

## Flowing from $\text{AdS}_5$ to $\text{AdS}_3$ with $T^{1,1}$

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**ABSTRACT:** We construct supersymmetric domain wall solutions of type IIB supergravity that interpolate between  $\text{AdS}_5 \times T^{1,1}$  in the UV and  $\text{AdS}_3 \times \mathbb{R}^2 \times S^2 \times S^3$  solutions in the IR. The  $\mathbb{R}^2$  factor can be replaced with a two-torus and then the solution describes a supersymmetric flow across dimensions, similar to wrapped brane solutions. While the domain wall solutions preserve  $(0, 2)$  supersymmetry, the  $\text{AdS}_3$  solutions in the IR have an enhanced  $(4, 2)$  superconformal supersymmetry and are related by two T-dualities to the  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  type IIB solutions which preserve a large  $(4, 4)$  superconformal supersymmetry. The domain wall solutions exist within the  $N = 4$   $D = 5$  gauged supergravity theory that is obtained from a consistent Kaluza-Klein truncation of type IIB supergravity on  $T^{1,1}$ ; a feature driving the flows is that two  $D = 5$  axion like fields, residing in the  $N = 4$  Betti multiplet, depend linearly on the two legs of the  $\mathbb{R}^2$  factor.

**KEYWORDS:** Gauge-gravity correspondence, AdS-CFT Correspondence, Supersymmetric gauge theory

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## 1 Introduction

Type II string theory possesses  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  solutions which are supported by magnetic three-form fluxes threading the  $S^3$  factors as well as electric three-form flux threading the  $S^3 \times S^3 \times S^1$  factor [1–5]. While these solutions have been known for a long time the dual field theory, which preserves a large  $(4, 4)$  super conformal symmetry, remains elusive. A detailed discussion of some of the issues is presented in [6] and we note that a recent proposal for the dual field theory appears in [7]. In this paper we will discuss some new results on these type II solutions, using a rather indirect approach.

Starting with the  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  solutions of type IIB string theory with the magnetic and electric three-form fluxes in the RR sector, we can carry out two T-dualities on two circles to obtain other type IIB solutions with an  $\text{AdS}_3$  factor. If we choose one of the two circles to be the explicit  $S^1$  factor and the other to be a diagonal of the two Hopf fibres of the  $S^3 \times S^3$ , then we obtain the  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  solutions with non-trivial RR five-form and three-form fluxes as well as NS three-form flux that were first

found in [8]. Here we will show that, as solutions of type IIB supergravity, these solutions preserve  $(4, 2)$  superconformal symmetry and not just the  $(0, 2)$  superconformal symmetry that was guaranteed from the original construction of [8].

A principal result of this paper is the construction of type IIB supergravity domain-wall solutions that interpolate between the  $\text{AdS}_5 \times T^{1,1}$  solution in the UV and approach these  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  solutions in the IR. The flow solutions preserve  $(0, 2)$  Poincaré symmetry which is enhanced to  $(4, 2)$  superconformal symmetry at the IR fixed point. The supergravity solutions are constructed directly in type IIB supergravity. However, they can also be constructed in an  $N = 4$   $D = 5$  gauged supergravity theory that can be obtained as a consistent KK truncation of type IIB supergravity on  $T^{1,1}$  [9–11] (extending [12–17]). This perspective is helpful in identifying the deformations of the  $N = 1$  SCFT that are needed to flow to the  $\text{AdS}_3$  fixed points.

As is well known, the  $\text{AdS}_5 \times T^{1,1}$  UV fixed point is dual to an  $N = 1$  SCFT in  $D = 4$  that arises on D3-branes sitting at the apex of the conifold [18]. Our constructions can be viewed as a variation of wrapped-brane solutions [19] (see [20] for a review and [21–23] for recent constructions with  $\text{AdS}_3$  factors), with the D3-branes wrapping a  $T^2$  and sitting at the apex of the conifold with particular deformations switched on. In particular, an important ingredient is that there are two axion like fields in the Betti multiplet of the gauged-supergravity which are linear in the  $T^2$  directions. This mechanism for preservation of supersymmetry differs from the usual one of activating  $R$ -symmetry currents, related to the spin connection of the cycle being wrapped, and also the constructions of [24, 25] where there are magnetic fluxes threading a  $T^2$ . In the most general solutions that we construct here, though, there is also a magnetic flux of the Betti vector field threading the  $T^2$  factor.

We also analyse the flux-quantisation for the  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  fixed point solutions. This turns out to be somewhat subtle due to the presence of Page charges. While the quantisation of Page charges have been discussed before [26–29], analysing our solutions reveals some new issues, which will also arise in the context of other classes of solutions. We explain our prescription for quantising the Page charges and use this to obtain the central charge of the dual SCFT.

The above discussion focussed on solutions that flow from  $\text{AdS}_5 \times T^{1,1}$  to  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$ . However, if one does not compactify two spatial dimensions then one has solutions flowing from  $\text{AdS}_5 \times T^{1,1}$  to  $\text{AdS}_3 \times \mathbb{R}^2 \times S^2 \times S^3$ . Such solutions may have interesting applications in the context of applied AdS/CFT, where there has been various studies on the emergence of  $\text{AdS}_3$  solutions after switching on magnetic fields, including examples preserving supersymmetry [24, 25]. Our solutions, utilising axions, provide an alternative approach<sup>1</sup> to hitting such fixed points. It is also worth commenting that our type IIB domain wall solutions share some similarities with supersymmetric solutions of  $D = 11$  supergravity that interpolate between  $\text{AdS}_4 \times Q^{111}$  in the UV and supersymmetric  $\text{AdS}_2 \times \mathbb{R}^2 \times S^2 \times S^2 \times S^2 \times S^1$  solutions in the IR [36]; a difference, however, is that those flows were driven by electric and magnetic baryonic fluxes.

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<sup>1</sup>Axions/massless fields that are linear in either null or spatial coordinates have been used in applied AdS/CFT in other works including [30–35].

The plan of the rest of the paper is as follows. In section 2 we describe the supersymmetric domain wall solutions in the simplest setting and then generalise them to a one-parameter family of flows in section 3. We briefly conclude in section 4. We have three appendices. In appendix A we review the charge quantisation of the  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  solutions. In appendix B we demonstrate that the fixed point solutions preserve (4, 2) supersymmetry and in appendix C we discuss some additional aspects of the quantisation of Page charges.

## 2 A flow from $\text{AdS}_5 \times T^{1,1}$ to $\text{AdS}_3 \times \mathbb{R}^2 \times S^2 \times S^3$

### 2.1 General set-up

We will construct supersymmetric solutions of type IIB supergravity [37, 38] using the conventions given in [39]. We will consider solutions with trivial ten dimensional axion and dilaton and hence the R-R and NS-NS three-forms can be combined into the complex three-form  $G = -dB - idC$ , where  $B, C$  are both two-forms. The Bianchi identities for  $G$  and the self-dual five-form  $F$ , satisfying  $F = *F$ , are given by

$$dG = 0, \quad dF = \frac{i}{2}G \wedge G^*, \quad (2.1)$$

while the equation of motion for  $G$  can be written

$$\nabla^\mu G_{\mu\nu\rho} = -\frac{i}{6}F_{\nu\rho\sigma_1\sigma_2\sigma_3}G^{\sigma_1\sigma_2\sigma_3}. \quad (2.2)$$

The Killing spinor equations take the form

$$\nabla_\mu \varepsilon + \frac{i}{16} \not{F} \Gamma_\mu \varepsilon + \frac{1}{16} (\Gamma_\mu \not{G} + 2\not{G} \Gamma_\mu) \varepsilon^c = 0, \quad (2.3)$$

$$\not{G} \varepsilon = 0. \quad (2.4)$$

We begin by recalling the standard  $\text{AdS}_5 \times T^{1,1}$  solution [40] of type IIB supergravity. The metric and the self-dual five-form are given by

$$\begin{aligned} \frac{1}{L^2} ds^2 &= e^{2\rho} (-dt^2 + dx^2 + dx_1^2 + dx_2^2) + d\rho^2 + \frac{1}{6}(ds_1^2 + ds_2^2) + \eta^2, \\ \frac{1}{L^4} F &= 4e^{4\rho} dt \wedge dx \wedge dx_1 \wedge dx_2 \wedge d\rho + \frac{1}{9}\eta \wedge \text{vol}_1 \wedge \text{vol}_2, \end{aligned} \quad (2.5)$$

where we have defined

$$\begin{aligned} ds_i^2 &= (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2), & \text{vol}_i &= \sin \theta_i d\theta_i \wedge d\phi_i, \\ \eta &= \frac{1}{3}(d\psi + P), & P &= P_1 + P_2, & P_i &= -\cos \theta_i d\phi_i, \end{aligned} \quad (2.6)$$

and  $dP_i = \text{vol}_i$ . Note that  $\eta$  is the Reeb one-form,  $\partial_\psi$  is the Reeb Killing vector and the period of  $\psi$  is  $4\pi$ . Also,  $L$  is a constant length scale fixed by flux quantisation (given in (2.26) below). This solution preserves four Poincaré and four superconformal supersymmetries. It is useful to record the explicit form of the Poincaré supersymmetries. Using the obvious

orthonormal frame (see (2.10) below) the Poincaré supersymmetries satisfy the following algebraic conditions<sup>2</sup>

$$\begin{aligned} i\Gamma^{0123}\varepsilon &= -\varepsilon, \\ \Gamma^{56}\varepsilon &= i\varepsilon, \quad \Gamma^{78}\varepsilon = i\varepsilon, \quad \Gamma^{49}\varepsilon = i\varepsilon. \end{aligned} \tag{2.7}$$

These conditions are equivalent to  $\Gamma^{5678}\varepsilon = -\varepsilon$ ,  $\Gamma^{5649}\varepsilon = -\varepsilon$ , corresponding to the conifold (the Calabi-Yau cone over  $T^{1,1}$ ), combined with  $i\Gamma^{0123}\varepsilon = -\varepsilon$  corresponding to putting a D3-brane at its apex. The four Poincaré Killing spinors can be written  $\varepsilon = e^{\rho/2}\varepsilon_0$  where  $\varepsilon_0$  satisfies

$$\hat{\nabla}_m \varepsilon_0 - \frac{1}{2}\Gamma^4 \Gamma_m \varepsilon_0 = 0, \tag{2.8}$$

where  $\hat{\nabla}$  is the Levi-Civita connection on  $T^{1,1}$  with coordinates  $y^m$ .

We are interested in constructing supersymmetric domain walls that approach  $\text{AdS}_5 \times T^{1,1}$  in the UV, and flow to particular  $\text{AdS}_3 \times M_7$  solutions in the IR. The ansatz that we shall consider first is given by

$$\begin{aligned} \frac{1}{L^2} ds^2 &= e^{2A}(-dt^2 + dx^2) + e^{2B}(dx_1^2 + dx_2^2) + d\rho^2 + \frac{1}{6}e^{2U}(ds_1^2 + ds_2^2) + e^{2V}\eta^2, \\ \frac{1}{L^4} F &= 4e^{2A+2B-V-4U} dt \wedge dx \wedge dx_1 \wedge dx_2 \wedge d\rho + \frac{1}{9}\eta \wedge \text{vol}_1 \wedge \text{vol}_2, \\ &+ \frac{\lambda^2}{12} [dx_1 \wedge dx_2 \wedge \eta \wedge (\text{vol}_1 + \text{vol}_2) + e^{2A-2B-V} dt \wedge dx \wedge d\rho \wedge (\text{vol}_1 + \text{vol}_2)], \\ \frac{1}{L^2} G &= \frac{\lambda}{6} (dx_1 - idx_2) \wedge (\text{vol}_1 - \text{vol}_2), \end{aligned} \tag{2.9}$$

where  $\lambda$  is a constant and  $A, B, U, V$  are functions of  $\rho$  only. We will discuss the dual SCFT interpretation of this ansatz in section 2.3 and discuss a generalisation in section 3.

Observe that, by construction, the ansatz has a self-dual five-form,  $F = *F$ , and that both the Bianchi identities (2.1) and the equation of motion for  $G$  (2.2) are satisfied. To analyse the conditions for preservation of supersymmetry, (2.3) and (2.4), we use the orthonormal frame

$$\begin{aligned} e^0 &= e^A dt, & e^1 &= e^A dx, & e^2 &= e^B dx_1, & e^3 &= e^B dx_2, & e^4 &= d\rho, \\ e^5 &= \frac{e^U}{\sqrt{6}} d\theta_1, & e^6 &= \frac{e^U}{\sqrt{6}} \sin\theta_1 d\phi_1, & e^7 &= \frac{e^U}{\sqrt{6}} d\theta_2, & e^8 &= \frac{e^U}{\sqrt{6}} \sin\theta_2 d\phi_2, & e^9 &= e^V \eta. \end{aligned} \tag{2.10}$$

We will continue to impose the algebraic conditions (2.7) and we will also impose  $\Gamma^{23}\varepsilon = i\varepsilon$  or equivalently

$$\Gamma^{01}\varepsilon = \varepsilon, \tag{2.11}$$

corresponding to a chiral  $(0, 2)$  Poincaré supersymmetry in  $d = 1 + 1$ . It is straightforward to see that (2.4) is automatically satisfied while equation (2.3) reduces to

$$\nabla_\mu \varepsilon + \frac{i}{16} \not{F} \Gamma_\mu \varepsilon = 0. \tag{2.12}$$

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<sup>2</sup>Note that our conventions are such that  $\Gamma_{0123456789}\varepsilon = -\varepsilon$  and also  $\varepsilon_{0123456789} = +1$ .

A calculation now shows that we can solve (2.12) provided that we choose  $\varepsilon = e^{A/2}\varepsilon_0$  with  $\varepsilon_0$  satisfying (2.8), and that the functions  $A, B, U, V$  satisfy the following coupled first order differential equations

$$\begin{aligned} A' - e^{-V-4U} - \frac{\lambda^2}{4}e^{-2B-V-2U} &= 0, \\ B' - e^{-V-4U} + \frac{\lambda^2}{4}e^{-2B-V-2U} &= 0, \\ U' + e^{-V-4U} - e^{V-2U} &= 0, \\ V' - 3e^{-V} + 2e^{V-2U} + e^{-V-4U} + \frac{\lambda^2}{4}e^{-2B-V-2U} &= 0. \end{aligned} \tag{2.13}$$

Since our ansatz satisfies  $F = *F$ , the Bianchi identities (2.1) and the equation of motion for  $G$  (2.2), we can conclude from the result in appendix D of [39], that any solution to these differential equations will also solve the type IIB Einstein equations and hence gives rise to a supersymmetric solution of type IIB supergravity preserving at least two supersymmetries.

We immediately recover the  $\text{AdS}_5 \times T^{1,1}$  solution (2.6) by setting  $\lambda = 0$  and

$$A = B = \rho, \quad U = V = 0. \tag{2.14}$$

When  $\lambda \neq 0$ , it is convenient to scale the coordinates  $x_i$  and shift the function  $B$  by a constant to set  $\lambda = 2$ , without loss of generality. We then find another exact solution to (2.13) corresponding to a solution with an  $\text{AdS}_3$  factor:

$$A = \frac{3^{3/4}}{\sqrt{2}}\rho, \quad B = \frac{1}{4}\ln\left(\frac{4}{3}\right), \quad U = \frac{1}{4}\ln\left(\frac{4}{3}\right), \quad V = -\frac{1}{4}\ln\left(\frac{4}{3}\right). \tag{2.15}$$

Indeed, if we substitute this solution into the ansatz (2.9) and scale  $x_i = \frac{1}{6^{1/2}}z_i$  we find that the metric can be written as

$$\begin{aligned} \frac{1}{L^2}ds^2 &= \frac{1}{3^{3/2}}\left(2ds^2(\text{AdS}_3) + dz_1^2 + dz_2^2 + ds_1^2 + ds_2^2 + \frac{1}{2}(d\psi + P)^2\right), \\ \frac{1}{L^4}F &= \frac{1}{27}\left(\text{vol}(\text{AdS}_3) \wedge [4dz_1 \wedge dz_2 + 2(\text{vol}_1 + \text{vol}_2)] \right. \\ &\quad \left. + (d\psi + P) \wedge \left[\text{vol}_1 \wedge \text{vol}_2 + \frac{1}{2}dz_1 \wedge dz_2 \wedge (\text{vol}_1 + \text{vol}_2)\right]\right), \\ \frac{1}{L^2}G &= \frac{1}{3^{3/2}2^{1/2}}(dz_1 - idz_2) \wedge (\text{vol}_1 - \text{vol}_2). \end{aligned} \tag{2.16}$$

This solution was first found<sup>3</sup> in section 3.1.2 of [8]. Observe that the topology of the internal five-dimensional compact space is unchanged from that of  $T^{1,1}$ , namely  $S^2 \times S^3$ . Thus the topology of the  $D = 10$  solution is  $\text{AdS}_3 \times \mathbb{R}^2 \times S^2 \times S^3$ , or, if we take  $x_i$  (or equivalently the  $z_i$ ) to parametrise a two-torus,  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$ .

<sup>3</sup>To compare we should set, in the notation of [8],  $l_1 = l_2 = 1$ ,  $m_1 = 1/2$  and also identify  $\psi = z$  and  $L^2 = (3^{3/2}/2^{1/2})L_{\text{there}}^2$ .

By construction this AdS<sub>3</sub> solution preserves (0, 2) Poincaré supersymmetry and this is supplemented by a further two supersymmetries to give (0, 2) superconformal symmetry. In fact this was already known from the construction in [8]. However, as we show in appendix B, and further discuss in section 2.5, the fixed point actually preserves an enhanced (4, 2) superconformal supersymmetry (i.e. twelve supersymmetries in total).

## 2.2 The supersymmetric flow

We would now like to construct, numerically, a supersymmetric flow from the AdS<sub>5</sub> × T<sup>1,1</sup> solution to the AdS<sub>3</sub> × ℝ<sup>2</sup> × S<sup>2</sup> × S<sup>3</sup> solution (2.16). We will develop a series expansion of the differential equations (2.13) about both the AdS<sub>5</sub> UV fixed point (2.14) and the AdS<sub>3</sub> IR fixed point (2.15) and then use a shooting technique to match them. We again set λ = 2.

By expanding about the AdS<sub>5</sub> UV fixed point (2.14) we can develop the following expansion as ρ → ∞:

$$\begin{aligned}
 A &= \rho - \frac{5}{12}e^{-2\rho} + \frac{287}{1152}e^{-4\rho} - \frac{5953}{34560}e^{-6\rho} + \dots, \\
 B &= \rho + \frac{7}{12}e^{-2\rho} - \frac{385}{1152}e^{-4\rho} + \frac{8267}{34560}e^{-6\rho} + \dots, \\
 U &= \frac{1}{12}e^{-2\rho} - \frac{13}{96}e^{-4\rho} + c_1e^{-6\rho} + \frac{3}{20}e^{-6\rho}\rho + \dots, \\
 V &= -\frac{1}{6}e^{-2\rho} + \frac{37}{96}e^{-4\rho} + \left(\frac{9023}{51840} - 4c_1\right)e^{-6\rho} - \frac{3}{5}e^{-6\rho}\rho + \dots.
 \end{aligned} \tag{2.17}$$

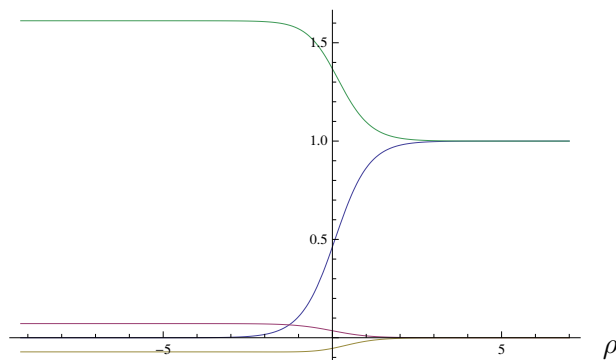
Here we have used the freedom to shift A by a constant in (2.13) to eliminate an integration constant. Notice that the expansion depends on one constant c<sub>1</sub>. We will comment on the dual D = 4 SCFT interpretation of this UV expansion in the next subsection.

We now consider the expansion about the AdS<sub>3</sub> fixed point (2.15). We find that as ρ → -∞ it is fixed by three integration constants, a<sub>0</sub>, s<sub>1</sub> and s<sub>2</sub>:

$$\begin{aligned}
 A &= a_0 + \rho/R + \frac{3s_1}{2}e^{\delta_1\rho/R} \dots + \frac{1}{4}(-3 + \sqrt{5})s_2e^{\delta_2\rho/R} + \dots, \\
 B &= \frac{1}{4}\ln\left(\frac{4}{3}\right) + s_1e^{\delta_1\rho/R} + \dots + s_2e^{\delta_2\rho/R} + \dots, \\
 U &= \frac{1}{4}\ln\left(\frac{4}{3}\right) - s_1e^{\delta_1\rho/R} + \dots + (2 - \sqrt{5})s_2e^{\delta_2\rho/R} + \dots, \\
 V &= -\frac{1}{4}\ln\left(\frac{4}{3}\right) - s_1e^{\delta_1\rho/R} + \dots + (-9 + 4\sqrt{5})s_2e^{\delta_2\rho/R} + \dots,
 \end{aligned} \tag{2.18}$$

where  $R = \frac{\sqrt{2}}{3^{3/4}}$ , δ<sub>1</sub> = 2 and δ<sub>2</sub> = -1 + √5. This expansion corresponds to shooting out with two irrelevant operators of the d = 2 IR SCFT of dimension Δ<sub>1</sub> = 4 and Δ<sub>2</sub> = 1 + √5.

Thus, we will obtain supersymmetric domain wall solutions interpolating between a deformation of AdS<sub>5</sub> × T<sup>1,1</sup> in the UV and the AdS<sub>3</sub> × ℝ<sup>2</sup> × S<sup>2</sup> × S<sup>3</sup> solution (2.16) in the IR, provided we can solve the differential equations (2.13) (with λ = 2), subject to the boundary conditions (2.17), (2.18). Using a numerical two-sided shooting method we were



**Figure 1.** Supersymmetric domain wall solutions interpolating between  $\text{AdS}_5 \times T^{1,1}$  and  $\text{AdS}_3 \times \mathbb{R}^2 \times S^2 \times S^3$  given by (2.9). From top to bottom in the figure, from the left, we have plotted the functions  $A'$  (green),  $U$  (red),  $B'$  (blue), and  $V$  (yellow).

able to match the two expansions provided that the constants take the values

$$c_1 = 0.105\dots, \quad a_0 = -0.130\dots, \quad s_1 = -0.210\dots, \quad s_2 = 0.480\dots \quad (2.19)$$

In figure 1 we have plotted the behaviour of the functions appearing in the supersymmetric domain wall solution.

### 2.3 $D = 5$ perspective and dual SCFT interpretation

The  $\text{AdS}_5 \times T^{1,1}$  solution is dual to an  $N = 1$   $d = 4$  SCFT described in [18]. The global symmetry is  $\text{SU}(2) \times \text{SU}(2) \times \text{U}(1)_R \times \text{U}(1)_B$  where  $\text{U}(1)_R$  is the  $R$ -symmetry and  $\text{U}(1)_B$  is the baryonic symmetry. The field content includes two gauge superfields,  $W_1$  and  $W_2$ , corresponding to the  $\text{SU}(N) \times \text{SU}(N)$  gauge group, as well as bi-fundamental chiral fields. The UV expansion given in (2.17) corresponds to deformations and expectation values of various operators in the dual SCFT, whose precise details require a careful treatment of holographic renormalisation. However, it is not difficult to extract the main features.

We first observe that the domain wall flow solutions we have constructed are actually contained within an  $D = 5$   $N = 4$  gauged supergravity theory arising from a consistent truncation of the Kaluza-Klein reduction on  $T^{1,1}$ . Recall that there is a consistent KK truncation of type IIB on a generic five-dimensional Sasaki-Einstein space to a  $D = 5$   $N = 4$  gauged supergravity with two  $N = 4$  vector multiplets [12–17]. Expanding about the  $\text{AdS}_5 \times T^{1,1}$  vacuum these fields give rise to  $\text{SU}(2, 2|1)$  multiplets, consisting of the gravity multiplet, a hypermultiplet, a massive gravitino multiplet and a massive vector multiplet. For the special case when  $\text{SE}_5 = T^{1,1}$ , the  $D = 5$   $N = 4$  gauged supergravity has an extra  $N = 4$  “Betti” vector multiplet [9, 10] (supersymmetry was discussed in [11]). Expanding about the  $\text{AdS}_5 \times T^{1,1}$  vacuum, the latter gives rise to a massless Betti vector multiplet, corresponding to the baryonic symmetry, as well as a Betti hypermultiplet. In fact the solutions that we have just constructed are actually solutions of a further truncation to an  $D = 5$   $N = 2$  gauged supergravity theory in which one discards the fields associated with the massive gravitino multiplet and also the massless Betti vector multiplet (while keeping the Betti hypermultiplet) [9].



A key feature of the UV expansion (2.17) that is driving the supersymmetric flow, is that two  $D = 5$  “axion” scalar fields (labelled by  $b^\Phi, c^\Phi$  in [9] and  $e_0^1, e_0^2$  in [11]) are equal to  $-\lambda x^1, \lambda x^2$ , respectively. These axions lie in the Betti hypermultiplet, which is identified with  $\text{Tr}(W_1^2 - W_2^2)$  in the dual SCFT [9, 41], and are dual to marginal operators with dimension  $\Delta = 4$ . The expansion (2.17) also has a free integration constant  $c_1$  which corresponds to an operator of dimension  $\Delta = 6$  acquiring an expectation value. This scalar operator lies in the massive vector multiplet, which is identified with  $\text{Tr}(W_1^2 \bar{W}_1^2 + W_2^2 \bar{W}_2^2) + \dots$  in the dual SCFT [9, 41].

It is worth highlighting the novelty of using  $D = 5$  axions in our construction. Indeed the standard way of obtaining an  $\text{AdS}_3 \times \mathbb{R}^2$  solution utilises massless  $D = 5$  vector fields carrying magnetic charges, with field strength proportional to  $\text{vol}(\mathbb{R}^2)$  [24, 25]. By contrast in the solutions that we have constructed both the R-symmetry vector field and the Betti vector field actually vanish identically.<sup>4</sup> The way in which the  $D = 5$  supersymmetry is being preserved for the solutions can be easily obtained using the results of [11].<sup>5</sup>

Finally, we note that in [42] a supersymmetric  $\text{AdS}_3 \times \mathbb{R}$  solution of  $N = 4$   $D = 4$  gauged supergravity was constructed in which the  $D = 4$  axion field is linear in the coordinate on  $\mathbb{R}$ . This solution was further discussed in [1]. It would be interesting to investigate the dimensional reduction of the  $D = 5$  gauged supergravity of [9, 10] on a circle to  $D = 4$  and the relationship between the two  $\text{AdS}_3$  solutions.

## 2.4 Flux quantisation and central charge

In this section we analyse the flux quantisation for the supersymmetric domain wall solutions that we just constructed, assuming that we have compactified the two spatial directions labelled by  $x_i$ . The flux quantisation involves Page charges and is somewhat subtle; some additional details are presented in appendix C. We will also obtain the central charge of the  $d = 2$  SCFT dual to the IR  $\text{AdS}_3$  solution.

Note that so far we have been working in the Einstein frame with  $\phi = 0$ . In this subsection, we view the metric as being in the string frame and, furthermore, we redefine our R-R fields  $F \rightarrow g_s F$  so that we are using similar conventions to [6] although we will not set  $2\pi l_s = 1$  as they do.

We begin by assuming that the  $x_i$  have period  $2\pi d_i$ ,

$$x_i = x_i + 2\pi d_i, \tag{2.20}$$

and parametrise a  $T^2$ . The topology of internal space is then  $T^2 \times S^2 \times S^3$ . The  $S^2 \times S^3$  is realised as a circle fibration over an  $S_1^2 \times S_2^2$  base space. A positive orientation on  $S^2 \times S^3$  is given by  $D\psi \wedge \text{vol}_1 \wedge \text{vol}_2$ . A smooth three-manifold,  $S_1^3$ , that can be used to generate

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<sup>4</sup>In section 3 we will construct more general solutions, still lying within the consistent truncation to  $N = 4$   $D = 5$  gauged supergravity of [9, 10] but not the further truncation to  $N = 2$  of [9], which have similar structure for the axion fields and, amongst other features, also have a magnetic field for the Betti vector field.

<sup>5</sup>See equation (28)–(31) and especially (33) of [11]. The connection terms in (33) are potentially problematic, but our solutions have  $A = \tau = \mathbb{A} = 0$  and we note that  $d\mathbb{A}$  is related to the linear axions via (15) of [11].

$H_3(S^2 \times S^3, \mathbb{Z})$  is provided by the circle bundle restricted to the  $S_1^2$  factor on the base space. We can also choose  $S_2^3$ , defined to be the circle bundle restricted to the  $S_2^2$  factor on the base space, with opposite orientation. To find a smooth manifold that can be used to generate  $H_2(S^2 \times S^3, \mathbb{Z})$  we consider any smooth manifold  $S$  on the base that represents the cycle  $[S] = [S_2^2] - [S_1^2]$ . Since the circle bundle is trivial over  $S$ , there is a section  $s$  and we can use  $s(S)$  to generate  $H_2(S^2 \times S^3, \mathbb{Z})$ . A more detailed discussion is presented in appendix C; here we record the values of the following integrals:

$$\begin{aligned} \int_{[S^3]} D\psi \wedge (\text{vol}_1 - \text{vol}_2) &= \int_{S_1^3} D\psi \wedge (\text{vol}_1) = \int_{S_2^3} D\psi \wedge (-\text{vol}_2) = 16\pi^2, \\ - \int_{s(S)} \text{vol}_1 &= \int_{s(S)} \text{vol}_2 = 4\pi. \end{aligned} \tag{2.21}$$

### 2.4.1 Flux quantisation

We begin with the three-form flux quantisation. We have

$$\begin{aligned} H &= dB = -\frac{L^2}{3} dx_1 \wedge (\text{vol}_1 - \text{vol}_2), \\ dC &= \frac{L^2}{3g_s} dx_2 \wedge (\text{vol}_1 - \text{vol}_2), \end{aligned} \tag{2.22}$$

and we note that both of these are globally defined and closed three-forms. We then demand that

$$\begin{aligned} \frac{1}{(2\pi l_s)^2} \int_{S_1^1 \times s(S)} H &= \frac{4d_1}{3} \left(\frac{L}{l_s}\right)^2 = Q_{N5} \in \mathbb{Z}, \\ \frac{1}{(2\pi l_s)^2} \int_{S_2^1 \times s(S)} dC^{(2)} &= -\frac{4d_2}{3g_s} \left(\frac{L}{l_s}\right)^2 = Q_{D5} \in \mathbb{Z}. \end{aligned} \tag{2.23}$$

Now we turn to the five-form. The relevant terms are

$$F = \frac{L^4}{27g_s} [D\psi \wedge \text{vol}_1 \wedge \text{vol}_2 + 3D\psi \wedge dx_1 \wedge dx_2 \wedge (\text{vol}_1 + \text{vol}_2)] + \dots, \tag{2.24}$$

which is globally defined since the one-form  $D\psi$  is. Recall that the Bianchi identity for the five-form is given by  $dF - H \wedge dC = 0$ . We will demand that a corresponding Page charge should be quantised. Specifically we demand that

$$\frac{1}{(2\pi l_s)^4} \int_{\Sigma_5} (F - B \wedge dC) \in \mathbb{Z}, \tag{2.25}$$

for any five-cycle  $\Sigma_5$ . As we will see there are some subtleties in imposing this condition. Furthermore, as will be clear from the subsequent discussion, the subtleties are not removed by having the Page charges defined by integrating, instead, the five-form  $F + C \wedge dB$ , for example.

There are two five-cycles to consider. For  $\Sigma_5 = S^2 \times S^3$  the gauge-dependent terms involving the two-form  $B$  do not contribute and we find

$$N \equiv \left(\frac{L}{l_s}\right)^4 \frac{\text{vol}(T^{1,1})}{g_s 4\pi^4} \in \mathbb{Z}, \tag{2.26}$$

where<sup>6</sup>  $\text{vol}(T^{1,1}) = 16\pi^3/27$ .

The delicate case to consider is the five-cycle that is the product of the  $T^2$  with the generator of  $H_3(S^2 \times S^3)$ . Recall that for the latter we can consider  $S_1^3$  which is the circle bundle over  $S_1^2$  at any fixed point on  $S_2^2$ . We can also consider  $S_2^3$  which is the circle bundle over  $S_2^2$  at any fixed point on  $S_1^2$ , but with opposite orientation. We first calculate that

$$\begin{aligned} \frac{1}{(2\pi l_s)^4} \int_{T^2 \times S_1^3} F &= \left(\frac{L}{l_s}\right)^4 \frac{4d_1 d_2}{9g_s}, \\ \frac{1}{(2\pi l_s)^4} \int_{T^2 \times S_2^3} F &= -\left(\frac{L}{l_s}\right)^4 \frac{4d_1 d_2}{9g_s}. \end{aligned} \tag{2.27}$$

These differ because  $F$  is not closed and hence does not define a cohomology class.

We now need to consider a suitable gauge for the two-form  $B$ . It does not seem possible to find a single gauge-choice for  $B$  that is well defined as a two-form for an arbitrary three manifold representing  $H_3(S^2 \times S^3)$ . However, it is possible to find a gauge for a specific representative. In particular, if we integrate over  $S_1^3$  we can choose the gauge  $B^{(1)} = \frac{L^2}{3} dx_1 \wedge (d\psi + P_1 - P_2)$ , where  $dP_i = \text{vol}_i$ , while if we integrate over  $S_2^3$  we can choose a different gauge  $B^{(2)} = \frac{L^2}{3} dx_1 \wedge (-d\psi + P_1 - P_2)$ . We will discuss this more carefully below. We then calculate

$$\begin{aligned} \frac{1}{(2\pi l_s)^4} \int_{T^2 \times S_1^3} -B^{(1)} \wedge dC &= \left(\frac{L}{l_s}\right)^4 \frac{4d_1 d_2}{9g_s}, \\ \frac{1}{(2\pi l_s)^4} \int_{T^2 \times S_2^3} -B^{(2)} \wedge dC &= -\left(\frac{L}{l_s}\right)^4 \frac{4d_1 d_2}{9g_s}. \end{aligned} \tag{2.28}$$

The quantisation condition that we will impose is given by

$$\bar{N} \equiv \left(\frac{L}{l_s}\right)^4 \frac{8d_1 d_2}{9g_s} \in \mathbb{Z}. \tag{2.29}$$

With this condition we see that the Page charge (2.25) when  $\Sigma_5 = T^2 \times S_1^3$  and  $\Sigma_5 = T^2 \times S_2^3$ , with the gauge-choices for  $B$  given above, are equal to  $\bar{N}$  and  $-\bar{N}$ , respectively. While these are quantised, one might be concerned that they are not equal given the two choices of  $\Sigma_5$  are homologous and that the integrand is closed. The key point is that the integrand is not a differential form since it changes under gauge-transformations, which we make precise below.

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<sup>6</sup>Recall that the central charge of the  $d = 4$  SCFT dual to  $\text{AdS}_5 \times T^{1,1}$  is given by  $a = (N^2/4)\pi^3/(\text{vol}(T^{1,1}))$ .

Note that the condition (2.29) is equivalent to the statement that the product of the three-form fluxes is constrained to be an even number:

$$2\bar{N} = -Q_{N_5}Q_{D_5}. \quad (2.30)$$

Let us now elaborate a little on the gauge choices for  $B$  that we made above. We first introduce four coordinate patches  $U_{NN}, U_{NS}, U_{SN}, U_{SS}$ , each isomorphic to  $\mathbb{R}^4 \times S^1$ , to cover  $S^2 \times S^3$ . We take  $U_{NN}$  to consist of the northern hemispheres of the two  $S^2$ 's on the base as well as a coordinate  $\psi_{NN}$  with period  $4\pi$ . Next,  $U_{NS}$  is the northern hemisphere of  $S_1^2$  and the southern hemisphere of  $S_2^2$  on the base, as well as a coordinate  $\psi_{NS}$  with period  $4\pi$ , and similarly for the rest. Now we know that the one-form  $D\psi \equiv d\psi + P$  is globally defined and we have

$$\begin{aligned} D\psi &= d\psi_{NN} + (1 - \cos \theta_1)d\phi_1 + (1 - \cos \theta_2)d\phi_2, \\ &= d\psi_{NS} + (1 - \cos \theta_1)d\phi_1 + (-1 - \cos \theta_2)d\phi_2, \\ &= d\psi_{SN} + (-1 - \cos \theta_1)d\phi_1 + (1 - \cos \theta_2)d\phi_2, \\ &= d\psi_{SS} + (-1 - \cos \theta_1)d\phi_1 + (-1 - \cos \theta_2)d\phi_2. \end{aligned} \quad (2.31)$$

On the overlaps of the patches we have

$$\psi_{NN} = \psi_{NS} - 2\phi_2 = \psi_{SN} - 2\phi_1 = \psi_{SS} - 2\phi_1 - 2\phi_2, \quad (2.32)$$

which shows that we have a good circle bundle: e.g.  $\psi_{NN}/2 = \psi_{NS}/2 + ie^{-i\phi_2}d(e^{i\phi_2})$  (and we note that the factors of  $1/2$  are present because  $\psi$  has period  $4\pi$ ).

For the five manifold  $T^2 \times S^2 \times S^3$  we can consider four coordinate patches, isomorphic to  $T^2 \times \mathbb{R}^4 \times S^1$ , labelled in the same way. In particular, as we will see, the  $T^2$  essentially just comes along for the ride. Now we consider the gauge for the two-form  $B$  given in the  $NN$  patch by

$$B^{(1)} = \frac{L^2}{3}dx_1 \wedge (d\psi_{NN} + (1 - \cos \theta_1)d\phi_1 - (1 - \cos \theta_2)d\phi_2), \quad (2.33)$$

which is clearly well defined in  $U_{NN}$ . We see that it is also well defined in  $U_{SN}$ , after using (2.32). Thus, it makes sense to integrate this over  $S_1^3$  which lies in the union of these two patches,<sup>7</sup> giving the result in (2.28). Observe that if we instead move to  $U_{NS}$  then we have

$$B^{(1)} = \frac{L^2}{3}dx_1 \wedge (d\psi_{NS} + (1 - \cos \theta_1)d\phi_1 + (1 + \cos \theta_2)d\phi_2) - \frac{4L^2}{3}dx_1 \wedge d\phi_2. \quad (2.34)$$

Now the first term on the right hand side is well defined in this patch, but the last term isn't. However, moving to this patch we can employ a gauge-transformation on the two-form given by

$$\delta B = \frac{4L^2}{3}dx_1 \wedge d\phi_2. \quad (2.35)$$

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<sup>7</sup>Observe that we have defined  $S_1^3$  here to be sitting at a fixed point on the northern hemisphere of the second two-sphere.

To see this is well defined, we recall that the definition of the integrality of the three-form curvature  $\bar{H}$  of a gerbe<sup>8</sup> connection (or “curving”)  $\bar{B}$  is given by  $\frac{1}{2\pi} \int \bar{H} \in \mathbb{Z}$ , and so we should absorb a factor of  $2\pi l_s^2$  in  $B$  and  $H$  and consider  $\frac{1}{2\pi l_s^2} \delta B$ . Using the flux quantisation condition (2.23) we find

$$\begin{aligned} \frac{1}{2\pi l_s^2} \delta B &= \frac{Q_{N_5}}{2\pi d_1} dx_1 \wedge d\phi_2, \\ &= -\frac{dx_1}{2\pi d_1} \wedge (ie^{-iQ_{N_5}\phi_2} de^{iQ_{N_5}\phi_2}), \end{aligned} \tag{2.36}$$

which is indeed a bona-fide gauge-transformation for the gerbe. Thus we have shown that  $B^{(1)}$  patches together to properly define a connection for the gerbe with curvature  $H$ , and furthermore  $B^{(1)}$  gives a well defined two-form on  $S_1^3$  and hence can be integrated over it.

Similarly, if we consider the gauge for the two-form  $B$  given by

$$B^{(2)} = \frac{L^2}{3g_s} dx_1 \wedge (-d\psi_{NN} + (1 - \cos\theta_1)d\phi_1 - (1 - \cos\theta_2)d\phi_2), \tag{2.37}$$

we see that it is well defined in  $U_{NN}$  and also on  $U_{NS}$ . Hence this is something that can be integrated on the manifold  $S_2^3$  (sitting at a point in the northern hemisphere of the first two sphere) leading to the result given in (2.28). To see that this is a well-defined gerbe connection we can calculate the difference between  $B^{(1)}$  and  $B^{(2)}$  on, say, the  $NN$  patch. We find

$$\frac{1}{2\pi l_s^2} (B^{(1)} - B^{(2)}) = \frac{dx_1}{2\pi d_1} \wedge (ie^{iQ_{N_5}\psi_{NN}/2} de^{-iQ_{N_5}\psi_{NN}/2}), \tag{2.38}$$

which is a good gauge-transformation since  $\psi_{NN}$  has period  $4\pi$ .

### 2.4.2 Central charge

These flux quantisation conditions we have just derived are valid for the entire domain wall flow solution. We can also calculate the central charge of the  $d = 2$  (0, 2) SCFT that is dual to the  $\text{AdS}_3$  solution (2.15). We use the standard formula

$$c = \frac{3R_{\text{AdS}_3}}{2G_3}, \tag{2.39}$$

where  $R_{\text{AdS}_3}$  is the  $\text{AdS}_3$  radius and  $G_3$  is the effective 3d Newton’s constant. Our  $D = 10$  Lagrangian in the string frame is of the form

$$\frac{1}{(2\pi)^7 g_s^2 l_s^8} \sqrt{-g} e^{-2\phi} R + \dots, \tag{2.40}$$

and a calculation leads to

$$\begin{aligned} c &= \frac{3}{2} |NQ_{N_5} Q_{D5}|, \\ &= 3|N\bar{N}|, \end{aligned} \tag{2.41}$$

where the second expression arises from (2.30).

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<sup>8</sup>As we will see in appendix C the essential aspects of these arguments don’t really involve gerbes but more familiar gauge-connections.

## 2.5 T-duality and enhanced supersymmetry

It was pointed out in [8] that, locally, the  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  IR solution (2.16) is related, after two T-dualities on the  $T^2$ , to the well known  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  solution of type IIB that is dual to a  $d = 2$  SCFT with large  $(4, 4)$  supersymmetry [1] (see [6] for a detailed discussion). To see this we carry out two T-dualities along the directions  $z_1, z_2$  (as in (2.16)) using, for example, the formulae in appendix B of [8]. After then introducing rescaled coordinates

$$\bar{z}_1 = \frac{2^{1/2} 3^{3/2}}{L^2} z_1, \quad \bar{z}_2 = \frac{3^{3/2}}{2^{1/2} L^2} z_2, \quad (2.42)$$

and defining

$$\alpha_1 = \frac{1}{2}(\psi - \bar{z}_1), \quad \alpha_2 = \frac{1}{2}(\psi + \bar{z}_1), \quad (2.43)$$

we obtain

$$\begin{aligned} ds^2 &= \frac{2L^2}{3^{3/2}} [ds^2(\text{AdS}_3) + 2ds^2(S_1^3) + 2ds^2(S_2^3) + d\bar{z}_2^2], \\ dC^2 &= e^{-\phi_0} \frac{2L^2}{3^{3/2}} [2 \text{vol}(\text{AdS}_3) + 4 \text{vol}(S_1^3) + 4 \text{vol}(S_2^3)], \\ e^{\phi_0} &= \frac{3^{3/2}}{L^2}, \end{aligned} \quad (2.44)$$

where

$$ds^2(S_i^3) = \frac{1}{4} [ds_i^2 + (d\alpha_i + P_i)^2]. \quad (2.45)$$

If  $\alpha_i$  are periodic coordinates with period  $4\pi$  then (2.45) is the metric on a round, unit radius three-sphere and we have the standard  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  solution of type IIB supergravity, which we briefly review in appendix A.

Observe that (2.43) implies that

$$\partial_\psi = \frac{1}{2}(\partial_{\alpha_1} + \partial_{\alpha_2}), \quad \partial_{\bar{z}_1} = \frac{1}{2}(-\partial_{\alpha_1} + \partial_{\alpha_2}). \quad (2.46)$$

Thus, locally, starting with the  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  solution we can obtain the  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  solution by carrying out a T-duality on the  $S^1$  factor, generated by  $\partial_{\bar{z}}$ , and on the diagonal of the two  $U(1)$  Hopf fibres generated by  $\frac{1}{2}(-\partial_{\alpha_1} + \partial_{\alpha_2})$ . Recalling that the  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  solution preserves 16 supersymmetries (8 Poincaré and 8 superconformal), this suggests that the  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  solution will preserve more than the obvious 4 supersymmetries. Indeed the explicit Killing spinors for the  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  solution, in a  $D = 11$  incarnation, were constructed in [3] and the corresponding superisometry algebra was found. Using the arguments in section 7 of [3] one can determine the Killing spinors which are left invariant under the action of the Lie derivative with respect to the Killing vector generating the diagonal  $U(1)$  on the  $S^3 \times S^3$  factor. We find that this action preserves all eight Killing spinors given by equation (48) of [3] and four of the eight given by equation (47) of [3] (in particular satisfying the projection  $(1 + \Gamma^{121'2'})\varepsilon = 0$  in the notation of that paper). We have verified this counting by a direct construction of the Killing spinors for the type IIB solutions in appendix B.

Using the results of [3] we can also deduce the superisometry algebra. It will be of the form  $D(2,1|\alpha) \times G$ , with  $G \subset D(2,1|\alpha)$  and the two factors having bosonic sub-algebras given by  $SL(2) \times SU(2) \times SU(2)$  and  $SL(2) \times U(1)^2$ , respectively.

A more careful examination of the global aspects of the T-duality will be left to future work. Note that the relevant  $AdS_3 \times S^3 \times S^3 \times S^1$  solution is fixed by two integers,  $Q_{D1}, Q_{D5}$  and the central charge is given by  $c = 3Q_{D1}Q_{D5}$ , which can be compared with the second expression in (2.41). However, the first expression in (2.41) suggests that orbifolds of  $AdS_3 \times S^3 \times S^3 \times S^1$  might need to be considered.

### 3 A more general class of flows

In this section we construct a more general class of flows interpolating between  $AdS_5 \times T^{1,1}$  and a one-parameter family of  $AdS_3 \times \mathbb{R}^2 \times S^2 \times S^3$  solutions found in [8]. The flows again preserve (0, 2) supersymmetry and we show in appendix B that the  $AdS_3$  fixed point solutions preserve (4, 2) superconformal symmetry.

Specifically, here we consider an ansatz for the type IIB fields given by

$$\begin{aligned}
 \frac{1}{L^2} ds^2 &= e^{2A} (-dt^2 + dx^2) + e^{2B} (dx_1^2 + dx_2^2) + d\rho^2 + \frac{1}{6} (e^{2U_1} ds_1^2 + e^{2U_2} ds_2^2) + e^{2V} \eta^2, \\
 \frac{1}{L^4} F &= 4e^{2A+2B-V-2U_1-2U_2} dt \wedge dx \wedge dx_1 \wedge dx_2 \wedge d\rho + \frac{1}{9} \eta \wedge \text{vol}_1 \wedge \text{vol}_2 \\
 &\quad + dx_1 \wedge dx_2 \wedge \eta \wedge \left[ \frac{(\lambda^2 - 4f^2)}{12} (\text{vol}_1 + \text{vol}_2) - \frac{\lambda(f + Q\lambda)}{3} (\text{vol}_1 - \text{vol}_2) \right] \\
 &\quad + e^{2A-2B-V} \frac{(\lambda^2 - 4f^2)}{12} dt \wedge dx \wedge d\rho \wedge (e^{2U_2-2U_1} \text{vol}_2 + e^{2U_1-2U_2} \text{vol}_1) \\
 &\quad - e^{2A-2B-V} \frac{\lambda(f + Q\lambda)}{3} dt \wedge dx \wedge d\rho \wedge (e^{2U_2-2U_1} \text{vol}_2 - e^{2U_1-2U_2} \text{vol}_1), \\
 \frac{1}{L^2} G &= (dx_1 - idx_2) \wedge \left( \frac{\lambda}{6} (\text{vol}_1 - \text{vol}_2) + d(f\eta) \right), \\
 &= (dx_1 - idx_2) \wedge \left( \frac{\lambda}{6} (\text{vol}_1 - \text{vol}_2) + f' d\rho \wedge \eta + \frac{f}{3} (\text{vol}_1 + \text{vol}_2) \right), \tag{3.1}
 \end{aligned}$$

with  $A, B, U_1, U_2, V, f$  all functions of  $\rho$ , and  $\lambda, Q$  are constants. The interpretation within the dual  $D = 4$  SCFT will be discussed below. One can check that the five-form is self-dual and that the Bianchi identities (2.1) are satisfied.

We find that if we again write the Killing spinors as  $\varepsilon = e^{A/2} \varepsilon_0$ , demand that they satisfy the projections (2.7), (2.11) and (2.8), then the Killing spinor equations (2.3), (2.4)

lead to the following system of first order differential equations

$$\begin{aligned}
& A' - e^{-V-2U_1-2U_2} + \frac{4f^2 - \lambda^2}{8} e^{-2B-V} (e^{-2U_1} + e^{-2U_2}) \\
& \quad + \frac{\lambda(f + Q\lambda)}{2} e^{-2B-V} (e^{-2U_1} - e^{-2U_2}) = 0, \\
& B' - e^{-V-2U_1-2U_2} - \frac{4f^2 - \lambda^2}{8} e^{-2B-V} (e^{-2U_1} + e^{-2U_2}) \\
& \quad - \frac{\lambda(f + Q\lambda)}{2} e^{-2B-V} (e^{-2U_1} - e^{-2U_2}) = 0, \\
& U_1' - e^{V-2U_1} + e^{-V-2U_1-2U_2} - \frac{4f^2 - \lambda^2}{8} e^{-2B-V} (e^{-2U_1} - e^{-2U_2}) \\
& \quad - \frac{\lambda(f + Q\lambda)}{2} e^{-2B-V} (e^{-2U_1} + e^{-2U_2}) = 0, \\
& U_2' - e^{V-2U_2} + e^{-V-2U_1-2U_2} + \frac{4f^2 - \lambda^2}{8} e^{-2B-V} (e^{-2U_1} - e^{-2U_2}) \\
& \quad + \frac{\lambda(f + Q\lambda)}{2} e^{-2B-V} (e^{-2U_1} + e^{-2U_2}) = 0, \\
& V' - 3e^{-V} + e^{V-2U_1} + e^{V-2U_2} + e^{-V-2U_1-2U_2} - \frac{4f^2 - \lambda^2}{8} e^{-2B-V} (e^{-2U_1} + e^{-2U_2}) \\
& \quad - \frac{\lambda(f + Q\lambda)}{2} e^{-2B-V} (e^{-2U_1} - e^{-2U_2}) = 0, \\
& f' + 2f (e^{V-2U_1} + e^{V-2U_2}) + \lambda (e^{V-2U_1} - e^{V-2U_2}) = 0.
\end{aligned} \tag{3.2}$$

The first five equations come from (2.3) while the sixth comes from (2.4). One can show that these equations imply that the equation of motion for the complex three-form  $G$ , given in (2.2), is satisfied.

Notice that if we set  $U_1 = U_2 = U$  and  $f = Q = 0$  we recover the equations (2.13) that we had in the last section.<sup>9</sup> In particular, the  $\text{AdS}_5 \times T^{1,1}$  solution is obtained via (2.14). The set of equations (3.2) also admits the following one parameter family of  $\text{AdS}_3$  solutions

$$\begin{aligned}
A = \rho/R &\equiv \frac{3^{3/4}}{\sqrt{2}(1-4Q^2)^{1/4}} \rho, & B = b_0 &\equiv \frac{1}{4} \ln \left[ \frac{\lambda^4}{12} (1-4Q^2) \right], \\
U_1 = u_1 &\equiv \frac{1}{4} \ln \left[ \frac{4}{3} \frac{1-2Q}{1+2Q} \right], & U_2 = u_2 &\equiv \frac{1}{4} \ln \left[ \frac{4}{3} \frac{1+2Q}{1-2Q} \right], \\
V = v &\equiv \frac{1}{4} \ln \left[ \frac{3}{4} (1-4Q^2) \right], & f &= -\lambda Q,
\end{aligned} \tag{3.3}$$

with

$$0 \leq Q < 1/2. \tag{3.4}$$

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<sup>9</sup>Observe that if we set  $f = 0$ , with  $\lambda \neq 0$ , we must have  $U_1 = U_2 = U$  and  $Q = 0$ .



After scaling  $x_i = \frac{2^{1/2}}{\lambda^{3/2}(1-4Q^2)^{1/4}} z_i$  the resulting type IIB solution can be written as

$$\begin{aligned}
\frac{1}{L^2} ds^2 &= \frac{(1-4Q^2)^{1/2}}{3^{3/2}} \left( 2 ds^2(\text{AdS}_3) + \frac{1}{(1-4Q^2)^{1/2}} (dz_1^2 + dz_2^2) \right. \\
&\quad \left. + \frac{1}{1+2Q} ds_1^2 + \frac{1}{1-2Q} ds_2^2 + \frac{1}{2} (d\psi + P)^2 \right), \\
\frac{1}{L^4} F &= \frac{1}{27} \left\{ \text{vol}(\text{AdS}_3) [4(1-4Q^2)^{1/2} dz_1 \wedge dz_2 + 2(1-2Q)^2 \text{vol}_1 + 2(1+2Q)^2 \text{vol}_2] \right. \\
&\quad \left. + (d\psi + P) \wedge \left[ \text{vol}_1 \wedge \text{vol}_2 + \frac{(1-4Q^2)^{1/2}}{2} dz_1 \wedge dz_2 \wedge (\text{vol}_1 + \text{vol}_2) \right] \right\}, \\
\frac{1}{L^2} G &= \frac{1}{3^{3/2} 2^{1/2} (1-4Q^2)^{1/4}} (dz_1 - i dz_2) \wedge [(1-2Q) \text{vol}_1 - (1+2Q) \text{vol}_2]. \tag{3.5}
\end{aligned}$$

which is precisely the same one-parameter family of solutions<sup>10</sup> found in section 3.1.2 of [8].

### 3.1 The supersymmetric flows

We now discuss the domain wall solutions that interpolate between  $\text{AdS}_5 \times T^{1,1}$  in the UV and this one-parameter family of  $\text{AdS}_3 \times \mathbb{R}^2 \times S^2 \times S^3$  solutions. There is an expansion about the  $\text{AdS}_5 \times T^{1,1}$  solution that involves three integration constants  $c_i$ , in addition to  $Q$  and the deformation parameter  $\lambda$ . The UV expansion analogous to (2.17) is rather long so we shall not write it out explicitly. The key feature is that as  $\rho \rightarrow \infty$  three integration constants  $c_i$  appear, schematically, as

$$\begin{aligned}
U_1 &= c_2 e^{-2\rho} + \dots + c_1 e^{-6\rho} + \dots \\
U_2 &= -c_2 e^{-2\rho} + \dots + c_1 e^{-6\rho} + \dots \\
V &= \dots - 4c_1 e^{-6\rho} + \dots \\
f &= \dots + c_3 e^{-4\rho} + \dots \tag{3.6}
\end{aligned}$$

The UV expansion that we had before, given in (2.17), is obtained by setting  $c_2 = c_3 = 0$ . We will discuss the holographic interpretation of the  $c_i$  below.

We next develop an expansion about the  $\text{AdS}_3$  solution (3.3) in the IR. We find that as  $\rho \rightarrow -\infty$  it can be constructed from four constants  $s_1, s_2, s_3$  and  $a_0$ :

$$\begin{aligned}
A &= a_0 + \rho/R + \frac{3}{2} s_1 e^{\delta_1 \rho/R} + w_1 s_2 e^{\delta_2 \rho/R} + w_2 s_3 e^{\delta_3 \rho/R} + \dots, \\
B &= b_0 + s_1 e^{\delta_1 \rho/R} + s_2 e^{\delta_2 \rho/R} + s_3 e^{\delta_3 \rho/R} + \dots, \\
U_1 &= u_1 - s_1 e^{\delta_1 \rho/R} - s_2 e^{\delta_2 \rho/R} + w_3 s_3 e^{\delta_3 \rho/R} + \dots, \\
U_2 &= u_2 - s_1 e^{\delta_1 \rho/R} + w_4 s_2 e^{\delta_2 \rho/R} - s_3 e^{\delta_3 \rho/R} + \dots, \\
f &= -\lambda Q + w_5 s_2 e^{\delta_1 \rho/R} + w_6 s_3 e^{\delta_3 \rho/R} + \dots, \\
V &= v - s_1 e^{\delta_1 \rho/R} + w_7 s_2 e^{\delta_2 \rho/R} + w_8 s_3 e^{\delta_3 \rho/R} + \dots, \tag{3.7}
\end{aligned}$$

<sup>10</sup>We should identify  $L^2 = [3^{3/2}(l_1+l_2)^{1/2}/2(l_1 l_2)^{1/2}] L_{\text{there}}^2$ ,  $Q = (l_1 - l_2)/2(l_1 + l_2)$ ,  $(z_1 - i z_2) = (l_1 l_2)^{1/4} u$  and  $\psi = z$ .

where  $\delta_1 = 2$ ,  $\delta_2 = -1 + \sqrt{5 - 8Q}$  and  $\delta_3 = -1 + \sqrt{5 + 8Q}$  and  $w_i$  are functions of  $Q$ . Explicitly we have

$$\begin{aligned}
 w_1 &= \frac{-3 + 8Q + \sqrt{5 - 8Q}}{(4 - 8Q)}, & w_2 &= \frac{-3 - 8Q + \sqrt{5 - 8Q}}{(4 + 8Q)}, \\
 w_3 &= \frac{-5 - 2Q + 2\sqrt{5 - 8Q}}{(-1 + 2Q)}, & w_4 &= \frac{5 - 2Q - 2\sqrt{5 - 8Q}}{(1 + 2Q)}, \\
 w_5 &= -2\lambda(-2 + 2Q + \sqrt{5 - 8Q}), & w_6 &= 2\lambda(-2 - 2Q + \sqrt{5 - 8Q}), \\
 w_7 &= \frac{-9 + 6Q + 4\sqrt{5 - 8Q}}{(1 + 2Q)}, & w_8 &= \frac{9 + 6Q - 4\sqrt{5 - 8Q}}{(-1 + 2Q)}. \tag{3.8}
 \end{aligned}$$

This expansion corresponds to shooting out with three IR irrelevant operators of dimension  $\Delta_1 = 4$ ,  $\Delta_2 = 1 + \sqrt{5 - 8Q}$  and  $\Delta_3 = 1 + \sqrt{5 + 8Q}$ . Observe that if we set  $Q = 0$  and in addition we also set  $s_2 = s_3$  then we recover the expansion (2.18) that we had in the last section. It is worth emphasising that when  $Q = 0$  the enlarged ansatz of this section, with  $U_1 \neq U_2$  and  $f \neq 0$ , leads to an extra irrelevant IR operator parametrised by, say,  $s_2 - s_3$ .

We now set  $\lambda = 2$ . Fixing  $0 \leq Q < 1/2$ , our UV expansion has three integration constants and our IR expansion has four. On the other hand our system of differential equations (3.2) is fixed by six integration constants. Thus, for each value of  $Q$ , including  $Q = 0$ , we expect to have a one parameter family of supersymmetric flows connecting the deformed  $\text{AdS}_5 \times T^{1,1}$  solution with the corresponding  $\text{AdS}_3 \times \mathbb{R}^2 \times S^2 \times S^3$  solution. We have constructed a couple of examples of such flows numerically, including when  $Q = 0$ . The  $Q = 0$  solutions of the last section are distinguished in this family by having  $U_1 = U_2$ , or equivalently  $c_2 = 0$  in the UV expansion (3.6), which is associated with a particular relevant operator with  $\Delta = 2$  being switched off (see below). In the next subsection we will argue that flux quantisation implies a rationality condition on  $Q$ .

The more general ansatz (3.1) that we are using for the supersymmetric flows is again contained within the  $N = 4$   $D = 5$  gauged supergravity obtained from a consistent Kaluza-Klein truncation on  $T^{1,1}$  [9–11]. As before, the two  $D = 5$  axion scalar fields in the Betti hypermultiplet, dual to  $\Delta = 4$  operators, are equal to  $-\lambda x^1$ ,  $\lambda x^2$ , respectively, and this deformation is driving the flow. We next note that the field strength of the massless vector field lying in the Betti vector multiplet (labelled  $d(a_1^\Phi)$  in [9] and  $r_2$  in [11]) is of the form  $\lambda(f + Q\lambda)dx_1 \wedge dx_2$ . This reveals that the  $Q$ -deformation corresponds to switching on a magnetic field for the massless gauge-field dual to the  $\Delta = 3$  current associated with the baryonic  $U(1)$  symmetry. The three constants  $c_i$  in (3.6) are related to various operators in the dual  $d = 4$  SCFT acquiring expectation values, which can be deduced from the results of [9, 11, 14]. The constant  $c_1$  is again associated with the scalar in the massive vector multiplet which is dual to an operator of dimension  $\Delta = 6$ . Similarly, the constant  $c_2$  is associated with the scalar in the Betti vector multiplet (labelled  $w$  in [9]) dual to an operator of dimension  $\Delta = 2$ . Finally, the constant  $c_3$  is associated with the massive one-forms (labelled  $b_1, c_1$  in [9]) appearing in the massive gravitino multiplet, dual to operators with dimension  $\Delta = 5$ .

### 3.2 Flux quantisation and central charge

Repeating the steps in section 2.4 (with  $\lambda = 2$ ) we find that the quantisation of the three-form flux leads to the same results, namely

$$\begin{aligned} Q_{N5} &= \frac{4d_1}{3} \left(\frac{L}{l_s}\right)^2 \in \mathbb{Z}, \\ Q_{D5} &= -\frac{4d_2}{3g_s} \left(\frac{L}{l_s}\right)^2 \in \mathbb{Z}. \end{aligned} \tag{3.9}$$

Similarly, integrating the five-form flux on the five-cycle  $\Sigma_5 = S^2 \times S^3$  implies

$$N \equiv \left(\frac{L}{l_s}\right)^4 \frac{4}{27g_s\pi} \in \mathbb{Z}, \tag{3.10}$$

as before.

The calculation of the Page charge for the five-cycle  $\Sigma_5 = T^2 \times S^3$ , however, exhibits some new features. We follow the same prescription that we deployed in section 2. To carry out the integral (2.25) over  $T^2 \times S_1^3$  and  $T^2 \times S_2^3$  we use the gauge choices:

$$\begin{aligned} B^{(1)} &= \frac{L^2}{3} dx_1 \wedge \left[ (d\psi + P_1 - P_2) + \frac{f}{3} D\psi \right], \\ B^{(2)} &= \frac{L^2}{3} dx_1 \wedge \left[ (-d\psi + P_1 - P_2) + \frac{f}{3} D\psi \right], \end{aligned} \tag{3.11}$$

respectively, and we obtain the two quantisation conditions

$$\begin{aligned} \bar{N}_1 &\equiv \left(\frac{L}{l_s}\right)^4 \frac{d_1 d_2}{9g_s} (8 - 4f^2 - 4f - 16Q) \in \mathbb{Z}, \\ \bar{N}_2 &\equiv -\left(\frac{L}{l_s}\right)^4 \frac{d_1 d_2}{9g_s} (8 - 4f^2 + 4f + 16Q) \in \mathbb{Z}, \end{aligned} \tag{3.12}$$

respectively.

Since  $f$  is a function of  $\rho$  we obviously cannot satisfy (3.12) throughout the whole flow. We can however, demand that the flux is properly quantised at the AdS<sub>5</sub> boundary, where  $f \rightarrow 0$ , and also at the AdS<sub>3</sub> fixed point, where  $f = -2Q$ , for suitable choices of rational  $Q$ . This would place additional constraints on the product  $Q_{N5} Q_{D5}$  generalising (2.30). The Page charge would then change along the radial flow, reminiscent of the flows in [43]. Further exploration of the Page charges will be left for future work.

By following a similar calculation as in section 2.4.2, we find the central charge for the AdS<sub>3</sub> fixed point solutions is given by

$$c = -\frac{3}{2} N Q_{N5} Q_{D5} (1 - 4Q^2). \tag{3.13}$$

### 3.3 T-duality and supersymmetry

Starting with (3.5) we introduce rescaled coordinates

$$\bar{z}_1 = \frac{2^{1/2} 3^{3/2} (1 - 4Q^2)^{1/4}}{L^2} z_1, \quad \bar{z}_2 = \frac{3^{3/2}}{2^{1/2} L^2 (1 - 4Q^2)^{1/4}} z_2, \tag{3.14}$$

and then carry out two T-dualities along the directions  $\bar{z}_1, \bar{z}_2$  using, for example, the formulae in appendix B of [8]. Making the further change of ordinates

$$\alpha_1 = \frac{1}{2}((1 + 2Q)\psi - \bar{z}_1), \quad \alpha_2 = \frac{1}{2}((1 - 2Q)\psi + \bar{z}_1), \quad (3.15)$$

we obtain

$$\begin{aligned} ds^2 &= \frac{2L^2(1 - 4Q^2)^{1/2}}{3^{3/2}} \left[ ds^2(\text{AdS}_3) + \frac{2}{1 + 2Q} ds^2(S_1^3) + \frac{2}{1 - 2Q} ds^2(S_2^3) + d\bar{z}_2^2 \right], \\ dC^2 &= e^{-\phi_0} \frac{2L^2(1 - 4Q^2)^{1/2}}{3^{3/2}} \left[ 2 \text{vol}(\text{AdS}_3) + \frac{4}{1 + 2Q} \text{vol}(S_1^3) + \frac{4}{1 - 2Q} \text{vol}(S_2^3) \right], \\ e^{\phi_0} &= \frac{3^{3/2}}{L^2}, \end{aligned} \quad (3.16)$$

where, as before,  $ds^2(S_i^3) = \frac{1}{4}[ds_i^2 + (d\alpha_i + P_i)^2]$ . When  $\alpha_i$  have period  $4\pi$  this is the general  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  type IIB solution reviewed in appendix A.

Observe that we again have  $\partial_{\bar{z}_1} = \frac{1}{2}(-\partial_{\alpha_1} + \partial_{\alpha_2})$  and hence following the same arguments as in section 2.5, we can conclude that the general  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  solutions (3.5) should preserve (4, 2) supersymmetry. This is confirmed in appendix B.

#### 4 Final comments

We have constructed a novel class of type IIB supergravity solutions, preserving (0, 2) supersymmetry, that interpolate between  $\text{AdS}_5 \times T^{1,1}$  in the UV and a class of  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  solutions in the IR. The IR solutions preserve (4, 2) superconformal supersymmetry and are related, locally, by two T-dualities to the well known  $\text{AdS}_3 \times S^3 \times S^3 \times S^1$  solutions. It would be interesting to establish in more detail how this T-duality works globally. We examined the quantisation of Page charges for the  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  solutions, finding some novel features. In particular, it does not seem possible to have the connection two-form  $B$  of the gerbe be well defined as a two-form on an arbitrary five-manifold, representing the homologically non-trivial five cycles, on which one wants to integrate in order to get the Page charge. However, for specific choices of the five-manifolds, we can find an appropriate connection, related by gauge transformations, so that it is well defined. Furthermore, we found that a placing a constraint on the three-form fluxes ensured that the Page charges obtained in different gauges were all integers. It would be helpful to investigate this in more detail as similar issues will arise in other supergravity solutions with fluxes. One approach, in the present setting, is to try and make a precise connection with the globally realised T-duality.

It is known that classical type IIB string theory on  $\text{AdS}_5 \times T^{1,1}$  is not integrable [44]. However, it seems likely that it will be integrable on the  $\text{AdS}_3 \times T^2 \times S^2 \times S^3$  solutions we have discussed here (for related discussion see [45–47]). It would be interesting to confirm this and also to investigate how the integrability emerges along the RG flow.

Another direction for further study would be to investigate whether the solutions that we have constructed here can be generalised to solutions that flow from more general

AdS<sub>5</sub> × SE<sub>5</sub> solutions, where SE<sub>5</sub> is a five-dimensional Sasaki-Einstein solution. It is reasonable to expect that if we choose SE<sub>5</sub> to be one of the  $Y^{p,q}$  spaces [48] then there will be flows to the various AdS<sub>3</sub> × T<sup>2</sup> solutions found in section 4 of [8] (global aspects of the T-dual solutions are discussed in [49]). These flow solutions might be difficult to construct explicitly, however, because unlike the case we have considered in this paper, there is not a known consistent KK truncation on  $Y^{p,q}$  analogous to the one on T<sup>1,1</sup>. These solutions would relate four dimensional superconformal field theories to two-dimensional superconformal field theories with (0, 2) supersymmetry, complementing other such examples [19, 21–25, 50–53].

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## A The AdS<sub>3</sub> × S<sup>3</sup> × S<sup>3</sup> × S<sup>1</sup> solution

Consider the standard AdS<sub>3</sub> × S<sup>3</sup> × S<sup>3</sup> × S<sup>1</sup> solution [1] (see also [6]) which is supported by RR fluxes. In the conventions of section 2.4 it can be written

$$\begin{aligned} ds^2 &= \bar{L}^2 [ds^2(\text{AdS}_3) + r_1^2 ds^2(S_1^3) + r_2^2 ds^2(S_2^3) + dy^2], \\ dC^2 &= \frac{1}{\bar{g}} \bar{L}^2 2 [\text{vol}(\text{AdS}_3) + r_1^2 \text{vol}(S_1^3) + r_2^2 \text{vol}(S_2^3)], \\ e^\phi &= 1, \end{aligned} \tag{A.1}$$

where  $ds^2(S_i^3)$  are the standard round metrics on three-spheres,  $y \cong y + \Delta y$  and, in the notation of (3.16),

$$r_1^2 = \frac{2}{1 + 2Q}, \quad r_2^2 = \frac{2}{1 - 2Q}, \tag{A.2}$$

with  $0 \leq Q < 1/2$ . Observe that  $r_1^2 + r_2^2 = r_1^2 r_2^2$ . We next quantise the flux. For the electric flux we have

$$Q_{D1} = \frac{1}{(2\pi l_s)^6} \int_{S_1^3 \times S_2^3 \times S^1} *dC^2 = \left(\frac{\bar{L}}{l_s}\right)^6 \frac{r_1^3 r_2^3 \Delta y}{8\bar{g}\pi^2} \in \mathbb{Z}. \tag{A.3}$$

For the magnetic flux we have

$$\begin{aligned} Q_{D5^{(1)}} &= \frac{1}{(2\pi l_s)^2} \int_{S_1^3} dC^2 = \left(\frac{\bar{L}}{l_s}\right)^2 \frac{r_1^2}{\bar{g}} \in \mathbb{Z}, \\ Q_{D5^{(2)}} &= \frac{1}{(2\pi l_s)^2} \int_{S_2^3} dC^2 = \left(\frac{\bar{L}}{l_s}\right)^2 \frac{r_2^2}{\bar{g}} \in \mathbb{Z}. \end{aligned} \tag{A.4}$$

Observe that we have

$$4\pi\bar{g}Q_{D1} = \frac{\bar{L}\Delta y}{2\pi l_s} \bar{g}Q_{D5^{(1)}} \bar{g}Q_{D5^{(2)}} \sqrt{\bar{g}Q_{D5^{(1)}} + \bar{g}Q_{D5^{(2)}}}, \quad (\text{A.5})$$

which agrees with (2.17) of [6] (they have set  $2\pi l_s = 1$ ), which shows that the radius of the circle is fixed by the RR charges  $gQ$ . Using (2.39), (2.40) we calculate the central charge as

$$c = 6Q_{D1} \frac{Q_{D5^{(1)}} Q_{D5^{(2)}}}{Q_{D5^{(1)}} + Q_{D5^{(2)}}}, \quad (\text{A.6})$$

again agreeing with (2.20) of [6].

## B Enhanced supersymmetry

In this appendix we show that the  $\text{AdS}_3 \times \mathbb{R}^2 \times S^2 \times S^3$  solutions given in (3.3) have a (4, 2) superconformal supersymmetry. We set  $L^2 = 3^{3/2}/(1 - 4Q^2)^{1/2}$  and use the orthonormal frame

$$\begin{aligned} e^\mu &= \sqrt{2}\bar{e}^\mu, & \mu &= 0, 1, 4, \\ e^2 &= \frac{1}{(1 - 4Q^2)^{1/4}} dz_1, & e^3 &= \frac{1}{(1 - 4Q^2)^{1/4}} dz_2, \\ e^5 &= \frac{1}{(1 + 2Q)^{1/2}} d\theta_1, & e^6 &= \frac{1}{(1 + 2Q)^{1/2}} \sin\theta_1 d\phi_1, \\ e^7 &= \frac{1}{(1 - 2Q)^{1/2}} d\theta_2, & e^8 &= \frac{1}{(1 - 2Q)^{1/2}} \sin\theta_2 d\phi_2 \\ e^9 &= \frac{1}{\sqrt{2}}(d\psi + P), \end{aligned} \quad (\text{B.1})$$

where  $\bar{e}^\mu$  is an orthonormal frame for a unit radius  $\text{AdS}_3$ . From (2.4) we deduce that

$$(1 - i\Gamma^{23})(1 + \Gamma^{5678})\varepsilon = 0. \quad (\text{B.2})$$

We thus write

$$\varepsilon = \varepsilon_1 + \varepsilon_2, \quad (\text{B.3})$$

with

$$\begin{aligned} \Gamma^{5678}\varepsilon_1 &= \varepsilon_1, & \Gamma^{5678}\varepsilon_2 &= -\varepsilon_2, \\ \Gamma^{23}\varepsilon_1 &= -i\varepsilon_1. \end{aligned} \quad (\text{B.4})$$

The Killing spinor equations (2.3) then take the following form. For the  $\mu = 0, 1, 4$  components we have

$$\bar{\nabla}_\mu(\varepsilon_1 + \varepsilon_2) - \frac{(1 - 4Q^2)^{1/2}}{4} \Gamma_\mu \Gamma^{256} \varepsilon_1^c - \frac{i}{4} \Gamma_\mu \Gamma^9 ((1 + 2Q)\Gamma^{2356})\varepsilon_1 - (1 - \Gamma^{2356})\varepsilon_2 = 0, \quad (\text{B.5})$$

where  $\bar{\nabla} \equiv \bar{e}^\mu \bar{\nabla}_\mu$  is the Levi-Civita connection on a unit radius  $\text{AdS}_3$ . For the 2,3 components we get

$$\begin{aligned}
 & (1 - 4Q^2)^{\frac{1}{4}} \partial_{z_1} (\varepsilon_1 + \varepsilon_2) + \frac{(1 - 4Q^2)^{\frac{1}{2}}}{4\sqrt{2}} \Gamma_2 \Gamma^{256} \varepsilon_1^c \\
 & + \frac{i}{4\sqrt{2}} \Gamma_2 \Gamma^9 ((-1 + 2Q \Gamma^{2356}) \varepsilon_1 - (1 + \Gamma^{2356}) \varepsilon_2) = 0, \\
 & (1 - 4Q^2)^{\frac{1}{4}} \partial_{z_2} (\varepsilon_1 + \varepsilon_2) + \frac{(1 - 4Q^2)^{\frac{1}{2}}}{4\sqrt{2}} \Gamma_3 \Gamma^{256} \varepsilon_1^c \\
 & + \frac{i}{4\sqrt{2}} \Gamma_3 \Gamma^9 ((-1 + 2Q \Gamma^{2356}) \varepsilon_1 - (1 + \Gamma^{2356}) \varepsilon_2) = 0. \tag{B.6}
 \end{aligned}$$

For the 5,6 components we get

$$\begin{aligned}
 & \left( (1 + 2Q)^{\frac{1}{2}} \partial_{\theta_1} + \frac{1 + 2Q}{4\sqrt{2}} \Gamma^{69} \right) (\varepsilon_1 + \varepsilon_2) + \frac{(1 - 4Q^2)^{\frac{1}{2}}}{4\sqrt{2}} \Gamma_5 \Gamma^{256} (\varepsilon_1^c + (1 - i\Gamma^{23}) \varepsilon_2^c) \\
 & \quad + \frac{i}{4\sqrt{2}} \Gamma_5 \Gamma^9 ((1 + \Gamma^{2356}) \varepsilon_1 - (1 - 2Q \Gamma^{2356}) \varepsilon_2) = 0, \\
 & \left( \frac{(1 + 2Q)^{\frac{1}{2}}}{\sin \theta_1} \partial_{\phi_1} + \frac{(1 + 2Q)^{\frac{1}{2}}}{2} \cot \theta_1 (2\partial_\psi - \Gamma^{56}) - \frac{1 + 2Q}{4\sqrt{2}} \Gamma^{59} \right) (\varepsilon_1 + \varepsilon_2) \\
 & + \frac{(1 - 4Q^2)^{\frac{1}{2}}}{4\sqrt{2}} \Gamma_6 \Gamma^{256} (\varepsilon_1^c + (1 - i\Gamma^{23}) \varepsilon_2^c) + \frac{i}{4\sqrt{2}} \Gamma_6 \Gamma^9 ((1 + \Gamma^{2356}) \varepsilon_1 - (1 - 2Q \Gamma^{2356}) \varepsilon_2) = 0. \tag{B.7}
 \end{aligned}$$

For the 7,8 components we get

$$\begin{aligned}
 & \left( (1 - 2Q)^{\frac{1}{2}} \partial_{\theta_2} + \frac{1 - 2Q}{4\sqrt{2}} \Gamma^{89} \right) (\varepsilon_1 + \varepsilon_2) + \frac{(1 - 4Q^2)^{\frac{1}{2}}}{4\sqrt{2}} \Gamma_7 \Gamma^{256} (\varepsilon_1^c - (1 - i\Gamma^{23}) \varepsilon_2^c) \\
 & \quad + \frac{i}{4\sqrt{2}} \Gamma_7 \Gamma^9 ((1 - \Gamma^{2356}) \varepsilon_1 - (1 + 2Q \Gamma^{2356}) \varepsilon_2) = 0, \\
 & \left( \frac{(1 - 4Q^2)^{\frac{1}{2}}}{\sin \theta_2} \partial_{\phi_2} + \frac{(1 - 4Q^2)^{\frac{1}{2}}}{2} \cot \theta_2 (2\partial_\psi - \Gamma^{78}) - \frac{1 - 2Q}{4\sqrt{2}} \Gamma^{79} \right) (\varepsilon_1 + \varepsilon_2) \\
 & + \frac{(1 - 4Q^2)^{\frac{1}{2}}}{4\sqrt{2}} \Gamma_8 \Gamma^{256} (\varepsilon_1^c - (1 - i\Gamma^{23}) \varepsilon_2^c) + \frac{i}{4\sqrt{2}} \Gamma_8 \Gamma^9 ((1 - \Gamma^{2356}) \varepsilon_1 - (1 + 2Q \Gamma^{2356}) \varepsilon_2) = 0. \tag{B.8}
 \end{aligned}$$

Finally, for the 9 component we get

$$\begin{aligned}
 & \partial_\psi (\varepsilon_1 + \varepsilon_2) - \frac{1}{4} \Gamma^{56} \varepsilon_2 - \frac{Q}{2} \Gamma^{56} \varepsilon_1 - \frac{(1 - 4Q^2)^{\frac{1}{2}}}{8} \Gamma_9 \Gamma^{256} \varepsilon_1^c \\
 & \quad + \frac{i}{8} ((1 + 2Q \Gamma^{2356}) \varepsilon_1 - (1 - \Gamma^{2356}) \varepsilon_2) = 0. \tag{B.9}
 \end{aligned}$$

By examining the integrability conditions for the two equations involving derivatives with respect to  $z_1, z_2$ , (B.6), we deduce that

$$\Gamma^{2569} \varepsilon_1 = \frac{i}{(1 - 4Q^2)^{\frac{1}{2}}} (1 - 2iQ \Gamma^{56}) \varepsilon_1^c, \tag{B.10}$$

and also that  $\partial_{z_i}\varepsilon_1 = 0$ . Examining (B.9) we also see that  $\partial_\psi\varepsilon_1 = 0$ . From (B.5) we can deduce that

$$\begin{aligned}\bar{\nabla}_\mu\varepsilon_1 - \frac{i}{2}\Gamma_\mu\Gamma^9\varepsilon_1 &= 0, \\ \bar{\nabla}_\mu\varepsilon_2 + \frac{i}{4}\Gamma_\mu\Gamma^9(1 - \Gamma^{2356})\varepsilon_2 &= 0,\end{aligned}\tag{B.11}$$

and the integrability conditions for the second line implies

$$\Gamma^{2356}\varepsilon_2 = -\varepsilon_2.\tag{B.12}$$

As a consequence we conclude from (B.6) that  $\partial_{z_i}\varepsilon_2 = 0$ .

It is now convenient to further decompose

$$\varepsilon_2 = \varepsilon_2^+ + \varepsilon_2^-, \quad i\Gamma^{23}\varepsilon_2^\pm = \pm\varepsilon_2^\pm.\tag{B.13}$$

After projecting out the equations using  $(1 \pm i\Gamma^{23})/2$  and  $(1 \pm i\Gamma^{56})/2$  we deduce the following general solution. Firstly,  $\varepsilon_1$  and  $\varepsilon_2^+$  are given by

$$\begin{aligned}\varepsilon_1 &= \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)[(1-2Q)^{1/2}c_1 - (1+2Q)^{1/2}\Gamma^{29}c_1^c] \\ &\quad + \cos\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)[(1-2Q)^{1/2}c_2 - (1+2Q)^{1/2}\Gamma^{29}c_2^c] \\ &\quad + \sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\Gamma^{68}[(1+2Q)^{1/2}c_2 + (1-2Q)^{1/2}\Gamma^{29}c_2^c] \\ &\quad - \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\Gamma^{68}[(1+2Q)^{1/2}c_1 + (1-2Q)^{1/2}\Gamma^{29}c_1^c],\end{aligned}\tag{B.14}$$

and

$$\begin{aligned}\varepsilon_2^+ &= \sqrt{2}\left[-\cos\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\Gamma^{89}c_1 + \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\Gamma^{89}c_2\right. \\ &\quad \left.+ \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\Gamma^{26}c_2^c + \sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)\Gamma^{26}c_1^c\right],\end{aligned}\tag{B.15}$$

with

$$c_1 = e^{-\frac{i}{2}(\phi_1 - \phi_2)}d_1, \quad c_2 = e^{-\frac{i}{2}(\phi_1 + \phi_2)}d_2,\tag{B.16}$$

and the two ten-dimensional spinors  $d_a$ ,  $a = 1, 2$ , satisfy the projections

$$i\Gamma^{23}d_a = d_a, \quad i\Gamma^{56}d_a = d_a, \quad i\Gamma^{78}d_a = -d_a, \quad a = 1, 2.\tag{B.17}$$

The  $d_a$  only depend on the coordinates on  $\text{AdS}_3$  and should satisfy

$$\left(\bar{\nabla}_\mu - \frac{i}{2}\Gamma_\mu\Gamma^9\right)d_a = 0, \quad a = 1, 2.\tag{B.18}$$

Secondly, the parameter  $\varepsilon_2^-$  takes the simple form

$$\varepsilon_2^- = e^{i\frac{\psi}{2}}d_3,\tag{B.19}$$



with

$$i\Gamma^{23}d_3 = d_3, \quad i\Gamma^{56}d_3 = d_3, \quad i\Gamma^{78}d_3 = d_3. \quad (\text{B.20})$$

Again  $d_3$  only depends on the coordinates on  $\text{AdS}_3$  and now should satisfy

$$\left(\bar{\nabla}_\mu + \frac{i}{2}\Gamma_\mu\Gamma^9\right)d_3 = 0. \quad (\text{B.21})$$

To solve (B.18), (B.21) we use the following frame and coordinates for  $\text{AdS}_3$ :

$$\bar{e}^0 = e^\rho dt, \quad \bar{e}^1 = e^\rho dx, \quad \bar{e}^4 = d\rho, \quad (\text{B.22})$$

to obtain

$$\begin{aligned} d_1 &= e^{\rho/2}\alpha_1^- + [e^{-\rho/2} - e^{\rho/2}(t\Gamma_0 + x\Gamma_1)\Gamma_4]\alpha_1^+, \\ d_2 &= e^{\rho/2}\alpha_2^- + [e^{-\rho/2} - e^{\rho/2}(t\Gamma_0 + x\Gamma_1)\Gamma_4]\alpha_2^+, \\ d_3 &= e^{\rho/2}\alpha_3^+ + [e^{-\rho/2} - e^{\rho/2}(t\Gamma_0 + x\Gamma_1)\Gamma_4]\alpha_3^-, \end{aligned} \quad (\text{B.23})$$

where  $\alpha_1^\pm$ ,  $\alpha_2^\pm$  and  $\alpha_3^\pm$  satisfy the projections (B.17) and (B.20), respectively, and in addition

$$\Gamma^{01}\alpha_a^\pm = \pm\alpha_a^\pm, \quad a = 1, 2, \quad \Gamma^{01}\alpha_3^\pm = \pm\alpha_3^\pm. \quad (\text{B.24})$$

We see that  $\alpha_3^+$  parametrises the  $(0, 2)$  Poincaré supersymmetry that is preserved throughout the whole flow of the domain wall solutions (recall (2.7) and (2.11)). The  $\alpha_a^-$  parametrise an enhancement of the Poincaré supersymmetry to  $(4, 2)$ . The remaining six supersymmetries, labelled by  $\alpha_3^-$  and  $\alpha_a^+$ , parametrise the superconformal supersymmetries.

As noted in the text, using the results of [3] we can deduce that the superisometry algebra is of the form  $D(2, 1|\alpha) \times G$ , where  $G \subset D(2, 1|\alpha)$  and has a bosonic sub-algebra given by  $\text{SL}(2, \mathbb{R}) \times \text{U}(1)^2$ . This could be verified using the explicit Killing spinors that we have constructed, but we shall not do that here.

## C Page charge quantisation

We will discuss the essential aspects of the quantisation of Page charges that we employed in the bulk of the paper in a simplified setting. We suppose that we have a manifold with topology  $S^2 \times S^3$  which is presented as a circle bundle fibred over  $S_1^2 \times S_2^2$  exactly as for  $T^{1,1}$ :

$$ds^2(S^2 \times S^3) = c_1^2 ds_1^2 + c_2^2 ds_2^2 + c_3^2 D\psi^2, \quad (\text{C.1})$$

where  $c_i$  are non-zero constants which won't be important, and

$$\begin{aligned} ds_i^2 &= d\theta_i^2 + \sin^2\theta_i d\phi_i^2, \\ d(D\psi) &= \text{vol}_1 + \text{vol}_2, \\ \text{vol}_i &= \sin\theta_i d\theta_i \wedge d\phi_i, \quad \text{no sum on } i, \end{aligned} \quad (\text{C.2})$$

and  $\psi$  has period  $4\pi$ . We note that  $D\psi$  is a globally defined one-form. A positive orientation on  $S^2 \times S^3$  is given by  $D\psi \wedge \text{vol}_1 \wedge \text{vol}_2$ .

**Topology.** A smooth manifold  $S_1^3$  that can be used to generate  $H_3(S^2 \times S^3, \mathbb{Z})$  is provided by the circle bundle restricted to the  $S_1^2$  factor on the base space. We can also choose the circle bundle restricted to the  $S_2^2$  factor on the base space, with *opposite orientation*, which we call  $S_2^3$ . Observe that  $D\psi \wedge (\text{vol}_1 - \text{vol}_2)$  is closed (since  $d(D\psi) = \text{vol}_1 + \text{vol}_2$ ) and hence when it is integrated over a three-cycle, it will only depend on the homology class of the cycle. We have:

$$\int_{[S^3]} D\psi \wedge (\text{vol}_1 - \text{vol}_2) = \int_{S_1^3} D\psi \wedge (\text{vol}_1) = \int_{S_2^3} D\psi \wedge (-\text{vol}_2) = 16\pi^2. \quad (\text{C.3})$$

To find a smooth manifold that can be used to generate  $H_2(S^2 \times S^3, \mathbb{Z})$  we consider any smooth manifold  $S$  on the base that represents the cycle  $[S] = [S_2^2] - [S_1^2]$ . Since the circle bundle is trivial over  $S$ , there is a section  $s$  and we can use  $s(S)$  to generate  $H_2(S^2 \times S^3, \mathbb{Z})$ . We can take, for example,  $\theta_1 = \theta_2$  and  $-\phi_1 = \phi_2$  at fixed  $\psi$ . In particular we have

$$-\int_{s(S)} \text{vol}_1 = \int_{s(S)} \text{vol}_2 = 4\pi. \quad (\text{C.4})$$

It can be helpful to explicitly identify the Poincaré duals of the above generators. A representative closed 3-form generator,  $\Phi_3$ , of  $H^3(S^2 \times S^3, \mathbb{Z})$  is given by  $\Phi_3 = (1/16\pi^2)D\psi \wedge (\text{vol}_1 - \text{vol}_2)$  with the property that  $\int_E \Phi_3 = 1$ , where  $E$  generates  $H_3(S^2 \times S^3, \mathbb{Z})$ . The three-form  $\Phi_3$  is Poincaré dual to  $[S]$  and we can use it to evaluate  $\int_{s(S)} \omega_2 = \int_{S^2 \times S^3} \omega_2 \wedge \Phi_3$  for any closed two-form  $\omega_2$ . In particular, we can check (C.4). A representative closed 2-form generator,  $\Phi_2$ , of  $H^2(S^2 \times S^3, \mathbb{Z})$  is given by  $\Phi_2 = -\frac{1}{8\pi}(\text{vol}_1 - \text{vol}_2)$  with the property that  $\int_{s(S)} \Phi_2 = 1$ . The two-form  $\Phi_2$  is Poincaré dual to  $[S^3]$  and we can use it to evaluate  $\int_{[S^3]} \omega_3 = \int_{S^2 \times S^3} \omega_3 \wedge \Phi_2$  for any closed three-form  $\omega_3$ . In particular, we can check (C.3). Note that  $\int_{S^2 \times S^3} \Phi_2 \wedge \Phi_3 = 1$ .

**Patches.** Let us introduce four coordinate patches to cover  $S^2 \times S^3$ . We consider four patches  $U_{NN}, U_{NS}, U_{SN}, U_{SS}$ , isomorphic to  $\mathbb{R}^4 \times S^1$ . We take  $U_{NN}$  to consist of the northern hemispheres of the two  $S^2$ 's on the base as well as a coordinate  $\psi_{NN}$  with period  $4\pi$ . Next,  $U_{NS}$  is the northern hemisphere of  $S_1^2$  and the southern hemisphere of  $S_2^2$  on the base, as well as a coordinate  $\psi_{NS}$  with period  $4\pi$ , and similarly for the rest. Now we know that the one-form  $D\psi \equiv d\psi + P$  is globally defined and we have

$$\begin{aligned} D\psi &= d\psi_{NN} + (1 - \cos \theta_1)d\phi_1 + (1 - \cos \theta_2)d\phi_2, \\ &= d\psi_{NS} + (1 - \cos \theta_1)d\phi_1 + (-1 - \cos \theta_2)d\phi_2, \\ &= d\psi_{SN} + (-1 - \cos \theta_1)d\phi_1 + (1 - \cos \theta_2)d\phi_2, \\ &= d\psi_{SS} + (-1 - \cos \theta_1)d\phi_1 + (-1 - \cos \theta_2)d\phi_2. \end{aligned} \quad (\text{C.5})$$

On the overlaps of the patches we have

$$\psi_{NN} = \psi_{NS} - 2\phi_2 = \psi_{SN} - 2\phi_1 = \psi_{SS} - 2\phi_1 - 2\phi_2, \quad (\text{C.6})$$

which shows that we have a good circle bundle, with the patching done with  $U(1)$  gauge-transformations: e.g.  $\psi_{NN}/2 = \psi_{NS}/2 = ie^{-i\phi_2}d(e^{i\phi_2})$  (the factors of two here are because  $\psi$  has period  $4\pi$ ).

**Fluxes and charges.** To illustrate the main features of the calculation in the text, we will consider a slightly simpler problem where we “forget” the  $T^2$  factor. We could imagine that we have carried out a dimensional reduction on the  $T^2$ , for example. The advantage of doing this is that the ambiguities in defining Page charges will just involve gauge-transformations of  $U(1)$  gauge-connections rather than gerbes.

We consider, therefore, the following globally defined fluxes

$$\begin{aligned} F_3 &= \frac{kl}{16} D\psi \wedge (\text{vol}_1 + \text{vol}_2), \\ F_2 &= -\frac{k}{4} (\text{vol}_1 - \text{vol}_2), \\ G_2 &= \frac{l}{4} (\text{vol}_1 - \text{vol}_2), \end{aligned} \tag{C.7}$$

with  $dF_2 = dG_2 = 0$  and  $dF_3 = F_2 \wedge G_2$ . We will assume that  $F_2$  and  $G_2$  are the curvature two-forms for two  $U(1)$  connections with integer Chern numbers. Thus we demand that  $k, l \in \mathbb{Z}$  so that that we have the quantisation conditions:

$$\frac{1}{2\pi} \int_{s(S)} F_2 = k, \quad \frac{1}{2\pi} \int_{s(S)} G_2 = -l. \tag{C.8}$$

If we write  $F_2 = dA_1$ , a natural Page charge to consider quantising is

$$\frac{1}{(2\pi)^2} \int_{[S^3]} (F_3 - A_1 \wedge G_2). \tag{C.9}$$

For definiteness we define  $S_1^3$  and  $S_2^3$  to sit at a fixed point on the northern hemisphere of the other two-sphere. We can calculate

$$\frac{1}{(2\pi)^2} \int_{S_1^3} F_3 = +\frac{kl}{4}, \quad \frac{1}{(2\pi)^2} \int_{S_2^3} F_3 = -\frac{kl}{4}, \tag{C.10}$$

which differ because  $F_3$  is not closed. We now introduce two gauge connections given by

$$\begin{aligned} A_1^{(1)} &= -\frac{k}{4} (d\psi_{NN} + (1 - \cos \theta_1) d\phi_1 - (1 - \cos \theta_2) d\phi_2), \\ A_1^{(2)} &= -\frac{k}{4} (-d\psi_{NN} + (1 - \cos \theta_1) d\phi_1 - (1 - \cos \theta_2) d\phi_2). \end{aligned} \tag{C.11}$$

Being connections with cohomologically non-trivial field strengths, these cannot be globally defined one-forms. However, they should patch together using  $U(1)$  gauge transformations. Let us first consider  $A_1^{(1)}$ . It is clearly defined on the  $NN$  patch. It is also well defined on the  $SN$  patch after using (C.6). Next, moving to the  $NS$  coordinate patch we get something that is well defined up to a  $U(1)$  gauge transformation:

$$\begin{aligned} A_1^{(1)} &= -\frac{k}{4} (d\psi_{NS} + (1 - \cos \theta_1) d\phi_1 + (1 + \cos \theta_2) d\phi_2) + kd\phi_2, \\ &= -\frac{k}{4} (d\psi_{NS} + (1 - \cos \theta_1) d\phi_1 + (1 + \cos \theta_2) d\phi_2) - ie^{-ik\phi_2} d(e^{ik\phi_2}), \end{aligned} \tag{C.12}$$

where we recall that  $\phi_2$  has period  $2\pi$ . Moving to  $SS$  is similar. Thus  $A_1^{(1)}$  is a  $U(1)$  gauge connection for  $F_2$ . Furthermore, we observe that  $A_1^{(1)}$  is a globally well defined one-form on  $S_1^3$  since for a fixed point on the northern hemisphere patch of the  $S_2^2$  we can switch to the  $SN$  patch in a regular manner using (C.6).

Similar comments apply to  $A_1^{(2)}$ . We calculate

$$A_1^{(1)} - A_1^{(2)} = -\frac{k}{2}d\psi_{NN} = -ie^{-ik\psi/2}d(e^{ik\psi/2}), \tag{C.13}$$

which shows that they are related by a good  $U(1)$  gauge transformation, since  $\psi$  has period  $4\pi$ . Thus  $A_1^{(2)}$  is also  $U(1)$  gauge connection for  $F_2$  and, in contrast to  $A_1^{(1)}$ , is now a well defined one-form on  $S_2^3$ .

We can now calculate:

$$\frac{1}{(2\pi)^2} \int_{S_1^3} -A_1^{(1)} \wedge G_2 = +\frac{kl}{4}, \quad \frac{1}{(2\pi)^2} \int_{S_2^3} -A_1^{(2)} \wedge G_2 = -\frac{kl}{4}, \tag{C.14}$$

and hence

$$\frac{1}{(2\pi)^2} \int_{S_1^3} (F_3 - A_1^{(1)} \wedge G_2) = \frac{kl}{2}, \quad \frac{1}{(2\pi)^2} \int_{S_2^3} (F_3 - A_1^{(2)} \wedge G_2) = -\frac{kl}{2}. \tag{C.15}$$

Now  $F_3 - A_1 \wedge G_2$  is closed, so we may naively have thought that these should be equal. However,  $A_1$  is connection, so  $F_3 - A_1 \wedge G_2$  is not a three-form and does not define a cohomology class. If we demand that  $kl = 2\bar{N}$  with  $\bar{N} \in \mathbb{Z}$  then both of these are integers.

In essence this is the flux quantisation procedure that we have adopted in the main text. An open issue, which we leave for the future, is to determine what happens if we choose other smooth three-manifolds  $\Sigma$  to represent  $H_3$ . What are the conditions for there to exist a connection one-form, related to  $A^{(i)}$  by a gauge transformation, which is well defined on  $\Sigma$  and, when it does exist, is the corresponding Page charge always an integer times  $\bar{N}$ ? We believe that similar issues will arise in other contexts.

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