it mixes tree-level and one-loop quantities [21]. A manifestation of this fact is that the left-hand side of Eq. (1) involves an integration, which is absent on the right-hand side of that equation. Our proof will show how this property arises naturally from the amplitudes. We also comment that in our derivation we will not be assuming the integrability of the theory.

Review and motivation.—In this section we review some important facts about the dilatation operator and Yangian symmetry. We will then motivate the calculation of the commutator $[Q,H]$ performed in the next section using the representation of the dilatation operator in terms of the amplitudes and form factors found in Ref. [13].

We consider single-trace local operators in $\mathcal{N}=4$ SYM theory of the form $\mathrm{Tr}(\Phi_1\cdots\Phi_L)(x)$, where the letters Φ are taken from the list $F^{\alpha\beta},\psi^{\alpha ABC},\phi^{[AB]},\bar\psi^{\dot\alpha A},\bar F^{\dot\alpha\dot\beta}$ (and symmetrized covariant derivatives acting on them), where $A=1,\ldots,4$ is a fundamental $SU(4)$ index.

It is well known that the operators can be described in terms of the spinor-helicity formalism [22]. The map to the letters introduced above is

$$\bar F\leftrightarrow\tilde\lambda\tilde\lambda,\qquad \bar\psi\leftrightarrow\tilde\lambda\eta,\qquad \phi\leftrightarrow\eta\eta,$$
$$\psi\leftrightarrow\lambda\eta\eta\eta,\qquad F\leftrightarrow\lambda\lambda\eta\eta\eta\eta, \tag{2}$$

while for derivatives $D\leftrightarrow\lambda\tilde\lambda$. As usual in $\mathcal{N}=4$ SYM theory, we combine the λ, $\tilde\lambda$, and η variables into a single object $\Lambda^a\coloneqq(\lambda^\alpha,\tilde\lambda^{\dot\alpha},\eta^A)$. In this formalism, a state is simply a polynomial in the Λ's satisfying the physical state condition of vanishing central charge at each spin-chain site; i.e., it has a sensible translation back to the letters $F^{\alpha\beta},\psi^{\alpha ABC},\phi^{[AB]},\bar\psi^{\dot\alpha A},\bar F^{\dot\alpha\dot\beta}$ (and symmetrized covariant derivatives acting on them), and we denote it as $P(\Lambda_1,\ldots,\Lambda_L)$. For instance, the Konishi operator is represented in this language as $\epsilon_{ABCD}(\eta_1^A\eta_1^B)(\eta_2^C\eta_2^D)$. We also note that in Ref. [14] it was observed that $P(\Lambda_1,\ldots,\Lambda_L)$ is nothing but the minimal form factor of the operator represented by the state via the dictionary (2). (The term "minimal" form factor was introduced in Ref. [23] to denote form factors where the state contains exactly as many particles as fields; i.e., the number of fields is the minimal number required to have a nonzero result at tree level.)

At one loop and in the planar limit, only two neighboring fields interact, and the one-loop dilatation operator H is the sum of densities H_{ii+1}, i.e., $H=\sum_{i=1}^L H_{ii+1}$, where L is the number of fields in the operator (or sites in the spin chain, of which H is the Hamiltonian), and H_{ii+1} acts only on fields at position i and $i+1$. The complete one-loop dilatation operator was derived in Ref. [2], with the result

$$H_{12}=\sum_{j=0}^{\infty}2h(j)\mathbb{P}_{12,j}. \tag{3}$$

Here $h(j)$ is the jth harmonic number and $\mathbb{P}_{12,j}$ projects onto a two-particle state with total spin j. The same paper also introduced an alternative representation of the dilatation operator termed "harmonic action," which can be rewritten in terms of spinor-helicity variables as [13]

$$H_{12}P(\Lambda_1,\Lambda_2)=-\frac{1}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi/2}d\theta\cot\theta[e^{2i\phi}P(\Lambda'_1,\Lambda'_2)-P(\Lambda_1,\Lambda_2)]. \tag{4}$$

Here, by $P(\Lambda_1,\Lambda_2)$ we mean $P(\cdots,\Lambda_1,\Lambda_2,\ldots)$, where the dots stand for all of the other fields in the state represented by P that are not involved in the interaction. Moreover, the Λ''s represent "rotated" spinor-helicity variables defined as

$$\begin{pmatrix}\lambda'_1\\ \lambda'_2\end{pmatrix}\coloneqq\mathcal{U}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix},\quad \begin{pmatrix}\tilde\lambda'_1\\ \tilde\lambda'_2\end{pmatrix}\coloneqq\mathcal{U}^*\begin{pmatrix}\tilde\lambda_1\\ \tilde\lambda_2\end{pmatrix},\quad \begin{pmatrix}\eta'_1\\ \eta'_2\end{pmatrix}\coloneqq\mathcal{U}^*\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \tag{5}$$

with the matrix $\mathcal{U}$ given by

$$\mathcal{U}\coloneqq\begin{pmatrix}\cos\theta & -e^{i\phi}\sin\theta\\ \sin\theta & e^{i\phi}\cos\theta\end{pmatrix}. \tag{6}$$

Note that while the state P satisfies the central charge condition, the rotated state in general violates it. The integration over ϕ in Eq. (4) is precisely enforcing the condition that the action of H_{12} on P returns a physical state.

As a final ingredient, we review an alternative form of Eq. (4) that was also discussed in Ref. [13], which will be particularly important for our analysis. [We note that Ref. [13] credits some unpublished work of Beisert for pointing out the connection between the rotating oscillator form of the harmonic action (4) and (7) below.] This representation for the action of the one-loop dilatation operator on a state $|1,2\rangle$ has the form [24]

$$H_{12}|1,2\rangle = \int d\Lambda A(1,2,3,4)[P(-4,-3) - rP(1,2)], \quad (7)$$

where momentum conservation reads $p_1 + p_2 + p_3 + p_4 = 0$. p_1 and p_2 are the external legs, while p_3 and p_4 are integrated over with the appropriate two-particle phase-space measure

$$d\Lambda = \prod_{i=3}^{4} d^2\lambda_i d^2\tilde{\lambda}_i d^4\eta_i. \quad (8)$$

Note that

$$A(1,2,3,4) = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \quad (9)$$

and the labels $1,\ldots,4$ are a shorthand notation for $\Lambda_1,\ldots,\Lambda_4$. We have also defined the ratio

$$r = \left(\frac{\langle 12\rangle}{\langle 34\rangle}\right)^2, \quad (10)$$

which allows us to write the two terms in Eq. (7) as integrated against the same tree-level amplitude, slightly departing from Refs. [13,14]. We find our presentation convenient, as it makes the infrared finiteness of Eq. (7) more manifest.

The relation between the two expressions for the dilatation operator (4) and (7) was shown in Ref. [13]. After integrating out the momentum conserving δ functions there are only two nontrivial integrals left, over θ and ϕ. The measures are then related by [13]

$$d\Lambda[A(1,2,3,4)r] \to -\frac{2}{2\pi} d\phi d\theta \cot\theta, \quad (11)$$

and we also have $r \to e^{-2i\phi}$, $\Lambda_3 \to -\Lambda'_2$, and $\Lambda_4 \to -\Lambda'_1$. These replacements take us from Eq. (7) to Eq. (4). As mentioned in Ref. [24], Eq. (11) is strictly only true up to a multiplicative numerical coefficient which will cancel in our final result.

Two observations are in order here. 1. An important feature of Eq. (7) is that it can be evaluated in four dimensions. The first term on the right-hand side of Eq. (7) has an infrared divergence which is canceled by the second term. This can be understood by observing that because of the four-point kinematics, the amplitude $A(1,2,3,4)$ develops a simple pole in the forward-scattering limit

$$p_4 = -p_1 \quad p_3 = -p_2, \quad (12)$$

which, in turn, generates infrared divergences in the first term of Eq. (7). It is then clear that the second term in Eq. (7) removes the pole in the integration. (Similar considerations were made in Ref. [9] in order to compute the dual conformal anomaly of one-loop superamplitudes with arbitrary helicity.) 2. The fact that Eq. (7) provides a representation of the complete one-loop dilatation operator of $\mathcal{N} = 4$ SYM theory may seem rather mysterious at this point. A neat physical interpretation of this result was found in Ref. [14]. In that paper it was observed that the first term on the right-hand side of Eq. (7) is nothing but the discontinuity (or two-particle cut) of a one-loop minimal form factor of a generic operator. This one-loop form factor is ultraviolet as well as infrared divergent, but the second term in Eq. (7) removes this infrared divergence, leaving only ultraviolet divergences. At one loop, the latter are entirely captured by a bubble integral whose discontinuity is a finite numerical constant. The coefficient of this discontinuity is minus the one-loop dilatation operator, and this is precisely the right-hand side of Eq. (7) [14].

Yangians and the commutation relation with level-one generators from amplitudes.—The action of Yangian symmetry in the context of the $\mathcal{N} = 4$ dilatation operator was first considered in Ref. [5]. The level-one generators are defined as [25]

$$Q^A \coloneqq \sum_{i<j} Q^A_{ij}, \qquad Q^A_{ij} = f^A_{CB} J^B_i J^C_j, \quad (13)$$

where $J^A = \sum_i J^A_i$ are level-zero (or superconformal) generators. Specifically, in Ref. [5] it was found that the commutator of Q with the complete one-loop dilatation operator is given by the following boundary term:

$$[Q^A, H] = 2(J^A_1 - J^A_L), \quad (14)$$

for a spin chain of length L. The main part of this Letter consists in evaluating this commutator $[Q,H]$ using the expression for H in terms of the amplitudes of Ref. [13] and the known action of Yangian generators on the amplitudes [6,10]. In this way we both give a very simple proof of Eq. (14) and at the same time further substantiate the connection between the spin chain and the amplitude Yangians.

In practice, one computes the commutator $[Q,H]|1,2\rangle$, where $|1,2\rangle$ is a two-particle state in the spin chain. As discussed in Ref. [5], the calculation of $[Q,H]|1,2\rangle$ boils down to that of the commutator $[Q_{12},H_{12}]|1,2\rangle$, which we address in this section.

We will now discuss the case of $Q = p^{(1)}$, namely, the generator corresponding to dual special conformal transformations K. The commutator in question is equal to

$$\begin{aligned}[Q_{12}, H_{12}]|1,2\rangle &= Q_{12}\int d\Lambda A(1,2,3,4)[P(-4,-3) - rP(1,2)] \\ &\quad - \int d\Lambda A(1,2,3,4)[Q_{-4,-3}P(-4,-3) - rQ_{12}P(1,2)],\end{aligned} \quad (15)$$

where [10]

$$(Q_{ij})_{\alpha\dot{\alpha}} = (m^{\gamma}_{j\alpha}\delta^{\dot{\gamma}}_{\dot{\alpha}} + \bar{m}^{\dot{\gamma}}_{j\dot{\alpha}}\delta^{\gamma}_{\alpha} - d_j\delta^{\gamma}_{\alpha}\delta^{\dot{\gamma}}_{\dot{\alpha}})p_{i\gamma\dot{\gamma}} + \bar{q}_{j\dot{\alpha}C}q^{C}_{i\alpha} - (i\leftrightarrow j). \quad (16)$$

The relevant generators are given by

$$d_i = \frac{1}{2}\left(\lambda^{\alpha}_i\frac{\partial}{\partial\lambda^{\alpha}_i} + \tilde{\lambda}^{\dot{\alpha}}_i\frac{\partial}{\partial\tilde{\lambda}^{\dot{\alpha}}_i}\right) + 1, \quad (17)$$

and

$$m_{\alpha\beta} = \lambda_{(\alpha}\partial_{\beta)}, \qquad \bar{m}_{\dot{\alpha}\dot{\beta}} = \tilde{\lambda}_{(\dot{\alpha}}\partial_{\dot{\beta})}, \qquad q^{A}_{\alpha} = \lambda_{\alpha}\eta^{A},$$
$$\bar{q}_{\dot{\alpha}A} = \tilde{\lambda}_{\dot{\alpha}}\partial_A, \qquad p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}. \quad (18)$$

We also note that $Q_{-4,-3} = Q_{34}$. Furthermore, in the second line, Q acts only on the form factor P, as required by the commutator.

We now describe our proof. First, we observe that we can rewrite Eq. (15) as

$$\begin{aligned}
&[Q_{12}, H_{12}]|1,2\rangle \\
&= \int d\Lambda[(Q_{12}+Q_{34})A(1,2,3,4)][P(-4,-3)-rP(1,2)] \\
&- \int d\Lambda[Q_{34}-(p_3-p_4)][A(1,2,3,4)[P(-4,-3)-rP(1,2)]] \\
&- P(1,2)\int d\Lambda[(\hat{Q}_{12}+\hat{Q}_{34})r]A(1,2,3,4) \\
&- P(1,2)\int d\Lambda(p_1-p_2-p_3+p_4)A(1,2,3,4)r. \quad (19)
\end{aligned}$$

In going from Eq. (15) to Eq. (19), we have performed an integration by parts, taking special care of the multiplicative part of Q_{ij}, obtained from taking the constant piece inside the dilatation operator. We have defined $\hat{Q}_{ij}$ to be the differential part of Q_{ij}, that is, $\hat{Q}_{ij} \coloneqq Q_{ij} + p_i - p_j$.

We will now show that the following statements concerning (19) are true: 1. The first line vanishes due to two reasons: first, $\sum_{i<j} Q_{ij}$ is the dual conformal generator K (up to a linear combination of level-zero generators, which annihilate the amplitude), which is a symmetry of the amplitudes; second, the nature of the supergroup $PSU(2,2|4)$, and specifically the vanishing of its dual Coxeter number. 2. The second line is a total derivative and integrates to zero. 3. We show that $(\hat{Q}_{12}+\hat{Q}_{34})r = 0$ and hence the third line vanishes. 4. The last line is the only nonzero contribution and provides the expected answer for the commutator. This is shown explicitly below.

1. We rewrite $Q_{12}+Q_{34} = \sum_{i<j} Q_{ij} - (Q_{13}+Q_{14}+Q_{23}+Q_{24})$. We then observe that $\sum_{i<j} Q_{ij}$ is precisely a Yangian generator, which annihilates the tree amplitude [10]. We can then recast the second term as

$$\begin{aligned}
(Q_{13}+Q_{14}+Q_{23}+Q_{24})^A &= f^A_{CB}(J_1+J_2)^B(J_3+J_4)^C \\
&= f^A_{CB}(J_1+J_2)^B J^C \\
&\quad - \frac{1}{2} f^A_{CB} f^{BC}_{D}(J_1+J_2)^D, \quad (20)
\end{aligned}$$

where $J \coloneqq J_1 + \cdots + J_4$. [We note the similarity between the right-hand side of Eq. (20) and Eq. (3) of Ref. [26].] The last term in Eq. (20) is proportional to the dual Coxeter number of $PSU(2,2|4)$ and hence vanishes. The penultimate term in Eq. (20) contains a level-zero generator J^C, which annihilates the amplitude. Thus,

$$(Q_{13}+Q_{14}+Q_{23}+Q_{24})A(1,2,3,4) = 0. \quad (21)$$

There is another way to appreciate this. Indeed, the fact that $Q_{13}+Q_{14}+Q_{23}+Q_{24}$ annihilates the amplitude is due to the fact that Yangian symmetry is compatible with the cyclicity of amplitudes. In more detail,

$$\sum_{1\le i<j\le 4} Q_{ij} - \sum_{3\le i<j\le 6} Q_{ij} = 2(Q_{13}+Q_{14}+Q_{23}+Q_{24}), \quad (22)$$

where we identify particle i with $i+4$. The two expressions $\sum_{1\le i<j\le 4} Q_{ij}$ and $\sum_{3\le i<j\le 6} Q_{ij}$ provide two representations of the level-one Yangian generator differing by a shift by two units of the particle labels. It is known from the work of Ref. [10] that the Yangian is consistent with the cyclicity of the scattering amplitudes; hence, both expressions annihilate the tree amplitude.

2. We consider the second term in Eq. (19), which contains the combination $Q_{34} - (p_3 - p_4)$, and show that it can be rewritten as a total derivative. Looking at the expression for Q_{ij} in Eq. (16), we note that the terms involving m, $\bar{m}$, and $\bar{q}q$ are total derivatives. We only need to focus on the term involving the tree-level dilatation operator d. To this end, we note that the relevant term is $-d_4 p_3 + d_3 p_4 - p_3 + p_4 = -(d_4+1)p_3 + (d_3+1)p_4$. We can then write its action on a function f as a total derivative,

$$\begin{aligned}
(1+d_i)f &= \left[2 + \frac{1}{2}\left(\lambda^{\alpha}_i\frac{\partial}{\partial\lambda^{\alpha}_i} + \tilde{\lambda}^{\dot{\alpha}}_i\frac{\partial}{\partial\tilde{\lambda}^{\dot{\alpha}}_i}\right)\right]f \\
&= \frac{1}{2}\left[\frac{\partial}{\partial\lambda^{\alpha}_i}(\lambda^{\alpha}_i f) + \frac{\partial}{\partial\tilde{\lambda}^{\dot{\alpha}}_i}(\tilde{\lambda}^{\dot{\alpha}}_i f)\right]. \quad (23)
\end{aligned}$$

The second line in Eq. (19) is then a boundary term which vanishes. Note that the integration can be carried out in four dimensions since the integral is finite.

3. A short calculation shows that the stronger statements

$$\hat{Q}_{12}r = \hat{Q}_{34}r = 0 \quad (24)$$

are true. Since $r = e^{-2i\phi}$ and the integration over ϕ imposes the vanishing of the central charge on the physical states, this condition should be equivalent to the fact that the central charge commutes with all generators of the algebra and hence also with $\hat{Q}$.

4. Finally, the last term is the only one that contributes to the commutator. We can now evaluate it using the parametrization introduced in Ref. [13]. All variables except θ and ϕ can be integrated trivially using δ functions, and one is left with the following effective parametrization for the loop momenta:

$$\begin{aligned}\lambda_3 &= \lambda_1 \sin\theta + e^{i\phi}\lambda_2\cos\theta,\\ \tilde{\lambda}_3 &= -(\tilde{\lambda}_1 \sin\theta + e^{-i\phi}\tilde{\lambda}_2\cos\theta),\\ \lambda_4 &= \lambda_1\cos\theta - e^{i\phi}\lambda_2\sin\theta,\\ \tilde{\lambda}_4 &= -(\tilde{\lambda}_1\cos\theta - e^{-i\phi}\tilde{\lambda}_2\sin\theta).\end{aligned} \tag{25}$$

We then find

$$\begin{aligned}p_3-p_4-(p_1-p_2)=2[&\sin^2\theta(p_2-p_1)\\ &-\cos\theta\sin\theta(\lambda_1\tilde{\lambda}_2 e^{-i\phi}+\lambda_2\tilde{\lambda}_1 e^{i\phi})].\end{aligned} \tag{26}$$

After integrating out all of the δ functions, the integration measure $d\Lambda A(1,2,3,4)$ in the last line of Eq. (19) becomes equal to the expression given in Eq. (11) [27], where $\theta \in (0,\pi/2)$ and $\phi \in (0,2\pi)$. Using Eqs. (26) and (11), one then finds

$$\int d\Lambda A(1,2,3,4) r[p_3-p_4-(p_1-p_2)] = 2(p_1-p_2), \tag{27}$$

where terms proportional to $e^{\pm i\phi}$ in Eq. (26) trivially integrate to zero. In conclusion, the right-hand side of Eq. (19) is

$$\begin{aligned}-P(1,2)&\int d\Lambda(p_1-p_2-p_3+p_4)A(1,2,3,4)r\\ &= 2(p_1-p_2)P(1,2),\end{aligned} \tag{28}$$

in agreement with Ref. [5]. This is the main result of the Letter.

A few comments are in order.

First, we observe that it is not necessary to check commutators with other level-one generators, given the invariance of H under the standard superconformal group. To see this, we note that $[Q_A, J_B] = f_{AB}{}^C Q_C$ and assume that Eq. (14) holds for Q_A. Therefore,

$$\begin{aligned}f_{BA}{}^C[H_{12}, Q_C] &= [[Q_A, J_B], H_{12}] = [J_B, [H_{12}, Q_A]]\\ &= 2[J_B, (J_A)_1 - (J_A)_2]\\ &= 2 f_{BA}{}^C[(J_C)_1 - (J_C)_2],\end{aligned} \tag{29}$$

where, in the second equality, we have used the fact that the level-zero generators commute with H_{12}. We have thus shown that Eq. (14) holds also for Q_C. We have also confirmed this fact by an explicit check for the level-one generator associated with supersymmetry q.

We also note that, in principle, one could try to compute the commutator $[Q_{12}, H_{12}]|1,2\rangle$ starting from Eq. (4); however, it is not immediate to extract the commutator directly and, in particular, to see the universal structure of the right-hand side of Eq. (14). It is precisely this feature that we have proved using the representation (7) provided by Ref. [13], and using the known action of Yangian generators on tree-level scattering amplitudes.

Finally, one should exercise some caution in the manipulations above, in particular in setting $K_{\alpha\dot{\alpha}}A = 0$. In fact, $K_{\alpha\dot{\alpha}}A$ contains an as yet unnoticed holomorphic anomaly [28] arising only in four-point kinematics. The key fact to notice is that [29]

$$K_{\alpha\dot{\alpha}}\frac{1}{\langle i i+1\rangle} = 2\pi\delta(\langle i i+1\rangle)\delta([i i+1])[i i+1](p_i+p_{i+1})_{\alpha\dot{\alpha}}. \tag{30}$$

The right-hand side of Eq. (30) vanishes, unless the $[ii+1]$ factor is compensated for by a corresponding pole, which indeed occurs in a four-point amplitude $A(1,2,3,4)$, when, for instance, the vanishing of $\langle 23\rangle$ implies the vanishing of $\langle 41\rangle$. Such a holomorphic anomaly could affect the first and second line of Eq. (19). However, thanks to the presence of the combination $P(-4,-3) - rP(1,2)$, which precisely vanishes on the support of the δ function, i.e., the forward-scattering kinematic configuration, these holomorphic anomalies cancel out.

To summarize, in this Letter we have presented the first concrete calculation showing that there is a single Yangian structure in $\mathcal{N}=4$ SYM theory. Yangian symmetry is believed to be a fundamental property of this theory, and yet the manifestations on the one-loop dilatation operator and the S matrix of the theory are vastly different. Here, we have solved the puzzle concerning the presence of these two contrasting realizations of the Yangian symmetry by providing a direct link between the two. We also note that, importantly, we have not assumed the integrability of the underlying theory. We expect that the ideas presented here will be useful in understanding the Yangian symmetry of the dilatation operator to higher loops.

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[25] Note that our definition of Q_{12} is identical to that of Ref. [10] and differs from that of Ref. [5] by a factor of $-1/2$, namely, $Q_{12}^{\mathrm{DNW}}=(-1/2)Q_{12}^{\mathrm{DHP}}$. The minus sign arises from having swapped the indices B and C in Eq. (13) compared to the corresponding definition in Ref. [5], while a factor of $1/2$ is introduced in lowering an index of the structure constants in the definition of the Yangian generators in Ref. [5].
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