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Mostow's Lattices and Cone Metrics on the Sphere

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Abstract. In his seminal paper of 1980, Mostow constructed a family of lattices in $\mathrm{PU}(2,1)$, the holomorphic isometry group of complex hyperbolic 2-space. In this paper, we use a description of these lattices given by Thurston in terms of cone metrics on the sphere, which is equivalent to Deligne and Mostow's description of them using monodromy of hypergeometric functions. We give an explicit fundamental domain for some of Mostow's lattices, specifically those with large phase shift. Our approach follows Parker's approach of describing Livné's lattices in terms of cone metrics on the sphere. The content of this paper is based on Boadi's PhD thesis.

1 Introduction

A *(Euclidean) cone metric* on the sphere is a flat Euclidean metric on the sphere with finitely many singularities. Each singularity locally looks like the apex of a cone and may be formed by identifying the sides of a sector. The angle between the sides of the sector, or equivalently at the apex of the cone, is the *cone angle*. If the cone angle at a vertex v is $2\pi - \alpha$ then α is the *curvature* at v. In this paper we only deal with the case where the cone angle lies in $[0, 2\pi)$ and so the curvature is positive. The sum of the curvatures at all singularities must be the total curvature of the sphere, which is 4π.

For example, a cube is a cone metric on the sphere with eight singularities. At each vertex three squares meet and so the cone angle is $3\pi/2$. Clearly the curvature at each vertex is $\pi/2$ and the sum over all eight vertices gives a total curvature of $8 \cdot \pi/2 = 4\pi$.

In [14] Thurston considered the following construction. Suppose we are given a fixed number of cone singularities with prescribed cone angles (chosen so the sum of the curvatures is 4π). Keeping these angles fixed but varying the location of the singularities gives a moduli space of cone metrics. One may move the cone points in this space along any non-trivial closed path in the moduli space (for example by performing a Dehn twist). In doing so one naturally obtains a modular group. The area of the cone metric is preserved by such a motion and so by all elements of the modular group. Thurston observed that the area gives an indefinite Hermitian form and one may embed the (projectivised) moduli space into complex hyperbolic space. For certain good choices of cone angle, the resulting group is a complex hyperbolic lattice. (See Weber [15] for an alternative point of view.)

Thurston's construction is an alternative point of view on the lattices constructed by Deligne and Mostow [2] and [10] via monodromy of hypergeometric functions in several variables. Thurston's good choices of angles correspond to ball N-tuples satisfying the condition ΣINT defined by Mostow in [10]. Among these groups are two families, one constructed by Mostow in [9] and the other constructed by Livné in [8]. In [12], Parker constructed Livné's lattices from first principles using Thurston's construction and he then went on to use this explicit construction to build fundamental domains for these groups. In his PhD thesis [1], Boadi extended Parker's construction to include some of the lattices constructed by Mostow in [9]. This paper is an account of the latter construction. We refer to [1] for many of the details. Deraux, Falbel and Paupert [3] have also given a construction of fundamental domains for Mostow's lattices. Below we give a description of the relationship between our constructions and those from [9] and [3].

We consider cone metrics on the sphere with five cone points with cone angles

$$(\pi-\theta+2\phi,\ \pi+\theta,\ \pi+\theta,\ \pi+\theta,\ 2\pi-2\theta-2\phi). \tag{1}$$

The angles θ and ϕ satisfy $\theta>0$, $\phi>0$ and $\theta+\phi<\pi$. Specifically, we consider the following values of θ and ϕ which Thurston showed yield discrete groups [14]:

θ	$2\pi/3$	$2\pi/3$	$2\pi/3$	$2\pi/4$	$2\pi/4$	$2\pi/5$	$2\pi/5$	$2\pi/6$	$2\pi/6$
ϕ	$\pi/4$	$\pi/5$	$\pi/6$	$\pi/3$	$\pi/4$	$\pi/2$	$\pi/3$	$\pi/2$	$\pi/3$

(2)

The groups corresponding to these cone angles with $\theta=2\pi/p$ for $p=3$, 4, 5 are the groups Mostow considered in [9] with large phase shift. We also consider two groups with $p=6$.

Certain automorphisms of our cone metrics yield unitary matrices R_1, R_2 and A_1. (The naming of these automorphisms follows Mostow, see his survey paper [11] for example.) Our goal is to show that the group Γ generated by these automorphisms is discrete. To do so, we construct a polyhedron D and use Poincaré's polyhedron theorem to show that Γ is discrete with fundamental polyhedron D. The vertices of D come from degenerate cone metrics where some of the cone points have coalesced, see Section 3.3 (again we follow Mostow [9] when naming these vertices). The polyhedron D has a simple description via a list of inequalities of distances in complex hyperbolic space to the vertices and their images under certain elements of the group generated by Γ. We state this in terms of the Hermitian form in order to be able to include the cases where one of these vertices lies on the boundary of complex hyperbolic space.

Theorem 1.1. *Let p_{12}, v_{123}, v_{231} and v_{312} be the vertices defined in Section 3.3. Let $P=R_1R_2$ and $J=R_1R_2A_1$. The polyhedron D is defined by points $p\in\mathbf{H}^2_{\mathbb{C}}$ satisfying the eight inequalities:*

$$\begin{aligned}
&\big|\langle p,p_{12}\rangle\big|<\big|\langle p,J^{-1}(p_{12})\rangle\big|, &&\big|\langle p,v_{123}\rangle\big|<\big|\langle p,P^{-1}(v_{231})\rangle\big|,\\
&\big|\langle p,v_{312}\rangle\big|<\big|\langle p,R_1^{-1}(v_{231})\rangle\big|, &&\big|\langle p,v_{231}\rangle\big|<\big|\langle p,R_1(v_{312})\rangle\big|,\\
&\big|\langle p,v_{231}\rangle\big|<\big|\langle p,P(v_{123})\rangle\big|, &&\big|\langle p,p_{12}\rangle\big|<\big|\langle p,J(p_{12})\rangle\big|,\\
&\big|\langle p,v_{123}\rangle\big|<\big|\langle p,R_2^{-1}(v_{312})\rangle\big|, &&\big|\langle p,v_{312}\rangle\big|<\big|\langle p,R_2(v_{123})\rangle\big|.
\end{aligned}$$

In particular, the boundary of D is made up of sides contained in bisectors where one of these inequalities has been replaced with an equality. Bisectors are hypersurfaces that are equidistant from two points in complex hyperbolic space, see Section 4 for more details. An equivalent description of D in terms of a well-chosen set of coordinates is given in (16). It is the simplicity of these two descriptions which make it straightforward for us to check the conditions of Poincaré's polyhedron theorem directly without the use of computers. The side pairing maps, which are part of the hypotheses of Poincaré's theorem, are given simply in terms of the maps R_1, R_2 and A_1. In Theorem 5.1, using Poincaré's polyhedron theorem, we prove that the group Γ generated by the side pairings of D is a discrete subgroup of $\mathrm{PU}(1,2)$ with fundamental domain D and presentation:

$$\Gamma = \left\langle J, P, R_1, R_2 : \begin{array}{c} J^3 = R_1^p = R_2^p = (P^{-1}J)^k = I, \\ R_2 = PR_1P^{-1} = JR_1J^{-1},\ P = R_1R_2 \end{array} \right\rangle.$$

The integers p and k are defined by $\theta = 2\pi/p$ and $\phi = \pi/k$ where θ and ϕ are given in table (2). Equivalently, in terms of R_1, R_2 and A_1 the presentation is

$$\Gamma = \left\langle R_1, R_2, A_1 : \begin{array}{c} R_1^p = R_2^p = A_1^k = (R_1R_2A_1)^3 = I, \\ R_1R_2R_1 = R_2R_1R_2,\ R_1A_1 = A_1R_1 \end{array} \right\rangle.$$

These presentations should be compared to the discussion in Mostow [11], particularly equation (5.3) and page 244. One may also write Γ as a two generator group. For example, as Mostow observes, it is easy to see that R_1 and J will generate Γ (which Mostow calls Γ_μ) but then the presentation is not so clean.

We now discuss how our paper relates to other similar constructions in the literature. Mostow, in his 1980 paper [9], gave a construction of fundamental domains for several complex hyperbolic lattices. These lattices were divided by him into two families, those with small phase shift and those with large phase shift. It is the lattices with large phase shift that are of interest to us. Mostow's construction relied heavily on the use of computers, and indeed his paper is a pioneering paper on the use of computers in mathematical proof. Another construction of fundamental domains for these lattices was given by Deraux, Falbel and Paupert [3], which again made use of computers in the proofs. Most of their paper gives details in the case of small phase shift, where the fundamental domain has more complicated combinatorics. However, [3] does contain a section giving the outline of their construction for lattices with large phase shift. Our fundamental domains have the same combinatorial structure as those constructed by Deraux, Falbel and Paupert (compare Figure 19 of [3] with our Figure 10). However, our construction gives a significant improvement in two ways. First, all the sides of our polyhedra are contained in bisectors. Secondly, and more importantly, by making a good choice of coordinates we are able to give a particularly simple description of the polyhedron, which means that the technical conditions of Poincaré's polyhedron theorem may be verified directly without the use of computers. This verification reduces to a large number of elementary manipulations of inequalities. We give a sample of these here and refer to Boadi's thesis [1] for the remainder.

In some special cases, there are further connections between our groups and some groups for which fundamental domains have been constructed. We now summarise these.

- The group corresponding to $(\theta, \phi) = (2\pi/6, \pi/2)$ is the Eisenstein-Picard modular group, considered by Falbel and Parker in [6].
- The group corresponding to $(\theta, \phi) = (2\pi/3, \pi/6)$ is the so called "sister" of the Eisenstein-Picard modular group, considered by Zhao in [16]. The isomorphism between these groups is described in Proposition 6.4 of [16].
- The group corresponding to $(\theta, \phi) = (2\pi/4, \pi/4)$ is an index two subgroup of the Gauss-Picard modular group, considered by Falbel, Francsics and Parker in [5]. See Section 9.3 of [5] for the relationship between these groups.

These three groups have parabolic maps and the fundamental domains constructed in [6], [16] and [5] are based on Ford domains centred at the parabolic fixed points. Again, our fundamental domain D is different from theirs. Further relationships between these groups can be found in Parker's survey paper [13].

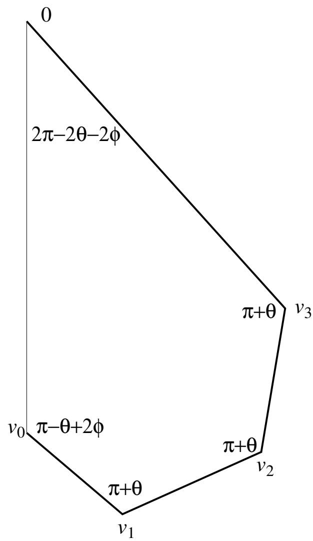

Figure 1. *The doubled pentagon. Note that the angles indicated are the cone angles, which are twice the internal angles of the pentagon. The octagon* Π *is obtained by cutting along the bold lines.*

The paper is arranged as follows: in Section 2 we consider the particular cone metrics on the sphere of interest here; in Section 3 we consider the cone metrics where the cone singularities coalesce, and which give rise to the vertices of our polyhedron; in Section 4 we construct the polyhedron D, which sets the stage for the final Section 5 which is a summary of the proof that D is a fundamental polyhedron for the group. Details of the work can be found in Boadi's thesis [1].

We would like to thank the referee for giving us many helpful suggestions, which have greatly improved the paper.

2 Construction of the Polygons and Automorphisms

2.1 The Fundamental Domain for the Cone Structures. We consider Euclidean cone metrics on the sphere with five cone points with cone angles given by (1) where θ and ϕ are given in table (2). Our goal is to find a unified construction for all these angles (and to verify that these angles correspond to discrete groups). If we cut the sphere open along a path through the five cone points, we obtain a Euclidean polygon Π. Conversely, if we glue the sides of Π together, we can reconstruct our cone metric on the sphere. We give an explicit parametrisation of such polygons in terms of three complex parameters (z_1, z_2, z_3). We show that, in terms of these parameters, the area of the polygon gives a Hermitian form of signature $(1, 2)$. Thurston [14] and Weber [15] describe different ways of doing this. We follow Parker's method from [12] which is different from the methods of Thurston and Weber.

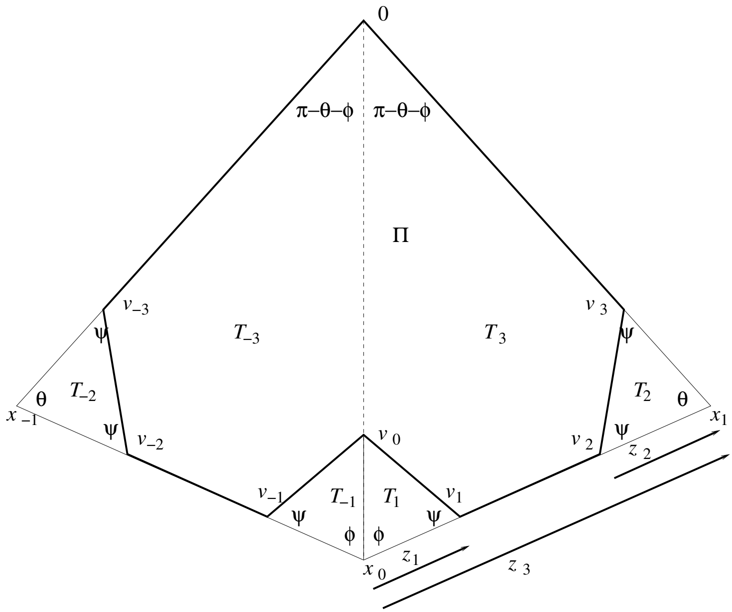

Figure 2. *The octagon* Π *when* z_1, z_2 *and* z_3 *are all real. Here* $\psi = (\pi - \theta)/2$. *Note that the arrows are* $z_j ie^{-i\phi}$.

We begin by looking at the case where the cone manifold is the double of a Euclidean pentagon. Cut the pentagon along four of its sides, as in Figure 1, with the first cut at the cone point with angle $2\pi - 2\theta - 2\phi$, then moving along the boundary of the pentagon through the three cone points v_3, v_2 and v_1 with cone angle $\pi + \theta$, ending at the cone point v_0 with cone angle $\pi - \theta + 2\phi$. When we cut the double pentagon this way, we get an octagon, which we call Π; see Figure 2. This octagon has a reflection symmetry. Using this symmetry to identify the boundary points reconstructs the doubled pentagon with which we began.

We now show how to construct Π geometrically. We start with a big triangle T_3 with angles θ, $\pi-\theta-\phi$ and ϕ. Recall that $\theta>0$, $\phi>0$ and $\theta+\phi<\pi$. We then take off two smaller triangles T_1 with angles ϕ, $\pi/2+\theta/2-\phi$ and $\pi/2-\theta/2$; and T_2 with angles θ, $\pi/2-\theta/2$ and $\pi/2-\theta/2$. The corners of the triangles T_1 and T_3 with angles ϕ are the same. The corners of the triangles T_2 and T_3 with angles θ are the same. The base vectors of T_1, T_2 and T_3 are $ie^{-i\phi}z_1$, $ie^{-i\phi}z_2$ and $ie^{-i\phi}z_3$. The parameters z_1, z_2 and z_3 are real for the doubled pentagon. See Figure 2 for the construction.

The vertices of the triangle T_1 are as follows:

$$\begin{aligned} v_0 &= \frac{i\sin\theta}{\sin\phi+\sin(\theta-\phi)}z_1-\frac{i\sin\theta}{\sin(\theta+\phi)}z_3, \\ x_0 &= \frac{-i\sin\theta}{\sin(\theta+\phi)}z_3, \\ v_1 &= ie^{-i\phi}z_1-\frac{i\sin\theta}{\sin(\theta+\phi)}z_3. \end{aligned}$$

The vertices of triangle T_2 are as follows:

$$\begin{aligned} v_2 &= -ie^{-i\phi}z_2+\frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)}z_3, \\ x_1 &= \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)}z_3, \\ v_3 &= -ie^{-i\theta-i\phi}z_2+\frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)}z_3. \end{aligned}$$

The vertices of triangle T_3 are also as follows:

$$\begin{aligned} & 0, \\ x_1 &= \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)}z_3, \\ x_0 &= \frac{-i\sin\theta}{\sin(\theta+\phi)}z_3. \end{aligned}$$

We have constructed a pentagon whose vertices are the vertex of T_3 and the two vertices of each of T_1 and T_2 not shared by one of the other triangles. This has one edge in common with each of T_1 and T_2. Consider the edge of this pentagon joining the vertices of T_1 with angle $\pi+\theta/2-\phi$ and the vertex of T_3 with angle $\pi-\theta-\phi$. Reflect the pentagon across this side to form an octagon, see Figure 2. The image of the triangle T_1 under this reflection will be a new triangle T_{-1}. Similarly, the images of T_2 and T_3 will be triangles T_{-2} and T_{-3}. The images of vertices v_1, v_2 and v_3 will be v_{-1}, v_{-2} and v_{-3} respectively. Our resulting octagon is preserved by reflection in the imaginary axis and we label its vertices so that this reflection interchanges v_j and v_{-j}. Moreover, gluing points of the boundary Π to their image under this reflection reconstructs the doubled pentagon

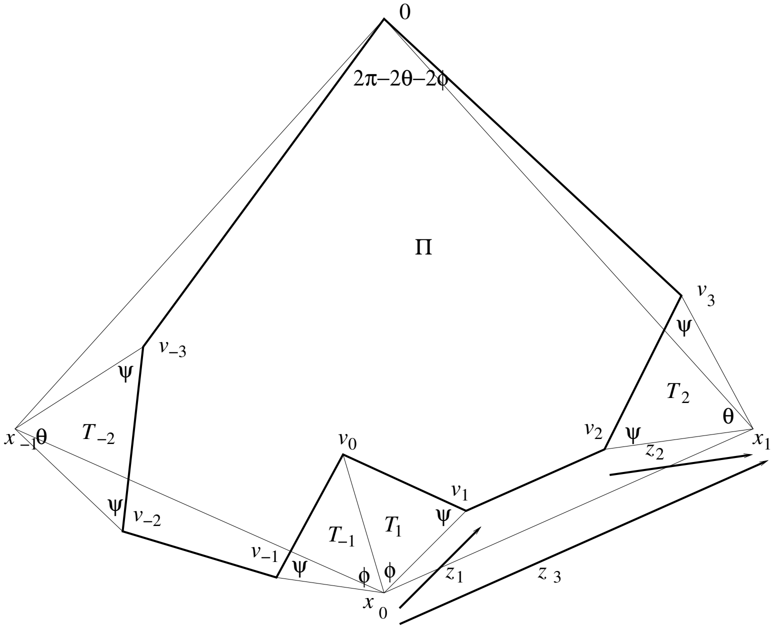

Figure 3. *The octagon* Π *with complex* z_1, z_2 *and* z_3. *Again* $\psi = (\pi - \theta)/2$ *and the arrows are* $z_j ie^{-i\phi}$.

we begun with. Below are the vertices of the octagon.

$$
\begin{aligned}
v_0 &= \frac{i\sin\theta}{\sin\phi + \sin(\theta-\phi)} z_1 - \frac{i\sin\theta}{\sin(\theta+\phi)} z_3, \\
v_1 &= ie^{-i\phi} z_1 - \frac{i\sin\theta}{\sin(\theta+\phi)} z_3, \\
v_2 &= -ie^{-i\phi} z_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)} z_3, \\
v_3 &= -ie^{-i\theta-i\phi} z_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}}{\sin(\theta+\phi)} z_3, \\
v_{-1} &= ie^{i\phi} z_1 - \frac{i\sin\theta}{\sin(\theta+\phi)} z_3, \\
v_{-2} &= -ie^{i\phi} z_2 + \frac{i\sin\phi\, e^{i\theta+i\phi}}{\sin(\theta+\phi)} z_3, \\
v_{-3} &= -ie^{i\theta+i\phi} z_2 + \frac{i\sin\phi\, e^{i\theta+i\phi}}{\sin(\theta+\phi)} z_3.
\end{aligned}
$$

The vertices of T_{-1} are v_0, x_0 and v_{-1}; the vertices of T_{-2} are v_{-2}, x_{-1} and v_{-3} and the vertices of T_3 are 0, x_{-1} and x_{-2}. We remark that substituting $\phi = \pi/2$ we recover the octagon from [12].

In the above construction, the octagon was formed by cutting a doubled pentagon along four of its edges. We now consider how to build an octagon associated with a more general cone metric on the sphere with five cone points whose angles are given by (1). We do this by following the above construction but allowing the parameters z_1, z_2, z_3 to be complex variables. A typical example of an octagon with complex parameters is shown in Figure 3. The triangles T_1, T_{-1}, T_3 and T_{-3} share the vertex x_0, the triangles T_2 and T_3 share the vertex x_1 and the triangles T_{-2} and T_{-3} share the vertex x_{-1}. The base of the triangle T_1 is still $v_1 - x_0 = ie^{-i\phi}z_1$, the base of T_2 is $x_1 - v_2 = ie^{-i\phi}z_2$ and the base of T_3 is $x_1 - x_0 = ie^{-i\phi}z_3$. In general these are no longer real multiples of each other. Similarly, the bases of T_{-1}, T_{-2} and T_{-3} are $v_{-1} - x_0 = ie^{i\phi}z_1$, $x_{-1} - v_{-2} = ie^{i\phi}z_2$ and $x_{-1} - x_0 = ie^{i\phi}z_3$ respectively. Note that v_{-j} is no longer the image of v_j under reflection in the imaginary axis.

Simple geometry shows that the areas of the triangles are as follows:

$$\begin{aligned}
\mathrm{Area}(T_1) &= \frac{\sin\theta\sin\phi}{2(\sin\phi+\sin(\theta-\phi))}\,|z_1|^2\,,\\
\mathrm{Area}(T_2) &= \frac{1}{2}\sin\theta\,|z_2|^2\,,\\
\mathrm{Area}(T_3) &= \frac{\sin\theta\sin\phi}{2\sin(\theta+\phi)}\,|z_3|^2\,.
\end{aligned}$$

When the parameters z_j are real it is easy to see that the area of octagon Π is

$$\mathrm{Area}(\Pi) = 2\mathrm{Area}(T_3) - 2\mathrm{Area}(T_1) - 2\mathrm{Area}(T_2).$$

When the z_j are complex, one can show by a simple cut and paste argument that this formula still holds. Therefore:

$$\begin{aligned}
\mathrm{Area}(\Pi) &= 2\mathrm{Area}(T_3) - 2\mathrm{Area}(T_1) - 2\mathrm{Area}(T_2)\\
&= \sin\theta\left(-\frac{\sin\phi}{(\sin\phi+\sin(\theta-\phi))}\,|z_1|^2 - |z_2|^2 + \frac{\sin\phi}{\sin(\theta+\phi)}\,|z_3|^2\right)\\
&= \sin\theta\begin{bmatrix}\overline{z_1} & \overline{z_2} & \overline{z_3}\end{bmatrix}\begin{bmatrix}\frac{-\sin\phi}{\sin\phi+\sin(\theta-\phi)} & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & \frac{\sin\phi}{\sin(\theta+\phi)}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}\\
&= \mathbf{z}^* H\mathbf{z},
\end{aligned}$$

where H is the Hermitian matrix:

$$H = \sin\theta\begin{bmatrix}-\sin\phi/\big(\sin\phi+\sin(\theta-\phi)\big) & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & \sin\phi/\sin(\theta+\phi)\end{bmatrix}. \tag{3}$$

We observe that the area gives a Hermitian form of signature (1,2) on $\mathbb{C}^3$. This leads to a complex hyperbolic structure on the moduli space of such polygons. This is a special case of Proposition 3.3 of Thurston [14].

There is a natural way to construct a particular Euclidean cone manifold from Π. The following, σ_j, are edge pairing maps of Π, which are Euclidean isometries which preserve orientation and so are completely determined on each edge by their value on the vertices v_j,v_{j+1}. The maps are:

$$\begin{array}{llll} \sigma_1(0) = 0, & \sigma_1(v_3) = v_{-3}; & \sigma_2(v_3) = v_{-3}, & \sigma_2(v_2) = v_{-2}; \\ \sigma_3(v_2) = v_{-2}, & \sigma_3(v_1) = v_{-1}; & \sigma_4(v_1) = v_{-1}, & \sigma_4(v_0) = v_0. \end{array}$$

Let M be the Euclidean cone manifold given by identifying the edges of Π using the maps σ_j. It is clear that the underlying topological space of the orbifold M is homeomorphic to a sphere and has five cone points corresponding to v_0, $v_{\pm 1}$, $v_{\pm 2}$, $v_{\pm 3}$, 0 with cone angles $\pi - \theta + 2\phi$, $\pi + \theta$, $\pi + \theta$, $\pi + \theta$, $2\pi - 2\theta - 2\phi$ respectively. These are the cone angles given in (1).

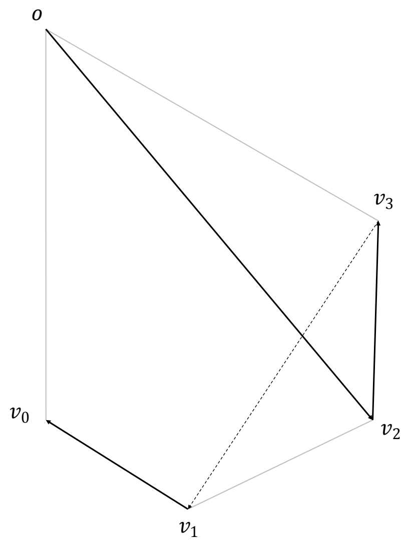

Figure 4. *The cut for move* R_1.

2.2 Moves on the cone structure. We define automorphisms which we call *moves* on such polygons in the spirit of Thurston [14]. These generalise the moves constructed by Parker in [12] in an obvious way.

We define them as follows. Our cone manifold has five cone points. The two corresponding to 0 and v_0 have cone angles $2\pi - 2\theta - 2\phi$ and $\pi + \phi - \theta$, respectively. The other three vertices have the same cone angle, which is $\pi + \theta$. In cutting our cone manifold to get back our octagon, there is no canonical ordering of these three vertices, hence we can change the order of the cut. This results in the moves we will consider namely R_1 and R_2. We introduce a third move A_1 along the same lines as Thurston's butterfly moves.

- **The move** R_1 The move R_1 fixes the vertex 0, v_0 and $v_{\pm1}$ and then interchanges $v_{\pm2}$ and $v_{\pm3}$. This corresponds to a Dehn twist along a simple closed curve through $v_{\pm2}$ and $v_{\pm3}$ that does not separate the other cone points. This is described on page 242 of Mostow's survey article [11]. When cutting open the cone manifold, one must begin cutting from 0 and then to $v_{\pm2}$, then to $v_{\pm3}$, and then to $v_{\pm1}$ and v_0; see Figure 4. When we cut open the double pentagon, we obtain an octagon, shown in Figure 5. Using cut and paste, one can obtain the new octagon from the old. The cut goes from 0 directly to v_2. Then the triangle 0, v_2, v_3 must be glued back on along the edge 0, v_{-3} according to the side identification σ_1. In the same way, the triangle v_{-1}, v_{-2}, v_{-3} must be glued by σ_3^{-1} to the side v_1, v_2; see Figure 5.
We now find the new parameters w_1, w_2, w_3 for the new polygon by analysing the vertices. We write the new vertices as v_j'. Then: $v_0' = v_0$, $v_1' = v_1$, $v_3' = v_2$. Thus:

$$\begin{aligned}
\frac{i\sin\theta\, w_1}{\sin\phi+\sin(\theta-\phi)} - \frac{i\sin\theta\, w_3}{\sin(\theta+\phi)} &= \frac{i\sin\theta\, z_1}{\sin\phi+\sin(\theta-\phi)} - \frac{i\sin\theta\, z_3}{\sin(\theta+\phi)},\\
ie^{-i\phi}w_1 - \frac{i\sin\theta\, w_3}{\sin(\theta+\phi)} &= ie^{-i\phi}z_1 - \frac{i\sin\theta\, z_3}{\sin(\theta+\phi)},\\
-ie^{-i\theta-i\phi}w_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}\, w_3}{\sin(\theta+\phi)} &= -ie^{-i\phi}z_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}\, z_3}{\sin\theta+\phi)}.
\end{aligned}$$

Solving these simultaneous equations gives the following:

$$w_1 = z_1, \quad w_2 = e^{i\theta}z_2, \quad w_3 = z_3.$$

In matrix form, R_1 is:

$$R_1 = \begin{bmatrix} 1 & 0 & 0\\ 0 & e^{i\theta} & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{4}$$

Since R_1 preserves the area of the octagon, it is unitary with respect to the Hermitian form H, that is $R_1^*HR_1 = H$. This can also be verified directly. (Recall H is given in equation (3).)

- **The move** R_2 The move R_2 is, in principle, very similar to R_1. However, in terms of coordinates it is more complicated. This move fixes 0, v_0 and $v_{\pm3}$ but interchanges $v_{\pm1}$ and $v_{\pm2}$. This corresponds to a Dehn twist along a simple closed curve through $v_{\pm1}$ and $v_{\pm2}$ that does not separate the other cone points. We obtain the octagon by cutting from 0 to $v_{\pm3}$, then to $v_{\pm1}$, to $v_{\pm2}$ and finally to v_0; see Figure 6.
Using cut and paste to obtain the new octagon from the old, we proceed as follows. The slit goes from 0 to v_3 and then directly to v_1. Hence the triangle v_1, v_2, v_3 should be glued by σ_2 to v_{-2}, v_{-3}. We also analyse the vertices to find the new coordinates; $v_0' = v_0$, $v_2' = v_1$, $v_3' = v_3$ as before:

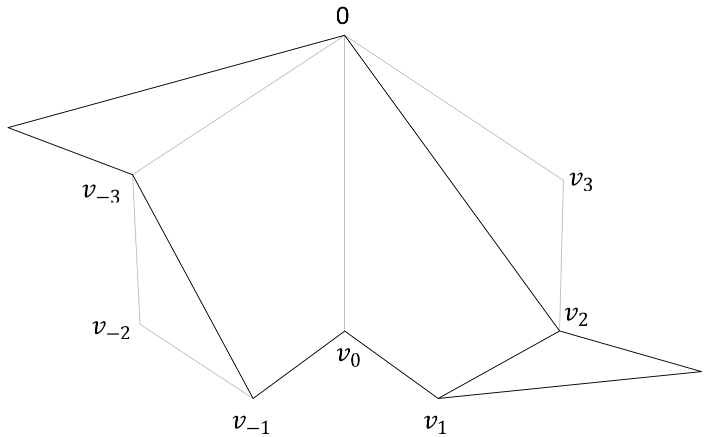

Figure 5. *The octagon obtained after performing move* R_1*.*

$$\begin{aligned}
\frac{i\sin\theta\, w_1}{\sin\phi + \sin(\theta-\phi)} - \frac{i\sin\theta\, w_3}{\sin(\theta+\phi)} &= \frac{i\sin\theta\, z_1}{\sin\phi + \sin(\theta-\phi)} - \frac{i\sin\theta\, z_3}{\sin(\theta+\phi)},\\
-ie^{-i\phi}w_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}\, w_3}{\sin\theta+\phi)} &= ie^{-i\phi}z_1 - \frac{i\sin\theta\, z_3}{\sin(\theta+\phi)},\\
-ie^{-i\theta-i\phi}w_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}\, w_3}{\sin(\theta+\phi)} &= -ie^{-i\theta-i\phi}z_2 + \frac{i\sin\phi\, e^{-i\theta-i\phi}\, z_3}{\sin(\theta+\phi)}
\end{aligned}$$

Solving these simultaneously results in the following matrix R_2 as the solution:

$$\begin{aligned}
&(1-e^{-i\theta})\sin(\phi)\, R_2\\
&= \begin{bmatrix} -\sin(\theta)e^{-i\phi} & -\sin(\phi)-\sin(\theta-\phi) & \sin(\phi)+\sin(\theta-\phi)\\ -\sin(\phi) & -\sin(\phi)e^{-i\theta} & \sin(\phi)\\ -\sin(\theta+\phi) & -\sin(\theta+\phi) & \sin(\phi)+\sin(\theta)e^{i\phi}\end{bmatrix}. \qquad (5)
\end{aligned}$$

Again, R_2 preserves area and so is unitary with respect to H.

- **The move** A_1 The third move is similar to the 'butterfly' move discussed by Thurston [14] and generalises the move I_1 in Parker [12]. In terms of monodromy, it is defined

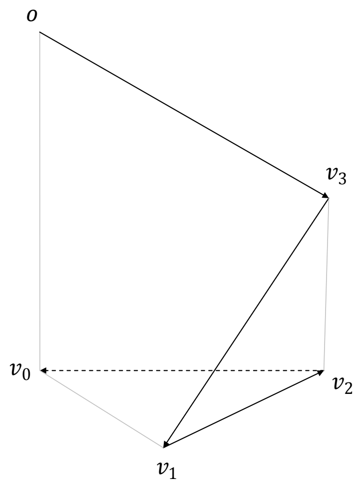

Figure 6. *The cut for move R_2*

in equations (5.2) and (5.3) and illustrated in Figure 1.5 of Mostow [11]. Thurston's butterfly operation moves one edge of the pentagon across a butterfly-shaped quadrilateral of zero signed area, yielding a new polygon of the same area. In our case, fix $v_{\pm 2}$, $v_{\pm 3}$ and we rotate the triangle T_1 so that $v_1' = v_{-1}$. The resulting octagon has a point of self intersection, but by using signed area we still preserve H. The move A_1 preserves the triangles T_2 and T_3 and so it fixes z_2 and z_3. The triangle T_1 is rotated by 2ϕ and so z_1 is sent to $e^{2i\phi}z_1$. That is, as a matrix, A_1 is given by:

$$A_1 = \begin{bmatrix} e^{2i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

Our goal will be to consider the group $\Gamma = \langle R_1, R_2, A_1 \rangle$ generated by the moves R_1, R_2 and A_1. We view these moves as the matrices given in (4), (5) and (6). All these moves preserve the (signed) area of Π and so the matrices are all unitary with respect to the Hermitian form given by the matrix H given in (3). We will show that Γ is discrete for the various values of θ and ϕ given in table (2).

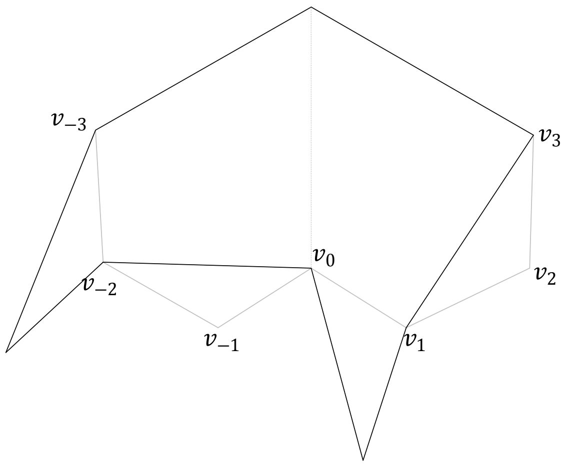

Figure 7. *The octagon obtained after performing move* R_2*.*

3 Coordinates and Special Points in the Space of Cone Metrics

3.1 Introduction. A key feature of our construction will be a good choice of coordinates on the space of polygons Π, that is the space of cone metrics on the sphere with our chosen cone angles. The initial set of coordinates are the parameters z_1, z_2 and z_3 used to describe the triangles T_1, T_2 and T_3 in the previous section. We have seen that with respect to these coordinates the automorphisms R_1 and A_1 are given by diagonal matrices. We shall give a second set of coordinates, called w_1, w_2 and w_3, with respect to which R_2 is given by a diagonal matrix. The interplay between these two sets of coordinates will help us construct the polyhedron D in the Section 4. Therefore we give the formulae for changing between the z_j and w_j coordinates. We will also be interested in certain degenerate cone metrics where two or more cone singularities come together to form a new one. These degenerate structures will be the vertices of our polyhedron. We conclude this section by describing these degenerate structures and giving then in terms of the z_j and w_j coordinates. This section is a transition from the cone metrics on the sphere discussed in Section 2 to the complex hyperbolic polyhedron and automorphisms discussed in the rest of the paper.

3.2 New Coordinates. Complex hyperbolic space can be defined to be the projectivisation of those points in the space of polygons with positive area. Since the area is given by a Hermitian form H of signature $(1,2)$, this is equivalent to the space of polygons for which the Hermitian H form is positive. We achieve the projectivisation by considering the section for which $z_3 = 1$. Using the definition of H in (3), we see that our model of

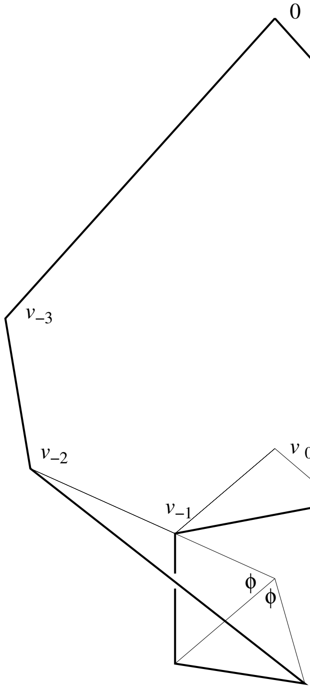

Figure 8. *The octagon obtained after performing move* A_1.

complex hyperbolic space is defined as

$$\mathbf{H}^2_{\mathbb{C}} = \left\{ \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} : \mathbf{z}^* H \mathbf{z} = \frac{-|z_1|^2 \sin\theta \sin\phi}{\big(\sin\phi + \sin(\theta-\phi)\big)} - |z_2|^2 \sin\theta + \frac{\sin\theta\sin\phi}{\sin(\theta+\phi)} > 0 \right\}. \tag{7}$$

There will be two elements of the group $\Gamma = \langle R_1, R_2, A_1\rangle$ that are of particular interest to us, namely $P = R_1R_2$ and $J = PA_1 = R_1R_2A_1$. Using the matrices for R_1, R_2 and A_1 in equations (4), (5) and (6), we find the following matrices for P and J:

$$\begin{aligned} &(1-e^{-i\theta})\sin(\phi)\,P \\ &= \begin{bmatrix} -\sin(\theta)e^{-i\phi} & -\sin(\phi)-\sin(\theta-\phi) & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi)e^{i\theta} & -\sin(\phi) & \sin(\phi)e^{i\theta} \\ -\sin(\theta+\phi) & -\sin(\theta+\phi) & \sin(\phi)+\sin(\theta)e^{i\phi} \end{bmatrix}, \end{aligned} \tag{8}$$

$$\begin{aligned} &(1-e^{-i\theta})\sin(\phi)\,J \\ &= \begin{bmatrix} -\sin(\theta)e^{i\phi} & -\sin(\phi)-\sin(\theta-\phi) & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi)e^{i(2\phi+\theta)} & -\sin(\phi) & \sin(\phi)e^{i\theta} \\ -\sin(\theta+\phi)e^{2i\phi} & -\sin(\theta+\phi) & \sin(\phi)+\sin(\theta)e^{i\phi} \end{bmatrix}. \end{aligned} \tag{9}$$

Note that $\mathrm{tr}(J) = 0$ and so, using Goldman's classification of elements of $\mathrm{SU}(1,2)$ by trace in Section 6.2.3 of [7], we see that J has order 3. Also, we have put the scalar factor of $(1 - e^{-i\theta})\sin(\phi)$ (which comes from R_2) on the left hand side of the equation. In our application we are only interested in projective classes of matrices. Therefore this factor may be dropped. We denote projective equality by $\sim$.

We remark that, using the action of R_1 and R_2 on the cone points, as discussed above, we can summarise the action of P on the cone points as follows:

Lemma 3.1. *The map $P = R_1R_2$ fixes the cone points 0 and v_0 and maps the other cone points as follows:*

$$P : v_{\pm 1} \longmapsto v_{\pm 3}, \qquad P : v_{\pm 2} \longmapsto v_{\pm 1}, \qquad P : v_{\pm 3} \longmapsto v_{\pm 2}.$$

We now define our second set of coordinates denoted by $\mathbf{w}$, which is the preimage under P of the first set of coordinates. Geometrically, the w_j-parameters have the following meaning. Before cutting the sphere to form an octagon, cyclically permute the cone points $v_{\pm j}$ as described in Lemma 3.1. The resulting octagon may be built up from three triangles in just the same way that Π was built up from the triangles T_1, T_2 and T_3. The coordinates w_1, w_2 and w_3 are then the bases of the new triangles. We consider the section with $w_3 = 1$.

In terms of parameters, the new coordinates are given by:

$$\begin{aligned} \mathbf{w} &= \begin{bmatrix} w_1 \\ w_2 \\ 1 \end{bmatrix} = \left[P^{-1}(\mathbf{z})\right] \\ &\sim \begin{bmatrix} -\sin(\theta)e^{i\phi} & -\big(\sin(\phi)+\sin(\theta-\phi)\big)e^{-i\theta} & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi) & -\sin(\phi) & \sin(\phi) \\ -\sin(\theta+\phi) & -\sin(\theta+\phi)e^{-i\theta} & \sin(\phi)+\sin(\theta)e^{-i\phi} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence finding w_1 and w_2 as rational functions of z_1 and z_2, we obtain

$$w_1 = \frac{-\sin(\theta)e^{i\phi}z_1 - \big(\sin(\phi)+\sin(\theta-\phi)\big)e^{-i\theta}z_2 + \sin(\phi)+\sin(\theta-\phi)}{-\sin(\theta+\phi)z_1 - \sin(\theta+\phi)e^{-i\theta}z_2 + \sin(\phi)+\sin(\theta)e^{-i\phi}}, \tag{10}$$

$$w_2 = \frac{-\sin(\phi)z_1 - \sin(\phi)z_2 + \sin(\phi)}{-\sin(\theta+\phi)z_1 - \sin(\theta+\phi)e^{-i\theta}z_2 + \sin(\phi)+\sin(\theta)e^{-i\phi}}. \tag{11}$$

By a similar procedure, we obtain z_1 and z_2 in terms of w_1 and w_2:

$$\begin{aligned} \mathbf{z} &= \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} = \left[P(\mathbf{w})\right] \\ &\sim \begin{bmatrix} -\sin(\theta)e^{-i\phi} & -\sin(\phi)-\sin(\theta-\phi) & \sin(\phi)+\sin(\theta-\phi) \\ -\sin(\phi)e^{i\theta} & -\sin(\phi) & \sin(\phi)e^{i\theta} \\ -\sin(\theta+\phi) & -\sin(\theta+\phi) & \sin(\phi)+\sin(\theta)e^{i\phi} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ 1 \end{bmatrix}. \end{aligned}$$

and hence

$$z_1 = \frac{-\sin(\theta)e^{-i\phi}w_1 - \big(\sin(\phi)+\sin(\theta-\phi)\big)w_2 + \sin(\phi)+\sin(\theta-\phi)}{-\sin(\theta+\phi)w_1 - \sin(\theta+\phi)w_2 + \sin(\phi)+\sin(\theta)e^{i\phi}}, \quad (12)$$

$$z_2 = \frac{-\sin(\phi)e^{i\theta}w_1 - \sin(\phi)w_2 + \sin(\phi)e^{i\theta}}{-\sin(\theta+\phi)w_1 - \sin(\theta+\phi)w_2 + \sin(\phi)+\sin(\theta)e^{i\phi}}. \quad (13)$$

Our reason of keeping track of two coordinates is that it gives a simple description of the polyhedron D in terms of the arguments of z_1, z_2, w_1 and w_2.

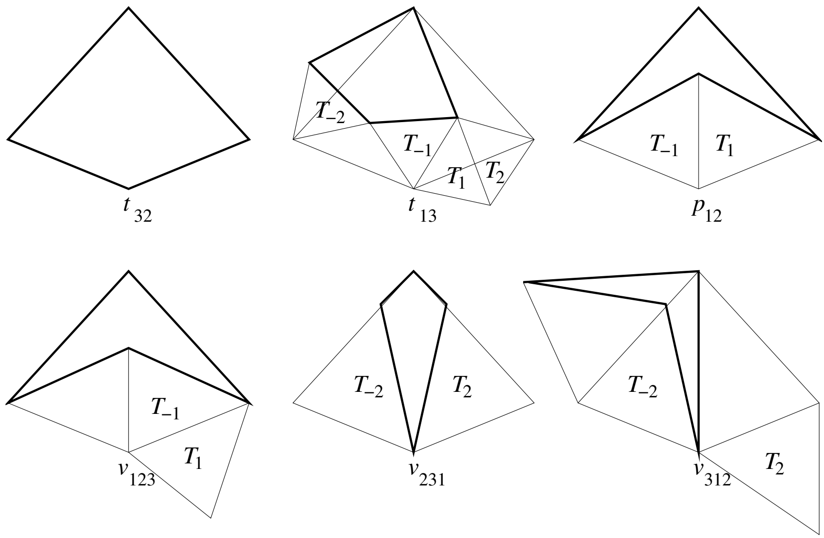

Figure 9. *Degenerate octagons corresponding to to vertices of D.*

3.3 Vertices. In this section, we obtain some distinguished points of $\mathbf{H}^2_{\mathbb{C}}$ which will be the vertices of our polyhedron. It will be useful to give these points in the two sets of coordinates $\mathbf{w}$ and $\mathbf{z}$ constructed in the previous section. The distinguished points (cone structures) are obtained by letting some of the cone points approach each other until in the limit they coalesce, and then result in a new point. The complementary angle (curvature) of this new cone point (that is 2π minus the cone angle) is the sum of the complementary angles of the cone points that have coalesced. Considering this from the view point of the octagon Π considered in Section 2, obtaining the new cone points is the same as either expanding or contracting the triangles T_1 and T_2 till some of the vertices become the same point. If such vertices are adjacent to each other then the edge between them has degenerated to a point. We define the following degenerate structures in terms of which cone points coalesce, and we give the cone angle at the resulting new vertex. The notation we use follows the notation established by Mostow in [9] and used by Deraux, Falbel and Paupert in [3].

Point	Cone Points	Angle	Cone Points	Angle
t_{32}	$v_0,\ v_{\pm1}$	2ϕ	$v_{\pm2},\ v_{\pm3}$	2θ
t_{13}	$v_0,\ v_{\pm3}$	2ϕ	$v_{\pm1},\ v_{\pm2}$	2θ
p_{12}	$v_{\pm1},\ v_{\pm2},\ v_{\pm3}$	$3\theta-\pi$		
v_{123}	$v_0,\ v_{\pm2},\ v_{\pm3}$	$\theta+2\phi-\pi$		
v_{231}	$v_0,\ v_{\pm1},\ v_{\pm2}$	$\theta+2\phi-\pi$		
v_{312}	$v_0,\ v_{\pm1},\ v_{\pm3}$	$\theta+2\phi-\pi$		

One can notice from the above table that $3\theta \geq \pi$ and $\theta+2\phi \geq \pi$. This will be the case for all the angles we are interested in; see Table (2). When $3\theta = \pi$ the vertex p_{12} will be on the boundary of complex hyperbolic space. Likewise, when $\theta+2\phi = \pi$ the vertices v_{312}, v_{123} and v_{231} will be on the boundary of complex hyperbolic space. We now describe the corresponding degenerate octagons in detail; see Figure 9.

t_{32}: (See page 226 of Mostow [9] or page 174 of Deraux, Falbel, Paupert [3].) When v_0 and $v_{\pm1}$ coalesce, the triangle T_1 shrinks to a point and so $z_1 = 0$. Likewise, when $v_{\pm2}$ and $v_{\pm3}$ coalesce then T_2 also shrinks to a point and so $z_2 = 0$. Thus t_{32} is given by $z_1 = z_2 = 0$. This is the origin in the z-coordinates. Putting $z_1 = z_2 = 0$ into (10) and (11) gives:

$$w_1 = \frac{\sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin\theta e^{-i\phi}}, \qquad w_2 = \frac{\sin\phi}{\sin\phi + \sin\theta e^{-i\phi}}.$$

t_{13}: When v_0 and v_3 coalesce and v_1 and v_2 coalesce, the triangles T_1 and T_2 share an edge. Also T_{-1} and T_{-2} share the vertex $v_{-1} = v_{-2}$. Alternatively, applying the map P permutes the cone points as in Lemma 3.1. This has the effect of making us use the w-coordinates. Therefore, we can repeat our argument for t_{32} to see that t_{13} corresponds to $w_1 = w_2 = 0$. Putting this into equations (12) and (13) gives:

$$z_1 = \frac{\sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin\theta e^{i\phi}}, \qquad z_2 = \frac{\sin\phi e^{i\phi}}{\sin\phi + \sin\theta e^{i\phi}}.$$

Note that (using $z_3 = 1$) when $v_{\pm1}$ and $v_{\pm2}$ coalesce we have $z_1 + z_2 = 1$. Likewise, when v_0 and v_3 coalesce we have

$$ie^{-i\phi} = \frac{i\sin\theta}{\sin\phi + \sin(\theta-\phi)} z_1 + ie^{-i\theta-i\phi} z_2. \tag{14}$$

These two equations yield the same solution for z_1 and z_2.

p_{12}: (See page 218 of Mostow [9] or page 166 of Deraux, Falbel, Paupert [3].) In this case, the cone points $v_{\pm1}$, $v_{\pm2}$ and $v_{\pm3}$ coalesce. Once again, as $v_{\pm1}$ and $v_{\pm2}$ coalesce we have $z_1 + z_2 = z_3 = 1$. Also, since $v_{\pm2}$ and $v_{\pm3}$ coalesce, the triangle T_2 shrinks to a point and we have $z_2 = 0$. Hence $z_1 = 1$. Either using Lemma 3.1 or substituting directly, we see that $w_1 = 1$ and $w_2 = 0$.

v_{123}: (See page 216 of Mostow [9] or page 170 of Deraux, Falbel, Paupert [3].) In this case v_0, $v_{\pm 2}$ and $v_{\pm 3}$ coalesce. Once again, when $v_{\pm 2}$ and $v_{\pm 3}$ coalesce we obtain $z_2 = 0$ and when v_0, v_3 coalesce we have equation (14). Putting this together gives

$$z_1 = \frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta e^{i\phi}}, \quad z_2 = 0.$$

In terms of the w-coordinates, when v_0 and v_3 coalesce then $w_1 = 0$ and when v_2 and v_3 coalesce we get $w_1 + e^{-i\theta} w_2 = 1$. Hence $w_1 = 0$ and $w_2 = e^{i\theta}$.

v_{231}: Here v_0, $v_{\pm 1}$ and $v_{\pm 2}$ coalesce. Arguing as before, we have $z_1 = 0$ and $z_1 + z_2 = 1$. Hence $z_2 = 1$. In terms of w-coordinates, (10) and (11) yield $w_2 = 0$ and

$$w_1 = \frac{(\sin\phi + \sin(\theta - \phi))(1 - e^{-i\theta})}{\sin\phi + \sin\theta e^{-i\phi} - \sin(\theta + \phi)e^{-i\theta}}.$$

Expanding, we see that $\sin\phi - \sin(\theta + \phi)e^{-i\theta} = -\sin\theta e^{-i\theta} e^{-i\phi}$. So cancelling $(1 - e^{-i\theta})$ from the top and bottom gives

$$w_1 = \frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta e^{-i\phi}}.$$

v_{312}: Here v_0, $v_{\pm 1}$ and $v_{\pm 3}$ coalesce. The above arguments yield $z_1 = 0$ and $z_1 + z_2 = e^{i\theta}$. Thus $z_2 = e^{i\theta}$. Also, $w_1 = 0$ and $w_1 + w_2 = 1$. Hence $w_2 = 1$.

In coordinates (normalising so $z_3 = w_3 = 1$) we have

Point	z_1	z_2	w_1	w_2
t_{32}	0	0	$\frac{\sin\phi+\sin(\theta-\phi)}{\sin\phi+\sin\theta e^{-i\phi}}$	$\frac{\sin\phi}{\sin\phi+\sin\theta e^{-i\phi}}$
t_{13}	$\frac{\sin\phi+\sin(\theta-\phi)}{\sin\phi+\sin\theta e^{i\phi}}$	$\frac{\sin\phi e^{i\theta}}{\sin\phi+\sin\theta e^{i\phi}}$	0	0
p_{12}	1	0	1	0
v_{123}	$\frac{\sin\phi+\sin(\theta-\phi)}{\sin\theta} e^{-i\phi}$	0	0	$e^{i\theta}$
v_{231}	0	1	$\frac{\sin\phi+\sin(\theta-\phi)}{\sin\theta} e^{i\phi}$	0
v_{312}	0	$e^{i\theta}$	0	1

In concluding this section, we show that the collection of vertices described above is symmetric with respect to an involution. The polyhedron D will also exhibit this symmetry when we get to Section 4. Let us consider the antiholomorphic isometry ι given by $\iota(\mathbf{z}) = R_1 R_2 R_1(\overline{\mathbf{z}})$, which is the same as $\iota(\mathbf{z}) = P R_1(\overline{\mathbf{z}})$. In coordinates:

$$\iota \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} \overline{w}_1 \\ \overline{w}_2 e^{i\theta} \\ 1 \end{bmatrix}. \tag{15}$$

Notice that $\sim$ refers to projective equality. The following lemma deduced from the above equation can be verified using the vertices obtained in the above table of vertices.

Lemma 3.2. *The isometry ι has order 2 and acts on the vertices by*

$$\iota(t_{32}) = t_{13}, \quad \iota(p_{12}) = p_{12}, \quad \iota(v_{123}) = v_{231}, \quad \iota(v_{312}) = v_{312}.$$

Proof. This follows by direct calculation. For example to see that ι fixes p_{12} observe

$$\iota\begin{bmatrix}1\\0\\1\end{bmatrix} = \frac{1}{(1-e^{-i\theta})\sin(\phi)}\begin{bmatrix}-\sin(\theta)e^{-i\phi}+\sin(\phi)+\sin(\theta-\phi)\\ -\sin(\phi)e^{i\theta}+\sin(\phi)e^{i\theta}\\ -\sin(\theta+\phi)+\sin(\phi)+\sin(\theta)e^{i\phi}\end{bmatrix} \sim \begin{bmatrix}1\\0\\1\end{bmatrix}.$$

where we have used

$$\sin(\theta-\phi)-\sin\theta e^{-i\phi} = -\cos\theta\sin\phi + i\sin\theta\sin\phi = -\sin(\theta+\phi)+\sin\theta e^{i\phi}.$$

Similarly to see that $\iota(v_{231}) = v_{123}$ observe

$$\iota\begin{bmatrix}0\\1\\1\end{bmatrix} = \frac{1}{(1-e^{-i\theta})\sin(\phi)}\begin{bmatrix}(1-e^{i\theta})(\sin\phi+\sin(\theta-\phi))\\ 0\\ \sin\phi+\sin\theta e^{i\phi}-\sin(\theta+\phi)e^{i\theta}\end{bmatrix} \sim \begin{bmatrix}\frac{\sin\phi+\sin(\theta-\phi)}{\sin\theta e^{i\phi}}\\ 0\\ 1\end{bmatrix}$$

where we have used

$$\sin\phi+\sin\theta e^{i\phi}-\sin(\theta+\phi)e^{i\theta} = \sin\theta e^{i\phi}(1-e^{i\theta}).$$

The other identities follow similarly by substituting and then simplifying using trigonometric formulae.

4 Construction of the Complex Hyperbolic Polyhedron D

4.1 Introduction. In this section we construct a polyhedron D in complex hyperbolic space. In the next section we will use Poincaré's polyhedron theorem to demonstrate that this is a fundamental polyhedron for $\Gamma = \langle R_1, R_2, A_1\rangle$. In particular, in Section 4.2 we prove Theorem 1.1 from the introduction.

The vertices of D will be the six special cone manifolds t_{32}, t_{13}, p_{12}, v_{123}, v_{231} and v_{312} constructed above in Section 3.3. Combinatorially, D is almost as simple as it can be: it will be the union of two four-simplices. The vertices of the first simplex are all the vertices except t_{13} and the vertices of the second are all except t_{32}.

The co-dimension 1 sides of D will be contained in bisectors. A bisector B is the locus of points equidistant from a given pair of points. They have been studied extensively and we will briefly summarise their properties. For more detail see Mostow [9] or Goldman [7]. If the bisector B is equidistant from points q_1 and q_2 then the complex line $\Sigma = \Sigma(B)$ spanned by q_1 and q_2 is called the *complex spine* of B. The geodesic $\sigma = \sigma(B)$ in Σ equidistant from q_1 and q_2 is called the *spine* of B. Bisectors are not totally geodesic but are foliated by totally geodesic subspaces in two different ways. First, if Π_Σ is orthogonal projection onto Σ then $B = \Pi_\Sigma^{-1}(\sigma)$. For each point s on σ, the fibre $\Pi_\Sigma^{-1}(\{s\})$ is a complex line, called a *slice* of Σ; see Mostow [9]. The slices foliate B. Secondly, B is the union of all totally real Lagrangian planes containing σ. Such a plane is called a *meridian*; see Goldman [7]. One property we will use is that each codimension 1 side of D will be a 3-simplex in a bisector with one edge in σ, two faces in meridians and one face in a slice. We will see, in Proposition 4.8, this has the consequence that the 1-skeleton of D is made up of geodesic arcs.

4.2 The polyhedron D. We define the polyhedron D to be those points of $\mathbf{H}^2_{\mathbb{C}}$ for which the arguments of z_1, z_2, w_1 and w_2 lie in the following intervals (compare Equation (17) of Parker [12]):

$$D = \left\{ \mathbf{z} = P(\mathbf{w}) \ : \ \begin{array}{ll} \arg(z_1) \in (-\phi, 0), & \arg(z_2) \in (0, \theta), \\ \arg(w_1) \in (0, \phi), & \arg(w_2) \in (0, \theta) \end{array} \right\}. \qquad (16)$$

We want to characterise D in terms of inequalities involving the Hermitian form. The following lemma generalises Lemmas 4.4 and 4.6 of [12].

Lemma 4.1. *Let $\langle \cdot, \cdot \rangle$ be the Hermitian form given by H. Using the coordinates $\mathbf{z}$ and $\mathbf{w}$ we have:*

- $\mathrm{Im}\,(z_1 e^{i\phi}) > 0$ *if and only if* $\big|\langle \mathbf{z}, p_{12}\rangle\big| < \big|\langle \mathbf{z}, J^{-1}(p_{12})\rangle\big|$;
- $\mathrm{Im}\,(z_1) < 0$ *if and only if* $\big|\langle \mathbf{z}, v_{123}\rangle\big| < \big|\langle \mathbf{z}, P^{-1}(v_{231})\rangle\big|$;
- $\mathrm{Im}\,(z_2) > 0$ *if and only if* $\big|\langle \mathbf{z}, v_{312}\rangle\big| < \big|\langle \mathbf{z}, R_1^{-1}(v_{231})\rangle\big|$;
- $\mathrm{Im}\,(z_2 e^{-i\theta}) < 0$ *if and only if* $\big|\langle \mathbf{z}, v_{231}\rangle\big| < \big|\langle \mathbf{z}, R_1(v_{312})\rangle\big|$;
- $\mathrm{Im}\,(w_1) > 0$ *if and only if* $\big|\langle \mathbf{w}, v_{231}\rangle\big| < \big|\langle \mathbf{w}, P(v_{123})\rangle\big|$;
- $\mathrm{Im}\,(w_1 e^{-i\phi}) < 0$ *if and only if* $\big|\langle \mathbf{w}, p_{12}\rangle\big| < \big|\langle \mathbf{w}, J(p_{12})\rangle\big|$;
- $\mathrm{Im}\,(w_2) > 0$ *if and only if* $\big|\langle \mathbf{w}, v_{123}\rangle\big| < \big|\langle \mathbf{w}, R_2^{-1}(v_{312})\rangle\big|$;
- $\mathrm{Im}\,(w_2 e^{-i\theta}) < 0$ *if and only if* $\big|\langle \mathbf{w}, v_{312}\rangle\big| < \big|\langle \mathbf{w}, R_2(v_{123})\rangle\big|$.

Proof. We illustrate one case of this theorem. In $\mathbf{z}$-coordinates

$$p_{12} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad J^{-1}(p_{12}) = \begin{bmatrix} e^{-2i\phi} \\ 0 \\ 1 \end{bmatrix}.$$

Therefore

$$\begin{aligned} \langle \mathbf{z}, p_{12}\rangle &= \frac{-\sin\phi}{\sin\phi + \sin(\theta-\phi)}\, z_1 + \frac{\sin\phi}{\sin(\theta+\phi)}, \\ \langle \mathbf{z}, J^{-1}(p_{12})\rangle &= \frac{-\sin\phi}{\sin\phi + \sin(\theta-\phi)}\, e^{2i\phi} z_1 + \frac{\sin\phi}{\sin(\theta+\phi)}. \end{aligned}$$

Hence $\big|\langle \mathbf{z}, p_{12}\rangle\big| < \big|\langle \mathbf{z}, J^{-1}(p_{12})\rangle\big|$ if and only if $-2\mathrm{Re}\,(z_1) < -2\mathrm{Re}\,(z_1 e^{2i\phi})$. This is true if and only if $\mathrm{Im}\,(z_1 e^{i\phi}) > 0$. This is the first part. The other parts are all similar.

In order to prove Theorem 1.1 stated in the introduction, we simply combine the description (16) of D and Lemma 4.1. For example, $\arg(z_1) \in (-\phi, 0)$ is equivalent to $\mathrm{Im}\,(z_1 e^{i\phi}) > 0$ and $\mathrm{Im}\,(z_1) < 0$. These two criteria are equivalent to $p \in \mathbf{H}^2_{\mathbb{C}}$ satisfying $\big|\langle p, p_{12}\rangle\big| < \big|\langle p, J^{-1}(p_{12})\rangle\big|$ and $\big|\langle p, v_{123}\rangle\big| < \big|\langle p, P^{-1}(v_{231})\rangle\big|$.

We refer to the codimension 1 facets of D as *sides*. Each side corresponds to one of the eight inequalities in Lemma 4.1 (or Theorem 1.1) being replaced with equality. Therefore each of the eight sides of D is contained in a bisector. Moreover, for each of of these bisectors B, we have:

(i) either the point t_{32} or t_{13} lies on B;
(ii) three of the four points p_{12}, v_{123}, v_{231} and v_{312} lie on B;
(iii) the fourth of these points lies on the complex spine Σ of B but not on B;
(iv) the spine σ of B passes through one of t_{32} and t_{13} and one of p_{12}, v_{123}, v_{231} or v_{312}.

We name the bisectors $B(X)$ where X is one of J, P, R_1, R_2 or their inverses so that the isometry X will send $B(X)$ to $B(X^{-1})$. For example, $B(R_1)$ is given by the equality

$$\big|\langle p, v_{312}\rangle\big| = \big|\langle p, R_1^{-1}(v_{231})\rangle\big|.$$

Applying R_1 sends v_{312} to $R_1(v_{312})$ and sends $R_1^{-1}(v_{231})$ to v_{231}. Therefore R_1 sends $B(R_1)$ to the bisector defined by

$$\big|\langle p, R_1(v_{312})\rangle\big| = \big|\langle p, v_{231}\rangle\big|.$$

This is $B(R_1^{-1})$.

The summary is:

Bisector	Definition	Equidistant from	Points on spine	Other points
$B(J)$	$\mathrm{Im}\,(z_1e^{i\phi})=0$	$p_{12},\ J^{-1}(p_{12})$	$t_{32},\ v_{123}$	$v_{231},\ v_{312}$
$B(J^{-1})$	$\mathrm{Im}\,(w_1e^{-i\phi})=0$	$p_{12},\ J(p_{12})$	$t_{13},\ v_{231}$	$v_{312},\ v_{123}$
$B(P)$	$\mathrm{Im}\,(z_1)=0$	$v_{123},\ P^{-1}(v_{231})$	$t_{32},\ p_{12}$	$v_{231},\ v_{312}$
$B(P^{-1})$	$\mathrm{Im}\,(w_1)=0$	$v_{231},\ P(v_{123})$	$t_{13},\ p_{12}$	$v_{312},\ v_{123}$
$B(R_1)$	$\mathrm{Im}\,(z_2)=0$	$v_{312},\ R_1^{-1}(v_{231})$	$t_{32},\ v_{231}$	$v_{123},\ p_{12}$
$B(R_1^{-1})$	$\mathrm{Im}\,(z_2e^{-i\theta})=0$	$v_{231},\ R_1(v_{312})$	$t_{32},\ v_{312}$	$v_{123},\ p_{12}$
$B(R_2)$	$\mathrm{Im}\,(w_2)=0$	$v_{123},\ R_2^{-1}(v_{312})$	$t_{13},\ v_{312}$	$v_{231},\ p_{12}$
$B(R_2^{-1})$	$\mathrm{Im}\,(w_2e^{-i\theta})=0$	$v_{312},\ R_2(v_{123})$	$t_{13},\ v_{123}$	$v_{231},\ p_{12}$

We go through these properties in two cases. The others are similar; see Boadi [1] for details.

(1) Consider $B(J)$. Then $z_1e^{i\phi}$ is real, so we write $z_1 = x_1e^{-i\phi}$ for $x_1 \in \mathbb{R}$. In **z**-coordinates $B(J)$ is given by

$$B(J) = \Big\{(x_1e^{-i\phi}, z_2) \in \mathbf{H}^2_{\mathbb{C}} \ :\ x_1 \in \mathbb{R},\ z_2 \in \mathbb{C}\Big\}.$$

The spine $\sigma(J)$ of $B(J)$ has **z**-coordinates

$$\sigma(J) = \Big\{(x_1e^{-i\phi}, 0) \in \mathbf{H}^2_{\mathbb{C}} \ :\ x_1 \in \mathbb{R}\Big\}.$$

The complex spine $\Sigma(J)$ of $B(J)$ has **z**-coordinates

$$\Sigma(J) = \Big\{(z_1, 0) \in \mathbf{H}^2_{\mathbb{C}} \ :\ z_1 \in \mathbb{C}\Big\}.$$

- t_{32} is given by $(z_1, z_2) = (0, 0)$. This clearly lies in $B(J)$ and $\sigma(J)$.
- p_{12} is given by $(z_1, z_2) = (1, 0)$ and $J^{-1}(p_{12})$ is given by $(z_1, z_2) = (e^{-2i\phi}, 0)$. These points clearly do not lie on $B(J)$ but do lie on $\Sigma(J)$, and $B(J)$ is equidistant from them.

- v_{123} is given by

$$(z_1, z_2) = \left(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta}\, e^{-i\phi}, 0 \right).$$

Since $\big(\sin\phi + \sin(\theta-\phi)\big)/\sin\theta$ is real, this lies on $B(J)$ and $\sigma(J)$.
- v_{231} is given by $(z_1, z_2) = (0, 1)$. This clearly lies on $B(J)$ but does not lie on $\sigma(J)$.
- v_{312} is given by $(z_1, z_2) = (0, e^{i\theta})$. This clearly lies on $B(J)$ but does not lie on $\sigma(J)$.

(2) Next, consider $B(R_2)$. Here w_2 is real, so in $\mathbf{w}$-coordinates $B(R_2)$ is given by

$$B(R_2) = \Big\{ (w_1, y_2) \in \mathbf{H}^2_{\mathbb{C}} \ :\ w_1 \in \mathbb{C},\ y_2 \in \mathbb{R} \Big\}.$$

The spine $\sigma(R_2)$ of $B(R_2)$ has $\mathbf{w}$ coordinates

$$\sigma(R_2) = \Big\{ (0, y_2) \in \mathbf{H}^2_{\mathbb{C}} \ :\ y_2 \in \mathbb{R} \Big\}.$$

The complex spine $\Sigma(R_2)$ of $B(R_2)$ has $\mathbf{w}$ coordinates

$$\Sigma(R_2) = \Big\{ (0, w_2) \in \mathbf{H}^2_{\mathbb{C}} \ :\ w_2 \in \mathbb{C} \Big\}.$$

- t_{13} is given by $(w_1, w_2) = (0, 0)$. This clearly lies in $B(R_2)$ and $\sigma(R_2)$.
- p_{12} is given by $(w_1, w_2) = (1, 0)$. This clearly lies in $B(R_2)$ but not in $\sigma(R_2)$.
- v_{231} is given by

$$(w_1, w_2) = \left(\frac{\sin\phi + \sin(\theta - \phi)}{\sin\theta}\, e^{i\phi}, 0 \right).$$

This clearly lies on $B(R_2)$ but not on $\sigma(R_2)$.
- v_{312} is given by $(w_1, w_2) = (0, 1)$. This lies on both $B(R_2)$ and $\sigma(R_2)$.
- v_{123} is given by $(w_1, w_2) = (0, e^{i\theta})$ and $R_2^{-1}(v_{312})$ is given by $(w_1, w_2) = (0, e^{-i\theta})$. These points clearly do not lie on $B(R_2)$. However, they do lie on $\Sigma(R_2)$, and $B(R_2)$ is equidistant from them.

4.3 Some useful inequalities. When we give details of the faces of D we will need to use the following lemmas which give inequalities satisfied by all points of $\mathbf{H}^2_{\mathbb{C}}$.

Lemma 4.2. *If* $\mathbf{z} \in \mathbf{H}^2_{\mathbb{C}}$ *then*

$$|z_1| < \frac{\sin\phi + \sin(\theta-\phi)}{\sin(\theta+\phi)}, \qquad |w_1| < \frac{\sin\phi + \sin(\theta-\phi)}{\sin(\theta+\phi)}.$$

Proof. We prove this by contradiction. If $|z_1| \geq \big(\sin\phi + \sin(\theta-\phi)\big)/\sin(\theta+\phi)$ then

$$\begin{aligned}
\mathbf{z}^* H\mathbf{z} &= \frac{-\sin\phi}{\sin\phi+\sin(\theta-\phi)}|z_1|^2 - |z_2|^2 + \frac{\sin\phi}{\sin(\theta+\phi)} \\
&\leq \frac{-\sin\phi\big(\sin\phi+\sin(\theta-\phi)\big)}{\sin^2(\theta+\phi)} - |z_2|^2 + \frac{\sin\phi}{\sin(\theta+\phi)} \\
&\leq \frac{\sin^2\phi\big(2\cos\theta-1\big)}{\sin^2(\theta+\phi)} - |z_2|^2 \leq 0,
\end{aligned}$$

where we have used $\cos\theta \leq 1/2$ on the last line. Thus $\mathbf{z}$ is not in $\mathbf{H}^2_{\mathbb{C}}$. A similar argument holds for $\mathbf{w}$.

Lemma 4.3. *If* $\mathbf{z} \in \mathbf{H}^2_{\mathbb{C}}$ *then*

$$|z_1|,\ |w_1| < \frac{\sin\theta}{\sin(\theta+\phi)}, \qquad |z_2|,\ |w_2| < \frac{\sin\phi}{\sin(\theta+\phi)},$$

Proof. In order to prove the lemma, first observe from the combination of ϕ and θ in Table (2) that we have $\theta + 2\phi \geq \pi$ and hence

$$\pi - (\theta+\phi) \leq \phi \leq \theta + \phi.$$

Therefore $\sin(\phi) \geq \sin(\theta+\phi)$.

Now we prove the inequality for z_1. If $|z_1| \geq \sin(\theta)/\sin(\theta+\phi)$ then from the area

$$\begin{aligned}
\mathbf{z}^* H\mathbf{z} &= \frac{-\sin\phi}{\sin\phi+\sin(\theta-\phi)}|z_1|^2 - |z_2|^2 + \frac{\sin\phi}{\sin(\theta+\phi)} \\
&\leq \frac{\sin\phi}{\sin(\theta+\phi)}\,\frac{\sin(\theta+\phi)\big(\sin\phi+\sin(\theta-\phi)\big)-\sin^2\theta}{\sin(\theta+\phi)\big(\sin\phi+\sin(\theta-\phi)\big)} - |z_2|^2 \\
&= \frac{\sin\phi}{\sin(\theta+\phi)}\,\frac{\sin(\theta+\phi)\sin\phi+\sin^2\theta\cos^2\phi-\cos^2\theta\sin^2\phi-\sin^2\theta}{\sin(\theta+\phi)\big(\sin\phi+\sin(\theta-\phi)\big)} - |z_2|^2 \\
&= \frac{\sin\phi}{\sin(\theta+\phi)}\,\frac{\sin(\theta+\phi)\sin\phi-\sin^2\phi}{\sin(\theta+\phi)\big(\sin\phi+\sin(\theta-\phi)\big)} - |z_2|^2 \\
&= -\frac{\sin^2\phi}{\sin^2(\theta+\phi)}\,\frac{\sin\phi-\sin(\theta+\phi)}{\sin\phi+\sin(\theta-\phi)} - |z_2|^2 \leq 0.
\end{aligned}$$

This a contradiction. Hence $|z_1| < \sin(\theta)/\sin(\theta+\phi)$. Proving the inequality for w_1 is identical.

Similarly, to prove the inequality for z_2 assume that $|z_2| \geq \sin\phi/\sin(\theta+\phi)$. Then

$$\begin{aligned}
\mathbf{z}^*H\mathbf{z} &= \frac{-\sin\phi}{\sin\phi+\sin(\theta-\phi)}|z_1|^2 - |z_2|^2 + \frac{\sin\phi}{\sin(\theta+\phi)} \\
&\leq \frac{-\sin\phi}{\sin\phi+\sin(\theta-\phi)}|z_1|^2 - \frac{\sin^2\phi}{\sin^2(\theta+\phi)} + \frac{\sin\phi}{\sin(\theta+\phi)} \\
&\leq \frac{-\sin\phi}{\sin\phi+\sin(\theta-\phi)}|z_1|^2 - \frac{\sin\phi}{\sin(\theta+\phi)}\left(\frac{\sin\phi}{\sin(\theta+\phi)} - 1\right) \\
&\leq 0,
\end{aligned}$$

as $\sin\phi/\sin(\theta+\phi) \geq 1$. The inequality for w_2 is similar.

4.4 Faces of the polyhedron. We refer to codimension 2 facets of the polyhedron D as *faces*. Each face is contained in the intersection of two of the bisectors defining the sides of D. The other bisectors determine further inequalities defining the edges bounding each face so that the face becomes a triangle in this bisector intersection. The faces come in three types: (a) faces contained in a common slice of two of the bisectors; (b) faces contained in a common meridian of two bisectors or (c) faces that are not contained in either a complex line or a Lagrangian plane. In fact the intersections of type (c) the two bisectors are coequidistant and so their intersection is contained in a *Giraud disc*, which is a particularly nice type of bisector intersection; see Theorem 8.3.3 of Goldman [7]. Specifically, for each of the four intersections contained in a Giraud disc, one of the vertices p_{12}, v_{123}, v_{231} or v_{312} is not contained in either bisector, but is contained in the intersection of their complex spines; see the table in Section 4.2.

We give representative examples of each type of bisector intersection. In Proposition 4.4 we give details of a face in a common slice, in Proposition 4.5 consider a face in a common meridian and in Propositions 4.6 and 4.7 we give two faces each in a Giraud disc. For the complete list, see Boadi's thesis [1].

First we give a face contained in a complex line that is a common slice of the two bisectors.

Proposition 4.4. *A point in the face of D contained in $B(J) \cap B(P)$ has coordinates $\mathbf{z} = (0, x + ye^{i\theta})$ and $\mathbf{w} = (w_1, w_2)$ with*

$$\begin{aligned}
w_1 &= \frac{(\sin(\phi)+\sin(\theta-\phi))(1-e^{-i\theta}x-y)}{\sin\phi+\sin\theta e^{-i\phi}-\sin(\theta+\phi)(e^{-i\theta}x+y)}, \\
w_2 &= \frac{\sin\phi(1-x-e^{i\theta}y)}{\sin\phi+\sin\theta e^{-i\phi}-\sin(\theta+\phi)(e^{-i\theta}x+y)},
\end{aligned}$$

where x, y are non-negative real numbers satisfying

$$0 \leq \sin\phi(1-x-y) - \sin(\theta+\phi)(x+y-x^2-y^2-2xy\cos\theta).$$

Proof. Since the point is on $B(J)$ we have $\mathrm{Im}\,(z_1e^{i\phi}) = 0$ and as it is on $B(P)$ we have $\mathrm{Im}\,(z_1) = 0$. Hence the intersection of these bisectors is the complex line $z_1 = 0$, which is a common slice. The conditions $\text{Im}\,(z_2) \geq 0$ and $\text{Im}\,(z_2 e^{-i\theta}) \leq 0$ mean that if we write $z_2 = x + ye^{i\theta}$ then $x \geq 0$ and $y \geq 0$. The expressions for w_1 and w_2 follow by substituting these values of z_1 and z_2 into (10) and (11).

It is not hard to check that $\text{Im}\,(w_1 e^{-i\phi}) \leq 0$ if and only if $\text{Im}\,(w_2) \geq 0$ if and only if

$$0 \leq \sin\phi(1 - x - y) - \sin(\theta+\phi)(x + y - x^2 - y^2 - 2xy\cos\theta).$$

Note that the curve given by equality in this expression intersects the line $y = 0$ (respectively $x = 0$) at $x = 1$ and $x = \sin\phi/\sin(\theta+\phi)$ (respectively $y = 1$ and $y = \sin\phi/\sin(\theta+\phi)$). The latter points are outside complex hyperbolic space, by Lemma 4.3. Therefore this curve and the lines $x = 0$, $y = 0$ bound a triangle with vertices $(x, y) = (0, 0)$, $(1, 0)$ and $(0, 1)$.

Finally, we must check that $\text{Im}\,(w_1) \geq 0$ and $\text{Im}\,(w_2 e^{-i\theta}) \leq 0$. It is easy to check that $\text{Im}\,(w_1) \geq 0$ if and only if

$$0 \leq 1 - y + x - 2\cos\theta x.$$

If $y = 1 + x - 2\cos\theta$ then

$$\begin{aligned} 0 &\leq \sin\phi(1 - x - y) - \sin(\theta+\phi)(x + y - x^2 - y^2 - 2xy\cos\theta) \\ &= -2(1 - \cos\theta)x(\sin\phi - \sin(\theta+\phi)x). \end{aligned}$$

Using Lemma 4.3 we see that $\sin\phi > \sin(\theta+\phi)x$ and so this expression is negative. Hence $\text{Im}\,(w_1 e^{-i\phi}) \leq 0$ implies $\text{Im}\,(w_1) \geq 0$.

Also, $\text{Im}\,(w_2 e^{-i\theta}) \leq 0$ if and only if

$$0 \leq (\sin\phi + \sin(\theta-\phi))(1 - x + y) - 2\cos\theta\sin\phi y.$$

A similar argument shows that $\text{Im}\,(w_2) \geq 0$ implies $\text{Im}\,(w_2 e^{-i\theta}) \leq 0$. We leave this to the reader.

Now we give a face contained in a Lagrangian plane, which is a common meridian of the two bisectors.

Proposition 4.5. *A point in the face of D contained in $B(J) \cap B(R_1)$ has coordinates $\mathbf{z} = (xe^{-i\phi}, y)$ and $\mathbf{w} = (w_1, w_2)$ with*

$$\begin{aligned} w_1 &= \frac{-\sin(\theta)x - \big(\sin(\phi) + \sin(\theta-\phi)\big)e^{-i\theta}y + \sin(\phi) + \sin(\theta-\phi)}{-\sin(\theta+\phi)e^{-i\phi}x - \sin(\theta+\phi)e^{-i\theta}y + \sin(\phi) + \sin(\theta)e^{-i\phi}}, \\ w_2 &= \frac{-\sin(\phi)e^{-i\phi}x - \sin(\phi)y + \sin(\phi)}{-\sin(\theta+\phi)e^{-i\phi}x - \sin(\theta+\phi)e^{-i\theta}y + \sin(\phi) + \sin(\theta)e^{-i\phi}}, \end{aligned}$$

where x and y are non-negative real numbers satisfying

$$0 \leq (\sin\phi + \sin(\theta-\phi))(1 - y) - \sin\theta x.$$

Proof. Since the point is on $B(J)$ we have Im $(z_1 e^{i\phi}) = 0$ and as it is on $B(R_1)$ we have Im $(z_2) = 0$. Hence the intersection of these bisectors is on the Lagrangian plane where $(z_1, z_2) = (xe^{-i\phi}, y)$ for real x and y. The conditions Im $(z_1) \leq 0$ and Im $(z_2 e^{-i\theta}) \leq 0$ imply $x \geq 0$ and $y \geq 0$. The expressions for w_1 and w_2 follow from (10) and (11).

It is not hard to check that Im $(w_1 e^{-i\phi}) \leq 0$ if and only if

$$0 \leq (\sin\phi - \sin(\theta+\phi)y)\big((\sin\phi + \sin(\theta-\phi))(1-y) - \sin\theta x\big).$$

Similarly Im $(w_2 e^{-i\theta}) \leq 0$ if and only if

$$0 \leq (\sin\theta - \sin(\theta+\phi)x)\big((\sin\phi + \sin(\theta-\phi))(1-y) - \sin\theta x\big).$$

Using Lemma 4.3 we see that $\sin\theta - \sin(\theta+\phi)x > 0$ and $\sin\phi - \sin(\theta+\phi)y > 0$. Finally, we must check Im $(w_1) \geq 0$ and Im $(w_2) \geq 0$. We leave this to the reader.

We now give details for a face contained in a Giraud disc. This disc is the intersection of $B(J)$, which is equidistant from p_{12} and $J(p_{12})$, and of $B(J^{-1})$, which is equidistant from p_{12} and $J(p_{12})$.

Proposition 4.6. *A point on the face of D contained in $\mathbf{z} \in B(J) \cap B(J^{-1})$ has coordinates $\mathbf{z} = (xe^{-i\phi}, z_2)$ and $\mathbf{w} = (ue^{i\phi}, w_2)$ where*

$$\begin{aligned} z_2 &= e^{i\theta}\frac{xu\sin(\theta+\phi) - u(e^{i\phi}\sin\phi + \sin\theta) - x\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - ue^{i\phi}\sin(\theta+\phi)}, \\ w_2 &= \frac{xu\sin(\theta+\phi) - x(e^{-i\phi}\sin\phi + \sin\theta) - u\sin\theta + \sin\phi + \sin(\theta-\phi)}{\sin\phi + \sin(\theta-\phi) - x\sin(\theta+\phi)e^{-i\phi}}, \end{aligned}$$

where x and u are non-negative real numbers satisfying

$$0 \leq (x\sin\theta - \sin\phi - \sin(\theta+\phi))(u\sin\theta - \sin\phi - \sin(\theta+\phi)) - xu\sin^2\phi.$$

Proof. Arguing as before, since the point is on $B(J)$ we have $z_1 = xe^{-i\phi}$ and as it is on $B(J)$ we have $w_1 = ue^{i\phi}$ for real x and u. Since Im $(z_1) \leq 0$ and Im $(w_1) \geq 0$ we have $x \geq 0$ and $y \geq 0$. Substituting these in (12) and (10) gives:

$$\begin{aligned} xe^{-i\phi} &= \frac{-u\sin\theta - (\sin\phi + \sin(\theta-\phi))w_2 + \sin\phi + \sin(\theta-\phi)}{-u\sin(\theta+\phi)e^{i\phi} - \sin(\theta+\phi)w_2 + \sin\phi + \sin\theta e^{i\phi}}, \\ ue^{i\phi} &= \frac{-x\sin\theta - e^{-i\theta}(\sin\phi + \sin(\theta-\phi))z_2 + \sin\phi + \sin(\theta-\phi)}{-x\sin(\theta+\phi)e^{-i\phi} - \sin(\theta+\phi)e^{-i\theta}z_2 + \sin\phi + e^{-i\phi}\sin\theta}. \end{aligned}$$

Solving for w_2 and z_2 gives the expressions in the statement of the proposition. The condition Im $(z_2) \geq 0$ is equivalent to

$$0 \leq \big(\sin\theta - u\sin(\theta+\phi)\big)p(u,x)$$

where

$$p(u,x) = (x\sin\theta - \sin\phi - \sin(\theta+\phi))(u\sin\theta - \sin\phi - \sin(\theta+\phi)) - xu\sin^2\phi.$$

The condition that $\mathrm{Im}\,(w_2e^{-i\theta}) \le 0$ is equivalent to

$$0 \le \big(\sin\theta - x\sin(\theta+\phi)\big)p(u,x).$$

From Lemma 4.3 we see that $\sin\theta - u\sin(\theta+\phi) > 0$ and $\sin\theta - x\sin(\theta+\phi) > 0$, thus we must have $p(u,x) \ge 0$ as claimed. Finally we must check $\mathrm{Im}\,(z_2e^{-i\theta}) \le 0$ and $\mathrm{Im}\,(w_2) \ge 0$. We leave this to the reader.

A similar method proves the following result, which describes another Giraud disc. This is the face in the intersection of $B(P)$, which is equidistant from v_{123} and $P^{-1}(v_{231})$, and of $B(R_2)$, which is equidistant from v_{123} and $R_2^{-1}(v_{312})$.

Proposition 4.7. *A point on the face of D contained in $\mathbf{z} \in B(P) \cap B(R_2)$ has coordinates $\mathbf{z} = (x, z_2)$ and $\mathbf{w} = (w_1, u)$ where*

$$\begin{aligned} z_2 &= \frac{\sin(\phi)(1-x) - (\sin\phi + \sin\theta e^{-i\phi})u + \sin(\theta+\phi)xu}{\sin\phi - \sin(\theta+\phi)e^{-i\theta}u}, \\ w_1 &= \frac{(\sin\phi + \sin(\theta-\phi))(1-u) - (\sin\phi + \sin\theta e^{i\phi})x + \sin(\theta+\phi)xu}{\sin\theta e^{-i\phi} - \sin(\theta+\phi)x}, \end{aligned}$$

where x and u are non-negative real numbers satisfying $u + x \le 1$.

We conclude this section by listing all the faces. We use the notation that $F(X,Y)$ is the face of D contained in the intersections of the bisectors $B(X)$ and $B(Y)$. In the following table, the letters S, M, G in the last column indicate whether the face is in a slice, a meridian or a Giraud disc.

Face	Vertices	Coordinates	
$F(J,J^{-1})$	$v_{312},\ v_{123},\ v_{231}$	$\mathrm{Im}\,(z_1e^{i\phi}) = \mathrm{Im}\,(w_1e^{-i\phi}) = 0$	G
$F(J,P)$	$t_{32},\ v_{231},\ v_{312}$	$\mathrm{Im}\,(z_1e^{i\phi}) = \mathrm{Im}\,(z_1) = 0$	S
$F(J,R_1)$	$t_{32},\ v_{231},\ v_{123}$	$\mathrm{Im}\,(z_1e^{i\phi}) = \mathrm{Im}\,(z_2) = 0$	M
$F(J,R_1^{-1})$	$t_{32},\ v_{312},\ v_{123}$	$\mathrm{Im}\,(z_1e^{i\phi}) = \mathrm{Im}\,(z_2e^{-i\theta}) = 0$	M
$F(J^{-1},P^{-1})$	$t_{13},\ v_{312},\ v_{123}$	$\mathrm{Im}\,(w_1e^{-i\phi}) = \mathrm{Im}\,(w_1) = 0$	S
$F(J^{-1},R_2)$	$t_{13},\ v_{231},\ v_{312}$	$\mathrm{Im}\,(w_1e^{-i\phi}) = \mathrm{Im}\,(w_2) = 0$	M
$F(J^{-1},R_2^{-1})$	$t_{13},\ v_{231},\ v_{123}$	$\mathrm{Im}\,(w_1e^{-i\phi}) = \mathrm{Im}\,(w_2e^{-i\theta}) = 0$	M
$F(P,R_1)$	$t_{32},\ v_{231},\ p_{12}$	$\mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2) = 0$	M
$F(P,R_1^{-1})$	$t_{32},\ v_{312},\ p_{12}$	$\mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2e^{-i\theta}) = 0$	M
$F(P,R_2)$	$v_{312},\ v_{231},\ p_{12}$	$\mathrm{Im}\,(z_1) = \mathrm{Im}\,(w_2) = 0$	G
$F(P^{-1},R_1^{-1})$	$v_{312},\ v_{123},\ p_{12}$	$\mathrm{Im}\,(w_1) = \mathrm{Im}\,(z_2e^{-i\theta}) = 0$	G
$F(P^{-1},R_2)$	$t_{13},\ v_{312},\ p_{12}$	$\mathrm{Im}\,(w_1) = \mathrm{Im}\,(w_2) = 0$	M
$F(P^{-1},R_2^{-1})$	$t_{13},\ v_{123},\ p_{12}$	$\mathrm{Im}\,(w_1) = \mathrm{Im}\,(w_2e^{-i\theta}) = 0$	M
$F(R_1,R_1^{-1})$	$t_{32},\ v_{123},\ p_{12}$	$\mathrm{Im}\,(z_2) = \mathrm{Im}\,(z_2e^{-i\theta}) = 0$	S
$F(R_1,R_2^{-1})$	$v_{123},\ v_{231},\ p_{12}$	$\mathrm{Im}\,(z_2) = \mathrm{Im}\,(w_2e^{-i\theta}) = 0$	G
$F(R_2,R_2^{-1})$	$t_{13},\ v_{231},\ p_{12}$	$\mathrm{Im}\,(w_2) = \mathrm{Im}\,(w_2e^{-i\theta}) = 0$	S

4.5 Other facets of D. We have discussed the vertices and faces of D, that is the facets of dimension 0 and 2. In this section we discuss the rest of the facets of D, that is the edges and sides of dimension 1 and 3 respectively. Figures 10 and 11 show the sides of D and their 1-skeletons. We begin with the *edges*, that is facets of dimension 1. We refer to the edge joining vertices a and b as $\gamma(a,b)$.

Proposition 4.8. *Each edge $\gamma(a,b)$ of D is a geodesic segment joining a pair of the vertices a and b.*

Proof. Each edge will be contained in either three or four of the faces listed in the previous section. We refer to faces contained in a common slice or a common meridian as S-faces or M-faces respectively. We now list the edges together with the faces containing them.

Edge	S-face	M-face	M-face
$\gamma(t_{32},v_{312})$	$F(J,P)$	$F(J,R_1^{-1})$	$F(P,R_1^{-1})$
$\gamma(t_{32},v_{123})$	$F(R_1,R_1^{-1})$	$F(J,R_1)$	$F(J,R_1^{-1})$
$\gamma(t_{32},v_{231})$	$F(J,P)$	$F(J,R_1)$	$F(P,R_1)$
$\gamma(t_{32},p_{12})$	$F(R_1,R_1^{-1})$	$F(P,R_1)$	$F(P,R_1^{-1})$
$\gamma(t_{13},v_{312})$	$F(J^{-1},P^{-1})$	$F(J^{-1},R_2)$	$F(P^{-1},R_2)$
$\gamma(t_{13},v_{123})$	$F(J^{-1},P^{-1})$	$F(J^{-1},R_2^{-1})$	$F(P^{-1},R_2^{-1})$
$\gamma(t_{13},v_{231})$	$F(R_2,R_2^{-1})$	$F(J^{-1},R_2)$	$F(J^{-1},R_2^{-1})$
$\gamma(t_{13},p_{12})$	$F(R_2,R_2^{-1})$	$F(P^{-1},R_2)$	$F(P^{-1},R_2^{-1})$
$\gamma(p_{12},v_{312})$		$F(P,R_1^{-1})$	$F(P^{-1},R_2)$
$\gamma(p_{12},v_{123})$	$F(R_1,R_1^{-1})$	$F(P^{-1},R_2^{-1})$	
$\gamma(p_{12},v_{231})$	$F(R_2,R_2^{-1})$	$F(P,R_1)$	
$\gamma(v_{312},v_{123})$	$F(J^{-1},P^{-1})$	$F(J,R_1^{-1})$	
$\gamma(v_{123},v_{231})$		$F(J,R_1)$	$F(J^{-1},R_2^{-1})$
$\gamma(v_{231},v_{312})$	$F(J,P)$	$F(J^{-1},R_2)$	

Each edge is contained in at least two totally geodesic faces, that is either a complex line or Lagrangian plane. This means that the edge is a geodesic segment. In particular:

(1) Each edge ending at either t_{32} or t_{13} is on one S-face and two M-faces. Moreover, the two M-faces are contained in meridians of the same bisector. Therefore this edge is a geodesic segment contained in the spine of this bisector.
(2) Each edge of the other edges is either in two M-faces or an S-face and an M-face (as well as two faces in Giraud discs that we do not list here).

This completes the proof.

Let $S(X)$ denote the side (that is codimension 1 facet) of D contained in the bisector $B(X)$. A consequence of the analysis we have done is:

Proposition 4.9. *Each side $S(X)$ of D is a 3-simplex (solid tetrahedron) contained in the bisector $B(X)$. It has one face contained in a slice of $B(X)$ and two faces contained in meridians of $B(X)$, intersecting in an arc of the spine $\sigma(X)$. The fourth face is contained in a Giraud disc.*

We also have a proposition describing the faces more precisely:

Proposition 4.10. *Each face $F(X,Y)$ of D is a 2-simplex (solid triangle) homeomorphic to an closed disc in $\mathbb{R}^2$ and the boundary of $F(X,Y)$ is made up of three geodesic segments joining the vertices.*

Note that in Section 4.4 there were some pairs of sides whose intersections we did not consider. Each of these pairs of sides intersect in and edge. For completeness, we now list them.

Side	Side	Edge
$S(J)$	$S(P^{-1})$	$\gamma(v_{123}, v_{312})$
$S(J)$	$S(R_2)$	$\gamma(v_{231}, v_{312})$
$S(J)$	$S(R_2^{-1})$	$\gamma(v_{123}, v_{231})$
$S(J^{-1})$	$S(P)$	$\gamma(v_{231}, v_{312})$
$S(J^{-1})$	$S(R_1)$	$\gamma(v_{123}, v_{231})$
$S(J^{-1})$	$S(R_1^{-1})$	$\gamma(v_{123}, v_{312})$
$S(P)$	$S(P^{-1})$	$\gamma(p_{12}, v_{312})$
$S(P)$	$S(R_2^{-1})$	$\gamma(p_{12}, v_{231})$
$S(P^{-1})$	$S(R_1)$	$\gamma(p_{12}, v_{123})$
$S(R_1)$	$S(R_2)$	$\gamma(p_{12}, v_{231})$
$S(R_1^{-1})$	$S(R_2)$	$\gamma(p_{12}, v_{312})$
$S(R_1^{-1})$	$S(R_2^{-1})$	$\gamma(p_{12}, v_{123})$

In order to see that these intersections are one dimensional, we follow a similar method to the one used to prove Propositions 4.4 to 4.6. For example, consider $S(P) \cap S(P^{-1})$. By definition, we have $z_1 = x$ and $w_1 = u$ where x and u are positive real numbers. It is not hard to see that for such a point

$$\begin{aligned}
\operatorname{Im}(z_2 e^{-i\theta}) &= \frac{\sin\theta\sin\phi(u-x)}{\sin\phi + \sin(\theta-\phi) - \sin(\theta+\phi)u}, \\
\operatorname{Im}(w_2) &= \frac{\sin\theta\sin\phi(u-x)}{\sin\phi + \sin(\theta-\phi) - \sin(\theta+\phi)x}.
\end{aligned}$$

Since $\operatorname{Im}(z_2 e^{-i\theta}) \le 0$ and $\operatorname{Im}(w_2) \ge 0$ we must have $u = x$. Substituting this back into the expressions for z_2 and w_2, it is not hard to see that we must have $0 \le x \le 1$.

5 Proof that D is a fundamental Polyhedron for the Group

5.1 Introduction. In this section we show that the group Γ generated by R_1, R_2 and A_1 is discrete group with fundamental polyhedron D. To do so, we use Poincaré's polyhedron theorem. For accounts of Poincaré's polyhedron theorem see Parker [12] or Mostow [9]. An account of Poincaré's theorem in the constant curvature setting is given by Epstein and Petronio [4]. Our proof is based on Parker's proof of a similar result for Livné's lattices. In particular, we use the statement of Poincaré's theorem as given in Section 4.1 of Parker

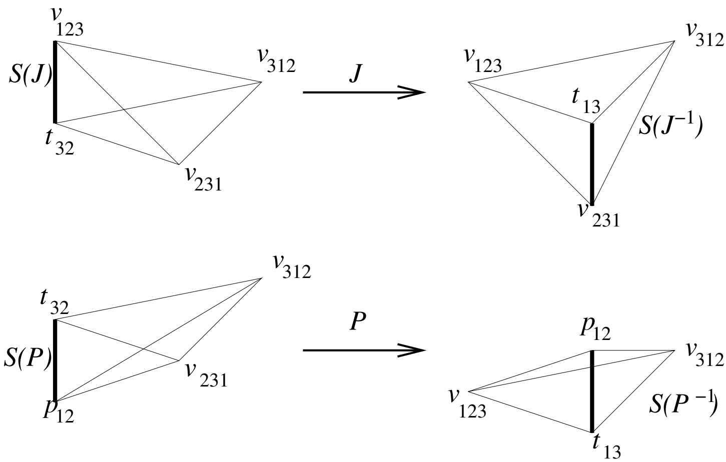

Figure 10. *The side pairing maps J and P. The solid lines denote edges in the spine of the relevant bisector.*

[12]. We will indicate how the proof goes and give sufficient details for the reader to be able to reconstruct out arguments, but we do not give full details, which are given in Boadi's thesis [1].

Our main theorem is

Theorem 5.1. *Let R_1, R_2, P and J be given by* (4), (5), (8) *and* (9) *respectively. The subgroup Γ of $\mathrm{PU}(H)$ generated by these maps is discrete and the polyhedron D constructed in Section* 4 *is a fundamental polyhedron for Γ. Moreover, Γ has the following presentation:*

$$\Gamma = \left\langle J, P, R_1, R_2 : \begin{array}{c} J^3 = R_1^p = R_2^p = (P^{-1}J)^k = I, \\ R_2 = PR_1P^{-1} = JR_1J^{-1},\ P = R_1R_2 \end{array} \right\rangle \tag{17}$$

where the values of p and k are given by:

p	3	3	3	4	4	5	5	6	6
k	4	5	6	3	4	2	3	2	3

We remark that, since $P = R_1R_2$ and $J = R_1R_2A_1$ the group Γ is the same as $\langle R_1,\ R_2,\ A_1\rangle$. Substituting this into the presentation (17) gives:

$$\Gamma = \left\langle R_1, R_2, A_1 : \begin{array}{c} R_1^p = R_2^p = A_1^k = (R_1R_2A_1)^3 = I, \\ R_1R_2R_1 = R_2R_1R_2,\ R_1A_1 = A_1R_1 \end{array} \right\rangle.$$

5.2 The side pairing maps. In this section we verify the side pairing conditions. There are eight sides of D, namely $S(J^{\pm 1})$, $S(P^{\pm 1})$, $S(R_1^{\pm 1})$ and $S(R_2^{\pm 1})$ as constructed in

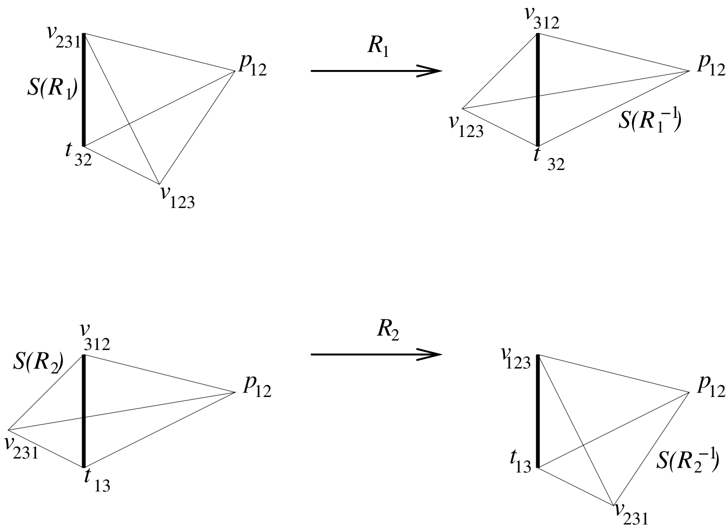

Figure 11. *The side pairing maps R_1 and R_2. The solid lines denote edges in the spine of the relevant bisector.*

Section 4. These sides are paired by the maps J, P, R_1 and R_2 as described in the following proposition. Recall that D, as defined in (16), is an open polyhedron. We use the convention that that sides, faces, edges and vertices are closed sets.

Proposition 5.2. *Let X be one of $J^{\pm 1}$, $P^{\pm 1}$, $R_1^{\pm 1}$ or $R_2^{\pm 1}$. Then:*

(i) X sends $S(X)$ bijectively to $S(X^{-1})$ sending vertices, edges and faces to vertices, edges and faces respectively.

(ii) $X^{-1}(D) \cap D = \emptyset$ and $X^{-1}(\overline{D}) \cap \overline{D} = S(X)$.

Proof. For each X the side $S(X)$ is contained in a bisector $B(X)$. By the construction of $B(X)$ in Section 4, for each X there are vertices a and b (possibly equal) so that a is not on $B(X)$ and $B(X)$ is equidistant from a and $X^{-1}(b)$. Moreover, D is contained in the half-space closer to a than to $X^{-1}(b)$.

Applying X we see that $X\big(B(X)\big)$ is equidistant from $X(a)$ and b. In other words, $X\big(B(X)\big) = B(X^{-1})$. Moreover $X(D)$ is contained in the half-space closer to $X(a)$ than to b. This is the opposite half-space to D. Thus $X(D) \cap D = \emptyset$ and $X(\overline{D}) \cap \overline{D} \subset B(X^{-1})$.

The rest of the proposition follows from the fact that $\overline{D} \cap B(X) = S(X)$, which comes out of our construction using the inequalities from Theorem 1.1.

The side pairing conditions (S.1), (S.2), (S.3) and (S.4) of Poincaré's theorem as given in Section 4.1 of Parker [12] follow directly from Proposition 5.2. The condition (S.5) saying D has finitely many faces, each with finite combinatorics follows from Proposition

4.9. Finally, the condition (S.6) is vacuous in this case, since every pair of sides intersects in at least an edge. Thus we have verified the side pairing conditions.

5.3 Face cycles. In this section we verify the face conditions of Poincaré's theorem. For each face $F(X,Y^{-1}) = S(X)\cap S(Y^{-1})$ we construct an element of Γ called a cycle transformation as follows. Let $X = X_1$. We know from Proposition 5.2 that $X = X_1$ sends faces of $S(X_1)$ to faces of $S(X_1^{-1})$. Suppose that

$$X_1\big(F(X_1,Y^{-1})\big) = F(X_2,X_1^{-1}) = S(X_2)\cap S(X_1^{-1}),$$

a face of $S(X_1^{-1})$. We can repeat this process, and consider $X_2\big(F(X_2,X_1^{-1})\big)$, which is a face $F(X_3,X_2^{-1})$ of $S(X_2^{-1})$. Eventually, we will find an n so that $X_n = Y$ and $Y\big(F(Y,X_{n-1}^{-1})\big) = F(X,Y^{-1})$, the first face we considered. In other words we have

$$F(X_1,X_n^{-1}) \xrightarrow{X_1} F(X_2,X_1^{-1}) \xrightarrow{X_2} \cdots \xrightarrow{X_{n-1}} F(X_n,X_{n-1}) \xrightarrow{X_n} F(X_1,X_n^{-1}).$$

The collection of faces $F(X_i,X_{i-1}^{-1})$, for $i=1$ to n with $X_0 = X_n$, is called the *face cycle* associated to $F(X_1,X_n^{-1}) = F(X,Y^{-1})$ and the composition $T = X_n\circ\cdots\circ X_2\circ X_1$ is called the *cycle transformation* associated to $F(X,Y^{-1})$.

Proposition 5.3. *Let $F(X_1,X_n^{-1})$ be a face of D with associated cycle transformation $T = X_n\circ\cdots\circ X_2\circ X_1$. Then:*

(i) There is an integer ℓ so that the restriction of T^ℓ to $F(X_1,X_n^{-1})$ is the identity.

(ii) There is an integer m so that $T^{\ell m} = (T^\ell)^m$ is the identity on the whole space.

(iii) For $i=0,\ldots,n-1$ and $j=0,\ldots,\ell m-1$ the images $T^{-j}\circ X_1^{-1}\circ\cdots\circ X_i^{-1}(D)$ are disjoint. (Here $i=0$ means the identity and so $T^{-j}\circ X_1^{-1}\circ\cdots\circ X_i^{-1}(D) = D$ when $i=j=0$.)

(iv) The union over $i=0,\ldots,n-1$ and $j=0,\ldots,\ell m-1$

$$\bigcup_{i,j} T^{-j}\circ X_1^{-1}\circ\cdots\circ X_i^{-1}(\overline{D})$$

covers a neighbourhood of the interior of $F(X_1,X_n^{-1})$.

Proof. We list the face cycles and the cycle transformation associated to the first face in the cycle and the integers l and m.

Face cycle				Transformation	ℓ	m
$F(J,J^{-1})$				J	3	1
$F(J,P)$	$F(P^{-1},J^{-1})$			$P^{-1}J$	1	k
$F(J,R_1)$	$F(R_2,J^{-1})$	$F(J^{-1},R_2^{-1})$	$F(R_1^{-1},J)$	$R_1^{-1}J^{-1}R_2J$	1	1
$F(P,R_1)$	$F(R_2,P^{-1})$	$F(P^{-1},R_2^{-1})$	$F(R_1^{-1},P)$	$R_1^{-1}P^{-1}R_2P$	1	1
$F(R_1,R_1^{-1})$				R_1	1	p
$F(R_2,R_2^{-1})$				R_2	1	p
$F(R_1,R_2^{-1})$	$F(P^{-1},R_1^{-1})$	$F(R_2,P)$		$R_2P^{-1}R_1$	1	1

This proves (i) and (ii). We now prove (iii) and (iv) in the cases where $F(X_1, X_n^{-1})$ is contained in a Giraud disc, a slice and a meridian. We do a single example in each case.

First consider faces contained in Giraud discs. We will give the details for the face $F(J, J^{-1})$. It is defined by

$$\big|\langle p, p_{12}\rangle\big| = \big|\langle p, J^{-1}(p_{12})\rangle\big| = \big|\langle p, J(p_{12}\rangle\big|.$$

There are three sectors around this face, each where one of these quantities is smallest. First, using Theorem 1.1, D is contained in the sector with

$$\big|\langle p, p_{12}\rangle\big| < \big|\langle p, J^{-1}(p_{12})\rangle\big|, \quad \big|\langle p, p_{12}\rangle\big| < \big|\langle p, J(p_{12})\rangle\big|.$$

Thus $J(D)$ is contained in the sector where

$$\big|\langle p, J(p_{12})\rangle\big| < \big|\langle p, p_{12}\rangle\big|, \quad \big|\langle p, J(p_{12})\rangle\big| < \big|\langle p, J^2(p_{12})\rangle\big|.$$

As J has order 3 we see that $J^2 = J^{-1}$. Likewise $J^{-1}(D)$ is contained in the sector where

$$\big|\langle p, J^{-1}(p_{12})\rangle\big| < \big|\langle p, J(p_{12})\rangle\big|, \quad \big|\langle p, J^{-1}(p_{12})\rangle\big| < \big|\langle p, p_{12}\rangle\big|.$$

It is easy to see that these three sectors are disjoint and that their closures cover a neighbourhood of $B(J) \cap B(J^{-1})$. Adding the extra inequalities defining the boundary of $F(J, J^{-1})$, we see that D, $J(D)$ and $J^{-1}(D)$ cover a neighbourhood of the interior of $F(J, J^{-1})$. This works because p_{12} is the intersection of the complex spines $\Sigma(J)$ and $\Sigma(J^{-1})$ of $B(J)$ and $B(J^{-1})$. For the other faces in Giraud discs we must modify the inequalities in Theorem 1.1 accordingly.

Now consider the face $F(J, P)$ contained in the complex line $z_1 = 0$. The polyhedron D is contained in the sector where $\arg(z_1) \in (-\phi, 0)$. Also note that D is contained in the sector $0 < \arg(w_1) < \phi$. since $\mathbf{w} = P^{-1}(\mathbf{z})$, we see $P^{-1}(D)$ is contained in the sector $\arg(z_1) < (0, \phi)$. Since $P^{-1}J = A_1$, which multiplies z_1 by $e^{2i\phi}$ we see that $J^{-1}(D) = (P^{-1}J)^{-1}P^{-1}(D)$ is contained in the sector with $\arg(z_1) < (-2\phi, -\phi)$. Continuing in this way we see see:

Image of D	Sector containing $\arg(z_1)$
$(P^{-1}J)^{-j}(D)$	$(-2j-1)\phi < \arg(z_1) < -2j\phi$
$(P^{-1}J)^{-j}J^{-1}(D)$	$(-2j-2)\phi < \arg(z_1) < (-2j-1)\phi$

Since $\phi = 2\pi/k$, clearly these sectors are disjoint and the union as j varies from 0 to $k-1$ of their closures covers a neighbourhood of $B(J) \cap B(P)$. Adding in the other inequalities shows that a neighbourhood of the interior of $F(J, P)$ is covered by

$$\bigcup_{j=0}^{k-1}\Big((P^{-1}J)^{-j}(\overline{D}) \cup (P^{-1}J)^{-j}J^{-1}(\overline{D})\Big).$$

Consider the face $F(J, R_1)$ contained in the Lagrangian plane where $\mathrm{Im}\,(z_1 e^{i\phi}) = 0$ and $\mathrm{Im}\,(z_2) = 0$. Similar arguments to those given above show that D is contained in the quadrant where $\mathrm{Im}\,(z_1 e^{i\phi}) > 0$ and $\mathrm{Im}\,(z_2) > 0$; $J^{-1}(D)$ is contained in the quadrant where $\mathrm{Im}\,(z_1 e^{i\phi}) < 0$ and $\mathrm{Im}\,(z_2) > 0$; $J^{-1}R_2^{-1}(D)$ is contained in the quadrant where $\mathrm{Im}\,(z_1 e^{i\phi}) < 0$ and $\mathrm{Im}\,(z_2) < 0$ and $J^{-1}R_2^{-1}J(D)$ is contained in the quadrant where $\mathrm{Im}\,(z_1 e^{i\phi}) > 0$ and $\mathrm{Im}\,(z_2) < 0$. Arguing as above, we see that these images of D are disjoint and their closures cover a neighbourhood of the interior of $F(J, R_1)$. (Compare this to Proposition 4.8 of Parker [12].)

Therefore we have proved the face conditions (F.1), (F.2) and (F.3) given in Section 4.1 of Parker [12]. Condition (F.1) was verified in Proposition 4.10, and conditions (F.2) and (F.3) are verified in Proposition 5.3.

We also have to be careful in the case where one of the vertices is on the ideal boundary. In this case, we must show that there is a *consistent horosphere* about this point; see Epstein and Petronio [4] or Section 5.1 of Falbel-Parker [6]. (This was not necessary for the groups considered in [12] as the lattices were cocompact.) In other words, there is a horosphere based at this point that is preserved under its stabiliser and mapped off itself by all other group elements. We show that the stabiliser of any of the vertices of D is generated by elliptic maps. Thus, when this vertex lies on the boundary of complex hyperbolic space, the stabiliser preserves all horospheres centred at this point. Since D has finitely many faces, we may shrink such a horosphere until it is disjoint from all its images.

(1) It is clear that t_{32} and t_{13} are never on the ideal boundary.

(2) The vertex p_{12} is on the ideal boundary exactly when $\theta = 2\pi/p = \pi/3$ and so $p = 6$. The stabiliser of p_{12} is generated by R_1 and R_2. Since these maps are elliptic they preserve all horospheres centred at p_{12}.

(3) The vertices v_{312}, v_{123} and v_{231} lie on the ideal boundary exactly when

$$\theta + 2\phi = 2\pi/p + 2\pi/k = \pi$$

and so (p, k) is one of $(3, 6)$, $(4, 4)$ or $(6, 3)$. The stabiliser of v_{231} is generated by R_2 and A_1. Since these maps are elliptic they preserve all horospheres centred at p_{12}.

We have verified the hypotheses of Poincaré's polyhedron theorem. Our only omission is that we have not fully proved the details of the intersection of D with the eight bisectors and the the intersections of each pair of sides and we have not proved all the details of face cycles and tessellation around faces. In Section 4.4 and in Proposition 5.3 we have given full details of the methods we use to analyse these sides, their intersections, and face cycles. We have given the details in representative cases. In all other cases, the details are straightforward to verify along the lines we have indicated.

Therefore the group generated by the side pairing transformations J, P, R_1 and R_2 is discrete, D is a fundamental polyhedron and the relations are generated by the cycle relations $T^{\ell m} = I$ for each face $F(X, Y^{-1})$ (there are no reflection relations in this case). Therefore the relations are generated by

$$J^3 = (P^{-1}J)^k = R_1^{-1}J^{-1}R_2J = R_1^{-1}P^{-1}R_2P = R_1^p = R_2^p = R_2P^{-1}R_1 = I.$$

It is clear that these may be rewritten to give the relations in (17). Therefore we have proved Theorem 5.1.

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